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On dynamics of $(2, 2)$ -rational mapping over $\mathbb{Q}_2$

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Abstract. This paper is devoted to the study of the discrete dynamical systems associated with a $(2, 2)$ -rational mapping $f_{a,b}(x) = \frac{ax^2+bx+1}{x^2+bx+a}$ over the field of 2-adic numbers $\mathbb{Q}_2$, where $a, b \in \mathbb{Q}_2$ are parameters. We fully characterize the sets of fixed and 2-periodic points of the given mapping depending on the 2-adic norms of the parameters. It is shown that the system can exhibit various dynamical behaviors: in the case $|a|_2 = |b|_2 = 1$, the system is regular and all trajectories converge to a unique fixed point; whereas for $|a|_2 = |b|_2 < 1$, the fixed point becomes a repeller and an attracting 2-cycle emerges. Furthermore, for the case $|a|_2 = |b|_2 > 1$, we establish the existence of invariant spheres and provide necessary and sufficient conditions for the ergodicity of the mapping on these spheres with respect to the normalized Haar measure. These results contribute to the classification of p -adic rational dynamics for higher-degree mappings.

Keywords: 2-adic numbers, rational mapping, fixed point, periodic point, trajectory, invariant sphere, ergodicity

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1. INTRODUCTION

The study of dynamical systems over non-Archimedean fields, especially over the field of p -adic numbers $\mathbb{Q}_p$, has become an active direction in modern dynamical systems theory. A systematic account of algebraic and p -adic dynamical systems was developed in the monograph of Anashin and Khrennikov, where ergodicity, measure-preserving transformations, and applications to mathematical physics and computer science are discussed in detail (see [1, 2, 3, 4, 5]). The ultrametric structure of $\mathbb{Q}_p$ makes the behavior of trajectories essentially different from the real and complex cases, since balls are simultaneously open and closed and the geometry of the phase space has a hierarchical nature [6].

A general theory of non-Archimedean dynamics has been extensively developed in the works of Benedetto [7, 8, 9] and others, including Rivera-Letelier [10] and Silverman [11]. These studies investigate periodic points, wandering domains, Fatou and Julia sets, and the role of good reduction for rational maps. In contrast to complex dynamics, non-Archimedean rational dynamics is strongly influenced by arithmetic properties of the coefficients and by the reduction of the map modulo powers of the prime.

Rational functions over $\mathbb{Q}_p$ form one of the central classes of p -adic dynamical systems [12]. Various families of rational maps have been studied from the point of view of fixed points, periodic orbits, Siegel disks, invariant spheres, attractors, and ergodic properties. In particular, Albeverio, Rozikov and Sattarov investigated $(2, 1)$ -rational p -adic dynamical systems, describing fixed points, invariant domains and attraction properties (see [13]). Later, Rozikov and Sattarov studied $(2, 2)$ -rational functions and obtained results on invariant spheres and ergodicity with respect to the Haar measure restricted to such spheres [14].

A substantial part of this theory is motivated by models of p -adic mathematical physics. In particular, rational mappings naturally arise from recurrence equations for Gibbs measures of p -adic Ising, Potts and related models on Cayley trees. Such mappings encode the structure of translation-invariant or periodic boundary laws, and their dynamical properties are closely connected with the existence, uniqueness and phase transition phenomena of corresponding p -adic Gibbs measures (see [15, 16, 17, 18, 19, 20, 21, 22, 23]).

Another important direction concerns ergodicity and minimality of p -adic transformations. For mappings on $\mathbb{Z}_p$ and on p -adic spheres, ergodicity is often studied with respect to the normalized Haar measure. In the non-Archimedean setting, minimality and ergodicity are frequently characterized by finite-level conditions modulo powers of p . For rational maps with good reduction, minimality and its relation to Haar ergodicity have also been studied (see [5, 24, 25, 26]).

Despite these developments, the asymptotic behavior of trajectories of higher-degree rational functions over $\mathbb{Q}_p$ is still far from being completely understood. In particular, for rational functions arising from statistical mechanics, it is important to describe not only fixed points and invariant spheres, but also the structure of ω -limit sets, the set of bad points, and the possible regularity of the system.

It is worth noting that the canonical form of a p -adic $(2, 2)$ -rational function is given by

$$f(x) = \frac{ax^2 + bx + c}{x^2 + dx + e},$$

where $a \neq 0$ and $|b - ad|_p + |c - ae|_p \neq 0$.

In [14, 27], the authors studied the dynamics of $(2, 2)$ -rational functions mainly over the field $\mathbb{C}_p$. According to the number of fixed points of such a function in $\mathbb{C}_p$, they reduced it, by means of topological conjugacy, to three types of rational dynamical systems. Using this classification of rational dynamics over $\mathbb{C}_p$, they then studied the corresponding rational dynamical systems over $\mathbb{Q}_p$. A delicate point is that the rational dynamical systems considered by the authors do not cover all $(2, 2)$ -rational dynamical systems over $\mathbb{Q}_p$. Nevertheless, it should be acknowledged that they obtained interesting results which enrich the theory of p -adic dynamical systems.

In this paper, we consider the following $(2, 2)$ -rational function, which arises in the theory of p -adic Gibbs measures:

$$f_{a,b}(x) = \frac{ax^2 + bx + 1}{x^2 + bx + a}. \quad (1.1)$$

Although functions of this form constitute only a subclass of $(2, 2)$ -rational functions, they are of practical importance in the study of the p -adic Vannimenus model. Moreover, under certain conditions on the parameters, this function may belong to a class which is not covered by the cases studied over $\mathbb{Q}_p$ in [14, 27]. The main object of the present paper is the function (1.1) over the field of 2-adic numbers.

Remark 1.1. We note that (1.1) represents a generalized form of the 2-adic Ising function, which has been thoroughly studied in [26, 28]. It is important to emphasize that (1.1) is not merely an object of abstract mathematical interest. Rather, this function naturally arises as a rational mapping in the construction of Gibbs measures associated with the Ising–Vannimenus model on a Cayley tree of order two. In this context, each periodic point of the map corresponds to a periodic Gibbs measure, thereby establishing a direct link between the dynamical properties of $f_{a,b}$ and the structure of Gibbs measures.

Depending on the parameters a and b , the map $f_{a,b}$ exhibits several distinct structural behaviors:

- it reduces to a $(1, 2)$ -rational function when $a = 0$;
- it becomes a $(1, 1)$ -rational function when $a \pm b + 1 = 0$;
- it degenerates to a constant function when $a = 1$;
- it coincides with an Ising-type function when $b^2 - 4a = 0$;
- it is conjugate to an Ising function in the case $b = 0$.

In the present paper, we focus exclusively on $(2, 2)$ -rational functions that are not of Ising type. Therefore, throughout the paper we impose the non-degeneracy condition

$$ab(a - 1)(a + b + 1)(a - b + 1)(b^2 - 4a) \neq 0. \quad (1.2)$$

2. PRELIMINARIES

Let $\mathbb{Q}_2$ denote the field of 2-adic numbers equipped with the standard non-Archimedean absolute value, called the 2-adic norm. It is well known that every nonzero element $x \in \mathbb{Q}_2$ admits a unique representation of the form

$$x = 2^{\text{ord}_2(x)} (1 + x_1 \cdot 2 + x_2 \cdot 2^2 + x_3 \cdot 2^3 + \cdots), \quad (2.1)$$

where $\text{ord}_2(x) \in \mathbb{Z}$ and the coefficients $x_j \in \{0, 1\}$ for all $j \in \mathbb{N}$. In this representation, the 2-adic norm of x is given by

$$|x|_2 = 2^{-\text{ord}_2(x)}.$$

For each $a \in \mathbb{Q}_2$, $r > 0$ we denote

$$B_r(a) = \{x \in \mathbb{Q}_2 : |x - a|_2 < r\}, \quad S_r(a) = \{x \in \mathbb{Q}_2 : |x - a|_2 = r\}.$$

We notice that $B_r(a)$ and $S_r(a)$ are clopen, compact subsets of $\mathbb{Q}_2$. Recall that $\mathbb{Z}_2 := B_2(0)$ and $\mathbb{Z}_2^* := S_1(0)$ are the set of all 2-adic integers and 2-adic units, respectively. One can check that for every $n \geq 1$ it holds:

$$S_{\frac{1}{2^n}}(1) = 1 + 2^n + 2^{n+1}\mathbb{Z}_2, \quad B_{\frac{1}{2^n}}(1) = 1 + 2^{n+1}\mathbb{Z}_2.$$

The proof of the following lemma is straightforward.

Lemma 2.1. *For any $x \in \mathbb{Z}_2^*$ one has $x^2 \in 1 + 8\mathbb{Z}_2$.*

Using Lemma 2.1, we can easily establish the following auxiliary result.

Lemma 2.2. *Let $a \in \mathbb{Q}_2$. Then $x^2 = a$ has a solution in $\mathbb{Q}_2$ iff $\text{ord}_2(a)$ is even and $\frac{a}{2^{\text{ord}_2(a)}} \in 1 + 8\mathbb{Z}_2$.*

A 2-adic dynamical system is a pair (X, f) , where $X \subset \mathbb{Q}_2$ and $f : X \rightarrow X$ is a continuous function with respect to the 2-adic topology. Throughout, we assume that the space X is either compact or equal to the entire field $\mathbb{Q}_2$. For a given point $x \in X$ the positive trajectory (or orbit) of x under iteration of f is defined as

$$\mathcal{O}^+(x, f) := \{x, f(x), f^2(x), \dots, f^n(x), \dots\}.$$

The central problem in the study of such systems is to describe the asymptotic behavior of this trajectory as $n \rightarrow \infty$. In other words, it is important to describe a set of all limiting point of the trajectory, which is denoted by $\omega_f(x)$ and is known as the omega-limit set of x .

A dynamical system (X, f) is said to be:

- *regular* if, for every $x \in X$, its omega-limit set is a singleton.
- *minimal* if the positive orbit of every point $x \in X$ is dense in X ; that is,

$$\overline{\mathcal{O}^+(x, f)} = X, \quad \forall x \in X.$$

- *topologically transitive* if for any pair of nonempty open sets $U, V \subset X$ there exists an integer $n \geq 0$ such that

$$f^n(U) \cap V \neq \emptyset.$$

It is obvious that every minimal dynamical system is topologically transitive. In general, a topologically transitive system need not be minimal. Transitivity only requires the existence of at least one point with a dense orbit, not that every point has this property.

3. FIXED POINTS AND PERIODIC POINTS OF $f_{a,b}$

Let us define the following sets:

$$\text{Fix}(f_{a,b}) = \{x \in \mathbb{Q}_2 : f_{a,b}(x) = x\},$$

$$\text{Per}_2(f_{a,b}) = \{x \in \mathbb{Q}_2 : f_{a,b}^2(x) = x\} \setminus \text{Fix}(f_{a,b}),$$

which are called the set of all fixed points and the set of all 2-periodic points of the function $f_{a,b}$, respectively.

In this section, we aim to describe these sets in detail.

The fixed point equation $f_{a,b}(x) = x$ reduces to

$$(x - 1)(x^2 + (b - a + 1)x + 1) = 0.$$

Hence, one solution is $x_0 = 1$. Denoting $D = (b - a + 1)^2 - 4$, we obtain that, whenever $\sqrt{D} \in \mathbb{Q}_2$, the remaining solutions are given by

$$x_{\pm} = \frac{a - b - 1 \pm \sqrt{D}}{2}. \quad (3.1)$$

We now analyze the condition $\sqrt{D} \in \mathbb{Q}_2$ in terms of the 2-adic norm of $b - a + 1$. To this end, we invoke Lemma 2.2.

Case $|b - a + 1|_2 \geq 1$. We observe that

$$\frac{D}{(b - a + 1)^2} \in \begin{cases} 1 + 16\mathbb{Z}_2, & \text{if } |b - a + 1|_2 > 1, \\ 1 + 4 + 8\mathbb{Z}_2, & \text{if } |b - a + 1|_2 = 1. \end{cases}$$

Consequently, $\sqrt{D} \in \mathbb{Q}_2$ whenever $|b - a + 1|_2 > 1$, whereas $\sqrt{D} \notin \mathbb{Q}_2$ in the case $|b - a + 1|_2 = 1$.

Case $|b - a + 1|_2 < 1$. First, assume that $|b - a + 1|_2 < \frac{1}{2}$. Then

$$\frac{D}{4} \in 1 + 2 + 4\mathbb{Z}_2,$$

which implies that $\sqrt{D} \notin \mathbb{Q}_2$.

Now consider the case $|b - a + 1|_2 = \frac{1}{2}$. Then there exists $\alpha \in \mathbb{Z}_2$ such that

$$b - a + 1 = 2 + 4\alpha.$$

Substituting this representation into the expression for D , we obtain

$$D = 16\alpha(1 + \alpha).$$

It follows that $\sqrt{D} \in \mathbb{Q}_2$ if and only if $\sqrt{\alpha(1 + \alpha)} \in \mathbb{Q}_2$. Since $\alpha \in \mathbb{Z}_2$, we have $|\alpha(1 + \alpha)|_2 < 1$. Therefore, by the characterization of quadratic residues in $\mathbb{Q}_2$, it follows that $\sqrt{\alpha(1 + \alpha)} \in \mathbb{Q}_2$ if and only if either $\alpha \in \{-1, 0\}$ or

$$\alpha(1 + \alpha) \in \bigcup_{n=1}^{\infty} 2^{2n} (1 + 8\mathbb{Z}_2). \quad (3.2)$$

Condition (3.2) follows directly from Lemma 2.2. We also note that, by Remark 1.1, one has $\alpha \neq 0$. Moreover, if $\alpha = -1$, then $x_- = x_+ = x_0$.

We now determine the canonical form of α under condition (3.2). To this end, we distinguish two cases: $\alpha \in \mathbb{Z}_2^*$ and $\alpha \in 2\mathbb{Z}_2$.

First, let $\alpha \in \mathbb{Z}_2^*$. Since $\alpha \neq -1$, there exist $k \geq 0$ and $\beta \in \mathbb{Z}_2$ such that

$$\alpha = \sum_{i=0}^k 2^i + 2^{k+2}\beta. \quad (3.3)$$

Then

$$\alpha(\alpha + 1) = 2^{k+1}(1 + 2\beta) \left(\sum_{i=0}^k 2^i + 2^{k+2}\beta \right). \quad (3.4)$$

By condition (3.2), the integer k must be odd. Hence, we write $k = 2n + 1$ for some $n \geq 0$. In this case, (3.4) becomes

$$\alpha(\alpha + 1) = 2^{2n+2} (-1 - 2\beta + 2^{2n+2} + 2^{2n+4}\beta(\beta + 1)). \quad (3.5)$$

Combining (3.2) with (3.5), we get

$$\beta \in \begin{cases} 1 + 4\mathbb{Z}_2, & \text{if } n = 0, \\ 1 + 2 + 4\mathbb{Z}_2, & \text{if } n > 0. \end{cases} \quad (3.6)$$

Substituting (3.6) into (3.3), we derive

$$\alpha \in \begin{cases} 1 + 2 + 8 + 32\mathbb{Z}_2, & \text{if } n = 0, \\ \sum_{i=0}^{2n+1} 2^i + 2^{2n+3} + 2^{2n+4} + 2^{2n+5}\mathbb{Z}_2, & \text{if } n > 0. \end{cases} \quad (3.7)$$

Next, assume that $\alpha \in 2\mathbb{Z}_2$. Since $\alpha \neq 0$, condition (3.2) implies that there exist $n \geq 1$ and $\beta \in \mathbb{Z}_2$ such that

$$\alpha = 2^{2n}(1 + 2\beta). \quad (3.8)$$

Consequently,

$$\alpha(\alpha + 1) = 2^{2n} (1 + 2\beta + 2^{2n} + 2^{2n+2}\beta(\beta + 1)). \quad (3.9)$$

Taking (3.2) into account, it follows from (3.9) that

$$\beta \in \begin{cases} 2 + 4\mathbb{Z}_2, & \text{if } n = 1, \\ 4\mathbb{Z}_2, & \text{if } n > 1. \end{cases} \quad (3.10)$$

Substituting (3.10) into (3.8), we obtain

$$\alpha \in \begin{cases} 4(1 + 4 + 8\mathbb{Z}_2), & \text{if } n = 1, \\ 2^{2n}(1 + 8\mathbb{Z}_2), & \text{if } n > 1. \end{cases} \quad (3.11)$$

Thus, by (3.7) and (3.11), condition (3.2) holds if and only if one of the following alternatives is satisfied:

- $\alpha \in 1 + 2 + 8 + 32\mathbb{Z}_2$;
- $\alpha \in \bigsqcup_{n=1}^{\infty} (1 + 2 + \dots + 2^{2n+1} + 2^{2n+3} + 2^{2n+4} + 2^{2n+5}\mathbb{Z}_2)$;
- $\alpha \in 4(1 + 4 + 8\mathbb{Z}_2)$;
- $\alpha \in \bigsqcup_{n=2}^{\infty} 2^{2n}(1 + 8\mathbb{Z}_2)$.

Keeping in mind that $b - a = 1 + 4\alpha$, we define the following sets:

$$\begin{cases} A_1 = 45 + 128\mathbb{Z}_2, \\ A_2 = \bigsqcup_{n=1}^{\infty} (1 + 2^2 + 2^3 + \dots + 2^{2n+3} + 2^{2n+5} + 2^{2n+6} + 2^{2n+7}\mathbb{Z}_2), \\ A_3 = 81 + 128\mathbb{Z}_2, \\ A_4 = \bigsqcup_{n=2}^{\infty} (1 + 2^{2n+2} + 2^{2n+5}\mathbb{Z}_2). \end{cases} \quad (3.12)$$

Thus, we obtain the following result:

Proposition 3.1. *Let $f_{a,b}$ be a $(2, 2)$ -rational function defined by (1.1). Then*

$$\text{Fix}(f_{a,b}) = \begin{cases} \{x_0, x_-, x_+\}, & \text{if } b - a \in \bigcup_{i=1}^5 A_i, \\ \{x_0\}, & \text{otherwise.} \end{cases}$$

Here A_1, A_2, A_3, A_4 are defined in (3.12), and $A_5 = \mathbb{Q}_2 \setminus \mathbb{Z}_2$.

Now we describe the set of all 2-periodic points of the function $f_{a,b}$. To this end, we consider the equation

$$\frac{f_{a,b}^2(x) - x}{f_{a,b}(x) - x} = 0,$$

which is equivalent to the quadratic equation

$$x^2 + \left(\frac{b-1}{a} + 1\right)x + 1 = 0. \quad (3.13)$$

The solutions of this equation are

$$x^{\pm} = \frac{-\frac{b-1}{a} - 1 \pm \sqrt{\left(\frac{b-1}{a} + 1\right)^2 - 4}}{2}, \quad (3.14)$$

whenever the square root exists in $\mathbb{Q}_2$.

By definition, $\text{Fix}(f_{a,b}) \cap \text{Per}_2(f_{a,b}) = \emptyset$. In particular, we exclude the case $\frac{b-1}{a} = -3$, since this would imply that the points x^+ and x^- coincide with the fixed point x_0 , i.e., $x^+ = x^- = x_0$.

Moreover, by condition (1.2), we have $\frac{b-1}{a} \neq 1$. Consequently, throughout this paper we impose the condition

$$\left(\frac{b-1}{a} + 1\right)^2 \neq 4.$$

Proposition 3.2. *Let $f_{a,b}$ be a $(2,2)$ -rational function defined by (1.1). Then*

$$\text{Per}_2(f_{a,b}) = \begin{cases} \{x^-, x^+\}, & \text{if } \frac{b-1}{a} \in \bigcup_{i=1}^5 A_i, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Here A_1, A_2, A_3, A_4 are defined in (3.12), and $A_5 = \mathbb{Q}_2 \setminus \mathbb{Z}_2$.

Remark 3.1. In this section, we have completely described the set of all fixed points and the set of all two-periodic points of the function $f_{a,b}$. It is worth emphasizing that this description is obtained for arbitrary values of the parameters a and b . Starting from the next section, we study the dynamical behaviour of $f_{a,b}$. Although we have explicit descriptions of the sets $\text{Fix}(f_{a,b})$ and $\text{Per}_2(f_{a,b})$ for general parameters, in what follows we restrict our attention to the case $|a|_2 = |b|_2$. The case $|a|_2 \neq |b|_2$ will be investigated in our future work. In general, the case $|a|_2 \neq |b|_2$ exhibits some similarities with the case $|a|_2 = |b|_2$. However, certain unexpected phenomena also arise. In particular, we have observed in some special cases that when $|a|_2 \neq |b|_2$, the dynamical system may display chaotic behaviour. Since a complete analysis of this case requires a separate and more detailed investigation, in the present paper we focus only on the case $|a|_2 = |b|_2$.

4. DYNAMICS OF THE FUNCTION $f_{a,b}$

In this section, we investigate the dynamical properties of the function $f_{a,b}$ in the case where $|a|_2 = |b|_2$.

We introduce the set

$$\mathcal{P}(f_{a,b}) = \left\{ x \in \mathbb{Q}_2 : \exists n \geq 0 \text{ such that } (f_{a,b}^n(x))^2 + b f_{a,b}^n(x) + a = 0 \right\},$$

which we call the set of preimages of the poles (or, equivalently, the full backward orbit of the poles) of the map $f_{a,b}$.

We note that $\mathcal{P}(f_{a,b}) \neq \emptyset$ if and only if $\sqrt{b^2 - 4a} \in \mathbb{Q}_2$. Whenever this condition holds, we define

$$\hat{x}_{\pm} := \frac{-b \pm \sqrt{b^2 - 4a}}{2}. \quad (4.1)$$

In the case $|a|_2 = |b|_2$ under consideration, there are the following three possibilities:

$$|a|_2 = |b|_2 = 1, \quad |a|_2 = |b|_2 < 1, \quad |a|_2 = |b|_2 > 1.$$

We investigate each case separately.

4.1. **Case** $|a|_2 = |b|_2 = 1$.

Remark 4.1. In this case, we have $|b - a|_2 < 1$ and $|b - 1|_2 < |a|_2$. Equivalently,

$$b - a \notin \bigcup_{i=1}^5 A_i, \quad \frac{b-1}{a} \notin \bigcup_{i=1}^5 A_i.$$

Therefore, by Propositions 3.1 and 3.2, we obtain

$$\text{Fix}(f_{a,b}) = \{x_0\}, \quad \text{Per}_2(f_{a,b}) = \emptyset.$$

Moreover, since $b^2 - 4a \in 5 + 8\mathbb{Z}_2$, it follows that $\sqrt{b^2 - 4a} \notin \mathbb{Q}_2$. Consequently, the function $f_{a,b}$ is well-defined on the entire field $\mathbb{Q}_2$.

Theorem 4.1. *Let $|a|_2 = |b|_2 = 1$. Then for any $x \in \mathbb{Q}_2$ it holds*

$$\omega_{f_{a,b}}(x) = \{x_0\}.$$

Proof: It is readily verified that

$$f_{a,b}(x) - 1 = \frac{(a-1)(x+1)(x-1)}{x^2 + bx + a}.$$

Using the estimates

$$|x+1|_2 \leq \begin{cases} |x|_2, & \text{if } |x|_2 > 1, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$|x^2 + bx + a|_2 = \begin{cases} |x|_2^2, & \text{if } |x|_2 > 1, \\ 1, & \text{otherwise,} \end{cases}$$

we obtain

$$\frac{|x+1|_2}{|x^2 + bx + a|_2} \leq 1.$$

Therefore,

$$|f_{a,b}(x) - 1|_2 \leq |(a-1)(x-1)|_2.$$

By iteration, it follows that

$$|f_{a,b}^n(x) - 1|_2 \leq |(a-1)^n(x-1)|_2.$$

Since $|a-1|_2 < 1$, we conclude that $f_{a,b}^n(x) \rightarrow 1$ as $n \rightarrow \infty$. □

4.2. **Case** $|a|_2 = |b|_2 = 2^m < 1$.

Remark 4.2. In this case, we have $|b - a|_2 < 1$ and $|b - 1|_2 > |a|_2$. Equivalently,

$$b - a \notin \bigcup_{i=1}^5 A_i, \quad \frac{b-1}{a} \in A_5.$$

Therefore, by Propositions 3.1 and 3.2, we obtain

$$\text{Fix}(f_{a,b}) = \{x_0\}, \quad \text{Per}_2(f_{a,b}) = \{x^-, x^+\}.$$

From (3.13) and Vieta's formula, we obtain

$$|x^- + x^+|_2 = \frac{1}{2^m}, \quad |x^- \cdot x^+|_2 = 1.$$

Hence the two roots have different 2-adic norms. More precisely, after relabelling the roots if necessary, we may assume that

$$x^- \in S_{2^m}(0), \quad x^+ \in S_{\frac{1}{2^m}}(0). \quad (4.2)$$

Theorem 4.2. Let $|a|_2 = |b|_2 = 2^m < 1$. Then for the initial point $x \in \mathbb{Q}_2$ it holds

$$\omega_{f_{a,b}}(x) = \begin{cases} \{x_0\} & \text{if } x \in \mathbb{Z}_2^*, \\ \{x^+, x^-\}, & \text{if } x \in \mathbb{Q}_2 \setminus (\mathbb{Z}_2^* \cup \mathcal{P}(f_{a,b})). \end{cases}$$

Moreover, $\mathcal{P}(f_{a,b})$ is finite.

Proof: A direct computation shows that

$$|f_{a,b}(x) - 1|_2 = |x^2 - 1|_2 < |x - 1|_2, \quad \forall x \in \mathbb{Z}_2^*.$$

Hence, $f_{a,b}(\mathbb{Z}_2^*) \subset \mathbb{Z}_2^*$ and

$$\lim_{n \rightarrow \infty} f_{a,b}^n(x) = 1, \quad \forall x \in \mathbb{Z}_2^*.$$

In order to analyse the dynamics of $f_{a,b}$ on $\mathbb{Q}_2 \setminus \mathbb{Z}_2^*$, we first establish the following preliminary facts:

$$(P_1) \quad S_{2^m}(0) \xrightarrow{f_{a,b}} S_{\frac{1}{2^m}}(0) \xrightarrow{f_{a,b}} S_{2^m}(0).$$

$$(P_2) \quad \text{for every } x \in 2\mathbb{Z}_2 \setminus \mathcal{P}(f_{a,b}) \text{ there exists } k \geq 1 \text{ such that } f_{a,b}^{2k}(x) \in S_{2^m}(0).$$

$$(P_3) \quad \text{for every } x \in \mathbb{Q}_2 \setminus (\mathbb{Z}_2 \cup \mathcal{P}(f_{a,b})) \text{ there exists } k \geq 1 \text{ such that } f_{a,b}^{2k}(x) \in S_{\frac{1}{2^m}}(0).$$

$$(P_4) \quad \mathcal{P}(f_{a,b}) \text{ is finite.}$$

Proof of (P_1) . Let $x \in S_{2^m}(0)$, i.e. $|x|_2 = 2^m$. Then by $|ax^2 + bx + 1|_2 = 1$ and $|x^2 + bx + a|_2 = 2^m$ we get $|f_{a,b}(x)|_2 = \frac{1}{2^m}$, i.e.

$$f_{a,b}(x) \in S_{\frac{1}{2^m}}(0).$$

Now pick $y \in S_{\frac{1}{2^m}}(0)$. Then by $|ay^2 + by + 1|_2 = \frac{1}{2^m}$ and $|y^2 + by + a|_2 = \frac{1}{4^m}$ we obtain $|f_{a,b}(y)|_2 = 2^m$, i.e.

$$f_{a,b}(y) \in S_{2^m}(0).$$

Proof of (P_2) . We note that, if $\hat{x}_-$ and $\hat{x}_+$ exist, then both of them lie on the sphere $S_{\sqrt{2^m}}(0)$, that is,

$$|\hat{x}_-|_2 = |\hat{x}_+|_2 = \sqrt{2^m}.$$

Let $x \in B_{\sqrt{2^m}}(0)$, i.e. $|x|_2 < 2^m$. Then we have

$$\begin{aligned} |ax^2 + bx + 1|_2 &= 1, \\ |x^2 + bx + a|_2 &= 2^m. \end{aligned}$$

Hence $f_{a,b}(x) \in S_{\frac{1}{2^m}}(0)$. By (P_1) we infer that $f_{a,b}^2(x) \in S_{2^m}(0)$. So, $k = 1$.

Take an arbitrary $y \in S_{\sqrt{2^m}}(0) \setminus \{\hat{x}_-, \hat{x}_+\}$. Then $|y^2|_2 = |a|_2 > |by|_2$. By strong triangle inequality,

$$|f_{a,b}(y)|_2 = \frac{1}{|y^2 + by + a|_2} > \frac{1}{2^m}.$$

Therefore, we have

$$\begin{aligned} |a(f_{a,b}(y))^2 + bf_{a,b}(y) + 1|_2 &= 2^m |f_{a,b}(y)|_2^2, \\ |(f_{a,b}(y))^2 + bf_{a,b}(y) + a|_2 &= |f_{a,b}(y)|_2^2, \end{aligned}$$

which implies that $f_{a,b}^2(y) \in S_{2^m}(0)$. So, $k = 1$.

Finally, pick $z \in 2\mathbb{Z}_2$ such that $|z|_2 > \sqrt{2^m}$. By strong triangle inequality we get $|f_{a,b}(z)|_2 = \frac{1}{|z^2|_2}$. Then using

$$\begin{aligned} |a(f_{a,b}(z))^2|_2 &= \frac{2^m}{|z^4|_2} < \frac{1}{|z^2|_2}, \\ |bf_{a,b}(z)|_2 &= \frac{2^m}{|z^2|_2} < 1, \end{aligned}$$

and $1 < \frac{1}{|z^2|_2}$ we get

$$|a(f_{a,b}(z))^2 + bf_{a,b}(z) + 1|_2 < \frac{1}{|z^2|_2}.$$

The last inequality together with $|(f_{a,b}(z))^2 + bf_{a,b}(z) + a|_2 = \frac{1}{|z^4|_2}$ implies that

$$|f_{a,b}^2(z)|_2 < |z^2|_2.$$

Thus, due to $|z|_2 < 1$ we have $|f_{a,b}^2(z)|_2 < |z|_2$. By the discreteness of the values of the 2-adic norm we conclude that there exists $i \geq 1$ such that $|f_{a,b}^{2i}(z)|_2 \leq \sqrt{2^m}$. Therefore, whenever $f_{a,b}^{2i}(z) \notin \{\hat{x}_-, \hat{x}_+\}$, choosing $k = i + 1$ gives

$$f_{a,b}^{2k}(z) \in S_{2^m}(0).$$

Proof of (P_3) . This follows directly by applying the identity

$$f_{a,b}\left(\frac{1}{x}\right) = \frac{1}{f_{a,b}(x)}$$

to properties (P_1) and (P_2) .

Proof of (P_4) . It follows from properties (P_1) – (P_3) that

$$\mathcal{P}(f_{a,b}) \subset \left\{ x \in \mathbb{Q}_2 : \sqrt{2^m} \leq |x|_2 \leq \frac{1}{\sqrt{2^m}} \right\}.$$

Taking into account that

$$\omega_{f_{a,b}}(x) = \{x_0\}, \quad \forall x \in \mathbb{Z}_2^*,$$

and that, for every y with $1 < |y|_2 \leq \frac{1}{\sqrt{2^m}}$, one has

$$|f_{a,b}(y)|_2 \leq \frac{1}{|y^2|_2},$$

we conclude that

$$\mathcal{P}(f_{a,b}) \subset \left\{ x \in \mathbb{Q}_2 : \sqrt{2^m} \leq |x|_2 \leq \frac{1}{\sqrt[4]{2^m}} \right\} \setminus \mathbb{Z}_2^*.$$

On the other hand, for every z satisfying $\sqrt{2^m} < |z|_2 < 1$, we have

$$|f_{a,b}^2(z)|_2 < |z|_2.$$

Therefore, by the symmetry relation

$$f_{a,b}\left(\frac{1}{x}\right) = \frac{1}{f_{a,b}(x)},$$

it follows that, for every t satisfying

$$1 < |t|_2 < \frac{1}{\sqrt[4]{2^m}},$$

one has

$$|f_{a,b}^2(t)|_2 > |t|_2.$$

Since the values of the 2-adic norm are discrete, the annulus

$$\left\{ x \in \mathbb{Q}_2 : \sqrt{2^m} \leq |x|_2 \leq \frac{1}{\sqrt[4]{2^m}} \right\}$$

contains only finitely many possible values of $|x|_2$. Moreover, on the part $\sqrt{2^m} < |x|_2 < 1$ the norm of the even iterates strictly decreases, whereas on the part $1 < |x|_2 \leq \frac{1}{\sqrt[4]{2^m}}$ the norm of the even iterates strictly increases. Hence any orbit which does not hit a pole enters the invariant set

$$X = S_{2^m}(0) \sqcup S_{\frac{1}{2^m}}(0) \tag{4.3}$$

after a uniformly bounded number of iterates. Therefore a point may belong to $\mathcal{P}(f_{a,b})$ only if it is mapped to one of the poles within a bounded number of iterates. Since each fixed iterate of the rational map has only finitely many preimages of the finite set of poles, the set $\mathcal{P}(f_{a,b})$ is finite.

Thus properties $(P_1) - (P_4)$ have been established. Moreover, they imply that

$$X \cap \mathcal{P}(f_{a,b}) = \emptyset. \quad (4.4)$$

Thus, the dynamics of the map $f_{a,b}$ on $\mathbb{Q}_2$ can in fact be studied on the invariant set (4.3). Moreover, since both spheres are invariant under the second iterate, it suffices to analyse the dynamics of the map $g := f_{a,b}^2$ on these spheres.

We first show that g is a contraction on $S_{\frac{1}{2^m}}(0)$. Then, in view of the symmetry relation

$$g\left(\frac{1}{x}\right) = \frac{1}{g(x)},$$

the same contraction property follows for g on the sphere $S_{2^m}(0)$.

Let $y, z \in S_{\frac{1}{2^m}}(0)$. Then

$$\begin{aligned} |g(y) - g(z)|_2 &= \frac{|f_{a,b}(y) - f_{a,b}(z)|_2}{|a|_2} \\ &= \frac{|y - z|_2 |b(yz + 1) + (a + 1)(y + z)|_2}{|a|_2 |y^2 + by + a|_2 |z^2 + bz + a|_2} \\ &= \frac{|y - z|_2 |byz|_2}{|a|_2 |y^2|_2 |z^2|_2} = \frac{|y - z|_2}{|yz|_2} = |a|_2^2 |y - z|_2 < |y - z|_2. \end{aligned}$$

Thus, g is a contraction on $S_{\frac{1}{2^m}}(0)$. Hence, for every $x \in X$ we have

$$\lim_{n \rightarrow \infty} g^n(x) = \begin{cases} x_*, & \text{if } x \in S_{2^m}(0), \\ x^*, & \text{if } x \in S_{\frac{1}{2^m}}(0). \end{cases}$$

On the other hand, by (4.2) we have $g(x^-) = x^-$ and $g(x^+) = x^+$. By the uniqueness of the fixed point of a contraction on each sphere, we get $x_* = x^-$ and $x^* = x^+$. Since every point

$$x \in \mathbb{Q}_2 \setminus (\mathbb{Z}_2^* \cup \mathcal{P}(f_{a,b}))$$

eventually enters the invariant set X , the two subsequences of its orbit under $f_{a,b}$ converge to the two points of the two-cycle $\{x^-, x^+\}$. Therefore

$$\omega_{f_{a,b}}(x) = \{x^-, x^+\}, \quad \forall x \in \mathbb{Q}_2 \setminus (\mathbb{Z}_2^* \cup \mathcal{P}(f_{a,b})).$$

This completes the proof. □

4.3. Case $|a|_2 = |b|_2 > 1$.

Remark 4.3. A direct computation shows that, in this regime, neither fixed points nor 2-periodic points (whenever they exist) can be attracting. More precisely, any existing 2-periodic point is necessarily indifferent. Among the fixed points, only $x_0 = 1$ may be either indifferent or repelling, depending on the parameters. Moreover, in this case, $\mathcal{P}(f_{a,b}) \neq \emptyset$ and

$$\mathcal{P}(f_{a,b}) \subset \mathbb{Z}_2^* \cup (\mathbb{Q}_2 \setminus B_{2^m}(0)).$$

First, we try to determine the invariant sets of the function $f_{a,b}$.

Proposition 4.1. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements hold:*

$$(i) \quad S_{\frac{1}{2^m}}(0) \xrightarrow{f_{a,b}} B_{\frac{1}{2^m}}(0) \xrightarrow{f_{a,b}} S_{\frac{1}{2^m}}(0);$$

(ii) if $m > 1$ then $S_{\frac{1}{2^n}}(0) \xrightarrow{f_{a,b}} S_{\frac{1}{2^n}}(0)$ for any $n \in \{1, \dots, m-1\}$.

Proof: (i) Let us take an arbitrary point $x \in S_{\frac{1}{2^m}}(0)$. Then

$$|f_{a,b}(x)|_2 = \frac{|ax^2 + bx + 1|_2}{|a|_2} < \frac{1}{2^m}.$$

Now pick an arbitrary point $y \in B_{\frac{1}{2^m}}(0)$. Since

$$|ay^2|_2 < |by|_2 < 1,$$

we obtain

$$|f_{a,b}(y)|_2 = \frac{|ay^2 + by + 1|_2}{|a|_2} = \frac{1}{2^m}.$$

(ii) Assume that $m > 1$. Let $n \in \{1, \dots, m-1\}$. Then, for every $x \in S_{\frac{1}{2^n}}(0)$, we have

$$|bx|_2 > \max\{|ax^2|_2, 1\}.$$

Therefore,

$$|f_{a,b}(x)|_2 = \frac{|bx|_2}{|a|_2} = |x|_2.$$

Hence

$$f_{a,b}\left(S_{\frac{1}{2^n}}(0)\right) \subset S_{\frac{1}{2^n}}(0).$$

□

Proposition 4.2. Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements hold:

- (i) The map $f_{a,b}$ is a bijective isometry on the closed ball $\overline{B}_{\frac{1}{2^m}}(0)$;
- (ii) If $m > 1$, then $f_{a,b}$ is a bijective isometry on each sphere $S_{\frac{1}{2^n}}(0)$ for all $n \in \{1, \dots, m-1\}$.

Proof: (i) By Proposition 4.1, we have

$$f_{a,b}(\overline{B}_{\frac{1}{2^m}}(0)) \subset \overline{B}_{\frac{1}{2^m}}(0).$$

Let $x, y \in \overline{B}_{\frac{1}{2^m}}(0)$. Then

$$\begin{aligned} |f_{a,b}(x) - f_{a,b}(y)|_2 &= \frac{|(x-y)(a-1)(bxy + (a+1)(x+y) + b)|_2}{|(x^2 + bx + a)(y^2 + by + a)|_2} \\ &= \frac{|ab(x-y)|_2}{|a^2|_2} = |x-y|_2, \end{aligned}$$

which shows that $f_{a,b}$ is an isometry on $\overline{B}_{\frac{1}{2^m}}(0)$.

We now prove surjectivity. Fix an arbitrary $\gamma \in \overline{B}_{\frac{1}{2^m}}(0)$ and consider the equation

$$f_{a,b}(t) = \gamma,$$

which is equivalent to

$$(a - \gamma)t^2 + b(1 - \gamma)t + (1 - a\gamma) = 0. \quad (4.5)$$

Observe that

$$\begin{aligned} |a - \gamma|_2 &= 2^m, \\ |b(1 - \gamma)|_2 &= 2^m, \\ |1 - a\gamma|_2 &\leq 1. \end{aligned}$$

Hence,

$$\frac{|4(a - \gamma)(1 - a\gamma)|_2}{|b^2(1 - \gamma)^2|_2} \leq \frac{1}{2^{m+2}} \leq \frac{1}{8}.$$

Then, by Lemma 2.1, the discriminant

$$b^2(1 - \gamma)^2 \left(1 - \frac{4(a - \gamma)(1 - a\gamma)}{b^2(1 - \gamma)^2} \right)$$

admits a square root in $\mathbb{Q}_2$. Hence, equation (4.5) has two solutions in $\mathbb{Q}_2$, denoted by t_1 and t_2 .

By Vieta's formulas,

$$|t_1 + t_2|_2 = 1, \quad |t_1 t_2|_2 \leq \frac{1}{2^m}.$$

It follows from the non-Archimedean property that exactly one of the roots has unit norm. Without loss of generality, let $|t_1|_2 = 1$, i.e., $t_1 \in \mathbb{Z}_2^*$, while the other root satisfies

$$|t_2|_2 \leq \frac{1}{2^m},$$

that is, $t_2 \in \overline{B}_{\frac{1}{2^m}}(0)$.

Thus, for every $\gamma \in \overline{B}_{\frac{1}{2^m}}(0)$, there exists $t \in \overline{B}_{\frac{1}{2^m}}(0)$ such that $f_{a,b}(t) = \gamma$. Hence, $f_{a,b}$ is surjective on $\overline{B}_{\frac{1}{2^m}}(0)$, and therefore a bijective isometry.

(ii) Since the proof of the fact that $f_{a,b}$ is an isometry on the sphere $S_{\frac{1}{2^n}}(0)$ is exactly the same as in case (i), we only verify surjectivity. To this end, we show that, for any $\gamma \in S_{\frac{1}{2^n}}(0)$, the quadratic equation (4.5) has a solution belonging to the same sphere.

We have

$$\begin{aligned} |a - \gamma|_2 &= 2^m, \\ |b(1 - \gamma)|_2 &= 2^m, \\ |1 - a\gamma|_2 &= 2^{m-n}. \end{aligned}$$

Hence,

$$\frac{|4(a - \gamma)(1 - a\gamma)|_2}{|b^2(1 - \gamma)^2|_2} = \frac{1}{2^{n+2}} \leq \frac{1}{8}.$$

Then, by Lemma 2.1, the discriminant

$$b^2(1 - \gamma)^2 \left(1 - \frac{4(a - \gamma)(1 - a\gamma)}{b^2(1 - \gamma)^2} \right)$$

admits a square root in $\mathbb{Q}_2$. Hence, equation (4.5) has two solutions in $\mathbb{Q}_2$, denoted by t_1 and t_2 .

By Vieta's formulas,

$$|t_1 + t_2|_2 = 1, \quad |t_1 t_2|_2 \leq \frac{1}{2^m}.$$

It follows from the non-Archimedean property that exactly one of the roots has unit norm. Without loss of generality, let $|t_1|_2 = 1$, i.e., $t_1 \in \mathbb{Z}_2^*$, while the other root satisfies

$$|t_2|_2 \leq \frac{1}{2^m},$$

that is, $t_2 \in \overline{B}_{\frac{1}{2^m}}(0)$. □

Using the relation

$$f_{a,b}\left(\frac{1}{x}\right) = \frac{1}{f_{a,b}(x)},$$

one can prove the following results in a similar manner to Proposition 4.1 and Proposition 4.2.

Proposition 4.3. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements hold:*

- (i) $S_{2^m}(0) \xrightarrow{f_{a,b}} \mathbb{Q}_2 \setminus \overline{B}_{2^m}(0) \xrightarrow{f_{a,b}} S_{2^m}(0)$;
- (ii) if $m > 1$ then $S_{2^n}(0) \xrightarrow{f_{a,b}} S_{2^n}(0)$ for any $n \in \{1, \dots, m-1\}$.

Proposition 4.4. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements hold:*

- (i) The map $f_{a,b}$ is a bijective isometry on the set $\mathbb{Q}_2 \setminus (B_{2^m}(0) \cup \mathcal{P}(f_{a,b}))$;
- (ii) If $m > 1$, then $f_{a,b}$ is a bijective isometry on each sphere $S_{2^n}(0)$ for all $n \in \{1, \dots, m-1\}$.

We now analyse the behaviour of $f_{a,b}$ on the unit sphere. It is worth noting that the unit sphere is not invariant under this function.

Lemma 4.1. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then there exists a partition $\{C_0, C_1, C_2\}$ of $\mathbb{Z}_2^*$ such that*

- (i) $f_{a,b}(C_0) = 2\mathbb{Z}_2$,
- (ii) $f_{a,b}(C_1) \subset \mathbb{Z}_2^*$,
- (iii) $f_{a,b}(C_2) = \mathbb{Q}_2 \setminus \mathbb{Z}_2$.

Proof: In the proof of the Proposition 4.2, we actually proved that for every $\gamma \in 2\mathbb{Z}_2$ the equation $f_{a,b}(t) = \gamma$ has a solution on $\mathbb{Z}_2^*$. So, we set

$$C_0 := f_{a,b}^{-1}(2\mathbb{Z}_2) \cap \mathbb{Z}_2^*.$$

Analogously, let

$$C_2 := f_{a,b}^{-1}(\mathbb{Q}_2 \setminus \mathbb{Z}_2) \cap \mathbb{Z}_2^*.$$

By construction,

$$C_0 \cap C_2 = \emptyset.$$

On the other hand, the point $x_0 = 1 \in \mathbb{Z}_2^*$ is a fixed point of $f_{a,b}$, hence it does not belong to $C_0 \cup C_2$ under the above classification. Therefore, the family $\{C_0, C_2\}$ does not form a partition of $\mathbb{Z}_2^*$.

Consequently, defining

$$C_1 := \mathbb{Z}_2^* \setminus (C_0 \cup C_2),$$

we obtain the desired decomposition, which completes the proof. $\square$

As a conclusion of the case $|a|_2 = |b|_2 > 1$ we formulate the following theorem.

Theorem 4.3. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then for any $x \in \mathbb{Q}_2 \setminus (\mathbb{Z}_2^* \cup \mathcal{P}(f_{a,b}))$ it holds*

$$\mathcal{O}^+(x, f_{a,b}) \subset \omega_{f_{a,b}}(x).$$

Proof: It is enough to check only for $x \in \overline{B}_{\frac{1}{2^m}}(0)$. By Proposition 4.2 we have

$$|f_{a,b}^2(x) - f_{a,b}(x)|_2 = |f_{a,b}(x) - x|_2.$$

Due to the strong triangle inequality, we obtain

$$|f_{a,b}^2(x) - x|_2 \leq \frac{|f_{a,b}(x) - x|_2}{2}.$$

Then by induction, we can easily check that

$$|f_{a,b}^{2^n}(x) - x|_2 \leq \frac{|f_{a,b}(x) - x|_2}{2^n}.$$

This means that $x \in \omega_{f_{a,b}}(x)$. More generally, if $k \geq 0$ then $f^{2^n+k}(x) \rightarrow f^k(x)$ as $n \rightarrow \infty$. Hence,

$$\mathcal{O}^+(x, f_{a,b}) \subset \omega_{f_{a,b}}(x).$$

$\square$

5. ERGODICITY OF $f_{a,b}$

In the previous section, we established that the dynamical system $(\mathbb{Q}_2, f)$ is regular (or cyclic) in the case $|a|_2 = |b|_2 \leq 1$, which implies that the asymptotic behavior of trajectories is completely predictable.

In contrast, the regime $|a|_2 = |b|_2 > 1$ gives rise to more complex dynamical behavior. In order to study properties such as minimality and ergodicity, it is crucial to identify and analyze invariant spheres associated with the dynamics.

We now recall the notions of measure-preserving transformations and ergodicity, both of which are formulated with respect to a given measure.

Definition 5.1. Let $(X, \mathcal{B}, \mu)$ be a measure space. A measurable mapping $f : X \rightarrow X$ is called *measure-preserving* if

$$\mu(f^{-1}(A)) = \mu(A) \quad (5.1)$$

for every $A \in \mathcal{B}$. In this case, we also say that f preserves the measure μ .

Equivalently, if (5.1) holds, then the measure μ is said to be invariant under the mapping f . It is important to note that the condition (5.1) is stated in terms of the preimage $f^{-1}(A)$ rather than the image $f(A)$. This distinction is essential, especially when the mapping f is not invertible.

Definition 5.2. Let $(X, \mathcal{B}, \mu)$ be a probability space and let $f : X \rightarrow X$ be a measure-preserving mapping. The mapping f is called *ergodic* with respect to the measure μ if every invariant measurable set is trivial with respect to μ ; that is, whenever $A \in \mathcal{B}$ satisfies $f^{-1}(A) = A$, we have

$$\mu(A) = 0 \quad \text{or} \quad \mu(A) = 1.$$

In this case, the dynamical system (X, f) is said to be ergodic with respect to μ .

The space $\mathbb{Z}_2$ is endowed with its natural probability measure, namely the normalized Haar measure μ , for which

$$\mu(\mathbb{Z}_2) = 1.$$

The elementary μ -measurable sets are the 2-adic balls. Recall that, for $n \geq 1$, the 2-adic open ball of radius 2^{-n+1} with centre c is given by

$$B_{\frac{1}{2^{n-1}}}(c) = c + 2^n \mathbb{Z}_2 = \{x \in \mathbb{Z}_2 : x \equiv c \pmod{2^n}\}.$$

Since every residue class modulo 2^n has a unique representative in the set $\{0, 1, \dots, 2^n - 1\}$, we may, without loss of generality, choose the centre c of the ball $c + 2^n \mathbb{Z}_2$ so that

$$c \in \{0, 1, \dots, 2^n - 1\}.$$

The volume of this ball is defined as

$$\mu(c + 2^n \mathbb{Z}_2) = 2^{-n}.$$

Every ball in $\mathbb{Z}_2$ can be viewed as a copy of $\mathbb{Z}_2$ itself. Indeed, if $B_{\frac{1}{2^n}}(c) \subset \mathbb{Z}_2$, then the map

$$h(x) = \frac{x - c}{2^n}$$

defines a homeomorphism from $B_{\frac{1}{2^n}}(c)$ onto $\mathbb{Z}_2$. This identification allows us to define a *normalized Haar measure* on the ball by pulling back the standard Haar measure on $\mathbb{Z}_2$. Moreover, since the Haar measure is locally finite and translation-invariant, it can be naturally restricted to any open compact subset $X \subset \mathbb{Z}_2$.

In [24], the author builds upon the work of Anashin and Yurova to establish ergodicity criteria for 2-adic rational mappings.

Theorem 5.1. [24] Let $P, Q : \mathbb{Z}_2^* \rightarrow \mathbb{Z}_2^*$ be polynomials whose coefficients are 2-adic integers.

Set $P(x) = \sum_i p_i x^i$, $Q(x) = \sum_i q_i x^i$, $P_1 = \sum_{i \text{ odd}} p_i$, $P_2 = \sum_{i \text{ even}} p_i$, $Q_1 = \sum_{i \text{ odd}} q_i$ and $Q_2 = \sum_{i \text{ even}} q_i$.

The rational function $\frac{P}{Q}$ is ergodic if and only if one of the following situations occur:

- (1) $P_1 \equiv 1(\text{mod } 4)$, $P_2 \equiv 2(\text{mod } 4)$, $Q_1 \equiv 0(\text{mod } 4)$ and $Q_2 \equiv 1(\text{mod } 4)$.
- (2) $P_1 \equiv 3(\text{mod } 4)$, $P_2 \equiv 2(\text{mod } 4)$, $Q_1 \equiv 0(\text{mod } 4)$ and $Q_2 \equiv 3(\text{mod } 4)$.
- (3) $P_1 \equiv 1(\text{mod } 4)$, $P_2 \equiv 0(\text{mod } 4)$, $Q_1 \equiv 2(\text{mod } 4)$ and $Q_2 \equiv 1(\text{mod } 4)$.
- (4) $P_1 \equiv 3(\text{mod } 4)$, $P_2 \equiv 0(\text{mod } 4)$, $Q_1 \equiv 2(\text{mod } 4)$ and $Q_2 \equiv 3(\text{mod } 4)$.
- (5) One of the previous cases with P and Q interchanged.

Lemma 5.1. Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements equivalent:

- (i) $f_{a,b}^2$ is ergodic on $S_{\frac{1}{2^m}}(0)$;
- (ii) $|b + a - 1|_2 = |2a|_2$.

Proof: By Proposition 4.1 we have $f_{a,b}^2 : S_{\frac{1}{2^m}}(0) \rightarrow S_{\frac{1}{2^m}}(0)$. A direct computation yields

$$f_{a,b}^2(x) = \frac{Ax^4 + Bx^3 + Cx^2 + Dx + E}{Ex^4 + Dx^3 + Cx^2 + Bx + A},$$

where

$$\begin{aligned} A &= a^3 + ab + 1, \\ B &= 2a^2b + ab^2 + b^2 + 2b, \\ C &= a^2b + 2a^2 + 2a + ab^2 + b^3 + b^2 + b, \\ D &= ab^2 + b^2 + 4ab, \\ E &= a^2 + ab + a. \end{aligned}$$

Consider the homeomorphism $h(x) = ax$ from $S_{\frac{1}{2^m}}(0)$ onto $\mathbb{Z}_2^*$, and define the conjugate map $g := h \circ f_{a,b}^2 \circ h^{-1}$. Then

$$g(x) = \frac{a^{-6}Ax^4 + a^{-5}Bx^3 + a^{-4}Cx^2 + a^{-3}Dx + a^{-2}E}{a^{-7}Ex^4 + a^{-6}Dx^3 + a^{-5}Cx^2 + a^{-4}Bx + a^{-3}A}.$$

Set

$$\begin{aligned} P(x) &= a^{-6}Ax^4 + a^{-5}Bx^3 + a^{-4}Cx^2 + a^{-3}Dx + a^{-2}E, \\ Q(x) &= a^{-7}Ex^4 + a^{-6}Dx^3 + a^{-5}Cx^2 + a^{-4}Bx + a^{-3}A. \end{aligned}$$

Then $P, Q : \mathbb{Z}_2^* \rightarrow \mathbb{Z}_2^*$, and the ergodicity of $(\mathbb{Z}_2^*, g)$ can be verified using Theorem 5.1. Using the estimates

$$|A|_2 = |B|_2 = |C|_2 = |D|_2 = |a^3|_2, \quad |E|_2 < |a^2|_2,$$

we compute modulo 4:

$$\begin{aligned} P_1 &\equiv \frac{B}{a^5} + \frac{D}{a^3} \equiv \frac{D}{a^3} \equiv 1 + \frac{1}{a}(\text{mod } 4), \\ P_2 &\equiv \frac{A}{a^6} + \frac{C}{a^4} + \frac{E}{a^2} \equiv 1 + \frac{b}{a}(\text{mod } 4), \\ Q_1 &\equiv \frac{D}{a^6} + \frac{B}{a^4} \equiv \frac{b^2}{a^3} \equiv \frac{1}{a}(\text{mod } 4), \\ Q_2 &\equiv \frac{E}{a^7} + \frac{C}{a^5} + \frac{A}{a^3} \equiv 1 + \frac{b}{a^2} \equiv 1 + \frac{1}{a}(\text{mod } 4). \end{aligned}$$

Therefore, by Theorem 5.1, the ergodicity condition is equivalent to

$$1 + \frac{b}{a} - \frac{1}{a} \equiv 2(\text{mod } 4).$$

This congruence holds if and only if

$$|b + a - 1|_2 = |2a|_2.$$

□

Lemma 5.2. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements are equivalent:*

- (i) $f_{a,b}$ is minimal on $\overline{B}_{\frac{1}{2^m}}(0)$;
- (ii) $f_{a,b}^2$ is minimal on $S_{\frac{1}{2^m}}(0)$;
- (iii) $f_{a,b}^2$ is minimal on $B_{\frac{1}{2^m}}(0)$.

Proof: By Proposition 4.1, the mapping $f_{a,b}$ sends $B_{\frac{1}{2^m}}(0)$ into $S_{\frac{1}{2^m}}(0)$ and $S_{\frac{1}{2^m}}(0)$ into $B_{\frac{1}{2^m}}(0)$. Therefore, both $B_{\frac{1}{2^m}}(0)$ and $S_{\frac{1}{2^m}}(0)$ are invariant under $f_{a,b}^2$.

We first prove (i) $\implies$ (ii). Let $x, y \in S_{\frac{1}{2^m}}(0)$ and $\varepsilon > 0$. Since $f_{a,b}$ is minimal on $\overline{B}_{\frac{1}{2^m}}(0)$, the positive orbit of x under $f_{a,b}$ is dense in $\overline{B}_{\frac{1}{2^m}}(0)$. Hence, for some $n \geq 0$, we have

$$|f_{a,b}^n(x) - y|_2 < \varepsilon.$$

Since $f_{a,b}$ maps $S_{\frac{1}{2^m}}(0)$ into $B_{\frac{1}{2^m}}(0)$ and $B_{\frac{1}{2^m}}(0)$ into $S_{\frac{1}{2^m}}(0)$, the point $f_{a,b}^n(x)$ belongs to $S_{\frac{1}{2^m}}(0)$ only when n is even. Thus $n = 2n_0$ for some $n_0 \geq 0$, and therefore

$$|f_{a,b}^{2n_0}(x) - y|_2 < \varepsilon.$$

This proves that $f_{a,b}^2$ is minimal on $S_{\frac{1}{2^m}}(0)$.

Now we prove (ii) $\implies$ (iii). Let $x, y \in B_{\frac{1}{2^m}}(0)$ and $\varepsilon > 0$. By Proposition 4.1,

$$f_{a,b}(x), f_{a,b}(y) \in S_{\frac{1}{2^m}}(0).$$

Since $f_{a,b}^2$ is minimal on $S_{\frac{1}{2^m}}(0)$, there exists $n_0 \geq 0$ such that

$$|f_{a,b}^{2n_0}(f_{a,b}(x)) - f_{a,b}(y)|_2 < \varepsilon. \quad (5.2)$$

Since

$$f_{a,b}^{2n_0}(f_{a,b}(x)) = f_{a,b}(f_{a,b}^{2n_0+1}(x)),$$

and since, by Proposition 4.2, $f_{a,b}$ is an isometry on $\overline{B}_{\frac{1}{2^m}}(0)$, from (5.2) we obtain

$$|f_{a,b}^{2n_0+1}(x) - y|_2 = |f_{a,b}(f_{a,b}^{2n_0}(x)) - f_{a,b}(y)|_2 < \varepsilon.$$

Thus the orbit of every point of $B_{\frac{1}{2^m}}(0)$ under $f_{a,b}^2$ is dense in $B_{\frac{1}{2^m}}(0)$. Hence $f_{a,b}^2$ is minimal on $B_{\frac{1}{2^m}}(0)$.

It remains to prove (iii) $\implies$ (i). Let $x, y \in \overline{B}_{\frac{1}{2^m}}(0)$ and let $\varepsilon > 0$ be arbitrary. We consider the following cases.

If $x, y \in B_{\frac{1}{2^m}}(0)$, then, by the minimality of $f_{a,b}^2$ on $B_{\frac{1}{2^m}}(0)$, there exists $n_0 \geq 0$ such that

$$|f_{a,b}^{2n_0}(x) - y|_2 < \varepsilon.$$

Let $x \in S_{\frac{1}{2^m}}(0)$ and $y \in B_{\frac{1}{2^m}}(0)$. Since $f_{a,b}(x) \in B_{\frac{1}{2^m}}(0)$, by the minimality of $f_{a,b}^2$ on $B_{\frac{1}{2^m}}(0)$, there exists $n_0 \geq 0$ such that

$$|f_{a,b}^{2n_0}(f_{a,b}(x)) - y|_2 < \varepsilon.$$

Equivalently,

$$|f_{a,b}^{2n_0+1}(x) - y|_2 < \varepsilon.$$

Now let $x \in B_{\frac{1}{2^m}}(0)$ and $y \in S_{\frac{1}{2^m}}(0)$. Since $f_{a,b}(y) \in B_{\frac{1}{2^m}}(0)$, there exists $n_0 \geq 1$ such that

$$|f_{a,b}^{2n_0}(x) - f_{a,b}(y)|_2 < \varepsilon.$$

Using the isometry of $f_{a,b}$ on $\overline{B}_{\frac{1}{2^m}}(0)$, we get

$$|f_{a,b}^{2n_0-1}(x) - y|_2 = |f_{a,b}(f_{a,b}^{2n_0-1}(x)) - f_{a,b}(y)|_2 < \varepsilon.$$

Finally, let $x, y \in S_{\frac{1}{2^m}}(0)$. Since $f_{a,b}(x), f_{a,b}(y) \in B_{\frac{1}{2^m}}(0)$, there exists $n_0 \geq 0$ such that

$$|f_{a,b}^{2n_0}(f_{a,b}(x)) - f_{a,b}(y)|_2 < \varepsilon.$$

That is,

$$|f_{a,b}(f_{a,b}^{2n_0}(x)) - f_{a,b}(y)|_2 < \varepsilon.$$

Again, by the isometry of $f_{a,b}$ on $\overline{B}_{\frac{1}{2^m}}(0)$, we obtain

$$|f_{a,b}^{2n_0}(x) - y|_2 < \varepsilon.$$

Thus, for arbitrary $x, y \in \overline{B}_{\frac{1}{2^m}}(0)$ and every $\varepsilon > 0$, the positive orbit of x under $f_{a,b}$ comes ε -close to y . Therefore,

$$\overline{\mathcal{O}^+(x, f_{a,b})} = \overline{B}_{\frac{1}{2^m}}(0)$$

for every $x \in \overline{B}_{\frac{1}{2^m}}(0)$. Hence $f_{a,b}$ is minimal on $\overline{B}_{\frac{1}{2^m}}(0)$. $\square$

Since ergodicity in the above sense is equivalent to minimality, the conclusions of Lemma 5.1 and Lemma 5.2 can be summarized in the following theorem.

Theorem 5.2. *Let $|a|_2 = |b|_2 = 2^m > 1$. Consider the probability space $(\overline{B}_{\frac{1}{2^m}}(0), \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ -algebra on the metric space $(\overline{B}_{\frac{1}{2^m}}(0), |\cdot|_2)$ and μ is the normalized Haar measure. Then the following statements equivalent:*

- (i) $f_{a,b}$ is ergodic on $\overline{B}_{\frac{1}{2^m}}(0)$;
- (ii) $|b + a - 1|_2 = |2a|_2$.

We now study the ergodicity of $f_{a,b}$ on each invariant sphere $S_{\frac{1}{2^n}}(0)$ under the assumption that case (ii) of Proposition 4.2 holds.

Theorem 5.3. *Let $|a|_2 = |b|_2 = 2^m$, $m > 1$. For $n \in \{1, \dots, m-1\}$ consider the probability space $(S_{\frac{1}{2^n}}(0), \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ -algebra on the metric space $(S_{\frac{1}{2^n}}(0), |\cdot|_2)$ and μ is the normalized Haar measure.*

Then the map $f_{a,b}$ is ergodic on $S_{\frac{1}{2^n}}(0)$ if and only if the following conditions hold:

- (a) $|b - a|_2 < |2a|_2$;
- (b) $n = m - 1$;
- (c) $n \neq -\text{ord}_2(b - a)$.

Proof: Let $n \in \{1, \dots, m-1\}$ and assume that $n \neq -\text{ord}_2(b - a)$. Define a homeomorphism $h : S_{\frac{1}{2^n}}(0) \rightarrow \mathbb{Z}_2^*$ by

$$h(x) = \frac{x}{2^n}.$$

Consider the conjugate map $g := h \circ f_{a,b} \circ h^{-1}$, that is,

$$g(x) = \frac{2^n x^2 + \frac{b}{a}x + \frac{1}{2^n a}}{\frac{4^n}{a}x^2 + \frac{2^n b}{a}x + 1}.$$

It is straightforward to verify that the set $\mathbb{Z}_2^*$ is invariant under both the numerator and denominator polynomials

$$P(x) = 2^n x^2 + \frac{b}{a}x + \frac{1}{2^n a}, \quad Q(x) = \frac{4^n}{a}x^2 + \frac{2^n b}{a}x + 1.$$

Moreover, one can check that

$$Q_1 \equiv \begin{cases} 0 \pmod{4}, & \text{if } n > 1, \\ 2 \pmod{4}, & \text{if } n = 1, \end{cases} \quad Q_2 \equiv 1 \pmod{4}.$$

Then, by Theorem 5.1, the dynamical system $(\mathbb{Z}_2^*, g)$ is ergodic if and only if

$$P_1 \equiv 1 \pmod{4},$$

and

$$P_2 \equiv \begin{cases} 0 \pmod{4}, & \text{if } n = 1, \\ 2 \pmod{4}, & \text{if } n > 1. \end{cases}$$

A direct computation shows that the condition $P_1 \equiv 1 \pmod{4}$ is equivalent to

$$|b - a|_2 < |2a|_2.$$

Furthermore, if $n = 1$, then $P_2 \equiv 0 \pmod{4}$ is equivalent to $|a|_2 = 4$, while for $n > 1$, the condition $P_2 \equiv 2 \pmod{4}$ is equivalent to

$$|a|_2 = 2^{n+1}.$$

Thus, we conclude that $(\mathbb{Z}_2^*, g)$ is ergodic if and only if the following conditions hold:

- $|b - a|_2 < |2a|_2$;
- $n = m - 1$.

Finally, suppose that $n = -\text{ord}_2(b - a)$. In this case, the sphere $S_{\frac{1}{2^n}}(0)$ contains a fixed point x_- . Therefore, the system cannot be ergodic on this sphere. $\square$

Remark 5.1. In general, using the identity

$$f_{a,b}\left(\frac{1}{x}\right) = \frac{1}{f_{a,b}(x)},$$

one can readily extend the results of Theorems 5.2 and 5.3 to characterize the ergodicity of the map $f_{a,b}$ on $\mathbb{Q}_2 \setminus \mathbb{Z}_2$.

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