

Optimal Control and Pursuit Game for Planar Rigid Segments with Inertial Dynamics

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Abstract. This paper studies time-optimal control and pursuit differential game problems for planar rigid segments with inertial dynamics, actuated by bounded forces applied perpendicularly at their endpoints. The objective of the control problem is to transfer a segment between prescribed configurations in minimum time. We derive closed-form expressions for the minimal transfer time, together with a complete analytical description of optimal motions. The obtained time-optimal steering results are then used to construct a capture strategy for the pursuit game, which guarantees finite-time capture.

Keywords: time-optimal control, pursuit differential game, Pontryagin's maximum principle, bang-bang control, inertial dynamics, rigid segment, finite-time capture

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1. INTRODUCTION

Time-optimal control problems constitute an important class of problems in optimal control theory [1, 2, 3]. The main tool for their analysis is Pontryagin's Maximum Principle [4, 5, 6], which often leads to bang-bang optimal controls [7, 8, 9]. Time-optimal problems for planar motion systems and related models on $SE(2)$ were studied in [10, 11, 12], where optimal trajectories were described with alternating rotational and translational phases. Optimal control problems for rigid body rotation under bounded controls were investigated in [13, 14, 15].

Recent studies have also addressed time-optimal and controllability problems in more general settings. In particular, time-optimal control problems with integral constraints for heat transfer models were studied in [16], while the controllability and stability properties of infinite-dimensional linear systems were investigated in [17].

The theory of pursuit-evasion differential games originated in the pioneering works of Isaacs [18] and further developed in [19, 20]. Simple-motion models with one evader and multiple pursuers were studied in [21]. Group pursuit-evasion problems on manifolds were considered in [22], while infinite-dimensional formulations were analyzed in [23, 24, 25, 26].

In this paper, we consider a planar rigid segment actuated by bounded forces applied perpendicularly at its endpoints. The dynamics are described by a second-order control system with coupled translational and rotational motions. Using Pontryagin's Maximum Principle, we show that singular arcs do not occur and that every time-optimal control is of bang-bang type. Furthermore, the structure of time-optimal trajectories is characterized, and an explicit analytical formula for the minimal transfer time is derived. Finally, the obtained optimal steering results are applied to a pursuit differential game for two rigid segments, and a constructive pursuit strategy guaranteeing finite-time capture is established under the assumption that the pursuer possesses a larger control resource than the evader.

2. TIME-OPTIMAL CONTROL PROBLEM

2.1. Problem formulation. Consider a uniform rigid segment A_0B_0 of fixed length $l > 0$ and total mass m_0 moving in the Euclidean plane $\mathbb{R}^2$. The mass is continuously distributed along the segment. The initial positions of the endpoints are given by

$$A_0 = (a_0, b_0), \quad B_0 = (a_0 + l, b_0),$$

and the desired terminal positions are

$$C = (0, 0), \quad D = (l, 0).$$

During the motion, the length of the segment remains constant. Let $A(t), B(t)$ denote the positions of the endpoints at time $t \geq 0$. The segment is actuated by bounded external forces $F_A(t)$ and $F_B(t)$, applied perpendicularly at the endpoints $A(t)$ and $B(t)$, respectively:

$$|F_A(t)| \leq \rho, \quad |F_B(t)| \leq \rho,$$

where $\rho > 0$ is a given constant representing the maximal admissible force. The objective is to transfer the segment from the initial configuration A_0B_0 to the terminal configuration CD in minimal time.

2.2. Reduction to the equivalent problem. To reduce the problem to an equivalent formulation, we introduce a coordinate representation of the segment. Let $\theta(t) \in \mathbb{R}$ be the orientation angle, i.e., the angle between the segment and the x -axis (see Fig. 1). Define the unit vectors

$$e(\theta) = (\cos \theta, \sin \theta), \quad e^\perp(\theta) = (-\sin \theta, \cos \theta),$$

representing the directions along and orthogonal to the segment, respectively.

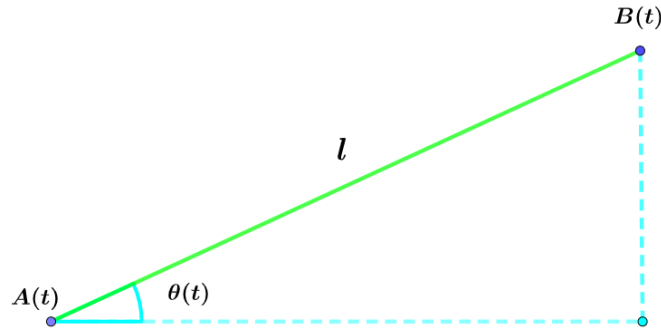


FIGURE 1. The orientation angle $\theta(t)$ of the rigid segment

The endpoints can be written as

$$A(t) = (x(t), y(t)), \quad B(t) = A(t) + l e(\theta(t)). \quad (2.1)$$

The midpoint (center of mass) of the segment is defined as

$$M(t) = \frac{1}{2}(A(t) + B(t)).$$

Using (2.1), this expression can be equivalently written as

$$M(t) = A(t) + \frac{l}{2} e(\theta(t)).$$

Hence, the initial and terminal positions of the midpoint are

$$M_0 = M(0) = \left(a_0 + \frac{l}{2}, b_0\right), \quad M_T = M(T) = \left(\frac{l}{2}, 0\right).$$

Since the forces are applied perpendicularly at the endpoints $A(t)$ and $B(t)$; therefore, the control inputs act along the direction $e^\perp(\theta(t))$ (see Fig. 2). Let $u_1(t)$ and $u_2(t)$ be scalar control functions associated with the endpoints $A(t)$ and $B(t)$, respectively. These controls satisfy

$$|u_1(t)| \leq \rho, \quad |u_2(t)| \leq \rho.$$

The corresponding control forces are given by

$$F_A(t) = u_1(t) e^\perp(\theta(t)), \quad F_B(t) = u_2(t) e^\perp(\theta(t)).$$

We now consider the following two components of motion.

Translational motion. By Newton's second law, the motion of the center of mass is governed by the resultant force:

$$m_0 \ddot{M}(t) = F_A(t) + F_B(t).$$

Substituting the expressions for the forces, we obtain

$$\ddot{M}(t) = \frac{u_1(t) + u_2(t)}{m_0} e^\perp(\theta(t)). \quad (2.2)$$

This equation describes the translational dynamics of the midpoint $M(t)$ of the segment.

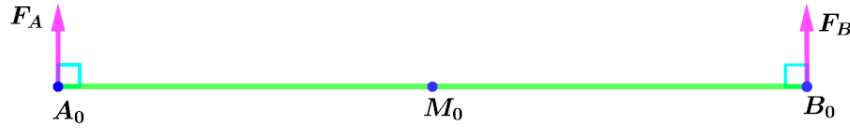


FIGURE 2. Control forces F_A and F_B acting on the segment endpoints

Rotational motion. The rotational dynamics are determined by the torque with respect to the midpoint $M(t)$. For planar motion, the torque is a scalar quantity given by

$$\tau = \overrightarrow{MA} \times F_A(t) + \overrightarrow{MB} \times F_B(t),$$

where

$$\overrightarrow{MA} = -\frac{l}{2} e(\theta(t)), \quad \overrightarrow{MB} = \frac{l}{2} e(\theta(t)),$$

and the symbol $\times$ denotes the scalar cross product in $\mathbb{R}^2$. Substituting the expressions for the forces and using the identity $e(\theta) \times e^\perp(\theta) = 1$, we obtain $\tau = \frac{l}{2}(u_2 - u_1)$. For a uniform segment of length l and mass m_0 , the moment of inertia about its midpoint is

$$J = \frac{m_0 l^2}{12}.$$

Hence, the rotational equation of motion takes the form

$$J \ddot{\theta}(t) = \tau,$$

which yields

$$\ddot{\theta}(t) = \frac{6}{m_0 l} (u_2(t) - u_1(t)). \quad (2.3)$$

Thus, the translational and rotational dynamics of the segment are completely determined by the scalar control inputs $u_1(t)$ and $u_2(t)$.

2.3. New Control Variables. The scalar controls $u_1(t)$ and $u_2(t)$ act along the same direction $e^\perp(\theta(t))$ and therefore produce only two independent effects: 1) a translation of the midpoint, 2) a rotation about the midpoint. To explicitly separate these effects, we introduce new control variables

$$s = \frac{1}{2}(u_1 + u_2), \quad a = \frac{1}{2}(u_2 - u_1). \quad (2.4)$$

The transformation (2.4) is linear and invertible, with its inverse given by $u_1 = s - a$, $u_2 = s + a$. Under this change of variables, the equations of motion (2.2), (2.3) take the form

$$\ddot{M}(t) = \frac{2}{m_0} s(t) e^\perp(\theta(t)), \quad \ddot{\theta}(t) = \frac{12}{m_0 l} a(t). \quad (2.5)$$

Thus, the control $s(t)$ governs the translational acceleration of the midpoint $M(t)$, while $a(t)$ governs the angular acceleration. This representation explicitly decouples the translational and rotational dynamics.

Since the original controls satisfy $|u_1(t)| \leq \rho$, $|u_2(t)| \leq \rho$, the admissible set in the (s, a) -coordinates is equivalently described by $|s - a| \leq \rho$, $|s + a| \leq \rho$. Using identities (2.4), we compute

$$|s| + |a| = \frac{1}{2}(|u_1 + u_2| + |u_2 - u_1|)$$

and obtain $|s| + |a| = \max\{|u_1|, |u_2|\} \leq \rho$. Therefore, the admissible control set can be written as

$$U = \{(s, a) \in \mathbb{R}^2 : |s| + |a| \leq \rho\}.$$

Geometrically, this set is a diamond-shaped region in the (s, a) -plane (see Fig. 3).

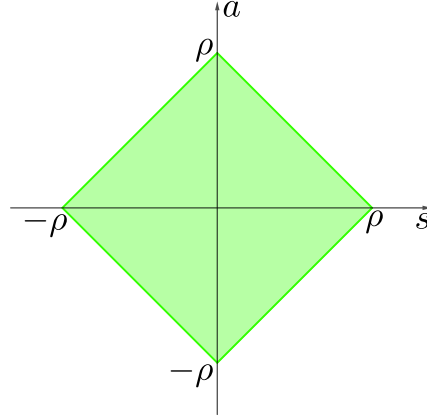


FIGURE 3. Admissible control set U in the transformed (s, a) plane

2.4. State-Space Formulation. Introduce the auxiliary variables

$$\dot{M} = m, \quad \dot{\theta} = \omega,$$

where m denotes the linear velocity of the midpoint, and ω denotes the angular velocity of the segment. The state vector is defined as

$$x = (M, m, \theta, \omega) \in \mathbb{R}^6, \quad M \in \mathbb{R}^2, \quad m \in \mathbb{R}^2.$$

The system can then be written in state-space form as

$$\begin{aligned} \dot{M} &= m, \\ \dot{m} &= \frac{2}{m_0} s e^\perp(\theta), \\ \dot{\theta} &= \omega, \\ \dot{\omega} &= \frac{12}{m_0 l} a, \end{aligned}$$

subject to the control constraint

$$|s| + |a| \leq \rho.$$

The initial conditions are

$$M(0) = M_0, \quad m(0) = 0, \quad \theta(0) = 0, \quad \omega(0) = 0,$$

and the terminal conditions are

$$M(T) = M_T, \quad m(T) = 0, \quad \theta(T) = 0, \quad \omega(T) = 0.$$

2.5. Optimal Control Problem. Based on the dynamic model, we formulate the time-optimal control problem. The control function

$$u(\cdot) = (s(\cdot), a(\cdot)) \in U$$

is to be chosen to minimize the transfer time T , subject to the system dynamics

$$\dot{x} = f(x, u) = \begin{pmatrix} m \\ \frac{2}{m_0} s e^\perp(\theta) \\ \omega \\ \frac{12}{m_0 l} a \end{pmatrix}$$

and the boundary conditions

$$x(0) = x_0 = (M_0, 0, 0, 0), \quad x(T) = x_T = (M_T, 0, 0, 0).$$

The following theorem describes the structure of time-optimal trajectories.

Theorem 2.1. *Every time-optimal trajectory consists of three phases of the form rotation $\rightarrow$ translation $\rightarrow$ rotation, and the minimal transfer time is given by*

$$T_{\min} = 2\sqrt{\frac{m_0 l |\theta^*|}{3\rho}} + \sqrt{\frac{2m_0 \sqrt{a_0^2 + b_0^2}}{\rho}},$$

where θ^* is the angle of the smallest absolute value such that

$$e^\perp(\theta^*) \parallel (M_T - M_0).$$

The proof of Theorem 2.1 is developed in Sections 2.6–2.8 using Pontryagin's Maximum Principle and a geometric analysis of optimal trajectories.

2.6. Pontryagin's Maximum Principle Analysis. Consider the control system

$$\dot{x} = f(x, u), \quad u(t) \in U, \quad x(0) = x_0, \quad x(T) = x_T.$$

To characterize optimal control and the corresponding trajectory, we apply Pontryagin's Maximum Principle (PMP), which provides necessary conditions for optimality in constrained control problems. If $u^*(\cdot)$ is an optimal control and $x^*(\cdot)$ is the corresponding optimal trajectory, then there exists a nontrivial absolutely continuous adjoint vector

$$p(\cdot) = (p_M(\cdot), p_m(\cdot), p_\theta(\cdot), p_\omega(\cdot)),$$

where $p_M, p_m \in \mathbb{R}^2$, $p_\theta, p_\omega \in \mathbb{R}$, and a constant $p_0 \leq 0$. For the normal minimum-time case, $p_0 = -1$. Moreover, $(p(\cdot), p_0) \neq 0$. The Hamiltonian is defined by

$$H(x, p, u) = p \cdot f(x, u) + p_0.$$

Necessary conditions of optimality:

1) Hamiltonian system:

$$\dot{p}_i = -\frac{\partial H}{\partial x_i},$$

or explicitly,

$$\begin{aligned} \dot{p}_M &= 0, \\ \dot{p}_m &= -p_M, \\ \dot{p}_\theta &= -s \frac{2}{m_0} \left\langle p_m, \frac{de^\perp(\theta)}{d\theta} \right\rangle = \frac{2}{m_0} s \langle p_m, e(\theta) \rangle, \\ \dot{p}_\omega &= -p_\theta. \end{aligned}$$

2) Maximum condition:

$$H(x^*(t), p(t), u^*(t)) = \max_{u \in U} H(x^*(t), p(t), u), \quad t \in [0, T].$$

3) Transversality condition:

$$H(x^*(t), p(t), u^*(t)) \equiv 0.$$

Define the switching functions:

$$\phi_1(t) = \frac{2}{m_0} \langle p_m(t), e^\perp(\theta(t)) \rangle, \quad \phi_2(t) = \frac{12}{m_0 l} p_\omega(t).$$

Then the Hamiltonian can be written as

$$H = -1 + p_M \cdot m + p_\theta \omega + s\phi_1 + a\phi_2.$$

The maximization problem

$$\max_{|s|+|a| \leq \rho} (s\phi_1 + a\phi_2)$$

leads to:

- If $|\phi_1| > |\phi_2|$, then $s = \rho \operatorname{sgn}(\phi_1)$, $a = 0$.
- If $|\phi_1| < |\phi_2|$, then $s = 0$, $a = \rho \operatorname{sgn}(\phi_2)$.
- If $|\phi_1| = |\phi_2|$, then $|s| + |a| = \rho$ (boundary control).

Thus, the optimal control is of bang–bang or boundary type.

2.7. Structure of Optimal Controls. We show that singular arcs do not occur for either control component.

Proposition 2.1. *No singular arc exists for either control component.*

Proof: **Absence of singular arcs for s .** Assume that $\phi_1(t) \equiv 0$ on a nontrivial interval I . Then

$$\langle p_m(t), e^\perp(\theta(t)) \rangle = 0,$$

which implies that $p_m(t)$ is collinear with $e(\theta(t))$, i.e.,

$$p_m(t) = \beta(t) e(\theta(t))$$

for some scalar function $\beta(t)$. From the adjoint system, we have $\dot{p}_m = -p_M$, $p_M = 0$, hence p_M is constant and $p_m(t) = -p_M t + C$, so that $p_m(t)$ is affine in time.

Thus, $p_m(t)$ is simultaneously an affine function of t and is aligned with the rotating vector $e(\theta(t))$. This is possible only if $p_m(t) \equiv 0$. Indeed, if $\theta(t)$ is nonconstant, the direction $e(\theta(t))$ rotates, which is incompatible with a nonzero affine function. If $\theta(t)$ is constant, then $a(t) = 0$, and the maximization condition implies a bang control unless $\phi_1 \equiv 0$, again leading to $p_m \equiv 0$.

From $\dot{p}_m = -p_M$, it follows that $p_M \equiv 0$, and consequently, all adjoint variables vanish. This contradicts the nontriviality condition of the adjoint vector. Hence, ϕ_1 cannot vanish on any nontrivial interval.

Absence of singular arcs for a . Assume that $\phi_2(t) \equiv 0$ on some interval I . Then $p_\omega(t) \equiv 0$. From $\dot{p}_\omega = -p_\theta$, it follows that $p_\theta(t) \equiv 0$. The adjoint equation for p_θ gives

$$\dot{p}_\theta = -\frac{2s}{m_0} \langle p_m, e(\theta) \rangle.$$

Hence, either $\langle p_m, e(\theta) \rangle \equiv 0$ or $s \equiv 0$. If $s \not\equiv 0$, then $p_m \equiv 0$, which again implies $p \equiv 0$, contradicting the nontriviality of the adjoint vector. If $s \equiv 0$, the system evolves under pure rotation. However, along a rotation arc, the switching function ϕ_2 cannot remain identically zero, since otherwise the maximum condition would be violated. This contradiction shows that ϕ_2 cannot vanish on any interval. Thus, no singular arcs exist for either control component, and every optimal control is of bang–bang type. $\square$

The Hamiltonian is affine with respect to the controls and the admissible control set

$$\{(s, a) : |s| + |a| \leq \rho\}$$

is a compact convex polygon with extreme points $(\pm\rho, 0)$ and $(0, \pm\rho)$. By the Pontryagin's Maximum Principle, optimal controls must attain their values at these extreme points. Therefore, optimal trajectories consist of a sequence of rotation and translation phases.

2.8. Optimal Trajectories and Time Computation. According to the maximum condition, the optimal control takes values in

$$(s, a) \in \{(\rho, 0), (-\rho, 0), (0, \rho), (0, -\rho)\}$$

almost everywhere. The selection is determined by the switching functions ϕ_1 and ϕ_2 . We consider the following cases separately.

Case A: Pure Rotation. If $|\phi_1(t)| < |\phi_2(t)|$, then the maximum condition $s(t) = 0$, $a(t) = \pm\rho$. The equations of motion (2.5) reduce to

$$\ddot{M}(t) = 0, \quad \ddot{\theta}(t) = \pm \frac{12\rho}{m_0 l}.$$

Under the initial conditions $M(0) = M_0$ and $\dot{M}(0) = 0$, it follows that $M(t) \equiv M_0$. Hence, the motion is a pure rotation about the midpoint M_0 . The angular acceleration is constant: $\alpha = \frac{12\rho}{m_0 l}$. The rotation angle θ^* is defined as the smallest angle in absolute value that satisfies

$$e^\perp(\theta^*) = \pm \frac{M_T - M_0}{\|M_T - M_0\|},$$

the direction of motion of the midpoint becomes aligned with the vector connecting the initial midpoint M_0 and the target midpoint M_T (see Fig. 4). By identifying components, we obtain

$$\sin \theta^* = \pm \frac{a_0}{\sqrt{a_0^2 + b_0^2}}, \quad \cos \theta^* = \mp \frac{b_0}{\sqrt{a_0^2 + b_0^2}}.$$

To achieve the rotation in minimal time with zero initial and final angular velocity, the optimal profile is symmetric bang–bang. Let $T_{\text{rot}} = 2t_1$ and

$$\ddot{\theta}(t) = \begin{cases} \alpha, & 0 \leq t \leq t_1, \\ -\alpha, & t_1 < t \leq 2t_1. \end{cases}$$

Integrating, we obtain $|\theta^*| = \alpha t_1^2$. Hence,

$$t_1 = \sqrt{\frac{m_0 l |\theta^*|}{12\rho}}, \quad T_{\text{rot}} = \sqrt{\frac{m_0 l |\theta^*|}{3\rho}}.$$

Case B: Pure Translation. If $|\phi_1(t)| > |\phi_2(t)|$, then the maximum condition yields $s(t) = \pm\rho$, $a(t) = 0$. The equations of motion (2.5) reduce to

$$\ddot{M}(t) = \pm \frac{2\rho}{m_0} e^\perp(\theta(t)), \quad \ddot{\theta}(t) = 0.$$

Hence, the angular velocity is constant. Since the translation phase starts after the rotation phase, we have $\dot{\theta}(T_{\text{rot}}) = 0$, $\theta(T_{\text{rot}}) = \theta^*$, and therefore $\theta(t) \equiv \theta^*$, $t \geq T_{\text{rot}}$. Thus, the motion reduces to a pure translation with fixed orientation θ^* along the direction $e^\perp(\theta^*)$. Let $d = |M_T - M_0| = \sqrt{a_0^2 + b_0^2}$. Introduce the scalar coordinate

$$x(t) = \langle M(t) - M_0, e^\perp(\theta^*) \rangle.$$

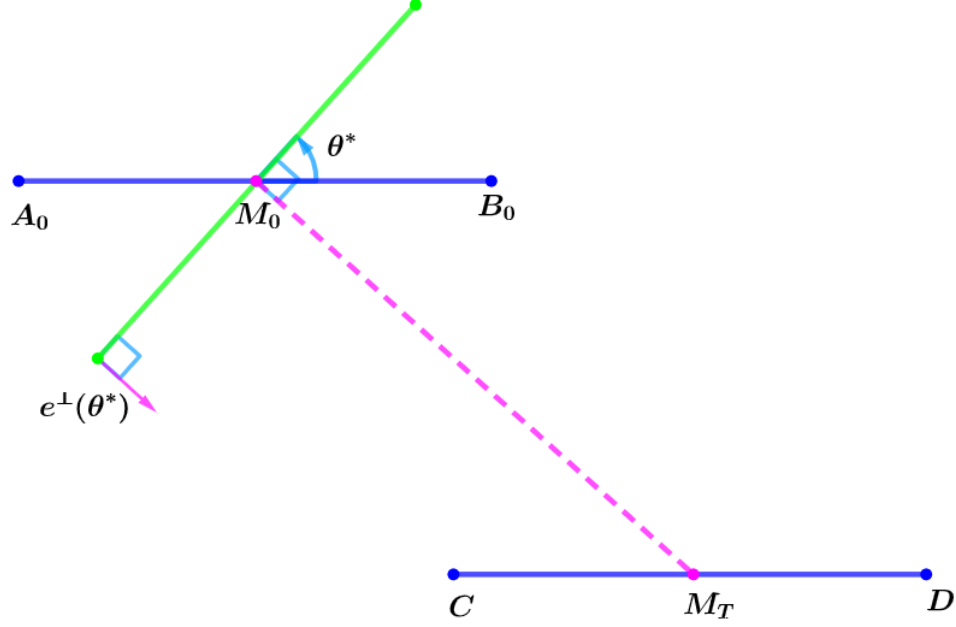


FIGURE 4. Bang-bang control structure for time-optimal rotation

Since $e^\perp(\theta^*)$ is a unit vector, we obtain

$$\ddot{x}(t) = \pm \frac{2\rho}{m_0},$$

with boundary conditions

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad x(T_1) = d, \quad \dot{x}(T_1) = 0.$$

This is exactly the time-optimal problem for the scalar double integrator already analyzed in Case A, with the substitutions

$$\theta \longrightarrow x, \quad \alpha \longrightarrow \frac{2\rho}{m_0}, \quad \theta^* \longrightarrow d.$$

By the same argument, the optimal control is bang-bang with one switching, and the minimal translation time is given by

$$T_{\text{trans}} = \sqrt{\frac{2m_0 d}{\rho}} = \sqrt{\frac{2m_0 \sqrt{a_0^2 + b_0^2}}{\rho}}.$$

The previous analysis implies that every time-optimal trajectory consists of exactly three phases: rotation $\rightarrow$ translation $\rightarrow$ rotation, namely (1) an initial rotation, (2) a pure translation, (3) a final rotation. Since the initial and final orientations coincide, the total rotation angle must vanish. Consequently, the trajectory necessarily contains two symmetric rotational phases of equal magnitude. From Case A, the minimal rotation time is

$$T_{\text{rot}} = \sqrt{\frac{m_0 l |\theta^*|}{3\rho}}.$$

Therefore, the minimal total transfer time is given by

$$T_{\text{min}} = 2T_{\text{rot}} + T_{\text{trans}} = 2\sqrt{\frac{m_0 l |\theta^*|}{3\rho}} + \sqrt{\frac{2m_0 \sqrt{a_0^2 + b_0^2}}{\rho}}.$$

This formula provides the minimal time required to steer the rigid segment from the initial configuration $A_0 B_0$ to the terminal configuration $C D$ under the bounded control constraint $|u_1|, |u_2| \leq \rho$. This completes the proof of Theorem 2.1.

Remark 2.1. The explicit formula for $T_{\min}$ holds under the assumption that the initial and terminal angular velocities vanish and that the initial and terminal orientations coincide. If nonzero boundary angular velocities are prescribed, additional switching patterns may arise, and the formula must be modified.

Example. For $m_0 = 1$, $l = 1$, $\rho = 1$, $a_0 = -5$, and $b_0 = 4$, the time-optimal motion of the rigid segment is illustrated in Fig. 5. The segment follows a rotation–translation–rotation strategy: it first undergoes a counterclockwise rotation with angle $\theta^* \approx 0.8961$ rad to align its direction with the target displacement; then moves along a straight translation path of length $d \approx 6.4031$; and finally performs a clockwise rotation with the identical angle, to reach the final configuration. The dashed curve represents the motion of the midpoint, while the solid lines show the movement of the endpoints. The minimal transfer time is $T_{\min} = 4.67$.

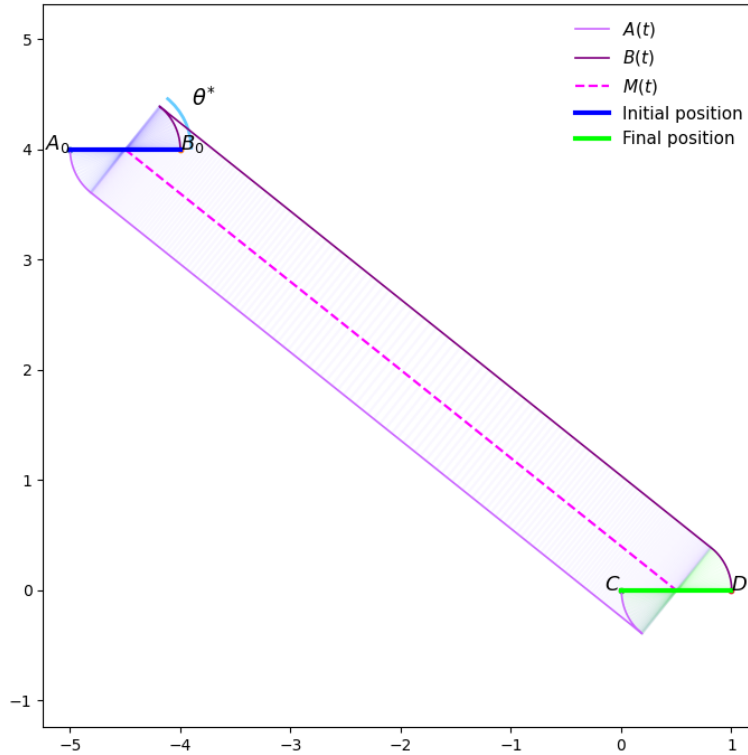


FIGURE 5. Time-optimal motion of the rigid segment from the initial configuration A_0B_0 to the terminal configuration CD

3. A PURSUIT DIFFERENTIAL GAME

Consider a planar pursuit differential game involving two rigid segments (the pursuer P and the evader E) with identical dynamics but different control resources. Let $M_P(t)$ and $M_E(t)$ denote the midpoints, and $\theta_P(t)$ and $\theta_E(t)$ denote the orientation angles of the pursuing and evading segments in $\mathbb{R}^2$, respectively.

The motions of the players are governed by the following system of differential equations:

$$\begin{aligned} \ddot{M}_P &= \frac{2}{m_0} s_P e^\perp(\theta_P), & \ddot{\theta}_P &= \frac{12}{m_0 l} a_P, \\ \ddot{M}_E &= \frac{2}{m_0} s_E e^\perp(\theta_E), & \ddot{\theta}_E &= \frac{12}{m_0 l} a_E, \end{aligned} \tag{3.1}$$

with the initial conditions

$$\begin{aligned} M_P(t_0) &= M_{P,1}, & \dot{M}_P(t_0) &= 0, & \theta_P(t_0) &= \theta_{P,0}, & \dot{\theta}_P(t_0) &= 0, \\ M_E(t_0) &= M_{E,1}, & \dot{M}_E(t_0) &= 0, & \theta_E(t_0) &= \theta_{E,0}, & \dot{\theta}_E(t_0) &= 0, \end{aligned} \quad (3.2)$$

where $(s_P(t), a_P(t))$ and $(s_E(t), a_E(t))$ are the translational and rotational control parameters of the pursuer P and the evader E , respectively.

Definition 3.1. The pairs of measurable functions $(s_P(t), a_P(t))$ and $(s_E(t), a_E(t))$ for $t \geq t_0$ are called the admissible translational and rotational controls of the pursuer P and the evader E , respectively, if they satisfy the following resource constraints:

$$|s_P(t)| + |a_P(t)| \leq \rho, \quad |s_E(t)| + |a_E(t)| \leq \sigma, \quad (3.3)$$

where ρ and σ are given numbers such that $\rho > \sigma > 0$ (i.e., the pursuer has a resource advantage).

Definition 3.2. We say that capture occurs in the game (3.1)-(3.3) if there exists a finite time $\tau > t_0$ such that the midpoints of the two rigid segments coincide, i.e., $M_P(\tau) = M_E(\tau)$.

The aim of the pursuer is to achieve capture, while the evader attempts to maintain the inequality $M_P(t) \neq M_E(t)$ for all $t > t_0$.

Problem 1. Construct a strategy for the pursuer that guarantees capture in the game (3.1)-(3.3) within a finite time.

A strategy for the pursuer. Motivated by the rotation-translation structure of time-optimal steering established in Theorem 2.1, we construct a stagewise pursuit strategy. Let $d_1 = |M_{P,1} - M_{E,1}|$.

First stage. The pursuer first performs a time-optimal rotation through the minimal angle $\theta_{P,1}$ such that $e^\perp(\theta_{P,0} + \theta_{P,1}) \parallel M_{E,1} - M_{P,1}$, and then executes a time-optimal translation toward the point $M_{E,1}$ (see Fig. 6). By Theorem 2.1, the duration of this maneuver is

$$\Delta t_1 = \sqrt{\frac{m_0 l |\theta_{P,1}|}{3\rho}} + \sqrt{\frac{2m_0 d_1}{\rho}}.$$

During the same interval Δt_1 , the evader applies an arbitrary admissible control and reaches a new position $M_{E,2}$. Denote $d_2 = |M_{E,2} - M_{E,1}|$. By Theorem 2.1, the minimal time required to transfer the evader from $M_{E,1}$ to $M_{E,2}$ under the control constraint σ is

$$\Delta \hat{t}_1 = \sqrt{\frac{m_0 l |\theta_{E,1}|}{3\sigma}} + \sqrt{\frac{2m_0 d_2}{\sigma}},$$

where $\theta_{E,1}$ denotes the time-optimal rotation angle of the evader. Since the evader performs this maneuver during the interval Δt_1 , we have $\Delta \hat{t}_1 \leq \Delta t_1$. At the end of the first stage, the pursuer reaches the previous position of the evader: $M_{P,2} = M_{E,1}$.

Second stage. The pursuer now repeats the same procedure. It performs a time-optimal rotation through the minimal angle $\theta_{P,2}$ such that

$$e^\perp(\theta_{P,0} + \theta_{P,1} + \theta_{P,2}) \parallel M_{E,2} - M_{P,2},$$

followed by a time-optimal translation toward the point $M_{E,2}$. The duration of the maneuver is

$$\Delta t_2 = \sqrt{\frac{m_0 l |\theta_{P,2}|}{3\rho}} + \sqrt{\frac{2m_0 d_2}{\rho}}.$$

During this interval, the evader reaches a new position $M_{E,3}$. Denoting $d_3 = |M_{E,3} - M_{E,2}|$, we obtain

$$\Delta \hat{t}_2 = \sqrt{\frac{m_0 l |\theta_{E,2}|}{3\sigma}} + \sqrt{\frac{2m_0 d_3}{\sigma}},$$

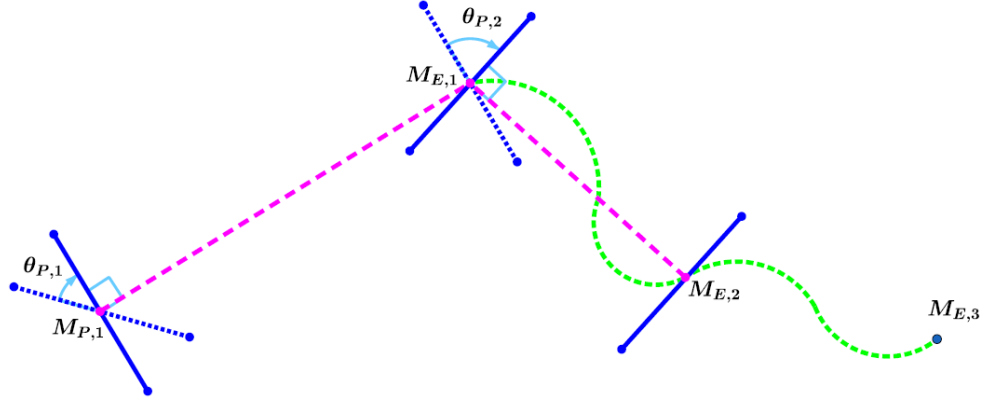


FIGURE 6. Pursuit strategy and trajectory convergence

with $\Delta \hat{t}_2 \leq \Delta t_2$. At the end of the second stage, $M_{P,3} = M_{E,2}$.

Proceeding inductively, let $M_{P,k}$ and $M_{E,k}$ denote the positions of the pursuer and the evader at the beginning of the k -th stage, and define $d_k = |M_{E,k} - M_{E,k-1}|$. At stage k , the pursuer rotates through the minimal angle $\theta_{P,k}$ such that

$$e^\perp \left(\sum_{j=0}^k \theta_{P,j} \right) \parallel M_{E,k} - M_{P,k},$$

and then performs a time-optimal translation toward $M_{E,k}$. The duration of the maneuver is

$$\Delta t_k = \sqrt{\frac{m_0 l |\theta_{P,k}|}{3\rho}} + \sqrt{\frac{2m_0 d_k}{\rho}}. \quad (3.4)$$

The next theorem shows that the above strategy is well-defined and leads to finite-time capture.

Theorem 3.1. *In game (3.1)–(3.3), the pursuer, under the constructed strategy, guarantees capture within the time*

$$T^* = \sqrt{\frac{m_0 l |\theta_{P,1}|}{3\rho}} + \sqrt{\frac{2m_0 d_1}{\rho}} + \frac{1}{\sqrt{\rho} - \sqrt{\sigma}} \left(\sqrt{\frac{m_0 l |\theta_{P,2}|}{3}} + \sqrt{2m_0 d_2} \right).$$

Proof: We prove the theorem by estimating the total capture time generated by the constructed stagewise strategy. At stage k , during the same interval Δt_k , the evader moves to a new position $M_{E,k+1}$. Denoting $d_{k+1} = |M_{E,k+1} - M_{E,k}|$, the minimal transfer time of the evader satisfies

$$\Delta \hat{t}_k = \sqrt{\frac{m_0 l |\theta_{E,k}|}{3\sigma}} + \sqrt{\frac{2m_0 d_{k+1}}{\sigma}}, \quad (3.5)$$

and consequently $\Delta \hat{t}_k \leq \Delta t_k$, where $\theta_{E,k}$ denotes the rotation angle appearing in the corresponding time-optimal steering maneuver. By construction, $M_{P,k+1} = M_{E,k}$, hence $d_k = |M_{E,k} - M_{P,k}|$, $k \geq 1$. Moreover, since at each stage the pursuer aligns its motion with the evader's previous displacement vector, and therefore

$$\theta_{P,k+1} = \theta_{E,k}, \quad k \geq 2. \quad (3.6)$$

Combining (3.4) and (3.5), and substituting (3.6), yields

$$\sqrt{\frac{m_0 l |\theta_{P,k}|}{3\rho}} + \sqrt{\frac{2m_0 d_k}{\rho}} \geq \sqrt{\frac{m_0 l |\theta_{P,k+1}|}{3\sigma}} + \sqrt{\frac{2m_0 d_{k+1}}{\sigma}}.$$

Introduce the quantity

$$S_k = \sqrt{\frac{m_0 l |\theta_{P,k}|}{3}} + \sqrt{2m_0 d_k}$$

and define $q = \sqrt{\frac{\sigma}{\rho}}$. Since $\rho > \sigma$, we have $0 < q < 1$, and the above inequality reduces to the recurrence relation $S_{k+1} \leq qS_k$, $k \geq 2$. Therefore, $S_k \leq q^{k-2}S_2$, $k \geq 3$. Thus, the sequence S_k decreases geometrically, which implies convergence of the total capture time. Using (3.4), we write

$$T_c = \sum_{k=1}^{\infty} \Delta t_k = \Delta t_1 + \Delta t_2 + \sum_{k=3}^{\infty} \Delta t_k.$$

From the definition of S_k , we obtain $\Delta t_k = \frac{S_k}{\sqrt{\rho}}$. Therefore,

$$T_c = \Delta t_1 + \Delta t_2 + \frac{1}{\sqrt{\rho}} \sum_{k=3}^{\infty} S_k \leq \Delta t_1 + \Delta t_2 \sum_{k=0}^{\infty} q^k.$$

Since

$$\sum_{k=0}^{\infty} q^k = \frac{\sqrt{\rho}}{\sqrt{\rho} - \sqrt{\sigma}},$$

we obtain

$$T_c \leq \Delta t_1 + \frac{\sqrt{\rho} \Delta t_2}{\sqrt{\rho} - \sqrt{\sigma}} = T^*.$$

Finally, since $S_k \rightarrow 0$ as $k \rightarrow \infty$, it follows that $d_k \rightarrow 0$. Consequently, the distance between the pursuer and the evader converges to zero, and the total capture time T_c is finite, which implies that capture occurs in a finite time T^* . This completes the proof of the theorem. $\square$

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