

Local and 2-local $\frac{1}{2}$ -derivations on filiform-type associative algebras

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Abstract. This paper is devoted to local and 2-local $\frac{1}{2}$ -derivations of null-filiform, filiform, and naturally graded quasi-filiform associative algebras. We obtain explicit descriptions of the corresponding $\frac{1}{2}$ -derivations and local $\frac{1}{2}$ -derivations and show that non-derivation local maps occur throughout these classes. For 2-local maps, however, a sharp dichotomy appears: every 2-local $\frac{1}{2}$ -derivation of a null-filiform associative algebra is a $\frac{1}{2}$ -derivation, whereas every filiform and naturally graded quasi-filiform family considered here admits a 2-local $\frac{1}{2}$ -derivation which is not a $\frac{1}{2}$ -derivation.

Keywords: Associative algebras, filiform associative algebras, quasi-filiform associative algebras, $\frac{1}{2}$ -derivation, local $\frac{1}{2}$ -derivation, 2-local $\frac{1}{2}$ -derivation

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1. INTRODUCTION

The concept of δ -derivations was introduced by V. Filippov for Lie algebras in [1, 2]. This class of linear mappings contains several important special cases, including ordinary derivations when $\delta = 1$, antiderivations when $\delta = -1$, and mappings related to the centroid. In [2], it was shown that prime Lie algebras usually do not possess nonzero δ -derivations for $\delta \neq 1, -1, 0, \frac{1}{2}$. In the exceptional case $\delta = \frac{1}{2}$, Filippov described all $\frac{1}{2}$ -derivations of prime Lie algebras satisfying suitable restrictions. In particular, if A is a central simple Lie algebra over a field of characteristic $p \neq 2, 3$ and admits a non-degenerate symmetric invariant bilinear form, then every $\frac{1}{2}$ -derivation of A is a scalar multiplication. Further results on δ -derivations of prime alternative and non-Lie Mal'tsev algebras were obtained in [3].

Local derivations form another important generalization of derivations. They were introduced by R. V. Kadison and independently by D. R. Larson and A. R. Sourour in the study of operator algebras [4, 5]. Since then, local derivations have been investigated for many classes of associative, Lie, Leibniz, and related algebras; see, for instance, [6, 7, 8, 9, 10, 11]. One of the central problems in this direction is to determine whether every local derivation on a given algebra is an ordinary derivation. For example, in [8] it was proved that each local derivation on a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero is a derivation. On the other hand, solvable and nilpotent algebras may admit local derivations which are not derivations. Such phenomena were studied for solvable Lie algebras in [10, 11], for conformal Galilei and Schrödinger type Lie algebras in [7, 12, 13], and for several classes of Leibniz algebras in [9, 6].

The finite-dimensional theory has also been complemented by a number of results for infinite-dimensional Lie algebras. In particular, local derivations on locally simple Lie algebras, generalized Witt algebras, Witt algebras, and Witt algebras over fields of prime characteristic were considered in [14, 15, 16]. In these settings, it was shown that, for the classes under consideration, every local derivation is a derivation.

A further nonlinear version of this idea was introduced by P. Šemrl in [17] in the form of 2-local automorphisms and 2-local derivations. This notion has attracted considerable attention in the theory of Lie algebras and related structures [18, 14, 19, 20, 21]. For instance, [18] proves that every 2-local derivation on a finite-dimensional semisimple Lie algebra is a derivation, while every finite-dimensional nilpotent Lie algebra of dimension greater than two admits a 2-local derivation which is not a derivation. Similar rigidity results have been obtained for several finite- and infinite-dimensional Lie algebras, including semisimple Lie algebras, Witt algebras, Virasoro algebras, Schrödinger-Virasoro algebras, Jacobson-Witt algebras, and planar Galilean conformal algebras.

Derivations, local derivations, and 2-local derivations on filiform associative algebras were studied in [22]. In that work, local and 2-local derivations of filiform associative algebras were introduced

and described. The study of local and 2-local δ -derivations was developed in [23], where local and 2-local $\frac{1}{2}$ -derivations of finite-dimensional solvable Lie algebras with filiform, Heisenberg, and abelian nilradicals were investigated. Moreover, local $\frac{1}{2}$ -derivations of oscillator Lie algebras, conformal perfect Lie algebras, and Schrödinger algebras were described. Related results for naturally graded quasi-filiform Leibniz algebras of type I were obtained by B. Yusupov, N. Vaisova, and T. Madrahimov in [24], where it was shown that such algebras may admit local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations. Local and 2-local $\frac{1}{2}$ -derivations of solvable Leibniz algebras were further studied in [25]. More recently, explicit descriptions of $\frac{1}{2}$ -derivations and local $\frac{1}{2}$ -derivations on naturally graded filiform Lie and Leibniz algebras were obtained in [26]. Naturally graded non-Lie p -filiform Leibniz algebras were considered in [27], where it was shown that, in general, they admit local and 2-local $\frac{1}{2}$ -derivations which are not ordinary $\frac{1}{2}$ -derivations.

The aim of the present paper is to investigate $\frac{1}{2}$ -derivations, local $\frac{1}{2}$ -derivations, and 2-local $\frac{1}{2}$ -derivations on null-filiform, filiform, and naturally graded quasi-filiform associative algebras. We obtain explicit descriptions of $\frac{1}{2}$ -derivations and local $\frac{1}{2}$ -derivations for these algebras. It is shown that, in general, these algebras admit local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations. We also prove that every 2-local $\frac{1}{2}$ -derivation of a null-filiform associative algebra is a $\frac{1}{2}$ -derivation, whereas filiform and naturally graded quasi-filiform associative algebras admit 2-local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations.

Thus the null-filiform case is rigid with respect to 2-local $\frac{1}{2}$ -derivations, while the filiform and quasi-filiform cases exhibit genuinely nonlinear 2-local phenomena. This distinction is one of the principal structural conclusions of the paper.

The paper is organized as follows. In Section 2, we describe the $\frac{1}{2}$ -derivations of null-filiform, filiform, and naturally graded quasi-filiform associative algebras. Section 3 is devoted to local and 2-local $\frac{1}{2}$ -derivations. In Subsection 3.1, we give descriptions of local $\frac{1}{2}$ -derivations and show the existence of local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations. In Subsection 3.2, we prove the corresponding results for 2-local $\frac{1}{2}$ -derivations.

1.1. Preliminaries. For an algebra $\mathbf{A}$ from an arbitrary variety, we consider the series

$$\mathbf{A}^1 = \mathbf{A}, \quad \mathbf{A}^{i+1} = \sum_{k=1}^i \mathbf{A}^k \mathbf{A}^{i+1-k}, \quad i \geq 1.$$

We say that an algebra $\mathbf{A}$ is nilpotent if $\mathbf{A}^i = 0$ for some $i \in \mathbb{N}$. The smallest integer satisfying $\mathbf{A}^i = 0$ is called the nilpotency index, or nilindex, of $\mathbf{A}$.

Definition 1.1. An n -dimensional algebra $\mathbf{A}$ is called null-filiform if $\dim \mathbf{A}^i = (n+1)-i$, $1 \leq i \leq n+1$.

It is easy to see that an algebra has a maximum nilpotency index if and only if it is null-filiform. For a nilpotent algebra, the condition of null-filiformity is equivalent to the condition that the algebra is one-generated.

All null-filiform associative algebras were described in [28, 29]:

Theorem 1.1. *An arbitrary n -dimensional null-filiform associative algebra is isomorphic to the algebra:*

$$\mu_0^n: \quad e_i e_j = e_{i+j}, \quad 2 \leq i+j \leq n,$$

where $\{e_1, e_2, \dots, e_n\}$ is a basis of the algebra μ_0^n .

The following classes of nilpotent algebras are filiform and quasi-filiform algebras whose nilindex equals n and $n-1$, respectively.

Definition 1.2. An n -dimensional algebra is called filiform if $\dim(\mathbf{A}^i) = n-i$, $2 \leq i \leq n$.

Definition 1.3. An n -dimensional associative algebra $\mathbf{A}$ is called a quasi-filiform algebra if $\mathbf{A}^{n-2} \neq 0$ and $\mathbf{A}^{n-1} = 0$.

Definition 1.4. Given a nilpotent associative algebra $\mathbf{A}$, put $\mathbf{A}_i = \mathbf{A}^i / \mathbf{A}^{i+1}$, $1 \leq i \leq k-1$, and define

$$\text{gr}(\mathbf{A}) = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \cdots \oplus \mathbf{A}_k.$$

The product induced by the multiplication of $\mathbf{A}$ satisfies $\mathbf{A}_i \mathbf{A}_j \subseteq \mathbf{A}_{i+j}$, and therefore $\text{gr}(\mathbf{A})$ is a graded algebra. If $\text{gr}(\mathbf{A}) \cong \mathbf{A}$, then $\mathbf{A}$ is called a naturally graded algebra.

All filiform and naturally graded quasi-filiform associative algebras were classified in [30].

Theorem 1.2 ([30]). *Every n -dimensional ($n > 3$) complex filiform associative algebra is isomorphic to one of the following pairwise non-isomorphic algebras with basis $\{e_1, e_2, \dots, e_n\}$:*

$$\begin{array}{ll} \mu_{1,1}^n(\mu_0^{n-1} \oplus \mathbb{C}) & : e_i e_j = e_{i+j}, \\ \mu_{1,2}^n & : e_i e_j = e_{i+j}, \quad e_n e_n = e_{n-1}, \\ \mu_{1,3}^n & : e_i e_j = e_{i+j}, \quad e_1 e_n = e_{n-1}, \\ \mu_{1,4}^n & : e_i e_j = e_{i+j}, \quad e_1 e_n = e_{n-1}, \quad e_n e_n = e_{n-1} \end{array}$$

where $2 \leq i+j \leq n-1$.

Theorem 1.3 ([30]). *Let $\mathbf{A}$ be an n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebra. Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{ll} \mu_{2,1}^n & : e_i e_j = e_{i+j}, \quad e_{n-1} e_1 = e_n \\ \mu_{2,2}^n(\alpha) & : e_i e_j = e_{i+j}, \quad e_1 e_{n-1} = e_n, \quad e_{n-1} e_1 = \alpha e_n \\ \mu_{2,3}^n & : e_i e_j = e_{i+j}, \quad e_{n-1} e_{n-1} = e_n \\ \mu_{2,4}^n & : e_i e_j = e_{i+j}, \quad e_1 e_{n-1} = e_n, \quad e_{n-1} e_{n-1} = e_n \end{array}$$

where $\alpha \in \mathbb{C}$ and $2 \leq i+j \leq n-2$.

Definition 1.5. Let A be an algebra over a field $\mathbb{F}$, and let $D : A \rightarrow A$ be a linear map. The map D is called a δ -derivation if

$$D(xy) = \delta(D(x)y + xD(y))$$

for all $x, y \in A$.

Note that a 1-derivation is an ordinary derivation, and a (-1) -derivation is called an antiderivation. The set of all δ -derivations of an algebra A is denoted by $\text{Der}_\delta(A)$.

Definition 1.6. A linear operator $\Delta : A \rightarrow A$ is called a local δ -derivation if, for any $x \in A$, there exists a δ -derivation $D_x : A \rightarrow A$ depending on x such that $\Delta(x) = D_x(x)$. The set of all local δ -derivations on A is denoted by $\text{LocDer}_\delta(A)$.

Definition 1.7. A map $\nabla : A \rightarrow A$ (not necessarily linear) is called a 2-local δ -derivation if, for any $x, y \in A$, there exists a δ -derivation $D_{x,y} \in \text{Der}_\delta(A)$ such that

$$\nabla(x) = D_{x,y}(x), \quad \nabla(y) = D_{x,y}(y).$$

It should be noted that every 2-local δ -derivation is homogeneous. Indeed, for any $x \in A$ and any scalar λ , we have

$$\nabla(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \nabla(x).$$

In this work, we focus on local and 2-local $\frac{1}{2}$ -derivations. The main examples of $\frac{1}{2}$ -derivations are scalar multiplications, that is, maps of the form $D(x) = \lambda x$ for all $x \in A$. Such $\frac{1}{2}$ -derivations are called trivial $\frac{1}{2}$ -derivations.

2. $\frac{1}{2}$ -DERIVATIONS ON NULL-FILIFORM, FILIFORM AND QUASI-FILIFORM ASSOCIATIVE ALGEBRAS

Now we study $\frac{1}{2}$ -derivations on n -dimensional null-filiform, n -dimensional ($n > 3$) complex filiform and n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebras.

Proposition 2.1. $\frac{1}{2}$ -derivations of n -dimensional complex null-filiform associative algebras are given as follows:

$$D(e_i) = \sum_{k=i}^n \alpha_{k-i+1} e_k, \quad 1 \leq i \leq n,$$

Proof: Write

$$D(e_1) = \sum_{r=1}^n \alpha_r e_r.$$

Since $e_2 = e_1 e_1$, the $\frac{1}{2}$ -derivation identity gives

$$\begin{aligned} D(e_2) &= \frac{1}{2}(D(e_1)e_1 + e_1 D(e_1)) \\ &= \frac{1}{2} \left(\sum_{r=1}^{n-1} \alpha_r e_{r+1} + \sum_{r=1}^{n-1} \alpha_r e_{r+1} \right) = \sum_{k=2}^n \alpha_{k-1} e_k. \end{aligned}$$

Assume inductively that

$$D(e_i) = \sum_{k=i}^n \alpha_{k-i+1} e_k$$

for some $1 \leq i < n$. Using $e_{i+1} = e_i e_1$, we obtain

$$\begin{aligned} D(e_{i+1}) &= \frac{1}{2}(D(e_i)e_1 + e_i D(e_1)) \\ &= \frac{1}{2} \left(\sum_{k=i}^{n-1} \alpha_{k-i+1} e_{k+1} + \sum_{r=1}^{n-i} \alpha_r e_{i+r} \right) = \sum_{k=i+1}^n \alpha_{k-i} e_k. \end{aligned}$$

This proves the displayed formula for all i .

Conversely, let D have the displayed form. If $i + j \leq n$, then

$$\frac{1}{2}(D(e_i)e_j + e_i D(e_j)) = \sum_{k=i+j}^n \alpha_{k-i-j+1} e_k = D(e_{i+j}) = D(e_i e_j).$$

If $i + j > n$, both sides vanish. By bilinearity, the identity holds for all elements of μ_0^n , and hence D is a $\frac{1}{2}$ -derivation. $\square$

Proposition 2.2. $\frac{1}{2}$ -derivations of the n -dimensional ($n > 3$) complex filiform associative algebras are given as follows:

	$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$	
$\mu_{1,1}^n :$	$D(e_i) = \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k,$	$2 \leq i \leq n-1,$
	$D(e_n) = \beta_{n-1} e_{n-1} + \beta_n e_n.$	
	$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$	
$\mu_{1,2}^n :$	$D(e_i) = \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k,$	$2 \leq i \leq n-1,$
	$D(e_n) = \alpha_n e_{n-2} + \beta_{n-1} e_{n-1} + \alpha_1 e_n.$	
	$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$	
$\mu_{1,3}^n :$	$D(e_2) = \sum_{k=2}^{n-2} \alpha_{k-1} e_k + (\alpha_{n-2} + \frac{1}{2}\alpha_n) e_{n-1},$	
	$D(e_i) = \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k,$	$3 \leq i \leq n-1,$
	$D(e_n) = \beta_{n-1} e_{n-1} + \alpha_1 e_n.$	

$$\begin{aligned} D(e_1) &= \sum_{i=1}^n \alpha_i e_i, \\ \mu_{1,4}^n : \quad D(e_2) &= \sum_{k=2}^{n-2} \alpha_{k-1} e_k + (\alpha_{n-2} + \frac{1}{2} \alpha_n) e_{n-1}, \\ D(e_i) &= \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k, & 3 \leq i \leq n-1, \\ D(e_n) &= -\alpha_n e_{n-2} + \beta_{n-1} e_{n-1} + \alpha_1 e_n. \end{aligned}$$

Proof: Let us prove the proposition for the algebra $\mu_{1,1}^n$, and for the algebras $\mu_{1,2}^n$, $\mu_{1,3}^n$, $\mu_{1,4}^n$ the proofs are similar. Since the algebra $\mu_{1,1}^n$ has two generators $\{e_1, e_n\}$, any $\frac{1}{2}$ -derivation D on $\mu_{1,1}^n$ is completely determined by $D(e_1)$ and $D(e_n)$.

Let

$$D(e_1) = \sum_{i=1}^n \alpha_i e_i, \text{ and } D(e_n) = \sum_{j=1}^n \beta_j e_j.$$

Applying the $\frac{1}{2}$ -derivation identity, we have

$$\begin{aligned} D(e_2) &= D(e_1 e_1) = \frac{1}{2} (D(e_1) e_1 + e_1 D(e_1)) = \frac{1}{2} \left(\sum_{i=1}^n \alpha_i e_i e_1 + \sum_{i=1}^n \alpha_i e_1 e_i \right) \\ &= \frac{1}{2} \left(\sum_{i=2}^{n-1} \alpha_{i-1} e_i + \sum_{i=2}^{n-1} \alpha_{i-1} e_i \right) = \sum_{i=2}^{n-1} \alpha_{i-1} e_i. \end{aligned}$$

$$\begin{aligned} D(e_3) &= D(e_2 e_1) = \frac{1}{2} (D(e_2) e_1 + e_2 D(e_1)) = \frac{1}{2} \left(\sum_{i=2}^{n-1} \alpha_{i-1} e_i e_1 + \sum_{i=1}^n \alpha_i e_2 e_i \right) \\ &= \frac{1}{2} \left(\sum_{i=3}^{n-1} \alpha_{i-2} e_i + \sum_{i=3}^{n-1} \alpha_{i-2} e_i \right) = \sum_{i=3}^{n-1} \alpha_{i-2} e_i. \end{aligned}$$

Generally, one can prove by induction

$$D(e_i) = \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k, \quad 2 \leq i \leq n-1.$$

Suppose the above is true for some i . Now we can determine for $i+1$.

$$\begin{aligned} D(e_{i+1}) &= D(e_i e_1) = \frac{1}{2} (D(e_i) e_1 + e_i D(e_1)) = \frac{1}{2} \left(\sum_{k=i}^{n-1} \alpha_{k-i+1} e_k e_1 + \sum_{k=1}^n \alpha_k e_i e_k \right) \\ &= \frac{1}{2} \left(\sum_{k=i+1}^{n-1} \alpha_{k-i} e_k + \sum_{k=i+1}^{n-1} \alpha_{k-i} e_k \right) = \sum_{k=i+1}^{n-1} \alpha_{k-i} e_k. \end{aligned}$$

Using the zero product $e_1 e_n = 0$, we obtain

$$0 = D(e_1 e_n) = \frac{1}{2} (D(e_1) e_n + e_1 D(e_n)) = \frac{1}{2} \left(\sum_{i=1}^n \alpha_i e_i e_n + \sum_{j=1}^n \beta_j e_1 e_j \right) = \frac{1}{2} \sum_{j=2}^{n-1} \beta_{j-1} e_j.$$

Therefore we have $\beta_j = 0$, $1 \leq j \leq n-2$. We complete the proof of Proposition. $\square$

Proposition 2.3. *The $\frac{1}{2}$ -derivations of the n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebras are given as follows:*

$\mu_{2,1}^n :$	$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$ $D(e_2) = \sum_{k=2}^{n-2} \alpha_{k-1} e_k + \frac{\alpha_{n-1}}{2} e_n,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 3 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n,$ $D(e_n) = \frac{1}{2}(\alpha_1 + \beta_{n-1}) e_n.$
$\mu_{2,2}^n(\alpha = 1) :$	$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$ $D(e_2) = \sum_{k=2}^{n-2} \alpha_{k-1} e_k + \alpha_{n-1} e_n,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 3 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-3} e_{n-3} + \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n,$ $D(e_n) = \frac{1}{2} \beta_{n-3} e_{n-2} + \frac{1}{2}(\alpha_1 + \beta_{n-1}) e_n.$
$\mu_{2,2}^n(\alpha = -1) :$	$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$ $D(e_2) = \sum_{k=2}^{n-2} \alpha_{k-1} e_k,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 3 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n,$ $D(e_n) = \frac{1}{2}(\alpha_1 + \beta_{n-1}) e_n.$
$\mu_{2,2}^n(\alpha \neq -1, 1) :$	$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$ $D(e_2) = \sum_{k=2}^{n-2} \alpha_{k-1} e_k + \frac{1}{2}(1 + \alpha) \alpha_{n-1} e_n,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 3 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n,$ $D(e_n) = \frac{1}{2}(\alpha_1 + \beta_{n-1}) e_n.$
$\mu_{2,3}^n :$	$D(e_1) = \sum_{i=1}^{n-2} \alpha_i e_i + \alpha_n e_n,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 2 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n,$ $D(e_n) = \beta_{n-1} e_n.$
$\mu_{2,4}^n :$	$D(e_1) = \sum_{i=1}^{n-2} \alpha_i e_i + \alpha_n e_n,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 2 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-2} e_{n-2} + \alpha_1 e_{n-1} + \beta_n e_n,$ $D(e_n) = \alpha_1 e_n.$

Proof: We give the details for $\mu_{2,1}^n$; the remaining cases are obtained by the same computation with their additional defining products. Write

$$D(e_1) = \sum_{r=1}^n \alpha_r e_r, \quad D(e_{n-1}) = \sum_{r=1}^n \gamma_r e_r.$$

Since $e_2 = e_1e_1$ and $e_{n-1}e_1 = e_n$, we obtain

$$\begin{aligned} D(e_2) &= \frac{1}{2}(D(e_1)e_1 + e_1D(e_1)) \\ &= \sum_{k=2}^{n-2} \alpha_{k-1}e_k + \frac{1}{2}\alpha_{n-1}e_n, \end{aligned}$$

and induction based on $e_i = e_{i-1}e_1$ gives

$$D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1}e_k, \quad 3 \leq i \leq n-2.$$

The zero product $e_1e_{n-1} = 0$ yields

$$0 = D(e_1e_{n-1}) = \frac{1}{2}(D(e_1)e_{n-1} + e_1D(e_{n-1})),$$

and hence $\gamma_1 = \dots = \gamma_{n-3} = 0$. Therefore

$$D(e_{n-1}) = \beta_{n-2}e_{n-2} + \beta_{n-1}e_{n-1} + \beta_n e_n.$$

Finally,

$$D(e_n) = D(e_{n-1}e_1) = \frac{1}{2}(D(e_{n-1})e_1 + e_{n-1}D(e_1)) = \frac{1}{2}(\alpha_1 + \beta_{n-1})e_n.$$

This is the first form in the statement.

For $\mu_{2,2}^n(\alpha)$, applying the identity to $e_1e_{n-1} = e_n$ and $e_{n-1}e_1 = \alpha e_n$ shows that

$$(\alpha - 1)\gamma_j = 0, \quad 1 \leq j \leq n-3.$$

If $\alpha = 1$, the zero products involving $e_{n-1}e_i$ for $i \geq 2$ leave only the possible extra coefficient γ_{n-3} ; if $\alpha \neq 1$, all these coefficients vanish. This gives the three $\mu_{2,2}^n(\alpha)$ rows displayed above. The defining products $e_{n-1}^2 = e_n$ and, respectively, $e_1e_{n-1} = e_n = e_{n-1}^2$ give the stated forms for $\mu_{2,3}^n$ and $\mu_{2,4}^n$.

Conversely, substitution into every nonzero defining product and into the zero products involving the generators verifies the $\frac{1}{2}$ -derivation identity. Bilinearity then completes the verification for arbitrary elements. $\square$

3. LOCAL AND 2-LOCAL $\frac{1}{2}$ -DERIVATIONS ON NULL-FILIFORM, FILIFORM AND QUASI-FILIFORM ASSOCIATIVE ALGEBRAS

In this section, we consider local and 2-local $\frac{1}{2}$ -derivations of null-filiform ($n > 2$), filiform ($n > 3$), and quasi-filiform ($n > 5$) algebras. All these algebras admit a local $\frac{1}{2}$ -derivation which is not a $\frac{1}{2}$ -derivation. At the same time we show that every 2-local $\frac{1}{2}$ -derivation of a null-filiform algebra is a $\frac{1}{2}$ -derivation. An example of a 2-local $\frac{1}{2}$ -derivation which is not a $\frac{1}{2}$ -derivation is given for filiform and quasi-filiform algebras.

3.1. Local $\frac{1}{2}$ -derivations on null-filiform, filiform and quasi-filiform associative algebras.

Now we study local $\frac{1}{2}$ -derivations on n -dimensional null-filiform, n -dimensional ($n > 3$) complex filiform and n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebras.

Theorem 3.1. *Let Δ be a linear map from a null-filiform associative algebra into itself. Then Δ is a local $\frac{1}{2}$ -derivation if and only if*

$$\Delta(e_i) = \sum_{j=i}^n c_{j,i}e_j, \quad 1 \leq i \leq n.$$

Proof: Let $\mathfrak{C}$ be the matrix of a local $\frac{1}{2}$ -derivation Δ on μ_0^n :

$$\mathfrak{C} = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n-1} & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n-1} & c_{2,n} \\ c_{3,1} & c_{3,2} & \cdots & c_{3,n-1} & c_{3,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-1,1} & c_{n-1,2} & \cdots & c_{n-1,n-1} & c_{n-1,n} \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n-1} & c_{n,n} \end{pmatrix}.$$

By the definition for every $x = \sum_{i=1}^n x_i e_i \in \mu_0^n$ there exists a $\frac{1}{2}$ -derivation D_x on μ_0^n such that

$$\Delta(x) = D_x(x).$$

By Proposition 2.1, the $\frac{1}{2}$ -derivation D_x has the following matrix form:

$$\mathfrak{C}_x = \begin{pmatrix} \alpha_1^x & 0 & 0 & \cdots & 0 & 0 \\ \alpha_2^x & \alpha_1^x & 0 & \cdots & 0 & 0 \\ \alpha_3^x & \alpha_2^x & \alpha_1^x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-2}^x & \alpha_{n-3}^x & \alpha_{n-4}^x & \cdots & 0 & 0 \\ \alpha_{n-1}^x & \alpha_{n-2}^x & \alpha_{n-3}^x & \cdots & \alpha_1^x & 0 \\ \alpha_n^x & \alpha_{n-1}^x & \alpha_{n-2}^x & \cdots & \alpha_2^x & \alpha_1^x \end{pmatrix}.$$

For the matrix $\mathfrak{C}$ of Δ by choosing subsequently $x = e_1, \dots, x = e_n$ and using $\Delta(x) = D_x(x)$, i.e. $\mathfrak{C}\bar{x} = D_x(\bar{x})$, where $\bar{x}$ is the vector corresponding to x , we obtain

$$\mathfrak{C} = \begin{pmatrix} c_{1,1} & 0 & \cdots & 0 & 0 \\ c_{2,1} & c_{2,2} & \cdots & 0 & 0 \\ c_{3,1} & c_{3,2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-2,1} & c_{n-2,2} & \cdots & 0 & 0 \\ c_{n-1,1} & c_{n-1,2} & \cdots & c_{n-1,n-1} & 0 \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n-1} & c_{n,n} \end{pmatrix}.$$

Conversely, suppose that the matrix of Δ has the above form and let us show that Δ is a local $\frac{1}{2}$ -derivation. For each x we must find a $\frac{1}{2}$ -derivation D_x such that $\Delta(x) = D_x(x)$. We have $\mathfrak{C}\bar{x} = \mathfrak{C}_x(\bar{x})$, where $\bar{x}$ is the coordinate vector of $x = \sum_{i=1}^n x_i e_i$. This implies the following system of equalities

$$\begin{aligned} c_{1,1}x_1 &= \alpha_1^x x_1, \\ c_{2,1}x_1 + c_{2,2}x_2 &= \alpha_2^x x_1 + \alpha_1^x x_2, \\ \cdots &\cdots \\ c_{n-1,1}x_1 + c_{n-1,2}x_2 + \cdots + c_{n-1,n-1}x_{n-1} &= \alpha_{n-1}^x x_1 + \alpha_{n-2}^x x_2 + \cdots + \alpha_1^x x_{n-1}, \\ c_{n,1}x_1 + c_{n,2}x_2 + \cdots + c_{n,n}x_n &= \alpha_n^x x_1 + \alpha_{n-1}^x x_2 + \cdots + \alpha_1^x x_n. \end{aligned}$$

Let us consider two cases separately:

Case 1: If $x_1 \neq 0$, then $\alpha_1^x = c_{1,1}$,

$$\alpha_i^x = c_{i,1} + \frac{1}{x_1} \left(\sum_{j=2}^i (c_{i,j} - \alpha_{i-j+1}^x) x_j \right), \quad 2 \leq i \leq n.$$

Case 2: If $x_1 = x_2 = \dots = x_{t-1} = 0$ and $x_t \neq 0$, then $\alpha_1^x = c_{t,t}$, $2 \leq t \leq n$

$$\alpha_{i-t+1}^x = c_{i,t} + \frac{1}{x_t} \left(\sum_{j=t+1}^i (c_{i,j} - \alpha_{i-j+1}^x) x_j \right) \quad t+1 \leq i \leq n.$$

Thus, the parameters α_i^x , $1 \leq i \leq n$, can be chosen so that the corresponding $\frac{1}{2}$ -derivation D_x satisfies $\Delta(x) = D_x(x)$. The proof is complete. $\square$

Theorem 3.2. *Let Δ be a linear map on an n -dimensional ($n > 3$) complex filiform associative algebra. Then Δ is a local $\frac{1}{2}$ -derivation if and only if*

- for the algebra $\mu_{1,1}^n$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) = \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_i) = \sum_{j=i}^{n-1} c_{j,i} e_j, \quad 2 \leq i \leq n-1 \\ \Delta(e_n) = c_{n-1,n} e_{n-1} + c_{n,n} e_n. \end{cases} \quad (3.1)$$

- for the algebra $\mu_{1,3}^n$, the same support conditions hold and

$$c_{i,i} = c_{n,n}, \quad 2 \leq i \leq n-2. \quad (3.2)$$

- for the algebras $\mu_{1,2}^n$, $\mu_{1,4}^n$

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) = \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_i) = \sum_{j=i}^{n-1} c_{j,i} e_j, \quad 2 \leq i \leq n-1 \\ \Delta(e_n) = c_{n-2,n} e_{n-2} + c_{n-1,n} e_{n-1} + c_{n,n} e_n. \\ c_{i,i} = c_{n,n}, \quad 2 \leq i \leq n-3. \end{cases} \quad (3.3)$$

Proof: Applying the local condition first to the basis vectors and using Proposition 2.2 gives the displayed support restrictions. Some additional relations are forced by sums of basis vectors. For example, in $\mu_{1,2}^n$, apply the local condition to $e_i + e_n$ with $2 \leq i \leq n-3$. In every $\frac{1}{2}$ -derivation, the coefficients of e_i in $D(e_i)$ and of e_n in $D(e_n)$ are both α_1 . Consequently,

$$c_{i,i} = c_{n,n}, \quad 2 \leq i \leq n-3.$$

The same argument applies to $\mu_{1,4}^n$. For $\mu_{1,3}^n$ there is no e_{n-2} component in $D(e_n)$ that can absorb the corresponding coefficient, so the relation extends to $i = n-2$. This proves necessity.

We now give the sufficiency argument in the two cases singled out above. For $\mu_{1,2}^n$, Proposition 2.2 gives

$$D(x) = x_1 \sum_{r=1}^n \alpha_r e_r + \sum_{i=2}^{n-1} x_i \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k + x_n (\alpha_n e_{n-2} + \beta_{n-1} e_{n-1} + \alpha_1 e_n).$$

If $x_1 \neq 0$, comparison with $\Delta(x)$ determines successively $\alpha_1, \alpha_2, \dots, \alpha_n$ and then β_{n-1} . Suppose $x_1 = 0$, and let t be the first index in $\{2, \dots, n-1\}$ for which $x_t \neq 0$. If $t \leq n-3$, set $\alpha_1 = c_{t,t} = c_{n,n}$ and solve successively for the remaining α 's; the free parameters α_n and β_{n-1} resolve the e_{n-2} and e_{n-1} equations when $x_n \neq 0$. If $t \in \{n-2, n-1\}$, first use the e_n equation when $x_n \neq 0$ and then the two preceding equations. If all $x_1, \dots, x_{n-1}$ vanish, choose $\alpha_1 = c_{n,n}$, $\alpha_n = c_{n-2,n}$, and $\beta_{n-1} = c_{n-1,n}$. Thus a suitable D_x exists in every case.

For $\mu_{1,4}^n$, the corresponding expression is

$$\begin{aligned} D(x) = & x_1 \sum_{r=1}^n \alpha_r e_r + x_2 \left(\sum_{k=2}^{n-2} \alpha_{k-1} e_k + \left(\alpha_{n-2} + \frac{1}{2} \alpha_n \right) e_{n-1} \right) \\ & + \sum_{i=3}^{n-1} x_i \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k + x_n (-\alpha_n e_{n-2} + \beta_{n-1} e_{n-1} + \alpha_1 e_n). \end{aligned}$$

The same successive solution works: α_n solves the last equation when $x_1 \neq 0$ and the e_{n-2} equation when $x_1 = 0$ and $x_n \neq 0$; β_{n-1} then solves the e_{n-1} equation. The relations $c_{i,i} = c_{n,n}$ are exactly the compatibility conditions in the remaining cases. The arguments for $\mu_{1,1}^n$ and $\mu_{1,3}^n$ are identical, with the latter using its stated extended diagonal relation. Hence every map in the displayed classes is local. $\square$

Theorem 3.3. *A linear map Δ on an n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebra is a local $\frac{1}{2}$ -derivation if and only if it has the corresponding form below:*

- for the algebra $\mu_{2,1}^n$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_2) &= \sum_{j=2}^{n-2} c_{j,2} e_j + c_{n,2} e_n, \\ \Delta(e_i) &= \sum_{j=i}^{n-2} c_{j,i} e_j, \quad 3 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-2,n-1} e_{n-2} + c_{n-1,n-1} e_{n-1} + c_{n,n-1} e_n, \\ \Delta(e_n) &= c_{n,n} e_n. \end{cases}$$

- for the algebra $\mu_{2,2}^n(\alpha = 1)$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_2) &= \sum_{j=2}^{n-2} c_{j,2} e_j + c_{n,2} e_n, \\ \Delta(e_i) &= \sum_{j=i}^{n-2} c_{j,i} e_j, \quad 3 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-3,n-1} e_{n-3} + c_{n-2,n-1} e_{n-2} \\ &\quad + c_{n-1,n-1} e_{n-1} + c_{n,n-1} e_n, \\ \Delta(e_n) &= c_{n-2,n} e_{n-2} + c_{n,n} e_n. \end{cases}$$

- for the algebra $\mu_{2,2}^n(\alpha = -1)$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_2) &= \sum_{j=2}^{n-2} c_{j,2} e_j, \\ \Delta(e_i) &= \sum_{j=i}^{n-2} c_{j,i} e_j, \quad 3 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-2,n-1} e_{n-2} + c_{n-1,n-1} e_{n-1} + c_{n,n-1} e_n, \\ \Delta(e_n) &= c_{n,n} e_n. \end{cases}$$

- for the algebra $\mu_{2,2}^n(\alpha \neq -1, 1)$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_2) &= \sum_{j=2}^{n-2} c_{j,2} e_j + c_{n,2} e_n, \\ \Delta(e_i) &= \sum_{j=i}^{n-2} c_{j,i} e_j, \quad 3 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-2,n-1} e_{n-2} + c_{n-1,n-1} e_{n-1} + c_{n,n-1} e_n, \\ \Delta(e_n) &= c_{n,n} e_n. \end{cases}$$

- for the algebra $\mu_{2,3}^n$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= \sum_{j=1}^{n-2} c_{j,1}e_j + c_{n,1}e_n, \\ \Delta(e_i) &= \sum_{j=i}^{n-2} c_{j,i}e_j, \quad 2 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-2,n-1}e_{n-2} + c_{n-1,n-1}e_{n-1} + c_{n,n-1}e_n, \\ \Delta(e_n) &= c_{n,n}e_n. \end{cases}$$

- for the algebra $\mu_{2,4}^n$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= c_{n,n}e_1 + \sum_{j=2}^{n-2} c_{j,1}e_j + c_{n,1}e_n, \\ \Delta(e_i) &= c_{n,n}e_i + \sum_{j=i+1}^{n-2} c_{j,i}e_j, \quad 2 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-2,n-1}e_{n-2} + c_{n,n}e_{n-1} + c_{n,n-1}e_n, \\ \Delta(e_n) &= c_{n,n}e_n. \end{cases}$$

Proof: For every basis vector e_i , choose a $\frac{1}{2}$ -derivation D_{e_i} such that $\Delta(e_i) = D_{e_i}(e_i)$. Proposition 2.3 then gives all the displayed support restrictions. In the case $\mu_{2,4}^n$, apply the local condition also to $e_i + e_n$ for $1 \leq i \leq n-2$ and to $e_{n-1} + e_n$. The diagonal coefficients of the two summands are governed by the same parameter α_1 ; hence

$$c_{1,1} = c_{2,2} = \cdots = c_{n-2,n-2} = c_{n-1,n-1} = c_{n,n}.$$

This proves necessity in every case.

Conversely, let Δ have one of the stated forms and fix $x = \sum_i x_i e_i$. Substituting the corresponding form from Proposition 2.3 into $D_x(x) = \Delta(x)$ gives a triangular system. Starting with the first nonzero coordinate among $x_1, \dots, x_{n-2}$ determines the chain parameters successively. The remaining equations involve only the free coefficients in $D_x(e_{n-1})$ and $D_x(e_n)$. For $\mu_{2,1}^n$ these are $\beta_{n-2}, \beta_{n-1}, \beta_n$; for $\mu_{2,2}^n(1)$ they are $\beta_{n-3}, \beta_{n-2}, \beta_{n-1}, \beta_n$; and in the other cases they are the three terminal coefficients displayed in Proposition 2.3. They solve the final two or three equations. If all chain coordinates vanish, those same terminal coefficients solve the system directly. In $\mu_{2,4}^n$, the common-diagonal relations above are precisely the compatibility conditions. Thus a suitable D_x exists for every x , proving sufficiency. $\square$

The dimensions of the spaces of $\frac{1}{2}$ -derivations and local $\frac{1}{2}$ -derivations are summarized in Table 1.

Corollary 3.1. *In each of the considered classes of null-filiform, filiform and naturally graded quasi-filiform non-split associative algebras, there exist local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations whenever the space of local $\frac{1}{2}$ -derivations is strictly larger than the space of $\frac{1}{2}$ -derivations.*

Remark 3.1. We also note that local $\frac{1}{2}$ -derivations of an arbitrary low-dimensional algebra can be similarly described using the fact that the $\frac{1}{2}$ -derivations have a common matrix form. The technique developed in this paper for constructing local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations can be applied to any low-dimensional algebra for which the $\frac{1}{2}$ -derivations have a common matrix form.

3.2. 2-local $\frac{1}{2}$ -derivations on null-filiform, filiform and quasi-filiform associative algebras. Now we study 2-local $\frac{1}{2}$ -derivations on n -dimensional null-filiform, n -dimensional ($n > 3$) complex filiform and n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebras.

Theorem 3.4. *Any 2-local $\frac{1}{2}$ -derivation on the null-filiform associative algebra μ_0^n is a $\frac{1}{2}$ -derivation.*

Proof: Consider a 2-local $\frac{1}{2}$ -derivation ∇ on μ_0^n such that $\nabla(e_1) = 0$.

TABLE 1. Dimensions of the spaces of $\frac{1}{2}$ -derivations and local $\frac{1}{2}$ -derivations.

Algebra	Dimension of the space of $\frac{1}{2}$ -derivations	Dimension of the space of local $\frac{1}{2}$ -derivations
μ_0^n	n	$\frac{n^2+n}{2}$
$\mu_{1,1}^n$	$n+2$	$\frac{n^2-n+6}{2}$
$\mu_{1,2}^n$	$n+1$	$\frac{n^2-3n+16}{2}$
$\mu_{1,3}^n$	$n+1$	$\frac{n^2-3n+12}{2}$
$\mu_{1,4}^n$	$n+1$	$\frac{n^2-3n+16}{2}$
$\mu_{2,1}^n$	$n+3$	$\frac{n^2-3n+16}{2}$
$\mu_{2,2}^n, \alpha = 1$	$n+4$	$\frac{n^2-3n+20}{2}$
$\mu_{2,2}^n, \alpha = -1$	$n+3$	$\frac{n^2-3n+14}{2}$
$\mu_{2,2}^n, \alpha \neq -1, 1$	$n+3$	$\frac{n^2-3n+16}{2}$
$\mu_{2,3}^n$	$n+2$	$\frac{n^2-3n+12}{2}$
$\mu_{2,4}^n$	$n+1$	$\frac{n^2-5n+14}{2}$

For the element $x = t_1 e_1 + t_2 e_2 + \cdots + t_n e_n \in \mu_0^n$, there is a $\frac{1}{2}$ -derivation $D_{e_1, x}$ such that

$$\nabla(e_1) = D_{e_1, x}(e_1), \quad \nabla(x) = D_{e_1, x}(x).$$

Then we have

$$0 = \nabla(e_1) = D_{e_1, x}(e_1) = \sum_{i=1}^n \alpha_i e_i.$$

Hence $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ and therefore $D_{e_1, x} = 0$. Since x is arbitrary, $\nabla(x) = 0$ for all $x \in \mu_0^n$, that is, $\nabla = 0$.

Now let ∇ be an arbitrary 2-local $\frac{1}{2}$ -derivation of the algebra μ_0^n . Then there is a $\frac{1}{2}$ -derivation D such that $\nabla(e_1) = D(e_1)$. The map $\nabla - D$ is also a 2-local $\frac{1}{2}$ -derivation and $(\nabla - D)(e_1) = 0$, which implies $\nabla \equiv D$. Hence ∇ is a $\frac{1}{2}$ -derivation. $\square$

Theorem 3.5. *The n -dimensional ($n > 3$) complex filiform associative algebras and the n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebras admit 2-local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations.*

Proof: We first prove the theorem for the algebra $\mu_{1,1}^n$. Let us define a homogeneous non-additive function f on $\mathbb{C}^2$ by

$$f(z_1, z_n) = \begin{cases} \frac{z_1^2}{z_n}, & \text{if } z_n \neq 0, \\ 0, & \text{if } z_n = 0. \end{cases}$$

where $(z_1, z_n) \in \mathbb{C}^2$. Consider the map $\nabla : \mu_{1,1}^n \rightarrow \mu_{1,1}^n$ defined by the rule

$$\nabla(x) = f(x_1, x_n)e_n, \quad \text{where } x = \sum_{i=1}^n x_i e_i \in \mu_{1,1}^n.$$

Since f is not additive, ∇ is not a $\frac{1}{2}$ -derivation.

Let us show that ∇ is a 2-local $\frac{1}{2}$ -derivation. For the elements

$$x = \sum_{i=1}^n x_i e_i, \quad y = \sum_{i=1}^n y_i e_i,$$

we look for a $\frac{1}{2}$ -derivation D in the form:

$$D(e_1) = \alpha_n e_n, \quad D(e_i) = 0, \quad 2 \leq i \leq n-1, \quad D(e_n) = \beta_n e_n.$$

Assume that $\nabla(x) = D(x)$ and $\nabla(y) = D(y)$. Then we obtain the following system of equations for α_n and β_n :

$$\begin{cases} x_1 \alpha_n + x_n \beta_n = f(x_1, x_n), \\ y_1 \alpha_n + y_n \beta_n = f(y_1, y_n). \end{cases} \quad (3.4)$$

Case 1. Let $x_1 y_n - x_n y_1 = 0$. Since f is homogeneous, the right-hand side of the system is compatible, and therefore the system has a solution.

Case 2. Let $x_1 y_n - x_n y_1 \neq 0$. Then the coefficient matrix of the system is nonsingular, and hence the system has a unique solution.

For the remaining algebras, the same construction works with the two coordinates and the common output vector shown below:

Algebra	coordinates	output vector
$\mu_{1,2}^n, \mu_{1,3}^n, \mu_{1,4}^n$	(x_1, x_n)	e_{n-1}
$\mu_{2,1}^n, \mu_{2,2}^n(\alpha), \mu_{2,3}^n, \mu_{2,4}^n$	(x_1, x_{n-1})	e_n

Indeed, Proposition 2.2 contains, for each filiform algebra in the first row, all maps

$$D(x) = (ax_1 + bx_n)e_{n-1},$$

and Proposition 2.3 contains, for each quasi-filiform algebra in the second row, all maps

$$D(x) = (ax_1 + bx_{n-1})e_n.$$

Replacing (x_1, x_n) in the definition of f by the indicated coordinate pair therefore gives a 2-local $\frac{1}{2}$ -derivation in every case. Since f is not additive, none of these maps is a $\frac{1}{2}$ -derivation. The proof is complete. $\square$

REFERENCES

- [1] Filippov V., On δ -derivations of Lie algebras. *Siberian Mathematical Journal*, (1998) Vol. 39, Iss. 6, P. 1218–1230.
- [2] Filippov V., δ -derivations of prime Lie algebras. *Siberian Mathematical Journal*, (1999) Vol. 40, Iss. 1, P. 174–184.
- [3] Filippov V., On δ -derivations of prime alternative and Mal'tsev algebras. *Algebra and Logic*, (2000) Vol. 39, Iss. 5, P. 354–358.
- [4] Kadison R. V., Local derivations. *Journal of Algebra*, (1990) Vol. 130, P. 494–509.
- [5] Larson D. R., Sourour A. R., Local derivations and local automorphisms of $B(X)$. *Proceedings of Symposia in Pure Mathematics*, (1990), P. 187–194.
- [6] Adashev J. Q., Yusupov B. B., Local derivations and automorphisms of direct sum null-filiform Leibniz algebras. *Lobachevskii Journal of Mathematics*, (2022) Vol. 43, Iss. 12, P. 3407–3413.
- [7] Alauadinov A. K., Yusupov B. B., Local derivations of conformal Galilei algebra. *Communications in Algebra*, (2024) Vol. 52, Iss. 6, P. 2489–2508.
- [8] Ayupov S., Kudaybergenov K., Local derivations on finite-dimensional Lie algebras. *Linear Algebra and its Applications*, (2016) Vol. 493, P. 381–398.
- [9] Ayupov S. A., Kudaybergenov K. K., Yusupov B. B., Local and 2-local derivations of p -filiform Leibniz algebras. *Journal of Mathematical Sciences*, (2020) Vol. 245, Iss. 3, P. 359–367.
- [10] Ayupov S., Khudoyberdiyev A., Local derivations on solvable Lie algebras. *Linear and Multilinear Algebra*, (2021) Vol. 69, Iss. 7, P. 1286–1301.
- [11] Kudaybergenov K., Omirov B., Kurbanbaev T., Local derivations on solvable Lie algebras of maximal rank. *Communications in Algebra*, (2022) Vol. 50, Iss. 9, P. 3816–3826.

- [12] Alauadinov A. K., Yusupov B. B., Local derivations on the Schrödinger Lie algebra. *Algebra Colloquium*, (2025) Vol. 32, Iss. 3, P. 467–480.
- [13] Alauadinov A. K., Yusupov B. B., Local derivations on the Schrödinger Lie algebra in $(n+1)$ -dimensional space-time. *Bulletin of the Karaganda University. Mathematics Series*, (2026), Iss. 1(121), P. 55–66.
- [14] Ayupov S. A., Kudaybergenov K. K., Yusupov B. B., Local and 2-local derivations of locally simple Lie algebras. *Journal of Mathematical Sciences*, (2024) Vol. 278, Iss. 4, P. 613–622.
- [15] Chen Y., Zhao K., Zhao Y., Local derivations on Witt algebras. *Linear and Multilinear Algebra*, (2022) Vol. 70, Iss. 6, P. 1159–1172.
- [16] Yao Y. F., Local derivations on the Witt algebra in prime characteristic. *Linear and Multilinear Algebra*, (2022) Vol. 70, Iss. 15, P. 2919–2933.
- [17] Šemrl P., Local automorphisms and derivations on $B(H)$. *Proceedings of the American Mathematical Society*, (1997) Vol. 125, P. 2677–2680.
- [18] Ayupov S., Kudaybergenov K., Rakhimov I., 2-local derivations on finite-dimensional Lie algebras. *Linear Algebra and its Applications*, (2015) Vol. 474, P. 1–11.
- [19] Ayupov S., Kudaybergenov K., Yusupov B., 2-local derivations on generalized Witt algebras. *Linear and Multilinear Algebra*, (2021) Vol. 69, Iss. 16, P. 3130–3140.
- [20] Ayupov S., Yusupov B., 2-local derivations of infinite-dimensional Lie algebras. *Journal of Algebra and Its Applications*, (2020) Vol. 19, Iss. 5, 2050100.
- [21] Chen Z., Wang D., 2-local automorphisms of finite-dimensional simple Lie algebras. *Linear Algebra and its Applications*, (2015) Vol. 486, P. 335–344.
- [22] Abdurasulov K. K., Ayupov S. A., Yusupov B. B., Local and 2-local derivations on filiform associative algebras. *Journal of Algebra and Its Applications*, (2025) Vol. 24, Iss. 10, 2550242.
- [23] Khudoyberdiyev A. K., Yusupov B. B., Local and 2-local $\frac{1}{2}$ -derivations on finite-dimensional Lie algebras. *Results in Mathematics*, (2024) Vol. 79, 210.
- [24] Yusupov B., Vaisova N., Madrahimov T., Local $\frac{1}{2}$ -derivation on n -dimensional naturally graded quasi-filiform Leibniz algebra of type I. *AIP Conference Proceedings*, (2024).
- [25] Mamadaliyev U. K., Sattarov A., Yusupov B. B., Local and 2-local $\frac{1}{2}$ -derivations of solvable Leibniz algebras. *Eurasian Mathematical Journal*, (2025) Vol. 16, Iss. 2, P. 42–54.
- [26] Yusupov B., Local $\frac{1}{2}$ -derivation on naturally graded filiform Leibniz algebra. *Journal of Mathematical Sciences*, (2025),
- [27] Yusupov B., Local and 2-local $\frac{1}{2}$ -derivation on naturally graded non-Lie p -filiform Leibniz algebras. *Journal of Algebra Combinatorics Discrete Structures and Applications*, (2025) Vol. 12, Iss. 3, P. 275–288.
- [28] Masutova K. K., Omirov B. A., On some zero-filiform algebras. *Ukrainian Mathematical Journal*, (2014) Vol. 66, Iss. 4, P. 541–552.
- [29] Dekimpe K., Ongenae V., Filiform left-symmetric algebras. *Geometriae Dedicata*, (1999) Vol. 74, Iss. 2, P. 165–199.
- [30] Karimjanov I. A., Ladra M., Some classes of nilpotent associative algebras. *Mediterranean Journal of Mathematics*, (2020) Vol. 17, 70.

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