

## On some existence theorems for quasilinear elliptic equations

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**Abstract.** Boundary value problems are considered for quasilinear elliptic equations with a principal quasilinear elliptic operator of the second order in the Sobolev space  $W_p^2(\Omega)$ . The main content of the work is to establish the maximum admissible growth of the subordinate nonlinear operator with respect to the corresponding lower-order derivatives. We consider a class of equations for which an a priori estimate of solutions in the norm of the space  $\|u\|_{L_l(\Omega)}$  ( $1 \leq l < \infty$ ) implies an a priori estimate of the solution in the norm of the space  $W_p^2(\Omega)$ . Based on the theorem on a priori estimates, a general solvability theorem for boundary value problems for quasi-linear elliptic equations is obtained under the condition of the existence of an intermediate a priori estimate  $\|u\|_{L_l(\Omega)}$  for the solutions of the corresponding family of boundary value problems.

**Keywords:** interpolation method, Sobolev space, quasilinear equation

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### 1. INTRODUCTION

Let  $R_+ \equiv \{t_0 \in R \mid t_0 \geq 0\}$ ,  $x = (x_1, \dots, x_N)$  be a point in the space  $R^N$ ,  $\Omega \subset R^N$  be a bounded domain with the boundary  $\partial\Omega$  of class  $C^2$ .

Throughout the paper, the functions are assumed to be real-valued. In the paper, we use the following function spaces (see [1, p. 107], [2] and [3]): the spaces of summable functions  $L_p(\Omega)$ ,  $p \geq 1$  with the norm

$$\|u\|_{p;\Omega} = \left( \int_{\Omega} |u(x)|^p dx \right)^{1/p};$$

the anisotropic Sobolev space  $W_p^2(\Omega)$  with the norm

$$\|u\|_{W_p^2(\Omega)} = \|u\|_{p;\Omega} + \sum_{i=1}^N \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|_{p;\Omega}.$$

By  $D^i u$  we denote a vector from the partial derivatives  $D^\alpha u$  of the function  $u(x)$ ,  $\alpha = (\alpha_1, \dots, \alpha_N)$ ,  $|\alpha| = i$  ( $i = 1, 2$ ).

Consider the following boundary value problem

$$\begin{cases} Lu = f(x, u, Du), & x \in \Omega, \\ u|_{\partial\Omega} = \varphi(x), & x \in \partial\Omega. \end{cases} \quad (1.1)$$

We consider this problem in the class of real functions from the Sobolev space  $W_p^2(\Omega)$  for  $p > 1$ , so that the boundary function  $\varphi(x)$  belongs to the space  $W_p^{2-1/p}(\partial\Omega)$  (see [2, p. 389]).

The paper provides a theorem on a priori estimates of the solutions  $\|u\|_{W_p^2(\Omega)}$  in the norm of the Sobolev space  $W_p^2(\Omega)$ , expressed by the norm  $\|u\|_{l;\Omega} \leq M$  ( $1 \leq l < \infty$ ), with some  $M > 0$ .

For the boundary value problem (1.1). We assume that the following conditions were fulfilled.

A.1) Let the function  $f(x, \xi_0, \xi_1)$  be defined on  $\bar{\Omega} \times R \times R^N$  with values in  $R$  and satisfy the Caratheodory condition, i.e. let it be measurable with respect to  $x$  for all  $(\xi_0, \xi_1) \in R \times R^N$  and continuous with respect to  $(\xi_0, \xi_1)$  for almost all  $x \in \Omega$ .

A.2) Let  $N > 2$  and

$$|f(x, t, \xi_0, \xi_1)| \leq b(x) + \sum_{i=0}^1 b_i(x) \cdot |\xi_i|^{\mu_i}$$

almost for all  $x \in \Omega$  and for all  $\xi_0 \in R$ ,  $\xi_1 \in R^N$ , with non-negative functions  $b(x)$  and  $b_i(x)$  ( $i = 0, 1$ ), such that the function  $b(x) \in L_p(\Omega)$ ,  $p > N$ , and  $b_i(x) \in L_{q_i}(\Omega)$ ,  $p < q_i \leq \infty$ ,  $i = 0, 1$ .

A.3) Let

$$\mu_i = \frac{1}{i + N/l} \cdot \left( 2 + \frac{N}{l} - \frac{N}{q_i} \right) \quad (i = 0, 1).$$

A.4) Let  $Lu$  be a quasilinear elliptic operator of second order of the form

$$Lu = \sum_{|\alpha|=2} a_\alpha(x, u, Du) D^\alpha u,$$

where the coefficients  $a_\alpha$  ( $|\alpha| = 2$ ) of the operator  $L$  are real and continuous functions on  $\bar{\Omega} \times R \times R^N$ .

Let  $L_v u$  linear with respect to  $u(x)$  operator equal to

$$L_v u = \sum_{|\alpha|=2} a_\alpha(x, v, Dv) D^\alpha u,$$

for any function  $v(x) \in C^1(\bar{\Omega})$  be a linear elliptic operator, with such real coefficients that the linear (with respect to  $u(x)$ ) boundary value problem

$$\begin{cases} L_v u = g(x), & x \in \Omega, \\ u|_{\partial\Omega} = \varphi(x), & x \in \partial\Omega \end{cases} \quad (1.2)$$

is coercive in the space  $W_p^2(\Omega)$ , i.e. a priori estimate

$$\|u\|_{W_p^2(\Omega)} \leq C \left( \|g\|_{p;\Omega} + \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} + \|u\|_{p;\Omega} \right),$$

with positive constant  $C$ , independent of  $g \in L_p(\Omega)$ ,  $\varphi(x) \in W_p^{2-1/p}(\partial\Omega)$  and on the solution  $u \in W_p^2(\Omega)$  of linear problem (1.2), is fulfilled. This time, the constant  $C$  can depend on the modulus of continuity of the coefficients  $a_\alpha$  ( $|\alpha| = 2$ ) and modules of continuity of functions  $v, Dv$  (see [4]).

For elliptic operators, this problem was considered in [5]-[6], [7]-[8]. In [5, 6, 7, 2], it is assumed that a nonlinear function  $f(x, u, Du)$  is continuous in all its arguments. In this case, the well-known S. N. Bernstein growth condition a non-linear function is sufficient for the a priori estimate in  $\|u\|_{\infty;\Omega}$ , to follow from the a priori estimate in  $\|Du\|_{\infty;\Omega}$  and, hence the estimate in  $\|u\|_{W_p^2(\Omega)}$ . In [9, 8], a boundary value problem for elliptic equations, respectively, was considered, where the non-linear function  $f$  belongs to the space  $L_p(\Omega)$ . In the present paper, for the non-linear function  $f(x, u, Du)$ , we adopt a different S. N. Bernstein separation [6] and the first an a priori estimate of  $\|u\|_{l;\Omega} \leq M$ ,  $1 \leq l < \infty$  and investigate the problem (1.1) using the interpolation method [9, 8]. In this context, we also refer to the work [10, 11, 12, 13].

To study problem (1.1), we will make essential use of the embedding theorem for spaces  $W_p^2(\Omega)$  and the multiplicative inequality that follows from it. Let us formulate the embedding theorem [1, p.278] in the form we need.

**Embedding theorems:** Assume that  $S_i \geq p$  ( $i = 0, 1$ ) and

$$\varkappa_1 = 1 - \frac{1}{2} \left[ N \left( \frac{1}{p} - \frac{1}{S_i} \right) + i \right] > 0, \quad i = 0, 1.$$

Then the space  $W_p^2(\Omega) \subset L_{S_i}(\Omega)$  ( $i = 0, 1$ ) is compactly embedded, and

$$\|D^i u\|_{S_i;\Omega} \leq C \cdot \|u\|_{W_p^2(\Omega)} \quad (i = 0, 1).$$

If  $S_i > l$  ( $i = 0, 1$ ), then there exists  $\delta_0 > 0$  such that for any  $0 < \delta \leq \delta_0$  and for function  $u \in W_p^2(\Omega)$

$$\|D^i u\|_{S_i;\Omega} \leq C \cdot \delta^{-\varkappa_0} \cdot \|u\|_{l;\Omega} + C \cdot \delta^{\varkappa_1} \cdot \|u\|_{W_p^2(\Omega)}, \quad (i = 0, 1), \quad (1.3)$$

where

$$\varkappa_0 = \frac{1}{2} \left[ N \left( \frac{1}{p} - \frac{1}{S_i} \right) + i \right], \quad i = 0, 1.$$

Here, the positive constant  $C$  does not depend on the function  $u(x)$  and  $\delta$ . Assume (see [1, p. 392])

$$\delta = \min \left\{ \left( \|u\|_{L:\Omega} \cdot \|u\|_{W_p^2(\Omega)}^{-1} \right)^{\frac{1}{\varkappa_0 + \varkappa_1}} ; \delta_0 \right\}.$$

Then of simple transformations we have

$$\|D^i u\|_{S_i:\Omega} \leq C_0 \cdot \|u\|_{W_p^2(\Omega)}^{1/\mu_i} \cdot \|u\|_{L:\Omega}^{1-1/\mu_i} + C_0 \cdot \|u\|_{L:\Omega}, \quad i = 0, 1. \quad (1.4)$$

Where  $S_i = \frac{\mu_i p q_i}{q_i - p}$ ,  $\frac{\varkappa_0}{\varkappa_0 + \varkappa_1} = \frac{1}{\mu_i}$ ,  $i = 0, 1$  and positive constants  $C_0$  independent of function  $u(x)$ . Note that under the condition on  $S_i > l$  ( $i = 0, 1$ ) the space  $W_p^2(\Omega) \subset L_{S_i}(\Omega) \subset L_l(\Omega)$  ( $i = 0, 1$ ), i.e.  $\|u\|_{L:\Omega} \leq \text{const} \cdot \|u\|_{W_p^2(\Omega)}$ .

These results can be formulated as follows.

**Lemma 1.1.** *Let the conditions on  $S_i \geq p$ ,  $S_i > l$ ,  $i = 0, 1$  and  $\varkappa_1 > 0$  be fulfilled. Then for an arbitrary function  $u(x) \in W_p^2(\Omega)$  the inequality (1.4) holds.*

Assume now the  $F(u)(x) \equiv f(x, u(x), Du(x))$ . Let  $u(x) \in W_p^2(\Omega)$ , the conditions A.1)–A.3),  $\mu_i > \frac{l(q_i - p)}{q_i p}$  ( $i = 0, 1$ ) and  $\varkappa_1 > 0$  be fulfilled. From condition A.2) it follows that

$$\|F(u)\|_{p:\Omega} \leq \|b\|_{p:\Omega} + \sum_{i=0}^1 \|b_i\|_{q_i:\Omega} \cdot \|D^i u\|_{S_i:\Omega}^{\mu_i},$$

where  $S_i = \frac{\mu_i p q_i}{q_i - p}$  ( $i = 0, 1$ ). We will consider the case  $S_i > l$ . From this we also obtain that  $\mu_i > \frac{l(q_i - p)}{q_i p}$  ( $i = 0, 1$ ). Then, based on the equalities A.3) and interpolation inequalities (1.4), we obtain

$$\begin{aligned} \|F(u)\|_{p:\Omega} &\leq \|b\|_{p:\Omega} + \sum_{i=0}^1 \|b_i\|_{q_i:\Omega} \cdot \left[ C_0 \cdot \|u\|_{W_p^2(\Omega)}^{1/\mu_i} \cdot \|u\|_{L:\Omega}^{1-1/\mu_i} + C_0 \cdot \|u\|_{L:\Omega} \right]^{\mu_i} \leq \\ &\leq \|b\|_{p:\Omega} + \sum_{i=0}^1 \|b_i\|_{q_i:\Omega} \cdot 2^{\mu_i - 1} \cdot C_0^{\mu_i} \cdot \|u\|_{L:\Omega}^{\mu_i - 1} \cdot \|u\|_{W_p^2(\Omega)} + \\ &\quad + \sum_{i=0}^1 \|b_i\|_{q_i:\Omega} \cdot 2^{\mu_i - 1} \cdot C_0^{\mu_i} \cdot \|u\|_{L:\Omega}^{\mu_i} = \\ &= \Phi_1(\|u\|_{L:\Omega}) \cdot \|u\|_{W_p^2(\Omega)} + \Phi_2(\|u\|_{L:\Omega}). \end{aligned} \quad (1.5)$$

Where

$$\Phi_1(\|u\|_{L:\Omega}) = \sum_{i=0}^1 \|b_i\|_{q_i:\Omega} \cdot 2^{\mu_i - 1} \cdot C_0^{\mu_i} \cdot \|u\|_{L:\Omega}^{\mu_i - 1},$$

$$\Phi_2(\|u\|_{L:\Omega}) = \|b\|_{p:\Omega} + \sum_{i=0}^1 \|b_i\|_{q_i:\Omega} \cdot 2^{\mu_i - 1} \cdot C_0^{\mu_i} \cdot \|u\|_{L:\Omega}^{\mu_i},$$

i.e.  $\Phi_1, \Phi_2 : R_+ \rightarrow R_+$  are increasing functions determined by the known data. This proves the boundedness of the operator  $F(u)$ .

On the other hand,  $p > N$ . Then, by virtue of the Sobolev embedding theorem [14], it follows that the embedding operator  $W_p^2(\Omega) \rightarrow L_{S_i}(\Omega)$  is completely continuous. By virtue of (1.5), the operator  $L_{S_i}(\Omega) \rightarrow L_p(\Omega)$  ( $i = 0, 1$ ) is bounded and, by the general property of the superposition operator, is continuous. Thus, the operator  $W_p^2(\Omega) \rightarrow L_p(\Omega)$  is completely continuous as a composition of completely continuous and continuous operators.

Thus, the following lemma is proved.

**Lemma 1.2.** *Let  $u(x) \in W_p^2(\Omega)$  and the conditions A.1) - A.3),  $\mu_i > \frac{l(q_i - p)}{p q_i}$ , ( $i = 0, 1$ ),  $\varkappa_1 > 0$  be fulfilled. Then the operator  $F(u)(x)$  is a bounded and continuous operator from  $W_p^2(\Omega)$  to  $L_p(\Omega)$ .*

**Theorem 1.3.** *Let conditions A.1) - A.3) be fulfilled and  $p > N$ . Then there exists a function  $\Phi : R_+^2 \rightarrow R_+$  increasing in each argument such for any possible solution  $u \in W_p^2(\Omega)$  of problem (1.1) the a priori estimate*

$$\|u\|_{W_p^2(\Omega)} \leq \Phi \left( M, \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \right)$$

holds. Where,  $\|u\|_{l;\Omega} \leq M$ ,  $l \in [1, +\infty)$  and  $R_+^2 = R_+ \times R_+$ .

*Proof:* It is evident that

$$Lu = f(x, u, Du) = \frac{f(x, u, Du)}{1 + |u|^{\mu_0} + |Du|^{\mu_1}} \cdot (1 + |u|^{\mu_0} + |Du|^{\mu_1}).$$

Hence, for the function  $u(x)$

$$\begin{cases} Lu - C(x)u = f_1(x) \cdot |u|^{\mu_0} + f_1(x) \cdot |Du|^{\mu_1} + f_0(x), & x \in \Omega, \\ u|_{\partial\Omega} = \varphi(x), & x \in \partial\Omega. \end{cases} \quad (1.6)$$

Where,  $C(x) = b(x) \geq 0$ ,  $C(x) \in L_p(\Omega)$ ,

$$f_1(x) = \frac{f(x, u, Du)}{1 + |u|^{\mu_0} + |Du|^{\mu_1}}, \quad f_0(x) = f_1(x) - C(x)u(x),$$

$f_0(x) \in L_p(\Omega)$ ,  $S_0 = \frac{\mu_0 p q_0}{q_0 - p}$  when  $f_1(x) \in L_{q_0}(\Omega)$ ,  $S_1 = \frac{\mu_1 p q_1}{q_1 - p}$  when  $f_1(x) \in L_{q_1}(\Omega)$ ;  $q_0 > p > N$ ,  $q_1 > p > N$  and  $\mu_i = \frac{1}{i+N/l} \cdot \left(2 + \frac{N}{l} - \frac{N}{q_i}\right)$  ( $i = 0, 1$ ).

Consider in the space  $W_p^2(\Omega)$  with  $p > N$  the boundary value problem.

$$\begin{cases} LV - C(x)V(x) = f_1(x) (|V|^{\mu_0} + |Du|^{\mu_1}) + \lambda \cdot f_0(x), & x \in \Omega, \\ V(x)|_{\partial\Omega} = \lambda \cdot \varphi(x), & x \in \partial\Omega, \end{cases} \quad (1.7)$$

with the problem  $\lambda \in [0, 1]$  and the functions  $C(x)$ ,  $\varphi(x)$ ,  $f_0(x)$  and  $f_1(x)$  defined above. It should be noted that a problem of the form (1.7) has also been considered in [5, 10, 9].  $\square$

**Lemma 1.4.** *For any fixed  $\lambda \in [0, 1]$ , the problem (1.7) has at most one solution  $u(x)$  from  $W_p^2(\Omega)$ ,  $p > N$ .*

*Proof:* Assume the contrary. Then for the difference  $\mathcal{W}(x) = V(x) - Z(x)$  of two possible solutions  $V(x)$  and  $Z(x)$ , we have

$$\begin{cases} L\mathcal{W}(x) - C(x)\mathcal{W}(x) = f_1(x) \left[ \sum_{i=1}^N h_i(x) D_i \mathcal{W}(x) + h(x) \mathcal{W}(x) \right], & x \in \Omega, \\ \mathcal{W}(x)|_{\partial\Omega} = 0, & x \in \partial\Omega. \end{cases}$$

Here,

$$h_i(x) = \int_0^1 H_i(x, \tau) d\tau, \quad h(x) = \int_0^1 H(x, \tau) d\tau.$$

$$H_i(x, \tau) = \mu_1 \left[ \sum_{k=1}^N (\tau \cdot D_k \mathcal{W} + D_k Z)^2 \right]^{\frac{\mu_1}{2} - 1} (\tau \cdot D_i \mathcal{W} + D_i Z)(x).$$

for

$$\sum_{k=1}^N (\tau \cdot D_k \mathcal{W} + D_k Z)^2(x) \neq 0;$$

$$H(x, \tau) = \mu_0 \left[ (\tau \mathcal{W} + Z)^2 \right]^{\frac{\mu_0}{2} - 1} (\tau \mathcal{W} + Z)(x) \quad \text{for} \quad (\tau \mathcal{W} + Z)^2 \neq 0$$

and

$$H_i(x, \tau) = 0 \quad \text{for} \quad \sum_{k=1}^N (\tau \cdot D_k \mathcal{W} + D_k Z)^2(x) = 0,$$

$$H(x, \tau) = 0 \quad \text{for} \quad (\tau \cdot \mathcal{W} + Z)^2(x) = 0.$$

Since  $C(x) \geq 0$ ,  $C(x) \in L_p(\Omega)$ ,  $h(x) \geq 0$ ,  $h(x) \cdot f_1(x) \in L_p(\Omega)$ ,  $f_1(x) \cdot h_i(x) \in L_{q_1}(\Omega)$  ( $i = 1, \dots, N$ ), and  $p > N$ , from the results of [15] it follows that  $\mathcal{W} \equiv 0$  in  $\Omega$ .

Lemma 1.4 is proved.  $\square$

Now let  $v_1(x)$  and  $v_2(x)$  be solutions to problem (1.7) corresponding to the values  $\lambda_1$  and  $\lambda_2$  ( $\lambda_2 > \lambda_1$ ) of the parameter  $\lambda$  respectively. Then for the difference  $\tilde{V}(x) = v_2(x) - v_1(x)$ , we have

$$\begin{cases} L\tilde{V} - C(x) \cdot \tilde{V}(x) = f_1(x) \left[ \sum_{i=1}^N \tilde{h}_i(x) D_i \tilde{V} + \tilde{h}(x) \tilde{V} \right] + \\ + (\lambda_2 - \lambda_1) f_0(x), \quad x \in \Omega, \\ \tilde{V}|_{\partial\Omega} = (\lambda_2 - \lambda_1) \cdot \varphi(x), \quad x \in \partial\Omega, \end{cases}$$

where

$$\tilde{h}_i(x) = \int_0^1 \tilde{H}_i(x, \tau) d\tau, \quad \tilde{h}(x) = \int_0^1 \tilde{H}(x, \tau) d\tau.$$

$$\tilde{H}_i(x, \tau) = \mu_1 \left[ \sum_{k=1}^N (\tau \cdot D_k \tilde{V} + D_k v_1)^2 \right]^{\frac{\mu_1}{2} - 1} \cdot (\tau \cdot D_i \tilde{V} + D_i v_1)(x),$$

for

$$\sum_{k=1}^N (\tau \cdot D_k \tilde{V} + D_k v_1)^2(x) \neq 0, \quad \tilde{H}_i(x, \tau) = 0$$

for

$$\sum_{k=1}^N (\tau \cdot D_k \tilde{V} + D_k v_1)^2(x) = 0;$$

$$\tilde{H}(x, \tau) = \mu_0 \left[ (\tau \tilde{V} + v_1)^2 \right]^{\frac{\mu_0}{2} - 1} \cdot (\tau \tilde{V} + v_1)(x).$$

for  $(\tau \cdot \tilde{V} + v_1)^2(x) \neq 0$ ,  $\tilde{H}(x, \tau) = 0$  for  $(\tau \cdot \tilde{V} + v_1)^2(x) = 0$ .

Assume

$$\mathbb{K} = (\lambda_2 - \lambda_1) (1 + \|u\|_{L(\Omega)}), \quad \lambda_2 > \lambda_1.$$

**Lemma 1.5.**  $\|\tilde{V}(x)\|_{L(\Omega)} \leq \mathbb{K}$ .

*Proof:* For the function  $(\tilde{V} - \mathbb{K})$ , we have

$$\begin{cases} L(\tilde{V} - \mathbb{K}) - C(x) (\tilde{V} - \mathbb{K}) = f_1(x) \left[ \sum_{i=1}^N \tilde{h}_i(x) D_i (\tilde{V} - \mathbb{K}) + \tilde{h}(x) (\tilde{V} - \mathbb{K}) \right] + \\ + (\lambda_2 - \lambda_1) f_0(x) + C(x) \mathbb{K} + f_1(x) \tilde{h}(x) \mathbb{K}, \quad x \in \Omega, \\ (\tilde{V} - \mathbb{K})|_{\partial\Omega} = (\lambda_2 - \lambda_1) \cdot \varphi(x) - \mathbb{K}, \quad x \in \partial\Omega. \end{cases}$$

Further,

$$\begin{aligned} & (\lambda_2 - \lambda_1) f_0(x) + C(x) \mathbb{K} + f_1(x) \tilde{h}(x) \mathbb{K} = \\ & = (\lambda_2 - \lambda_1) \cdot \left[ f_0(x) + (C(x) + f_1(x) \tilde{h}(x)) (1 + \|u\|_{L(\Omega)}) \right] = \end{aligned}$$

$$\begin{aligned}
 &= (\lambda_2 - \lambda_1) \cdot \left[ f_1(x) - C(x)u(x) + \left( C(x) + f_1(x)\tilde{h}(x) \right) (1 + \|u\|_{L^1(\Omega)}) \right] = \\
 &= (\lambda_2 - \lambda_1) \cdot \left[ C(x) (\|u\|_{L^1(\Omega)} - u(x)) + f_1(x) + \right. \\
 &\quad \left. + C(x) + f_1(x)\tilde{h}(x) (1 + \|u\|_{L^1(\Omega)}) \right] \geq 0, \quad x \in \Omega; \\
 &\left( \tilde{V} - \mathbb{K} \right) \Big|_{\partial\Omega} = (\lambda_2 - \lambda_1) \varphi(x) - (\lambda_2 - \lambda_1) (1 + \|u\|_{L^1(\Omega)}) \leq \\
 &\leq (\lambda_2 - \lambda_1) (\varphi(x) - \|u\|_{L^1(\Omega)}) \leq 0, \quad x \in \partial\Omega.
 \end{aligned}$$

Then from [15] it follows that  $\tilde{V}(x) \leq \mathbb{K}$  in  $\bar{\Omega}$ . Similarly, inequality  $\tilde{V}(x) \geq -\mathbb{K}$  is proved in  $\bar{\Omega}$ . Lemma 1.5 is proved.  $\square$

Consider a parametric family of problems (1.7). It should be noted that, by Lemma 1.4, the solution of problem (1.6) and (in view of the adopted notation) with the solution of problem (1.1).

Let  $v_1(x)$  and  $v_2(x)$  be solutions to problem (1.7), corresponding to the values  $\lambda_1$  and  $\lambda_2$  ( $\lambda_2 > \lambda_1$ ) of the parameter  $\lambda$ . Then, by Lemma 1.5 we have

$$\|v_2 - v_1\|_{L^1(\Omega)} \leq (\lambda_2 - \lambda_1) (1 + \|u\|_{L^1(\Omega)}). \quad (1.8)$$

On the other hand, this function is a solution to the problem

$$\begin{cases} L\tilde{V} - C(x)\tilde{V} = f_1(x) [ (|v_2|^{\mu_0} - |v_1|^{\mu_0}) + (|Dv_2|^{\mu_1} - |Dv_1|^{\mu_1}) ] + \\ + (\lambda_2 - \lambda_1) f_0(x), \quad x \in \Omega, \\ \tilde{V} \Big|_{\partial\Omega} = (\lambda_2 - \lambda_1) \varphi(x), \quad x \in \partial\Omega. \end{cases}$$

Consequently,

$$\begin{aligned}
 &\left\| L\tilde{V} - C(x)\tilde{V} \right\|_{p;\Omega} \leq 2^{\mu_0-1} \cdot \|f_1\|_{q_0;\Omega} \cdot \left\| \tilde{V} \right\|_{S_0;\Omega}^{\mu_0} + \\
 &+ 2^{\mu_1-1} \cdot \|f_1\|_{q_1;\Omega} \cdot \left\| D\tilde{V} \right\|_{S_1;\Omega}^{\mu_1} + 2^{\mu_0} \cdot \|f_1\|_{q_0;\Omega} \cdot \|v_1\|_{S_0;\Omega}^{\mu_0} + \\
 &+ 2^{\mu_1} \cdot \|f_1\|_{q_1;\Omega} \cdot \|Dv_1\|_{S_1;\Omega}^{\mu_1} + (\lambda_2 - \lambda_1) \|f_0\|_{p;\Omega}, \quad (1.9)
 \end{aligned}$$

$$\left\| \tilde{V} \right\|_{W_p^{2-1/p}(\partial\Omega)} \leq (\lambda_2 - \lambda_1) \cdot \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}. \quad (1.10)$$

On the other hand, it follows from (1.4) that

$$\left\| D^i \tilde{V} \right\|_{S_i;\Omega} \leq C_0 \cdot \left\| \tilde{V} \right\|_{W_p^2(\Omega)}^{1/\mu_i} \cdot \left\| \tilde{V} \right\|_{L^1(\Omega)}^{1-1/\mu_i} + C_0 \cdot \left\| \tilde{V} \right\|_{L^1(\Omega)}, \quad i = 0, 1. \quad (1.11)$$

By virtue of the well-known linear theory of elliptic problems, the inequality

$$\left\| \tilde{V} \right\|_{W_p^2(\Omega)} \leq C_2 \left( \left\| L\tilde{V} - C(x)\tilde{V} \right\|_{p;\Omega} + \left\| \tilde{V} \right\|_{W_p^{2-1/p}(\partial\Omega)} \right).$$

Here,  $C_2 = C_2(\Omega, p, q_0, q_1, N)$  is a positive constant. Then, using inequalities (1.8)–(1.11) we obtain

$$\begin{aligned}
 &\left\| \tilde{V} \right\|_{W_p^2(\Omega)} \leq C_2 \cdot 2^{\mu_0-1} \cdot \|f_1\|_{q_0;\Omega} \cdot \left\| \tilde{V} \right\|_{S_0;\Omega}^{\mu_0} + \\
 &\quad + C_2 \cdot 2^{\mu_1-1} \cdot \|f_1\|_{q_1;\Omega} \cdot \left\| D\tilde{V} \right\|_{S_1;\Omega}^{\mu_1} + \\
 &+ C_2 \left( \|f_1\|_{q_0;\Omega} \cdot 2^{\mu_0} \cdot \|v_1\|_{S_0;\Omega}^{\mu_0} + \|f_1\|_{q_1;\Omega} \cdot 2^{\mu_1} \cdot \|Dv_1\|_{S_1;\Omega}^{\mu_1} \right) + \\
 &\quad + C_2 (\lambda_2 - \lambda_1) \left( \|f_0\|_{p;\Omega} + \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \right) \leq
 \end{aligned}$$

$$\begin{aligned}
&\leq C_2 \cdot 2^{\mu_0-1} \cdot \|f_1\|_{q_0:\Omega} \cdot \left[ C_0 \cdot \|\tilde{V}\|_{W_p^2(\Omega)}^{1/\mu_0} \cdot \|\tilde{V}\|_{l:\Omega}^{1-1/\mu_0} + C_0 \cdot \|\tilde{V}\|_{l:\Omega} \right]^{\mu_0} + \\
&+ C_2 \cdot 2^{\mu_1-1} \cdot \|f_1\|_{q_1:\Omega} \cdot \left[ C_0 \cdot \|\tilde{V}\|_{W_p^2(\Omega)}^{1/\mu_1} \cdot \|\tilde{V}\|_{l:\Omega}^{1-1/\mu_1} + C_0 \cdot \|\tilde{V}\|_{l:\Omega} \right]^{\mu_1} + \\
&+ C_2 \left( \|f_1\|_{q_0:\Omega} \cdot 2^{\mu_0} \cdot \|v_1\|_{S_0:\Omega}^{\mu_0} + 2^{\mu_1} \cdot \|f_1\|_{q_1:\Omega} \cdot \|Dv_1\|_{S_1:\Omega}^{\mu_1} \right) + \\
&+ C_2 (\lambda_2 - \lambda_1) \left( \|f_0\|_{p:\Omega} + \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \right) \leq \\
&\leq C_2 \cdot \|\tilde{V}\|_{W_p^2(\Omega)} \cdot \left[ \|f_1\|_{q_0:\Omega} \cdot 2^{2\mu_0-2} \cdot C_0^{\mu_0} \cdot (\lambda_2 - \lambda_1)^{\mu_0-1} \cdot (1 + \|u\|_{l:\Omega})^{\mu_0-1} + \right. \\
&+ \|f_1\|_{q_1:\Omega} \cdot 2^{2\mu_1-2} \cdot C_0^{\mu_1} \cdot (\lambda_2 - \lambda_1)^{\mu_1-1} \cdot (1 + \|u\|_{l:\Omega})^{\mu_1-1} \left. \right] + \\
&+ C_2 \left( \|f_1\|_{q_0:\Omega} \cdot 2^{\mu_0} \cdot \|v_1\|_{S_0:\Omega}^{\mu_0} + \|f_1\|_{q_1:\Omega} \cdot 2^{\mu_1} \cdot \|Dv_1\|_{S_1:\Omega}^{\mu_1} \right) + \\
&+ C_2 \cdot 2^{2\mu_0-2} \cdot \|f_1\|_{q_0:\Omega} \cdot C_0^{\mu_0} \cdot (\lambda_2 - \lambda_1)^{\mu_0} \cdot (1 + \|u\|_{l:\Omega})^{\mu_0} + \\
&+ C_2 \cdot 2^{2\mu_1-2} \cdot \|f_1\|_{q_1:\Omega} \cdot C_0^{\mu_1} \cdot (\lambda_2 - \lambda_1)^{\mu_1} \cdot (1 + \|u\|_{l:\Omega})^{\mu_1} + \\
&+ C_2 (\lambda_2 - \lambda_1) \left( \|f_0\|_{p:\Omega} + \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \right).
\end{aligned}$$

Let

$$\begin{aligned}
&C_2 \cdot \left( \|f_1\|_{q_0:\Omega} \cdot 2^{2\mu_0-2} \cdot C_0^{\mu_0} \cdot (\lambda_2 - \lambda_1)^{\mu_0-1} \cdot (1 + \|u\|_{l:\Omega})^{\mu_0-1} + \right. \\
&+ \|f_1\|_{q_1:\Omega} \cdot 2^{2\mu_1-2} \cdot C_0^{\mu_1} \cdot (\lambda_2 - \lambda_1)^{\mu_1-1} \cdot (1 + \|u\|_{l:\Omega})^{\mu_1-1} \left. \right) \leq \frac{1}{2}.
\end{aligned}$$

Then, if

$$0 < \lambda_2 - \lambda_1 \leq h \quad (1.12)$$

holds, we obtain the inequality

$$\|\tilde{V}\|_{W_p^2(\Omega)} \leq \Phi_1 + \Phi_2 \cdot \|v_1\|_{S_0:\Omega}^{\mu_0} + \Phi_3 \cdot \|Dv_1\|_{S_1:\Omega}^{\mu_1}. \quad (1.13)$$

Here, the functions  $\Phi_1, \Phi_2, \Phi_3 : R_+ \rightarrow R_+$  are increasing functions defined by the data of the problem (1.1), the quantities  $\lambda_1, \lambda_2 \in [0, 1]$  satisfy the condition (1.12), and the functions  $v_1(x), v_2(x) \in W_p^2(\Omega)$  are solutions corresponding to the quantities  $\lambda_1$  and  $\lambda_2$  of the problem (1.7).

Denote  $\lambda^{(k-1)} = \lambda_1$ ,  $\lambda^{(k)} = \lambda_2$ ,  $v^{(k-1)} = v_1$ , and  $v^{(k)} = v_2$ . From the inequality (1.13), we obtain

$$\|v^{(k)}\|_{W_p^2(\Omega)} \leq \Phi_1 + \Phi_2 \cdot \|v^{(k-1)}\|_{S_0:\Omega}^{\mu_0} + \Phi_3 \cdot \|Dv^{(k-1)}\|_{S_1:\Omega}^{\mu_1} + \|v^{(k-1)}\|_{W_p^2(\Omega)},$$

for  $0 < \lambda^{(k)} - \lambda^{(k-1)} \leq h$ ,  $\lambda^{(k-1)}, \lambda^{(k)} \in [0, 1]$ . Hence, by virtue of the independence of  $h$  of the indicated values  $\lambda^{(k-1)}, \lambda^{(k)}$  and the embedding inequality (see [14]) with  $p > N$

$$\|v^{(k-1)}\|_{S_0:\Omega} \leq \text{const} \cdot \|v^{(k-1)}\|_{W_p^2(\Omega)},$$

$$\|Dv^{(k-1)}\|_{S_1:\Omega} \leq \text{const} \cdot \|v^{(k-1)}\|_{W_p^2(\Omega)},$$

after a finite number of iterations, we obtain the assertion of Theorem 1.3/ In this case, the first iteration ( $k = 1$ ) corresponds to  $\lambda^{(0)} = 0$  and  $v^{(0)} = 0$ .

Theorem 1.3 is proved.

## 2. SOLVABILITY THEORY

Let us consider boundary value problem (1.1) with real Sobolev space  $W_p^2(\Omega)$  with  $p > N$  provided that there exists a priori estimate  $\|u\|_{l,\Omega} \leq M$  independent of  $\lambda \in [0, 1]$ , for solving parametric family of problem

$$\begin{cases} Lu = \lambda \cdot f(x, u, Du), & x \in \Omega, \\ u|_{\partial\Omega} = \varphi(x), & x \in \partial\Omega \end{cases} \quad (2.1)$$

for  $l \in [1, \infty)$  and  $M > 0$ .

For the operator  $L$  and boundary function  $\varphi(x)$  we suppose that the following condition is fulfilled.

A.5) Let  $L$  be a quasilinear elliptic operator with real continuous coefficients, Suppose that the operator

$$Lu = \sum_{|\alpha|=2} a_\alpha(x, u, Du) D^\alpha u$$

is a second-order  $\varphi(x) \in W_p^{2-1/p}(\partial\Omega)$  and linear boundary value problem (1.2) for any  $g(x) \in L_p(\Omega)$  is uniquely solvable in space  $W_p^2(\Omega)$  so that estimation

$$\|u\|_{W_p^2(\Omega)} \leq C \left( \|g\|_{L_p(\Omega)} + \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \right)$$

with positive constant  $C$ , independent of  $g \in L_p(\Omega)$  and of the solution  $u(x) \in W_p^2(\Omega)$  is fulfilled.

**Theorem 2.1.** *Let conditions A.1)-A.3), A.5) be fulfilled, and for parametric family of problems (2.1) for  $\lambda \in [0, 1]$  there exist a priori estimate  $\|u\|_{l,\Omega}$ , independent of  $\lambda$  with  $l \in [1, +\infty)$  and  $M > 0$ . Then there exists a solution of problem (1.1) in the space  $W_p^2(\Omega)$  with  $p > N$ .*

**Proof.** Let us consider a parametric family of problem (2.1) for  $\lambda \in [0, 1]$ . For this family of problems, the conditions of Theorem 1.3 are fulfilled, by virtue of which there exists such a constant  $C_1$ , independent of  $\lambda$ , that for any possible solution  $W_p^2(\Omega)$  of problem (2.1) the inequality

$$\|u\|_{W_p^2(\Omega)} \leq C_1, \quad \forall \lambda \in [0, 1] \quad (2.2)$$

is fulfilled.

From condition A.5) it follows that linear boundary value problem (1.2) is uniquely solvable and the following representation is valid for its solutions  $\tilde{u}(x)$

$$\tilde{u}(x) = V(x) + \Phi(x), \quad (2.3)$$

where  $V(x), \Phi(x) \in W_p^2(\Omega)$  are such that

$$\begin{cases} L_v V = g(x), & x \in \Omega, \\ V|_{\partial\Omega} = 0, & x \in \partial\Omega, \end{cases} \quad (2.4)$$

and

$$\begin{cases} L_v \Phi = 0, & x \in \Omega, \\ \Phi|_{\partial\Omega} = \varphi(x), & x \in \partial\Omega. \end{cases}$$

Linear problem (2.4) is uniquely solvable for any function  $g(x) \in L_p(\Omega)$ , so that  $V = Ag$ , where  $A$  is a linear continuous operator from  $L_p(\Omega)$  to  $W_p^2(\Omega)$ . Then, the boundary value problem (2.1) is equivalent to the following operator equation

$$\mathcal{W} = \lambda \cdot T(\mathcal{W}), \quad \lambda \in [0, 1] \quad (2.5)$$

consider in the space

$$X = \left\{ \mathcal{W}(x) \in W_p^2(\Omega) \mid \mathcal{W}|_{\partial\Omega} = 0 \right\}.$$

Here,

$$T(\mathcal{W}) = A\tilde{f}(V + \Phi), \quad \tilde{f}(u)(x) = f(x, u(x), Du(x))$$

The operator  $\tilde{f}$  was determined in the space  $W_p^2(\Omega)$  with the values in  $L_p(\Omega)$  and is a completely continuous operator from  $W_p^2(\Omega)$  to  $L_p(\Omega)$ . This fact follows from the Sobolev-Kondrashev embedding theorem [14] based on the conditions of Theorem 2.1. Therefore, the operator  $T$  is a completely continuous operator acting in the space  $X$ . For possible solutions  $\mathcal{W}(x)$  from the space  $X$ , by virtue of estimation (2.2) and representation (2.3) the following inequality is fulfilled

$$\|\mathcal{W}\|_X \leq C_2, \quad \forall \lambda \in [0, 1]$$

where,  $C_2$  is a positive constant independent neither of  $W(x)$ , nor of  $\lambda$ . Then, by virtue of the Leray-Schauder theorem [3, p.135] equations (2.5) for  $\lambda = 1$  and consequently boundary value problem (1.1) have the solution  $u_0(x) \in W_p^2(\Omega)$ .

The proof of Theorem 2.1 is complete.

#### REFERENCES

- [1] Besov O. A., Ilin V. A., Nikolski S. M., Integral representations of functions and embedding theorems. N., M., (1996).
- [2] Ladyzhenskaya O. A., Uraltseva N. N., Linear and quasilinear equations of elliptic type. Nauka, Moscow, (1973).
- [3] Bilalov B. T., Sadiqova S. R., On solvability in the small of higher order elliptic equations in rearrangement invariant spaces. Siberian j. math. (2022) Vol. 63, P. 425–437.
- [4] Agmon S., Douglis A., Nirenberg L., Estimations of the solutions of elliptic equations near the boundary. N., M., (1962).
- [5] Amann H., Grandall M. G., On some existence theoremes for semilinear elliptic equations. Ind. Univ. Math. j. (1978) Vol. 27, Iss. 5, P. 779–790.
- [6] Bernstein S. N., On the equations of the calculus of variations. Collerted works. USSR Academy of Sciences, (1960), P. 191–241.
- [7] Kazdan A. I., Kramer R. I., Invariant criteria for existence of solutions to second order quasilinear elliptic equations. Comm. Pure Appl. Math. (1978) Vol. 31, Iss. 5, P. 319–645.
- [8] Pohozhaev S. I., On the solvability of quasilinear elliptic equations of arbitrary order. Mathem. Sb. (1982) Vol. 117(159), Iss. 2, P. 251–265.
- [9] Pohozhaev S. I., On the equations of the form  $\Delta u = f(x, u, Du)$ . Matem. Sb. (1980) Vol. 113 (155), Iss. 2 (10), P. 324–338.
- [10] Amanov R. A., Ismailov A. I., On equations of the form  $\Delta u - \frac{\partial u}{\partial t} = f(x, t, u, Du)$ . Azerbaijan Journal of Mathematics, (2023) Vol. 17, Iss. 2, P. 137–158.
- [11] Amanova N. R., On quasilinear parabolic equations of higher order. Azerbaijan Journal of Mathematics, (2025) Vol. 15, Iss. 1, P. 191–209.
- [12] Amanov R. A., Amanova N. R., On solvability of some quasilinear parabolic equations of higher order. Azerbaijan Journal of Mathematics, (2025) Vol. 15, Iss. 2, P. 240–263.
- [13] Khalilov V. S., Amanova N. R., On the cristence of strong solutions of higher order quasilinear. Uzbek Mathematical journal, (2024) Vol. 68, Iss. 1, P. 11–18.
- [14] Sobolev S. L., Some applications of functional analysis in mathematical physics. SO AN SSSR, Novosibirsk, (1962).
- [15] Aleksandrov A. D., Uniqueness conditions and estimates of the solution to the Dirichlet problem. Vestnik LGU, ser. math. mech. and astr. (1963) Vol. 3, Iss. 13, P. 5–29.

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