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Solvability of a nonlinear elliptic problem involving $p-$Laplacian with Dirichlet condition

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Abstract. In this work, we study a class of nonlinear elliptic equations of the form $Au = -\Delta_p u + div\,(Vu) = R(x,u)$ in a bounded domain $\Omega \subset \mathbb{R}^n$, where $-\Delta_p u = -div\left(|\nabla u|^{p-2}\nabla u\right)$ denotes the p-Laplacian operator, $V(x)$ is a given vector field, and $R(x)$ represents a source term. We establish the existence and uniqueness of weak solutions under suitable assumptions on V and R.To establish the existence of weak solutions to the nonlinear elliptic problem we employ a fixed point approach based on the compactness of the associated operator.The existence result is obtained by proving that the composition $A^{-1}N_R$ is a compact and continuous mapping that admits at least one fixed point, where N_R is the Nemyskii operator corresponding to a Carathéodory function R . This ensures the existence of a weak solution $u \in W_0^{1,p}(\Omega)$. For the uniqueness of solutions, we rely on the strict monotonicity of the p-Laplacian operator and appropriate growth and Lipschitz conditions on $R(x,u)$. Under these assumptions, one can show that if u_1 and u_2 are two weak solutions, then their difference satisfies an energy inequality leading to $u_1 = u_2$ almost everywhere in Ω.

Keywords: $p-$Laplacian, Fixed point, Variational formulation, Carathéodory function, Nemytskii operator.
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1. Introduction

Partial differential equations (PDEs) play a fundamental role in modeling a wide range of physical, biological, and engineering phenomena. They provide an essential mathematical framework for describing diffusion, transport, and nonlinear interactions that arise in various natural and applied processes. Among them, nonlinear elliptic equations [1, 2], particularly those involving the p-Laplacian operator, have attracted considerable attention due to their rich mathematical structure and broad spectrum of applications in physics, mechanics, and materials science Albo [3, 4, 5, 6, 7, 8, 9, 10].

It is more investigated problem with $divVu = 0$ [11, 4, 5, 6, 7], and the references therein. In this regard we mention the work of [4] et al in which they considered the problem:

$$-div(a(x)\,|\nabla u|^{p-2}\nabla u) = f(x,u,\nabla u), \qquad u = 0 \text{ on } \partial\Omega.$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ and $p \in (1,n)$.In this paper, we investigate the well-posedness of the nonlinear elliptic boundary value problem

$$\begin{cases} Au = -\Delta_p u + div\,(Vu) = R & \text{in } \Omega, \quad p \geq 2, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ_p denotes the p-Laplacian operator, $V:\Omega \longrightarrow \mathbb{R}^n$ is a bounded vector field, $V \in L^\infty(\Omega)^n$, and $R{:}\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying suitable growth and continuity conditions. Such equations naturally arise in models of nonlinear diffusion and convection processes in heterogeneous media.

The existence of weak solutions is established via a priori estimates and fixed-point techniques applied to the compact operator $T = A^{-1}N_R : W_0^{1,p}(\Omega) \longrightarrow W_0^{1,p}(\Omega)$, where $Au = -\Delta_p u + div\,(Vu)$, and N_R denotes the Nemytskii operator associated with the Carathéodory function R, defined by $N_R(x,u) = R(x,u)$.[12] et al. The compactness and continuity of N_R play a crucial role in applying fixed-point theorems to ensure the existence of at least one weak solution. Uniqueness is derived using monotone operator theory by introducing a Carathéodory mapping $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$-div\, b(x,u,\nabla u) = -\Delta_p u + div\,(Vu)\,.$$

Problems of this type appear in the modeling of non-Newtonian fluids, nonlinear elasticity, porous media flow, and other nonlinear diffusion–convection phenomena. The present study contributes to the theoretical understanding of such equations and provides new results on the existence and uniqueness of weak solutions.This formulation allows the use of monotone operator theory and variational techniques to prove that the associated operator is strictly monotone under appropriate hypotheses on R and V, there by guaranteeing both existence and uniqueness of the solution to problem (1.1). From an applied perspective, nonlinear elliptic equations involving the p-Laplacian arise in numerous contexts such as non-Newtonian fluid dynamics, nonlinear elasticity, porous medium flow, electro-rheological and thermo-rheological materials, and population dynamics. The convection term $divVu$ introduces significant analytical challenges, as it models transport or drift phenomena in heterogeneous media. Consequently, the study of such equations not only deepens the theoretical understanding of nonlinear PDEs with nonstandard growth but also contributes to the rigorous modeling of complex processes observed in real-world systems.

The paper is organized as follows. In the second section, we define the problem that serves as the focus of our work and we recall some mathematical tools. The Third section will be consecrated to the study of the problem that consists in establishing existence and unicity results.

The organization of this paper is as follows. In Section 2, the considered problem is stated. Sections 3 deals with the solvability of the stated problem. While section 4 gives the uniqueness of the solution of the considered problem.

For the study of this problem, we first recall the following results.

In the sequel, Ω be an open bounded domain of $\mathbb{R}^n$, $n \geqslant 2$ with Lipschitz continuous boundary and $p \geq 1$. We shall use the standard notations:

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega),\ \frac{\partial u}{\partial x_i} \in L^p(\Omega),\ i = 1, ..., n \right\}.$$

equipped with the norm:

$$\|u\|^p_{W^{1,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + \sum_{i=1}^{i=n} \left\| \frac{\partial u}{\partial x_i} \right\|^p_{L^p(\Omega)},$$

where $\|.\|_{L^p(\Omega)}$ is the usual norm on $L^p(\Omega)$. It is well known that $(W^{1,p}(\Omega); \|.\|_{W^{1p,}(\Omega)})$ is separable, reflexive and uniformly convex [13, 14].We need the space $W^{1,p}_0(\Omega)$ the closure of the space $C^\infty_0(\Omega)$ in the space $W^{1,p}(\Omega)$

$$W^{1,p}_0(\Omega) = \left\{ u \in W^{1,p}(\Omega),\ u \mid_{\partial\Omega} = 0 \right\}.$$

The dual space $\left(W^{1,p}_0(\Omega)\right)^*$ will be denoted by $W^{-1,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. For each $u \in W^{1,p}(\Omega)$, we put

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n} \right), \qquad |\nabla u|^2 = \sum_{i=1}^{i=n} \left| \frac{\partial u}{\partial x_i} \right|^2.$$

Definition 1.1. [15] Let X be real Banach space; and let $L : X \longrightarrow X^*$ be an operator Then :

(i) L is called monotone if $\langle Lu - Lv, u - v \rangle \geqslant 0$ for all $u, v \in X$.

(ii) L is called stictly monotone if: $\langle Lu - Lv, u - v \rangle > 0$ for all $u, v \in X$ with $u \neq v$.

(iii) L is called coercive if $\lim\limits_{\|u\| \longrightarrow \infty} \dfrac{\langle Lu, u \rangle}{\|u\|} = +\infty$.

(iv) L is hemicontinuous, i.e., the map $t \longmapsto \langle L(u + tv), w \rangle$ is continuous on $[0, 1]$ for all $u, v, w \in X$.

Definition 1.2. The function $R : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function if for each $s \in \mathbb{R}$, the function $x \longmapsto R(x, s)$ is measurable in Ω and for a.e. $x \in \Omega$, the function $s \longmapsto R(x, s)$ is continuous in $\mathbb{R}$.

Theorem 1.3. *[16](Leray–Schauder Principle). Suppose that*

(i) *The operator* $T : X \longrightarrow X$ *is compact, with* X *a Banach space;*

(ii) *There existe an* $r > 0$ *such that if* $x = tT(x)$ *with* $0 < t < 1$ *then* $\|x\| \leqslant r$.
Then the equation $x = T(x)$ *is solvable.*

2. Problem Statement

In the domain Ω we consider the problem find a solution u such that:

$$\begin{cases} Lu = -\Delta_p u + div\,(Vu) = R, & \text{in } \Omega, \quad p \geq 2, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $-\Delta_p$ is the $p-$Laplacian acting from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ defined by

$$\langle -\Delta_p u, \varphi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \varphi dx, \quad \forall u, \varphi \in W_0^{1,p}(\Omega), \tag{2.2}$$

The function $V : \Omega \longrightarrow \mathbb{R}^n$ is a vector field $\in L^\infty(\Omega)$ such that

$$divV \geq 0, \tag{2.3}$$

and $R \in W^{-1,q'}(\Omega)$ is a Carathéodory function and it satisfies the growth condition :

$$|R(x,s)| \leqslant a(x)\,|s|^{q-2} + b(x) \quad \text{p.p } x \in \Omega \quad \text{and} \quad s \in \mathbb{R}, \tag{2.4}$$

where $a \in L^\infty(\Omega)$, $b \in L^{q'}(\Omega)$, $\dfrac{1}{q} + \dfrac{1}{q'} = 1$ and $q \in]1, p^*[$ such that :

$$p^* = \begin{cases} \dfrac{np}{n-p} & \text{if } p < n, \\ +\infty & \text{if } p \geqslant n. \end{cases} \tag{2.5}$$

Proposition 2.1. *If the function* R *is a Carathéodory function, then for each* $u \in M$, *where* M *be the set of all measurable function* $u : \Omega \longrightarrow \mathbb{R}$. *The function* $N_R u : \Omega \longrightarrow \mathbb{R}$ *defined by*

$$(N_R u)(x) = R(x, u(x)) \quad \textit{for } x \in \Omega \tag{2.6}$$

is measurable in Ω. *In view of this proposition, a Carathéodory function* $R : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ *defines an operator* $N_R : M \longrightarrow M$, *which is called Nemytskii operator.*

Definition 2.2. An element $u \in W_0^{1,p}(\Omega)$ is a solution of problem (2.1) if

$$-\Delta_p u + div\,(Vu) = (N_R u), \tag{2.7}$$

in the sens of $W^{-1,p'}(\Omega)$, i.e

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \varphi dx - \int_\Omega Vu.\nabla \varphi dx = \int_\Omega R(x,u)\,\varphi dx, \quad \forall u, \varphi \in W_0^{1,p}(\Omega). \tag{2.8}$$

3. Solvability of Problem (2.1)

In this section, the solvability of the problem (2.1) will be proved using the Leray-Schauder principle.

Theorem 3.1. *If the operator* L *is monotone, coercive and hemicontinuous then* L *is surjective.*

Proof: (1) The operator $L : W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ defined by :

$$Lu = -\Delta_p u + div\,(Vu), \tag{3.1}$$

is bounded.
For $\varphi \in W_0^{1,p}(\Omega)$ we have

$$\langle Lu, \varphi\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx - \int_\Omega uV\nabla\varphi dx.$$

Using Hölder's inequality we get

$$|\langle Lu, \varphi\rangle| \le \int_\Omega |\nabla u|^{p-1} . |\nabla\varphi|\, dx + \int_\Omega |u|\,|V|\,|\nabla\varphi|\, dx$$

$$\le \left(\int_\Omega |\nabla u|^p\, dx\right)^{\frac{p-1}{p}} . \left(\int_\Omega |\nabla\varphi|^p\, dx\right)^{\frac{1}{p}} + \|V\|_{L^\infty(\Omega)} \left(\int_\Omega |u|^p\, dx\right)^{\frac{1}{p}} \left(\int_\Omega |\nabla\varphi|^{p'}\, dx\right)^{\frac{1}{p'}},$$

where $\frac{1}{p}+\frac{1}{p'}=1$, using the fact that $p \geq 2$ and Pioncaré's inequality, we obtain

$$|\langle Lu, \varphi\rangle| \le c(\|u\|_{W_0^{1,p}(\Omega)}, \|V\|_{L^\infty(\Omega)}) \|\varphi\|_{W_0^{1,p}(\Omega)}, \tag{3.2}$$

Thus, the operator L is bounded.

(2) If $p \in [2, +\infty[$, then operator L is monotone.
From definition 1.1, the operator L is monotone if and only if $\langle Lu - L\varphi, u-\varphi\rangle \geqslant 0,\ \forall u, \varphi \in W_0^{1,p}(\Omega)$, integrating by parts we obtain

$$\langle Lu - L\varphi, u - \varphi\rangle = \langle -\Delta_p u + div\,(Vu) + \Delta_p\varphi - div\,(V\varphi)\,, u-\varphi\rangle =$$

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla\varphi|^{p-2}\nabla\varphi\right)\nabla(u-\varphi)\,dx + \frac{1}{2}\int_\Omega (divV).(u-\varphi)^2\,dx, \tag{3.3}$$

Using the fact that

$$\left(|a|^{p-2}a - |b|^{p-2}b\right)(a-b) \ge 2^{-1}\left(|a|^{p-2}+|b|^{p-2}\right)|a-b|^2 \geqslant 2^{2-p}|a-b|^p, \quad \forall a, b \in \mathbb{R}^n, \tag{3.4}$$

then (3.3) becomes

$$\langle Lu - L\varphi, u-\varphi\rangle \ge 2^{2-p}\int_\Omega |\nabla u - \nabla\varphi|^p\, dx + \frac{1}{2}\int_\Omega (divV).(u-\varphi)^2\, dx \tag{3.5}$$

According to (2.3) we arrive at

$$\langle Lu - L\varphi, u-\varphi\rangle \ge 0,$$

So, the operator L is monotone.

(3) The operator L is coercive. From (3.5), we have

$$\langle Lu, u\rangle = \langle -\Delta_p u + div\,(Vu)\,, u\rangle \ge 2^{2-p}\int_\Omega |\nabla u|^p\, dx + \int_\Omega u.V.\nabla u dx. \tag{3.6}$$

According to Hölder's inequality, the second term in the right hand sid of (3.6) can be controlled by

$$\int_\Omega uV.\nabla u dx \le |\Omega|^{\frac{p-2}{p}} \|V\|_\infty \left(\int_\Omega |u|^p\, dx\right)^{\frac{1}{p}} \left(\int_\Omega |\nabla u|^p\, dx\right)^{\frac{1}{p}},$$

then from (3.6), we get

$$\langle Lu, u\rangle \geq 2^{2-p} \int_{\Omega} |\nabla u|^p \, dx - c(\Omega, \|V\|_{\infty}) \left(\int_{\Omega} |\nabla u|^p \, dx \right)^{\dfrac{2}{p}}.$$

consequently we have

$$\lim_{\|u\|_{1,p} \longrightarrow +\infty} \frac{\langle Lu, u\rangle}{\|u\|_{1,p}} \longrightarrow +\infty.$$

According to definition (1.1), we conclude that L is coercive.

(4) The operator L is hemicontinuous.
The operator L is hemicontinuous of $W_0^{1,p}(\Omega)$ in $W^{-1,p'}(\Omega)$ if the mapping $t \longmapsto \langle L(u + t_n\varphi), w\rangle$ is continuous in $\mathbb{R}$. Either $u, \varphi, w \in W_0^{1,p}(\Omega)$, $(t_n)_{n\in\mathbb{N}}$ a real sequence that converges to l, we will show that $\langle L(u + t_n\varphi), w\rangle \longrightarrow \langle L(u + l\varphi), w\rangle$.
L is hemicontinuous :

$$\langle L(u + t_n\varphi), w\rangle = \int_{\Omega} |\nabla (u + t_n\varphi)|^{p-2} \nabla (u + t_n\varphi).\nabla w \, dx - \int_{\Omega} (u + t_n\varphi) V.\nabla w \, dx,$$

we have :

$$|\langle L(u + t_n\varphi), w\rangle| \leq \int_{\Omega} |\nabla (u + t_n\varphi)|^{p-1} |\nabla w| \, dx + \int_{\Omega} |u| |V| |\nabla w| + |t_n| \int_{\Omega} |\varphi| |V| |\nabla w| \, dx.$$

Using the fact that

$$(a + b)^p \leq 2^{p-1} (a^p + b^p), \text{ for all } a \geq 0, \ b \geq 0 \text{ and } p \geq 1.$$

we have

$$|\langle L(u + t_n\varphi), w\rangle| \leq 2^{p-1} \int_{\Omega} \left(|\nabla u|^{p-1} + |t_n|^{p-1} |\nabla \varphi|^{p-1} \right) |\nabla w| \, dx + \int_{\Omega} |u| |V| |\nabla w| + |t_n| \int_{\Omega} |\varphi| |V| |\nabla w| \, dx. \tag{3.7}$$

Since $t_n \underset{n\longrightarrow\infty}{\longrightarrow} l$, then there exist $M > 0$ such that $|t_n| \leq M$, using Hölder's inequality, the right hand side of (3.7) becomes

$$|\langle L(u + t_n\varphi), w\rangle| \leq 2^{p-1} 2^{\overline{p-1}^{\,1}} \int_{\Omega} (|\nabla u|^p + |M|^p |\nabla \varphi|^p) \left(\int_{\Omega} |\nabla w|^p \, dx \right)^{\dfrac{1}{p}}$$

$$+ \|V\|_{\infty} \left(\int_{\Omega} |u|^p \, dx \right)^{\dfrac{1}{p}} \left(\int_{\Omega} |\nabla w|^{p'} \, dx \right)^{\dfrac{1}{p'}} + M \|V\|_{\infty} \left(\int_{\Omega} |\varphi|^p \, dx \right)^{\dfrac{1}{p}} \left(\int_{\Omega} |\nabla w|^{p'} \, dx \right)^{\dfrac{1}{p'}}.$$

According to Lebesgue's dominated convergence theorem, the operator L is hemicontinuous. Then the problem $Lu = g(x)$ has a unique solution for $g \in W^{-1,p'}(\Omega)$ i.e the operator L defines one−to−one correspondence between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, with inverse L^{-1} monotone, bounded and continuous. Since $q \in (1, q^*)$, then the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^q$ is compact. Consequently we have the diagram $W_0^{1,p}(\Omega) \overset{I_d}{\hookrightarrow} L^q(\Omega) \overset{N_R}{\hookrightarrow} L^{q'}(\Omega) \overset{I_d^*}{\hookrightarrow} W_0^{-1,p'}(\Omega)$.

□

Theorem 3.2. *If the Carathéodory function $R : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the condition (2.4) with $q < p$, then the operator $L^{-1}N_R$ has a fixed point in $W_0^{1,p}(\Omega)$ via Leray-Schauder principe.*

Proof: Using the theorem of Leray-Schauder 1.3, we will show tha the set

$$Z = \left\{ u \in W_0^{1,p}(\Omega) \ / \ \exists \lambda \in [0,1], \text{ such that } u = \lambda L^{-1} N_R u = \lambda T u \right\},$$

is bounded in $W_0^{1,p}(\Omega)$. From (2.7), it is obvious to show that for arbitrary $u \in W_0^{1,p}(\Omega)$, it is obvious that :

$$\|Tu\|^p_{W_0^{1,p}(\Omega)} = \langle -\Delta_p u + div(Vu), Tu\rangle = \langle N_R u, Tu\rangle = \int_\Omega R(x,u)\, Tu\, dx, \tag{3.8}$$

Using (2.4), the right hand side of the previous equality be controlled by

$$\int_\Omega R(x,u)\, Tu\, dx \leq \int_\Omega \left| a(x)\, |u|^{q-2} + b(x) \right| |Tu|\, dx.$$

Furthermore, for $u \in Z$ i.e. $u = \lambda Tu$ with some $\lambda \in [0,1]$, by using Hölder's inequality we get

$$\int_\Omega R(x,u)\, Tu\, dx \leq \int_\Omega |a(x)|\, |\lambda Tu|^{q-2}\, |Tu|\, dx + \int_\Omega |b(x)|\, |Tu|\, dx$$

$$\leq \lambda^{q-1} \|a(x)\|_{L^\infty(\Omega)} \|Tu\|^{q-1}_{L^{q-1}(\Omega)} + \left(\int_\Omega |b(x)|^{q'} dx \right)^{\frac{1}{q'}} \left(\int_\Omega |Tu|^p\, dx \right)^{\frac{1}{p}}. \tag{3.9}$$

Combining (3.8) with (3.9) we have :

$$\|Tu\|^p_{1,p} \leqslant K_1 \|Tu\|^{q-1}_{L^q(\Omega)} + K_2 \|Tu\|_{L^q(\Omega)}, \tag{3.10}$$

where

$$K_1 = \lambda^{q-2} \|a(x)\|_{L^\infty(\Omega)},$$

and

$$K_2 = \|b(x)\|_{L^{q'}(\Omega)}.$$

Thus, from (3.10) we obtain

$$\|Tu\|^p_{1,p} - K_1 \|Tu\|^{q-1}_{1,p} - K_2 \|Tu\|_{1,p} \leq 0.$$

Then, there is a constant $d \geqslant 0$ such that

$$\|Tu\|_{1,p} \leqslant d.$$

Since

$$\|u\|_{1,p} = \|\lambda Tu\|_{1,p} = \lambda \|Tu\|_{1,p} \leqslant \lambda d \leqslant d.$$

Then, the bounddeness of Z would de proved, so the problem $-\Delta_p u + div(Vu) = R(x,u)$ has a fixed point $u \in W_0^{1,p}(\Omega)$. □

4. A uniqueness result

In this section we address the problem of the uniqueness of the solution of the problem (2.1):

Theorem 4.1. *For $R \in W^{-1,p'}(\Omega)$, the problem (2.1) has a unique solution $u \in W_0^{1,p}(\Omega)$.*

Proof: Let $b(x,u,\nabla u) = |\nabla u|^{p-2} \nabla u - Vu$. It is obvious to show that

(H1) The function $b : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a Carathéodory function, i.e. :

a) For each (s,ξ), the function $x \longmapsto b(x,s,\xi)$ is measurable.

b) For a.e. x, the function $(s,\xi) \longmapsto b(x,s,\xi)$ is continuous.

(H2) (Monotonie)

$$\langle b(x,s,\xi) - b(x,s,\eta), \xi - \eta \rangle = \left(|\nabla\xi|^{p-2} \nabla\xi - |\nabla\eta|^{p-2} \nabla\eta \right) (\nabla\xi - \nabla\eta) \geqslant C_p |\xi - \eta|^p .$$

(H3) For almost all $x \in \Omega$ and for each $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ with $\xi \neq \eta$

$$\langle b(x,s,\xi) - b(x,s,\eta), \xi - \eta \rangle > 0.$$

(H4) According (3.4), there exists $\beta > 0$ such that

$$\langle b(x,s,\xi) - b(x,s,\eta), \xi - \eta \rangle \geqslant \beta |\xi - \eta|^2 \left(|\xi|^{p-2} + |\eta|^{p-2} \right).$$

(H5) There exists $C > 0$ and $f \in L^{p'}(\Omega)$ such that :

$$|b(x,s,\xi) - b(x,t,\xi)| \leqslant C |s-t| \left(f(x) + |\xi|^{p-1} + |s|^{p-1} + |t|^{p-1} \right).$$

Let u_1, u_2 be two solutions of problem (2.1) then we have

$$\int_\Omega (b(x,,u_1,\nabla u_1) - b(x,u_1,\nabla u_2)) \nabla\varphi dx = \int_\Omega (b(x,,u_2,\nabla u_2) - b(x,u_1,\nabla u_2)) \nabla\varphi dx,$$

$T_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ is the truncation function $T_\varepsilon(r) = \max(\min(\varepsilon, r), -\varepsilon)$, for $\varphi = T_\varepsilon u$ and $v = u_1 - u_2$. Since

$$\nabla T_\varepsilon(v) = \nabla\varphi = 1_{A_\varepsilon} \nabla v,$$

where

$$A_\varepsilon = \{x \in \Omega,\ 0 < |u_1 - u_2| \leqslant \varepsilon\},$$

Then, applying **(H4)** and **(H5)** we get:

$$\begin{aligned} &\beta \int_\Omega |\nabla(u_1 - u_2)|^2 \left(|\nabla u_1|^{p-2} + |\nabla u_2|^{p-2} \right) dx \\ &\leq C \int_\Omega |u_1 - u_2| \left(f(x) + |\nabla u_2|^{p-1} + |u_1|^{p-1} + |u_2|^{p-1} \right) |\nabla(u_1 - u_2)| dx \\ &\leq C\varepsilon \int_{A_\varepsilon} F |\nabla T_\varepsilon(u_1 - u_2)| dx, \end{aligned}$$

where

$$F = \left(f(x) + |\nabla u_2|^{p-1} + |u_1|^{p-1} + |u_2|^{p-1} \right) \in L^{p'}(\Omega).$$

Since

$$\int_{A_\varepsilon} |\nabla(u_1 - u_2)|^p dx = \int_\Omega |1_{A_\varepsilon} \nabla(u_1 - u_2)|^p dx,$$

and

$$\int_\Omega |\nabla T_\varepsilon(u_1 - u_2)|^p dx = \|T_\varepsilon(u_1 - u_2)\|^p_{W_0^{1,p}(\Omega)}.$$

Using the fact that

$$\left| |z|^{p-2} z - |y|^{p-2} y \right| \leq |z - y| |z - y|^{p-2},$$

and (3.4), we obtain

$$\beta \int_{A_\varepsilon} |\nabla (u_1 - u_2)|^p \, dx \leqslant \beta \int_{A_\varepsilon} |\nabla (u_1 - u_2)|^2 \left(|\nabla u_1|^{p-2} + |\nabla u_2|^{p-2} \right) dx,$$

Consequently from the previous inequality we have

$$\beta \left\| T_\varepsilon (u_1 - u_2) \right\|^p_{W_0^{1,p}(\Omega)} \leqslant C\varepsilon \left\| F \right\|_{L^{p'}(\Omega)} \left\| T_\varepsilon (u_1 - u_2) \right\|_{W_0^{1,p}(\Omega)} .$$

The Poincaré (for $p = 1$) and Cauchy-Schwarz inequalities allow us to obtain

$$\begin{aligned} \frac{1}{\varepsilon} \int_{A_\varepsilon} |T_\varepsilon (u_1 - u_2)| \, dx &\leq \frac{C}{\varepsilon} \int_{A_\varepsilon} |\nabla T_\varepsilon (u_1 - u_2)| \, dx \\ &\leq \frac{C}{\varepsilon} \left\| T_\varepsilon (u_1 - u_2) \right\|_{W_0^{1,p}(\Omega)} |A_\varepsilon|^{\frac{1}{p'}} \leq C \, |A_\varepsilon|^{\frac{1}{p'}} \end{aligned} .$$

Gold $\cap_{\varepsilon>0} A_\varepsilon = \emptyset$ and $(A_\varepsilon)_{\varepsilon>0}$ is decreasing when $\varepsilon \longrightarrow 0$. Therefore $|A_\varepsilon| \longrightarrow 0$ when $\varepsilon \longrightarrow 0$.We choose $r > 0$ anything and $\varepsilon < r$, then

$$\begin{aligned} |\{x : \ |u_1 - u_2| \geq r\}| &\leq |\{x : \ |u_1 - u_2| \geq \varepsilon\}| \\ &\leq |\{x : \ |T_\varepsilon (u_1 - u_2)| \geq \varepsilon\}| \\ &\leq \frac{1}{\varepsilon} \int_\Omega |T_\varepsilon (u_1 - u_2)| \, dx \longrightarrow 0, \qquad \text{when } \varepsilon \longrightarrow 0. \end{aligned}$$

So, for everything $r > 0$,$|u_1 - u_2| < r$ almost everywhere. Therefore we conclude that $u_1 = u_2$ almost everywhere. □

Proposition 4.2. *If $R(x, s) \geqslant 0$ a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$. If $u \in W_0^{1,p}(\Omega)$ is solution of problem (2.1) then $u \geqslant 0$.*

Proof: Let $u \in W_0^{1,p}(\Omega)$ be a solution of problem (2.1) and let us denoted $\Omega^- = \{x \in \Omega : \ u(x) < 0\}$, $\Omega^+ = \Omega \diagdown \Omega^-$ and $u_- = \max\{-u, \ 0\}$, then, it is known that $u_- \in W_0^{1,p}(\Omega)$ and

$$\nabla u_- = \begin{cases} \nabla u & \text{in } \Omega_-, \\ 0 & \text{in } \Omega_+. \end{cases}$$

Integrating by parts over Ω we have

$$\begin{aligned} \langle -\Delta_p u + div(Vu), u_- \rangle &= \int_\Omega |\nabla u|^{p-2} \nabla u . \nabla u_- dx _- \int_\Omega V u . \nabla u_- dx \\ &= -\int_\Omega |\nabla u|^p \, dx + \int_\Omega V u . \nabla u dx \\ &= -\int_{\Omega_-} |\nabla u|^p \, dx - \int_{\Omega_-} div(V) \frac{u^2}{2} dx = -\int_{\Omega_-} R(x, u) \, u dx \geqslant 0 \end{aligned}$$

Then, we have

$$\int_{\Omega_-} |\nabla u|^p \, dx + \int_{\Omega_-} div(V) \frac{u^2}{2} dx = \int_{\Omega_-} R(x, u) \, u dx \leqslant 0.$$

So

$$\int_{\Omega_-} |\nabla u|^p \, dx + \int_{\Omega_-} div(V) . \frac{u^2}{2} dx = 0.$$

then $u = 0$ everywhere. Wich allows as to achieve the proof. □

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