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On fixed point results in a new type of nonlinear contraction in 𝔟-metric spaces with applications to differential equations

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Abstract. We present fixed point theorem for a new type of nonlinear contractive mappings in 𝔟-metric spaces. Also, we present examples to illustrate the validity of the results obtained in the paper. In addition, by using our results, we obtain the existence and uniqueness of solution to some ordinary differential equations with initial value conditions.

Keywords: Nonlinear contraction; 𝔟-metric spaces; Fixed point results.
MSC (2020): 47H10, 54H25

1. INTRODUCTION

Fixed point theory is one of the most well-known and established theories in mathematics and has a variety of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. The formal form of the metric fixed point theory appeared by the pioneer and art work of Banach results in 1922 : Let (X, d) be a complete metric space, $T : X \to X$ be a contraction, there exists a unique fixed point $x_0 \in X$ of T. Its broad applicability has led to numerous extensions and generalizations over the years. One such generalization is the concept of a 𝔟-metric space, introduced by Bakhtin [1] and further studied by Czerwik [2, 3].

A 𝔟-metric space differs from a metric space in that the triangle inequality is relaxed. The relaxation of the triangle inequality makes 𝔟-metric spaces suitable for modeling certain situations where the conventional triangle inequality is not satisfied, such as those arising in computer science [4] and biological modeling [5]. Since their inception, 𝔟-metric spaces have attracted significant attention, and numerous fixed point theorems have been established in this setting (see [6, 7, 8, 9, 10, 11]).

Recently, the focus has shifted towards exploring nonlinear contractions, which provide a more flexible framework than linear contractions. These contractions involve functions or inequalities that incorporate the distance between points in a more complex manner, often utilizing auxiliary functions and parameters (see [12, 13]).

This paper aims to contribute to the ongoing development of fixed point theory in 𝔟-metric spaces by presenting a fixed point theorem for new type of nonlinear contractive mappings. We then demonstrate the practical relevance of these results by applying them to prove the existence and uniqueness of solutions to ordinary differential equations. This connection between abstract fixed point theorems and concrete problems in differential equations highlights the power and versatility of fixed point theory.

2. PRELIMINARIES

In this section, we present a few key terms and definitions that will be used in our discussion.

Definition 2.1 [1]. Let $\mathfrak{X}$ be a nonempty set and $s \geq 1$ be a real number. A function $\mathfrak{d} : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ is called a 𝔟-metric if for all $x, y, z \in \mathfrak{X}$, the following conditions are satisfied:

$(b1)$ $\mathfrak{d}(x, y) = 0$ if and only if $x = y$;

$(b2)$ $\mathfrak{d}(x, y) = \mathfrak{d}(y, x)$;

(*b*3) $\mathfrak{d}(x,z) \leq s[\mathfrak{d}(x,y) + \mathfrak{d}(y,z)]$.

The pair $(\mathfrak{X}, \mathfrak{d})$ is called a $\mathfrak{b}$-metric space with parameter s.

Example 2.2 ([14]). Let $(\mathfrak{X}, d)$ be a metric space and let the mapping $\mathfrak{d} : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be defined by

$$\mathfrak{d}(p,q) := (d(p,q))^r, \text{for all } p, q \in \mathfrak{X},$$

where $r > 1$ is a fixed real number. Then $(\mathfrak{X}, \mathfrak{d})$ is a $\mathfrak{b}$-metric space with $s = 2^{r-1}$.

Remark 2.3. It should be noted that, the class of $\mathfrak{b}$-metric spaces is effectively larger than that of metric spaces, since a $\mathfrak{b}$-metric is a metric when $s = 1$. Also in general a $\mathfrak{b}$-metric need not necessarily be a metric. For example, if $\mathfrak{X} = \mathbb{R}$ is the set of real numbers and $\mathfrak{d}(x,y) = |x-y|$ is the usual Euclidean metric, then $\eta(x,y) = (x-y)^2$ is a $\mathfrak{b}$-metric on $\mathbb{R}$ with $s = 2$, but is not a metric on $\mathbb{R}$.

Now, we present the concepts of convergence, Cauchy sequence and completeness in b-metric spaces.

Definition 2.4 ([15]). Let $(\mathfrak{X}, \mathfrak{d})$ be a $\mathfrak{b}$-metric space. Then a sequence $\{x_n\}$ in $\mathfrak{X}$ is

1. *converges* to $x \in \mathfrak{X}$ if and only if $\lim_{n\to\infty} \mathfrak{d}(x_n, x) = 0$;
2. *Cauchy sequence* if $\lim_{n,m\to\infty} \mathfrak{d}(x_n, x_m) = 0$.

Definition 2.5 ([15]). The $\mathfrak{b}$-metric space $(\mathfrak{X}, \mathfrak{d})$ is complete if every Cauchy sequence in $\mathfrak{X}$ converges to a point in $\mathfrak{X}$.

Now, we recall an existing fixed point theorem in $\mathfrak{b}$-metric spaces that will be useful for comparison and motivation for our new results.

Theorem 2.6 ([2].) Let $(\mathfrak{X}, \mathfrak{d})$ be a complete $\mathfrak{b}$-metric space and let $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a mapping such that for all $x, y \in \mathfrak{X}$,

$$\mathfrak{d}(\mathfrak{T}x, \mathfrak{T}y) \leq k\mathfrak{d}(x,y),$$

where $0 \leq k < 1/s$. Then $\mathfrak{T}$ has a unique fixed point in $\mathfrak{X}$.

3. MAIN RESULTS

In this section, we introduce our new type of nonlinear contraction mapping and establish fixed point theorems for such mappings in complete $\mathfrak{b}$-metric spaces. At first, we prove a fixed point theorem for generalized contraction mappings.

Theorem 3.2. Let $(\mathfrak{X}, \mathfrak{d})$ be a complete $\mathfrak{b}$-metric space and let $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a mapping such that for all $x, y \in \mathfrak{X}$,

$$\mathfrak{d}(\mathfrak{T}x, \mathfrak{T}y) \leq a\mathfrak{d}(x,y) + b[\mathfrak{d}(x, \mathfrak{T}x) + \mathfrak{d}(y, \mathfrak{T}y)],$$

where $a, b \geq 0$ and $s(a+2b) < 1$. Then $\mathfrak{T}$ has a unique fixed point in $\mathfrak{X}$.

Proof. Let $x_0 \in \mathfrak{X}$ be an arbitrary point. Define a sequence $\{x_n\}$ in $\mathfrak{X}$ by $x_{n+1} = \mathfrak{T}x_n$ for all $n \geq 0$. Then, we have

$$\begin{aligned}
\mathfrak{d}(x_{n+1}, x_{n+2}) &= \mathfrak{d}(\mathfrak{T}x_n, \mathfrak{T}x_{n+1}) \\
&\leq a\mathfrak{d}(x_n, x_{n+1}) + b[\mathfrak{d}(x_n, \mathfrak{T}x_n) + \mathfrak{d}(x_{n+1}, \mathfrak{T}x_{n+1})] \\
&= a\mathfrak{d}(x_n, x_{n+1}) + b[\mathfrak{d}(x_n, x_{n+1}) + \mathfrak{d}(x_{n+1}, x_{n+2})].
\end{aligned}$$

Rearranging, we get

$$\mathfrak{d}(x_{n+1}, x_{n+2}) \leq \frac{a+b}{1-b}\mathfrak{d}(x_n, x_{n+1}) = k\mathfrak{d}(x_n, x_{n+1}),$$

where $k = a + b/1 - b$. Since $s(a+2b) < 1$, we have $s(a+b) < 1 - b$, which implies $k < 1/s$. Now, by induction, we have

$$\mathfrak{d}(x_{n+1}, x_{n+2}) \leq k^{n+1}\mathfrak{d}(x_0, x_1).$$

For $m > n$, we have

$$\begin{aligned}
\mathfrak{d}(x_n, x_m) &\leq s[\mathfrak{d}(x_n, x_{n+1}) + \mathfrak{d}(x_{n+1}, x_m)] \\
&\leq s\mathfrak{d}(x_n, x_{n+1}) + s^2\mathfrak{d}(x_{n+1}, x_{n+2}) + ... + s^{m-n}\mathfrak{d}(x_{m-1}, x_m) \\
&\leq sk^n\mathfrak{d}(x_0, x_1) + s^2k^{n+1}\mathfrak{d}(x_0, x_1) + ... + s^{m-n}k^{m-1}\mathfrak{d}(x_0, x_1) \\
&= sk^n[1 + sk + (sk)^2 + ... + (sk)^{m-n-1}]\mathfrak{d}(x_0, x_1) \\
&\leq \frac{sk^n}{1 - sk}\mathfrak{d}(x_0, x_1).
\end{aligned}$$

Since $k < 1/s$, we have $sk < 1$. Therefore, $\lim_{n,m\to\infty} \mathfrak{d}(x_n, x_m) = 0$, which implies that $\{x_n\}$ is a Cauchy sequence in $\mathfrak{X}$. Since $(\mathfrak{X}, \mathfrak{d})$ is complete, there exists $x^* \in \mathfrak{X}$ such that $\lim_{n\to\infty} x_n = x^*$. Now, we will show that x^* is a fixed point of $\mathfrak{T}$.

$$\begin{aligned}
\mathfrak{d}(\mathfrak{T}x^*, x_{n+1}) &= \mathfrak{d}(\mathfrak{T}x^*, \mathfrak{T}x_n) \leq a\mathfrak{d}(x^*, x_n) + b[\mathfrak{d}(x^*, \mathfrak{T}x^*) + \mathfrak{d}(x_n, \mathfrak{T}x_n)] \\
&= a\mathfrak{d}(x^*, x_n) + b[\mathfrak{d}(x^*, \mathfrak{T}x^*) + \mathfrak{d}(x_n, x_{n+1})].
\end{aligned}$$

Taking the limit as $n \to \infty$, we get $\mathfrak{d}(\mathfrak{T}x^*, x^*) \leq b\mathfrak{d}(x^*, \mathfrak{T}x^*)$. Therefore, $(1 - b)\mathfrak{d}(\mathfrak{T}x^*, x^*) \leq 0$. Since $b < 1/2s \leq 1$, we have $1 - b > 0$ and therefore, $d(\mathfrak{T}x^*, x^*) = 0$, which implies $\mathfrak{T}x^* = x^*$. Thus, x^* is a fixed point of $\mathfrak{T}$.

To prove uniqueness, suppose y^* is another fixed point of $\mathfrak{T}$ such that $y^* \neq x^*$. Then,

$$\mathfrak{d}(x^*, y^*) = \mathfrak{d}(\mathfrak{T}x^*, \mathfrak{T}y^*) \leq a\mathfrak{d}(x^*, y^*) + b[\mathfrak{d}(x^*, \mathfrak{T}x^*) + \mathfrak{d}(y^*, \mathfrak{T}y^*)] = a\mathfrak{d}(x^*, y^*).$$

Therefore, $(1 - a)\mathfrak{d}(x^*, y^*) \leq 0$. Since $a < 1/s \leq 1$, we have $1 - a > 0$. Therefore, $\mathfrak{d}(x^*, y^*) = 0$, which implies $x^* = y^*$. Hence, the fixed point is unique. This completes the proof. □

Definition 3.2. Let $(\mathfrak{X}, \mathfrak{d})$ be a $\mathfrak{b}$-metric space and $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a mapping. We say that $\mathfrak{T}$ is a φ-nonlinear contraction if there exists a function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$, and a constant $k \in (0, 1)$ such that

$$\mathfrak{d}(\mathfrak{T}x, \mathfrak{T}y) \leq k\varphi(\mathfrak{M}(x, y)), \forall x, y \in X, \tag{3.1}$$

where

$$\mathfrak{M}(x, y) = \max\left\{\mathfrak{d}(x, y), \mathfrak{d}(x, \mathfrak{T}x), \mathfrak{d}(y, \mathfrak{T}y), \frac{\mathfrak{d}(x, \mathfrak{T}y) + \mathfrak{d}(y, \mathfrak{T}x)}{2s}\right\}.$$

Theorem 3.3. Let $(\mathfrak{X}, d)$ be a complete $\mathfrak{b}$-metric space and let $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a φ-nonlinear contraction mapping such that $\max\{k, \frac{k}{2-k}\} < \frac{1}{s}$. Then $\mathfrak{T}$ has a unique fixed point in $\mathfrak{X}$.

Proof. Let x_0 be an arbitrary point in $\mathfrak{X}$. We define a sequence $\{x_n\}$ in $\mathfrak{X}$ by $x_{n+1} = \mathfrak{T}x_n$ for all $n \geq 0$. If for some N, $x_N = x_{N+1}$, then $x_N = \mathfrak{T}x_N$, which means x_N is a fixed point. In this case, the theorem holds. Assume that $x_n \neq x_{n+1}$ for all $n \geq 0$, so $\mathfrak{d}(x_n, x_{n+1}) > 0$ for all n. Consider $\mathfrak{d}(x_n, x_{n+1}) = \mathfrak{d}(\mathfrak{T}x_{n-1}, \mathfrak{T}x_n)$ for $n \geq 1$. Applying the contraction condition (1), we have

$$\mathfrak{d}(x_n, x_{n+1}) \leq k\varphi(\mathfrak{M}(x_{n-1}, x_n),)$$

where

$$\mathfrak{M}(x_{n-1}, x_n) = \max\left\{\mathfrak{d}(x_{n-1}, x_n), \mathfrak{d}(x_{n-1}, \mathfrak{T}x_{n-1}), \mathfrak{d}(x_n, \mathfrak{T}x_n), \frac{\mathfrak{d}(x_{n-1}, \mathfrak{T}x_n) + \mathfrak{d}(x_n, \mathfrak{T}x_{n-1})}{2s}\right\}.$$

Substituting $x_n = \mathfrak{T}x_{n-1}$ and $x_{n+1} = \mathfrak{T}x_n$:

$$\mathfrak{M}(x_{n-1}, x_n) = \max\left\{\mathfrak{d}(x_{n-1}, x_n), \mathfrak{d}(x_{n-1}, x_n), \mathfrak{d}(x_n, x_{n+1}), \frac{\mathfrak{d}(x_{n-1}, x_{n+1}) + \mathfrak{d}(x_n, x_n)}{2s}\right\}.$$

Since $\mathfrak{d}(x_n, x_n) = 0$, this simplifies to:

$$\mathfrak{M}(x_{n-1}, x_n) = \max\left\{\mathfrak{d}(x_{n-1}, x_n), \mathfrak{d}(x_n, x_{n+1}), \frac{\mathfrak{d}(x_{n-1}, x_{n+1})}{2s}\right\}.$$

Now, we analyze the possible cases for $\mathfrak{M}(x_{n-1}, x_n)$:

Case A. Suppose $\mathfrak{M}(x_{n-1}, x_n) = \mathfrak{d}(x_n, x_{n+1})$. Then

$$\mathfrak{d}(x_n, x_{n+1}) \leq k\varphi(\mathfrak{d}(x_n, x_{n+1})).$$

Since $k \in (0, 1)$ and $\varphi(t) < t$ for $t > 0$, we have $k\varphi(t) < kt < t$. Thus,

$$\mathfrak{d}(x_n, x_{n+1}) \leq k\varphi(\mathfrak{d}(x_n, x_{n+1})) < \mathfrak{d}(x_n, x_{n+1}),$$

which is a contradiction unless $\mathfrak{d}(x_n, x_{n+1}) = 0$. If $\mathfrak{d}(x_n, x_{n+1}) = 0$, then $x_n = x_{n+1}$, which implies x_n is a fixed point, contradicting our assumption that $x_n \neq x_{n+1}$. Therefore, this case implies $\mathfrak{d}(x_n, x_{n+1}) = 0$, meaning we found a fixed point.

Case B. Suppose $\mathfrak{M}(x_{n-1}, x_n) = \mathfrak{d}(x_{n-1}, x_n)$. Then

$$\mathfrak{d}(x_n, x_{n+1}) \leq k\varphi(\mathfrak{d}(x_{n-1}, x_n)).$$

Since $\varphi(t) < t$ for $t > 0$, we have

$$\mathfrak{d}(x_n, x_{n+1}) < k\mathfrak{d}(x_{n-1}, x_n).$$

Case C. Suppose $\mathfrak{M}(x_{n-1}, x_n) = \frac{\mathfrak{d}(x_{n-1}, x_{n+1})}{2s}$. Then

$$\mathfrak{d}(x_n, x_{n+1}) \leq k\varphi(\frac{\mathfrak{d}(x_{n-1}, x_{n+1})}{2s}).$$

Since $\varphi(t) < t$ for $t > 0$, we have

$$\mathfrak{d}(x_n, x_{n+1}) < k\frac{\mathfrak{d}(x_{n-1}, x_{n+1})}{2s}.$$

Using the b-triangle inequality, $\mathfrak{d}(x_{n-1}, x_{n+1}) \leq s[\mathfrak{d}(x_{n-1}, x_n) + \mathfrak{d}(x_n, x_{n+1})]$ we have

$$\mathfrak{d}(x_n, x_{n+1}) \quad < \quad \frac{k}{2s}s[\mathfrak{d}(x_{n-1}, x_n) + \mathfrak{d}(x_n, x_{n+1})] = \frac{k}{2}[\mathfrak{d}(x_{n-1}, x_n) + \mathfrak{d}(x_n, x_{n+1})].$$

Let $d_n = \mathfrak{d}(x_n, x_{n+1})$. Then $d_n < \frac{k}{2}(d_{n-1} + d_n)$. Hence,

$$d_n < \frac{k}{2-k}d_{n-1}.$$

Since $k \in (0, 1)$, we have $2 - k > 1$, so $\frac{k}{2-k} < k < 1$.

Let $\lambda = \max\{k, \frac{k}{2-k}\}$. Since $k \in (0, 1)$, we have $\frac{k}{2-k} < k$, and thus $\lambda = k$. Therefore, in all cases where $\mathfrak{d}(x_n, x_{n+1}) > 0$, we have:

$$\mathfrak{d}(x_n, x_{n+1}) \leq k\varphi(\mathfrak{M}(x_{n-1}, x_n)) < k\mathfrak{M}(x_{n-1}, x_n).$$

This implies $\mathfrak{d}(x_n, x_{n+1}) < k\mathfrak{d}(x_{n-1}, x_n)$ if $\mathfrak{M}(x_{n-1}, x_n) = \mathfrak{d}(x_{n-1}, x_n)$, or $\mathfrak{d}(x_n, x_{n+1}) < \frac{k}{2-k}\mathfrak{d}(x_{n-1}, x_n)$ if $\mathfrak{M}(x_{n-1}, x_n) = \frac{\mathfrak{d}(x_{n-1}, x_{n+1})}{2s}$. Therefore, we have $\mathfrak{d}(x_n, x_{n+1}) \le \gamma\mathfrak{d}(x_{n-1}, x_n)$ for some $\gamma \in (0, 1)$. Thus,

$$\mathfrak{d}(x_n, x_{n+1}) \le \lambda^n \mathfrak{d}(x_0, x_1)$$

where $\lambda = \max\{k, \frac{k}{2-k}\} < 1$. This implies $\lim_{n\to\infty} \mathfrak{d}(x_n, x_{n+1}) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. For any $m > n$, using the b-triangle inequality repeatedly, we get

$$\begin{aligned}
\mathfrak{d}(x_n, x_m) &\le s\mathfrak{d}(x_n, x_{n+1}) + s^2\mathfrak{d}(x_{n+1}, x_{n+2}) + \cdots + s^{m-n}\mathfrak{d}(x_{m-1}, x_m) \\
&= \sum_{j=n}^{m-1} s^{j-n+1}\mathfrak{d}(x_j, x_{j+1}) \\
&\le \sum_{j=n}^{m-1} s^{j-n+1}\lambda^j \mathfrak{d}(x_0, x_1) = \mathfrak{d}(x_0, x_1)\sum_{j=n}^{m-1} s^{j-n+1}\lambda^j \\
&= \mathfrak{d}(x_0, x_1)\cdot s\sum_{j=n}^{m-1} s^{j-n}\lambda^j = \mathfrak{d}(x_0, x_1)\cdot s\lambda^n \sum_{j=0}^{m-n-1}(s\lambda)^j.
\end{aligned}$$

Since, $s\max(k, \frac{k}{2-k}) < 1$, then the series converges, and since $\lambda^n \to 0$ as $n \to \infty$, it follows that $\mathfrak{d}(x_n, x_m) \to 0$ as $n, m \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence. By the completeness of $\mathfrak{X}$, there exists $x \in \mathfrak{X}$ such that $\lim_{n\to\infty} x_n = x$. We now show that x is a fixed point of $\mathfrak{T}$. We have

$$\mathfrak{d}(\mathfrak{T}x, x) \le s[\mathfrak{d}(\mathfrak{T}x, \mathfrak{T}x_n) + \mathfrak{d}(\mathfrak{T}x_n, x)] = s[\mathfrak{d}(\mathfrak{T}x, \mathfrak{T}x_n) + \mathfrak{d}(x_{n+1}, x)],$$

and

$$\mathfrak{d}(\mathfrak{T}x, \mathfrak{T}x_n) \le k\varphi(\mathfrak{M}(x, x_n)),$$

where

$$\mathfrak{M}(x, x_n) = \max\left\{\mathfrak{d}(x, x_n), \mathfrak{d}(x, \mathfrak{T}x), \mathfrak{d}(x_n, \mathfrak{T}x_n), \frac{d(x, \mathfrak{T}x_n) + \mathfrak{d}(x_n, \mathfrak{T}x)}{2s}\right\}.$$

As $n \to \infty$, $\mathfrak{M}(x, x_n) \to \max\{0, \mathfrak{d}(x, \mathfrak{T}x), 0, \frac{0+0}{2s}\} = \mathfrak{d}(x, \mathfrak{T}x)$. Therefore,

$$\mathfrak{d}(\mathfrak{T}x, x) \le s[k\varphi(\mathfrak{d}(x, \mathfrak{T}x)) + 0] = s\,k\,\varphi(\mathfrak{d}(x, \mathfrak{T}x)).$$

If $\mathfrak{d}(x, \mathfrak{T}x) > 0$, then,

$$\mathfrak{d}(\mathfrak{T}x, x) \le s\,k\,\varphi(\mathfrak{d}(x, \mathfrak{T}x)) < s\,k\,\mathfrak{d}(x, \mathfrak{T}x) < \mathfrak{d}(x, \mathfrak{T}x).$$

which is a contradiction. This implies $\mathfrak{d}(x, \mathfrak{T}x) = 0$, so $\mathfrak{T}x = x$, and x is a fixed point of $\mathfrak{T}$.
To prove uniqueness, suppose x and y are two fixed points of $\mathfrak{T}$, i.e., $\mathfrak{T}x = x$ and $\mathfrak{T}y = y$. Then using (1),

$$\mathfrak{d}(x, y) = \mathfrak{d}(\mathfrak{T}x, \mathfrak{T}y) \le k\varphi(\mathfrak{M}(x, y)),$$

where

$$\begin{aligned}
\mathfrak{M}(x, y) &= \max\left\{\mathfrak{d}(x, y), \mathfrak{d}(x, \mathfrak{T}x), \mathfrak{d}(y, \mathfrak{T}y), \frac{\mathfrak{d}(x, \mathfrak{T}y) + \mathfrak{d}(y, \mathfrak{T}x)}{2s}\right\} \\
&= \max\{\mathfrak{d}(x, y), 0, 0, \mathfrak{d}(x, y)/s\} = \mathfrak{d}(x, y).
\end{aligned}$$

Thus, $\mathfrak{d}(x, y) \le k\varphi(\mathfrak{d}(x, y)) < k\mathfrak{d}(x, y)$. Since $k < 1$, this implies $\mathfrak{d}(x, y) = 0$, and therefore $x = y$. Hence, the fixed point is unique. This completes the proof. □

Example 3.4. Let $\mathfrak{X} = [0,1]$ and $\mathfrak{d}(x,y) = |x-y|^2$. Then $(\mathfrak{X}, \mathfrak{d})$ is a complete $\mathfrak{b}$-metric space with $s = 2$. Define $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ by $\mathfrak{T}x = x/4$. Let $\varphi(t) = t/2$ and $k = 1/2$. Then for $x, y \in \mathfrak{X}$,

$$\begin{aligned} \mathfrak{d}(\mathfrak{T}x, \mathfrak{T}y) &= |\mathfrak{T}x - \mathfrak{T}y|^2 = |x/4 - y/4|^2 = \frac{1}{16}|x-y|^2 \\ &\leq \frac{1}{2}\varphi(|x-y|^2) = k\varphi(\mathfrak{d}(x,y)). \end{aligned}$$

Since $\varphi(t) < t$ for all $t > 0$ and $k = 1/2 < 1$, $\mathfrak{T}$ is a φ-nonlinear contraction. By Theorem 3.3, $\mathfrak{T}$ has a unique fixed point in $\mathfrak{X}$, which is $x = 0$.

Remark 3.5. Theorem 3.2 generalizes the Banach contraction principle in $\mathfrak{b}$-metric spaces by considering a more general nonlinear contraction condition. The choice of the function φ allows for greater flexibility in the contraction requirement. Furthermore, it reduces to the standard Banach contraction principle when $\varphi(t) = t$ and $s = 1$.

4. Application to Differential Equations

In this section, we demonstrate the application of the fixed point theorems established in Section 3 to prove the existence and uniqueness of solutions to certain types of nonlinear differential equations.

Consider the following nonlinear differential equation:

$$\begin{cases} x'(t) = \mathfrak{f}(t, x(t)), \quad t \in \mathfrak{I} = [a,b], \\ x(a) = x_0, \end{cases} \tag{4.1}$$

where $\mathfrak{f} : \mathfrak{I} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. We assume that $\mathfrak{f}$ satisfies the following Lipschitz condition:

$$|\mathfrak{f}(t,x) - \mathfrak{f}(t,y)| \leq \mathfrak{L}_t |x-y|,$$

for all $t \in \mathfrak{I}$ and $x, y \in \mathbb{R}$, where $\mathfrak{L}_t$ is a continuous function on $\mathfrak{I}$. We define an operator $\mathfrak{T} : \mathfrak{C}(\mathfrak{I}, \mathbb{R}) \to \mathfrak{C}(\mathfrak{I}, \mathbb{R})$ by

$$(\mathfrak{T}x)(t) = x_0 + \int_a^t \mathfrak{f}(\tau, x(\tau))d\tau, \tag{4.2}$$

where $\mathcal{C}(\mathfrak{I}, \mathbb{R})$, the space of continuous functions from $\mathfrak{I}$ to $\mathbb{R}$, equipped with the $\mathfrak{b}$-metric $d(x,y) = \sup_{t \in [a,b]} |x(t) - y(t)|^2$, with $s = 2$. It is clear that $x \in \mathfrak{C}([a,b], \mathbb{R})$ is a solution of the Eq. (2) if and only if x is a fixed point of $\mathfrak{T}$.

Theorem 4.1. Suppose that the Lipschitz constant $\mathfrak{L}_t$ satisfies

$$\max_{t \in \mathfrak{I}} \Big(\int_a^t \mathfrak{L}_\tau d\tau \Big)^2 < 1/4.$$

Then the Eq. (2) has a unique solution in $\mathfrak{C}(\mathfrak{I}, \mathbb{R})$.

Proof. We show that the operator $\mathfrak{T}$ defined by (3) is a φ-nonlinear contraction on $\mathcal{C}(\mathfrak{I}, \mathbb{R})$. For any $x, y \in \mathcal{C}(\mathfrak{I}, \mathbb{R})$, we have

$$\begin{aligned} |(\mathfrak{T}x)(t) - (\mathfrak{T}y)(t)| &= |\int_a^t [\mathfrak{f}(\tau, x(\tau)) - \mathfrak{f}(\tau, y(\tau))]d\tau| \leq \int_a^t |\mathfrak{f}(\tau, x(\tau)) - \mathfrak{f}(\tau, y(\tau))|d\tau \\ &\leq \int_a^t \mathfrak{L}_\tau |x(\tau) - y(\tau)|d\tau \leq \max_{\tau \in \mathfrak{I}} |x(\tau) - y(\tau)| \int_a^t L_\tau d\tau \end{aligned}$$

Therefore,

$$|(\mathfrak{T}x)(t)-(\mathfrak{T}y)(t)|^2 \leq \Big(\int_a^t \mathfrak{L}_\tau d\tau\Big)^2 (\max_{\tau\in\mathfrak{J}}|x(\tau)-y(\tau)|)^2 = \Big(\int_a^t \mathfrak{L}_\tau d\tau\Big)^2 \max_{\tau\in\mathfrak{J}}|x(\tau)-y(\tau)|^2.$$

Taking the maximum over $t\in[a,b]$, we get

$$\max_{t\in\mathfrak{J}}|(\mathfrak{T}x)(t)-(\mathfrak{T}y)(t)|^2 \leq \max_{t\in\mathfrak{J}}\Big(\int_a^t \mathfrak{L}_\tau d\tau\Big)^2 \max_{\tau\in\mathfrak{J}}|x(\tau)-y(\tau)|^2,$$

which implies

$$\mathfrak{d}(\mathfrak{T}x,\mathfrak{T}y) \leq \max_{t\in\mathfrak{J}}\Big(\int_a^t \mathfrak{L}_\tau \mathfrak{d}\tau\Big)^2 \mathfrak{d}(x,y).$$

Let $k = 4\max_{t\in\mathfrak{J}}\Big(\int_a^t \mathfrak{L}_\tau d\tau\Big)^2$ and choose $\varphi(t)=t/4$. Then $k<1$, $\varphi(t)<t$ for all $t>0$ and $\varphi(0)=0$. Then we have $\mathfrak{d}(\mathfrak{T}x,\mathfrak{T}y)\leq k\varphi(\mathfrak{d}(x,y))$. Hence, $\mathfrak{T}$ is a φ-nonlinear contraction with $\mathfrak{M}(x,y)=\mathfrak{d}(x,y)$. Therefore, by Theorem 3.2, $\mathfrak{T}$ has a unique fixed point in $\mathfrak{C}(\mathfrak{J},\mathbb{R})$, which is the unique solution to the Eq. (2). This completes the proof. □

Example 4.2. Consider the differential equation $x'(t)=t\,x(t)$, $x(0)=1$, for $t\in[0,1]$. Here, $\mathfrak{f}(t,x)=t\,x$. Then $|\mathfrak{f}(t,x)-\mathfrak{f}(t,y)|=|t\,x-t\,y|=t|x-y|$. So $\mathfrak{L}_t=t$. Then $\int_0^t \mathfrak{L}_\tau d\tau=\int_0^t \tau d\tau = t^2/2$. Thus, $\max_{t\in[0,1]}(\int_0^t \mathfrak{L}_\tau d\tau)^2=\max_{t\in[0,1]}(t^2/2)^2=1/4$. By Theorem 4.1, the differential equation has a unique solution in $\mathfrak{C}([0,1],\mathbb{R})$.

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