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Mixed piecewise constant argument and the Runge-Kutta method for numerically solving first-order differential equations

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Abstract. This work introduces an efficient methodology for approximating solutions to first-order non-linear differential equations. The approach is based on formulating a differential equation with piecewise constant arguments associated with the parameters of the classical Runge-Kutta (RK) discrete equations depended on a positive integer parameter n. It is demonstrated that the constructed equation has a unique piecewise-smooth solution and for sufficiently large values of n, this solution approximating solution to the original problem. Numerical results are provided through examples, demonstrating the efficiency and high accuracy of the proposed method.

Keywords: Initial value problem, Differential equation with piecewise constant argument, Approximated solution, Runge-Kutta method, Absolute error
MSC (2020): 34A12, 34A34, 65L05, 65L06

1. Introduction

The finite element approach has been widely applied to nonlinear equations [1]. Both mesh-based and meshless methods have advanced the solving of linear and nonlinear ODEs, PDEs, and fractional differential equations [2], [3], [4].

A finite-time stability for hybrid dynamical systems with deviating argument is explored using a hybrid control scheme in [5]. The reliability of the Non-Fixed Step-Size Algorithm offering efficient integration over varying intervals for stiff differential systems is confirmed through validation of its properties [6], [7], [8]. Finding exact periodic solutions of first-order non-homogeneous differential equations with piecewise constant arguments, providing conditions and explicit solutions are given in [9], [10].

Recent published papers has demonstrated numerical methods for solving initial value problems using various approaches, including Euler's method, the Taylor series method, the Adomian decomposition method, and the Runge-Kutta method [11]. In addition, conventional Taylor and fourth-order Runge-Kutta (RK4) methods have been compared in [12].

In [13], the approximation of initial value problems was addressed through the introduction of an innovative and enhanced method inspired by Taylor's approach, which outperforms the Runge-Kutta (RK) method in terms of accuracy.

In this paper, we propose an effective method for constructing an approximate solution to a class of nonlinear ordinary differential equations of the form

$$y'(t) = f(t, y(t)), \qquad y(a) = y_0, \qquad t \in [a, b], \tag{1.1}$$

where $f(\cdot,\cdot)$ is a real-valued function continuous in $t \in [a, b]$ and Lipschitz continuous in $y \in \mathbb{R}$ (with Lipschitz constant independent from t), and y_0 is a given real constant. The approach involves constructing the initial value problem as a differential equation with piecewise constant arguments by using parameters of the seventh-order RK discrete equation, parametrized by a positive integer n. It is established that the resulting equation admits a unique piecewise-smooth solution, which serves as an approximation to the original problem when n becomes sufficiently large.

To demonstrate the practical effectiveness of the proposed method, several numerical experiments are conducted. The results are compared with the results of [11], [12] and [13] showing that the suggested approach yields high accuracy and exhibits rapid convergence.

2. Differential equations with piecewise constant arguments

Let n be a positive integer, and define the partition points as $t_k = kh + a$, where $h = \frac{b-a}{n}$ and $k = 0, 1, 2, \ldots, n$.

This study focuses on the constructing of differential equation with piecewise constant arguments using the coefficients of a seventh-order Runge-Kutta method comprising thirteen stages for solving initial value problems [14]. The main idea is based on the following sketch: the numbers $K(t_k)$ in (2.1) are constructed so that the function $F_n(t)$, defined in (2.4) converges uniformly to $f(t, y(t))$ on $[a, b]$ as $n \to \infty$, where $y(t)$ is a regular solution of (1.1).

The numerical approximation of the solution to (1.1) using the classical seventh-order Runge–Kutta method is expressed as [15], [14]:

$$\frac{y_k - y_{k-1}}{h} = K(t_k), \tag{2.1}$$

where h denotes the step size, and $K(t_k)$ represents the Runge-Kutta increment function evaluated at t_k, where

$$K(t_k) = \frac{31}{720}k_1(t_k) + \frac{16}{75}k_6(t_k) + \frac{16807}{79200}k_7(t_k) + \frac{16807}{79200}k_8(t_k) + \frac{243}{1760}k_9(t_k) + \frac{243}{1760}k_{12}(t_k) + \frac{31}{720}k_{13}(t_k), \tag{2.2}$$

$k_1(t_k) = f(t_k, y(t_k))$,

$k_2(t_k) = f(t_k + \frac{1}{4}h, y(t_k) + h(\frac{1}{4}k_1(t_k)))$,

$k_3(t_k) = f(t_k + \frac{1}{12}h, y(t_k) + h(\frac{5}{72}k_1(t_k) + \frac{1}{72}k_2(t_k)))$,

$k_4(t_k) = f(t_k + \frac{1}{8}h, y(t_k) + h(\frac{1}{32}k_1(t_k) + \frac{3}{32}k_3(t_k)))$,

$k_5(t_k) = f(t_k + \frac{2}{5}h, y(t_k) + h(\frac{106}{125}k_1(t_k) - \frac{408}{125}k_3(t_k) + \frac{352}{125}k_4(t_k)))$,

$k_6(t_k) = f(t_k + \frac{1}{2}h, y(t_k) + h(\frac{1}{48}k_1(t_k) + \frac{8}{33}k_4(t_k) + \frac{125}{528}k_5(t_k)))$,

$k_7(t_k) = f(t_k + \frac{6}{7}h, y(t_k) + h(-\frac{1263}{2401}k_1(t_k) + \frac{39936}{26411}k_4(t_k) - \frac{64125}{26411}k_5(t_k) + \frac{5520}{2401}k_6(t_k)))$,

$k_8(t_k) = f(t_k + \frac{1}{7}h, y(t_k) + h(\frac{37}{392}k_1(t_k) + \frac{1625}{9408}k_5(t_k) - \frac{2}{15}k_6(t_k) + \frac{61}{6720}k_7(t_k)))$,

$k_9(t_k) = f(t_k + \frac{2}{3}h, y(t_k) + h(\frac{17176}{25515}k_1(t_k) - \frac{47104}{25515}k_4(t_k) + \frac{1325}{504}k_5(t_k) - \frac{41792}{25515}k_6(t_k) + \frac{20237}{145800}k_7(t_k) + \frac{4312}{6075}k_8(t_k)))$,

$k_{10}(t_k) = f(t_k + \frac{2}{7}h, y(t_k) + h(-\frac{23834}{180075}k_1(t_k) - \frac{77824}{1980825}k_4(t_k) - \frac{636635}{633864}k_5(t_k) + \frac{254048}{300125}k_6(t_k) - \frac{183}{7000}k_7(t_k) + \frac{8}{11}k_8(t_k) - \frac{324}{3773}k_9(t_k)))$,

$k_{11}(t_k) = f(t_k + h, y(t_k) + h(\frac{12733}{7600}k_1(t_k) - \frac{20032}{5225}k_4(t_k) + \frac{456485}{80256}k_5(t_k) - \frac{42599}{7125}k_6(t_k) + \frac{339227}{912000}k_7(t_k) - \frac{1029}{4180}k_8(t_k) + \frac{1701}{1408}k_9(t_k) + \frac{5145}{2432}k_{10}(t_k)))$,

$k_{12}(t_k) = f(t_k + \frac{1}{3}h, y(t_k) + h(-\frac{27061}{204120}k_1(t_k) + \frac{40448}{280665}k_4(t_k) - \frac{1353775}{1197504}k_5(t_k) + \frac{17662}{25515}k_6(t_k) - \frac{71687}{1166400}k_7(t_k) + \frac{98}{225}k_8(t_k) + \frac{1}{16}k_9(t_k) + \frac{3773}{11664}k_{10}(t_k)))$,

$k_{13}(t_k) = f(t_k + h, y(t_k) + h(\frac{11203}{8680}k_1(t_k) - \frac{38144}{11935}k_4(t_k) + \frac{2354425}{458304}k_5(t_k) - \frac{84046}{16275}k_6(t_k) + \frac{673309}{1636800}k_7(t_k) + \frac{4704}{8525}k_8(t_k) + \frac{9477}{10912}k_9(t_k) - \frac{1029}{992}k_{10}(t_k) + \frac{729}{341}k_{12}(t_k)))$.

For each positive integer n, we construct a differential equation incorporating a piecewise constant argument, which serves as an approximation to the nonlinear differential equation given in (1.1), and is associated with the corresponding initial value problem.

$$y'(t) = F_n(t), \quad y(t_0) = y_0, \quad t \in (a, b), \tag{2.3}$$

where

$$F_n(t) = K(t_{k-1}) \quad \text{for} \quad t \in [t_{k-1}, t_k), \quad k = 1, \ldots, n, \tag{2.4}$$

$$y_k = y(t_k) = \lim_{t \to t_k - 0} \int_{t_{k-1}}^{t} F_n(s) ds = K(t_{k-1}) \frac{1}{n},$$

By the construction of $K(t_{k-1})$, we have

$$K(t_{k-1}) - f(t_{k-1}, y_{k-1}) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.5}$$

A solution to equation (2.3) is characterized in the following manner, as described in [16], [17].

Definition 2.1. A function $y(t) := y_n(t)$ is said to be a solution of the initial value problem (2.3) if the following conditions are fulfilled:

(i) $y(t)$ is continuous on the interval $[a, b]$;

(ii) The derivative $y'(t)$ exists and is continuous on $[a, b]$, except possibly at the points $t = t_k$, where $k = 0, 1, \ldots, n$, at which the one-sided derivatives exist;

(iii) The function $y(t)$ satisfies the differential equation in (2.3) for all $t \in (a, b)$, possibly excluding the points $t = t_k$, where $k = 0, 1, \ldots, n$.

Theorem 2.2. *For every positive integer n, the initial value problem* (2.3) *admits a unique solution $y(t) := y_n(t)$, as defined in* (3.5).

Let $y(t)$ be a solution of the initial value problem defined in (1.1). Then there exists $d > 0$ such that it satisfies the boundedness condition

$$|y(t)| \leq d.$$

Moreover, we assume that the function $f(t, y)$ has compact support $D = [a, b] \times [-2d, 2d]$, that is,

$$f(t, y) = 0 \quad \text{for} \quad (t, y) \notin D. \tag{2.6}$$

We note that under the given assumptions, the function $f(t, y)$ is bounded. Consequently, the corresponding solutions $y_n(t)$ are uniformly bounded as well. That is, there exists a positive constant C such that

$$|y_n(t)| \leq C. \tag{2.7}$$

The next theorem establishes that the function $y_n(t)$ serves as an approximate solution to the initial value problem (1.1).

Theorem 2.3. *For any given $\varepsilon > 0$, there exists a positive integer $n_0 = n(\varepsilon)$ such that for all integers $n > n_0$, the following inequality holds:*

$$\sup_{t \in [a,b]} |y_n'(t) - f(t, y_n(t))| < \varepsilon, \tag{2.8}$$

where $y_n(t)$ denotes the solution of the initial value problem (2.3).

3. Proof of the Main results

Proof of Theorem 2.2. Let n be a fixed positive integer. For $t \in [t_0, t_1)$, equation (2.3) takes the form:

$$y'(t) = K(t_0). \tag{3.1}$$

Integrating both sides of (3.1) yields:

$$y(t) = \int_{t_0}^{t} K(t_0)\, ds + y(t_0), \qquad t \in [t_0, t_1). \tag{3.2}$$

Since the function $y(t)$ is continuous on the interval $[t_0, t_1)$, the left-hand limit at $t = t_1$ exists

$$y(t_1) = \lim_{t \to t_1 - 0} y(t). \tag{3.3}$$

Now, let $y(t)$ denote the solution to equation (2.3) on the interval $[t_{k-2}, t_{k-1})$ with the initial condition $y(t_{k-2}) = \lim\limits_{t \to t_{k-2}-0} y(t)$.

Let

$$y(t_{k-1}) = \lim_{t \to t_{k-1} - 0} y(t). \tag{3.4}$$

Further, using the initial condition (3.4), we integrate equation (2.3) over the interval $[t_{k-1}, t_k)$:

$$y(t) = K(t_{k-1})(t - t_{k-1}) + y(t_{k-1}), \qquad t \in [t_{k-1}, t_k), \tag{3.5}$$

for $k = 2, \ldots, n$.

By construction, the function $y(t)$ defined by (3.5) is differentiable on each subinterval

$$(a, t_1) \cup (t_1, t_2) \cup \cdots \cup (t_{n-1}, b),$$

and continuous on $[a, b]$. Hence, by construction, the function $y(t)$ is the unique solution to equation (2.3).

Proof of Theorem 2.3. Let y_n be the solution of equation (2.3). Then, for $t \in [t_k, t_{k+1})$, the following identity holds:

$$y_n'(t) - f(t, y_n(t)) = K(t_{k-1}) - f(t, y_n(t)).$$

We estimate the right-hand side as follows:

$$|K(t_{k-1}) - f(t, y_n(t))| \le |K(t_{k-1}) - f(t_{k-1}, y_n(t_{k-1}))| + |f(t_{k-1}, y_n(t_{k-1})) - f(t, y_n(t))|.$$

According to equation (2.5), there exists a constant $C > 0$ such that

$$|K(t_{k-1}) - f(t_{k-1}, y_n(t_{k-1}))| \le \frac{C}{n}$$

for sufficiently large n.

By equation (3.5), the solution $y_n(t)$ on $[t_k, t_{k+1})$ can be written as:

$$y_n(t) = K(t_{k-1})(t - t_{k-1}) + y(t_{k-1}),$$

so that

$$f\big(t, y_n(t)\big) = f\big(t, K(t_{k-1})(t - t_{k-1}) + y(t_{k-1})\big), \quad t \in [t_k, t_{k+1}).$$

Since the function f is differentiable on the domain D, there exists a constant $F > 0$ such that

$$|f(t, y_n(t_{k-1})) - f(t, y_n(t))| \le \frac{F}{n}, \tag{3.6}$$

for all $t \in [t_{k-1}, t_k)$ and sufficiently large n.

Combining the above estimates yields:

$$\big|y_n'(t) - f(t, y_n(t))\big| \le \frac{C + F}{n}, \tag{3.7}$$

for all $t \in [t_{k-1}, t_k)$, $k = 1, \ldots, n$, and sufficiently large n.

This completes the proof of the theorem.

4. Algorithmic Summary

The following steps outline the computational procedure for approximating the solution of the initial value problem:

Input: Prescribed initial condition.

Step 1: Compute the value of $K(t_0)$ using equation (2.2);

Step 2: Determine the function $y_n(t)$ over the interval $t \in [t_0, t_1)$ via equation (3.2);

Step 3: Evaluate the one-sided limit $y_n(t_1) = \lim_{t \to t_1 - 0} y_n(t)$;

Step 4: For each subinterval, calculate $K(t_{k-1})$ using equation (2.2);

Step 5: Apply formula (3.5) to compute $y_n(t)$ on the interval $t \in [t_{k-1}, t_k)$ for $k = 2, \dots, n$, and evaluate the left-hand limit $y_n(t_{k-1}) = \lim_{t \to t_{k-1} - 0} y_n(t)$;

Step 6: Construct the approximate solution $y_n(t)$ using the piecewise-defined formula (3.5);

Step 7: Evaluate the maximum absolute error between the exact solution $y(t_{k-1})$ and the numerical approximation $y_n(t_{k-1})$ at the mesh points:

$$\text{MaxErr} = \max_{1 \le k \le n} \left| y(t_{k-1}) - y_n(t_{k-1}) \right|;$$

Output: If the computed maximum error is within the prescribed tolerance, the procedure terminates successfully.

5. Numerical Results

To obtain an approximate solution of the initial value problem (1.1) based on Theorem 2.3, it is essential to compute the function y_n, which is explicitly defined by formula (3.5).

Example 1. Consider the initial value problem described in [13]:

$$y'(t) = te^{y(t)}, \quad 0 \le t \le 0.7, \quad y(0) = 1.$$

The parameters are $N = 10$, $h = 0.07$, and $t_i = 0.07i$. The exact solution corresponding to this problem is given by

$$y(t) = -\ln\left(e^{-1} - \frac{1}{2}t^2\right).$$

The absolute error results by using a hybrid technique combining the piecewise constant argument approach with the Runge-Kutta method are given in Table 1. The comparison with the numerical results obtained via the classical Runge-Kutta (RK) method, the Obreschkoff method, and our proposed approach highlights the accuracy and efficiency of the proposed scheme.

Table 1. Comparison of absolute errors in the Runge-Kutta (RK) method of order 6 [13], the Obreschkoff method (OM) of order 8 [13] and present method applied in Example 1 with step size $h = 0.07$.

t_i	**RK Error [13]**	**OM Error [13]**	**Present Method**
0.0	0.0000000	0.0000000	0.0000000
0.07	7.9991×10^{-10}	1.9000×10^{-13}	8.2486×10^{-16}
0.14	1.8513×10^{-9}	9.5000×10^{-13}	4.1093×10^{-15}
0.21	8.5394×10^{-9}	2.9800×10^{-12}	1.1712×10^{-14}
0.28	2.0817×10^{-8}	8.5900×10^{-12}	2.8994×10^{-14}
0.35	4.1761×10^{-8}	2.5740×10^{-11}	7.1300×10^{-14}
0.42	7.7709×10^{-8}	8.6570×10^{-11}	1.8795×10^{-13}
0.49	1.3675×10^{-7}	3.5061×10^{-10}	5.6835×10^{-13}
0.56	2.3260×10^{-7}	1.8740×10^{-9}	2.1447×10^{-12}
0.64	3.1537×10^{-7}	1.5364×10^{-8}	1.1346×10^{-11}
0.70	7.3768×10^{-7}	2.6095×10^{-7}	8.8373×10^{-11}

Example 2. Consider the Abel-type nonlinear differential equation [11]

$$y'(t) = t + 3ty + 3ty^2 + ty^3,$$

subject to the initial condition $y(0) = 0$. The exact solution to this problem is expressed as

$$y(t) = -1 + \frac{1}{\sqrt{1-t^2}}.$$

TABLE 2. The table of absolute errors at the point t_i obtained by Euler, Taylor, Adomian, RK th4 [11] and Present methods for Example 2.

t	Exact	Abs. Err Euler [11]	Abs. Err Taylor [11]	Abs. Err Adomian [11]	Abs. Err RK 4th [11]	Abs. Err Present method
0.1	5.0378e-03	1.0000e+00	7.5063e-03	2.7352e-07	6.1705e-06	7.9062e-15
0.2	2.0621e-02	5.1505e-01	1.9136e-02	1.7543e-05	6.4452e-06	5.2515e-14
0.3	4.8285e-02	3.6614e-01	3.1291e-02	2.0138e-04	7.2163e-06	1.5983e-13
0.4	9.1089e-02	3.0348e-01	4.5789e-02	1.1539e-03	8.6288e-06	4.1544e-13
0.5	1.5470e-01	2.7892e-01	6.4632e-02	4.5820e-03	1.0966e-05	1.2183e-12
0.6	2.5000e-01	2.7911e-01	9.1203e-02	1.4693e-02	1.4457e-05	6.574e-12
0.7	4.0028e-01	3.0334e-01	1.3228e-01	4.1529e-02	1.8686e-05	1.0107e-10
0.8	6.6667e-01	3.6210e-01	2.0385e-01	1.0966e-01	1.8517e-05	4.3633e-09
0.9	1.2942e+00	4.9242e-01	3.5428e-01	2.8525e-01	1.0507e-03	9.3657e-07

Example 3. The Generalized Taylor method is employed to approximate the solution of the initial value problem [12]:

$$y'(t) = e^{t-y(t)}, \quad 0 \le t \le 1, \quad y(0) = 1,$$

where the parameters are specified as $N = 10$, $h = 0.1$, and $t_i = 0.1i$. The exact analytical solution for this problem is given by

$$y(t) = \ln(e^t + e^{-1} - 1).$$

Moreover, the performance of the proposed approximation is compared against several classical numerical methods to assess its accuracy and efficiency.

TABLE 3. Comparison of absolute errors for the Modified Euler (MEM), Trapezoid (TM), Midpoint (MM), Taylor and RK methods of order 4, alongside the Generalized Taylor method (GTM) [12] and Present method for Example 3, with a step size of $h = 0.1$.

t_i	MEM [12]	TM [12]	MM [12]	Taylor [12]	RKM [12]	GTM [12]	Present method
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	2.8×10^{-5}	2.8×10^{-5}	8.7×10^{-6}	9.0×10^{-9}	4.4×10^{-9}	5.2×10^{-10}	2.9×10^{-12}
0.2	5.6×10^{-5}	5.6×10^{-5}	1.8×10^{-5}	1.6×10^{-8}	9.2×10^{-9}	9.5×10^{-10}	3.1×10^{-12}
0.3	8.3×10^{-5}	8.3×10^{-5}	2.9×10^{-5}	2.2×10^{-8}	1.3×10^{-8}	1.3×10^{-9}	2.9×10^{-12}
0.4	1.1×10^{-4}	1.1×10^{-4}	4.1×10^{-5}	2.6×10^{-8}	2.0×10^{-8}	1.4×10^{-9}	2.5×10^{-12}
0.5	1.4×10^{-4}	1.4×10^{-4}	5.3×10^{-5}	2.7×10^{-8}	2.5×10^{-8}	1.5×10^{-9}	2.2×10^{-12}
0.6	1.6×10^{-4}	1.6×10^{-4}	6.7×10^{-5}	2.6×10^{-8}	3.1×10^{-8}	1.4×10^{-9}	1.9×10^{-12}
0.7	1.9×10^{-4}	1.9×10^{-4}	8.0×10^{-5}	2.3×10^{-8}	3.8×10^{-8}	1.2×10^{-9}	1.6×10^{-12}
0.8	2.1×10^{-4}	2.1×10^{-4}	9.4×10^{-5}	1.8×10^{-8}	4.4×10^{-8}	8.8×10^{-10}	1.4×10^{-12}
0.9	2.3×10^{-4}	2.3×10^{-4}	1.1×10^{-4}	1.2×10^{-8}	5.0×10^{-8}	4.7×10^{-10}	1.2×10^{-12}
1.0	2.5×10^{-4}	2.5×10^{-4}	1.2×10^{-4}	4.2×10^{-9}	5.6×10^{-8}	1.9×10^{-11}	1.1×10^{-12}

It is evident that the present method yields substantially higher accuracy in comparison to the other numerical schemes under consideration. The comparative analysis of the approximate solutions obtained via the present method and alternative approaches, along with their corresponding absolute errors, is presented in Table 3.

Example 4. Let

$$y'(t) = e^{y^2(t)} + \cos(t) - e^{\sin^2(t)}, \quad 0 \le t \le 1, \quad y(0) = 0.$$

The exact solution of this IVP is

$$y(t) = \sin(t).$$

The results for $N = 10$, $h = 0.1$, and $t_k = 0.1k$ using the proposed method are presented in Table 4.

TABLE 4. The absolute errors in the Runge-Kutta (RK) method of order 7 in Example 4 with step size $h = 0.1$.

t_i	**Exact**	**Approximate**	**Absolute error**
0.0	0.00000000000000	0.00000000000000	0.0000000
0.1	0.0998334166468281	0.0998334166468285	3.5143×10^{-16}
0.2	0.1986693307950612	0.1986693307950617	5.1645×10^{-16}
0.3	0.2955202066613396	0.2955202066613394	1.8762×10^{-16}
0.4	0.3894183423086505	0.3894183423086475	3.0056×10^{-15}
0.5	0.4794255386042030	0.4794255386041950	7.9623×10^{-15}
0.6	0.5646424733950354	0.5646424733950275	7.8677×10^{-15}
0.7	0.6442176872376911	0.6442176872377125	2.1443×10^{-14}
0.8	0.7173560908995228	0.7173560908996445	1.2174×10^{-13}
0.9	0.7833269096274834	0.7833269096278118	3.2845×10^{-13}
1.0	0.8414709848078965	0.8414709848086330	7.3648×10^{-13}

6. Conclusion

A novel numerical scheme is introduced for approximating solutions of differential equations. The proposed methodology is based on constructing an associated differential equation incorporating a piecewise constant argument, using the parameters seventh-order RK method, which corresponds to the original problem. A solution to such ordinary differential equations with piecewise constant arguments is defined as a piecewise smooth function, depended on a positive integer n. It is demonstrated that, for sufficiently large values of n, the solution of the constructed equation closely approximates the solution of the original initial value problem.

In Examples 1-4, using the proposed method, the maximum absolute errors between the exact and the approximate solutions at the points t_k, are demonstrated, with the results of the recently published papers, in Tables 1-4, respectively. The comparisons show the accuracy and efficiency of the proposed technique.

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