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Schrödinger equation is a weak form of the classical Euler-Lagrange equation

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Abstract. Inspired by quantum mechanics, we introduce a weak form of solutions for differential equations. We show that Schrödinger equation is a weak form of the classical Euler-Lagrange equation.

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1. Introduction

For partial differential equations various forms of weak solutions are defined, [1]. In this paper, we define a new form of weak solutions for differential equations. Motivation comes from quantum mechanics where Newton equation is replaced with the Schrödinger equation. In classical mechanics, Newton equation for a particle is

$$m\ddot{x}(t) + \nabla U(x(t)) = 0,$$

where m is a positive constant, $x(t)$ is a curve in $\mathbb{R}^3$ and $U : \mathbb{R}^3 \to \mathbb{R}$. In quantum mechanics this equation is replaced with the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla\psi + U\psi, \tag{1.1}$$

where $\psi(t,x) \in \mathbb{C}, t \in \mathbb{R}, x \in \mathbb{R}^3$. There is a big radical change in passing from Newton equation to Schrödinger equation. We try to bring these two different equations close to each other. To do this task, we first rewrite Newton equation as follows

$$\dot{x}(t) = v(t), \tag{1.2}$$

and

$$m\dot{v}(t) + \nabla U(x(t)) = 0. \tag{1.3}$$

The first equation just means that the variable $v(t)$ is the derivative of $x(t)$. Next, using the polar decomposing of the complex function $\psi(t,x) = R(t,x)e^{\frac{i}{\hbar}S(t,x)}$ and setting $\rho(t,x) := R^2(t,x), V(t,x) := \frac{1}{m}\nabla S(t,x)$ we rewrite the Schrödinger equation as follows

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho V) = 0, \tag{1.4}$$

and

$$m\Big(\frac{\partial V}{\partial t} + (V\cdot\nabla)V\Big) + \nabla U + \nabla Q = 0,$$

where the term Q is called quantum potential and it is known that $\int \rho\nabla Q dx = 0$, [2]. Thus

$$\int \rho\Big(m(\frac{\partial V}{\partial t} + (V\cdot\nabla)V) + \nabla U\Big)dx = 0. \tag{1.5}$$

We call equations (1.4) and (1.5) as *weak form* of equations (1.2) and (1.3). This observation suggests to replace a curve $x(t) \in \mathbb{R}^3$ and its derivative $v(t) = dx/dt$ by probability density $\rho(t,x)$ and vector field $V(t,x)$ satisfying (1.4). We will call the pair $(x(t), v(t))$ as a *strong differentiable function* and the pair $(\rho(t,x), V(t,x))$ as a *weak differentiable function.*

Ambrosio et al., [3], have studied differential calculus over a metric measure space and in particular over Wasserstein space of probability measures. We briefly recall this theory. First, let (S, d) be a complete metric space. For any curve $\rho : (a, b) \subseteq \mathbb{R} \to S$ the limit

$$|\rho'|(t) := \lim_{h\to 0} \frac{d(\rho(t+h), \rho(t))}{|h|},$$

if exists, is called the **metric derivative** of ρ. For any absolutely continuous curve $\rho : (a, b) \subseteq \mathbb{R} \to S$ the metric derivative exists for Lebesgue-a.e. $t \in (a, b)$ and we have $d(\rho(s), \rho(t)) \leq \int_s^t |\rho'|(r)dr$ for any interval $(s, t) \subseteq (a, b)$. In the case where S is a Banach space, a curve $\rho : (a, b) \to S$ is absolutely continuous if and only it is differentiable in the ordinary sense for Lebesgue-a.e. $t \in (a, b)$ and we have $\|\rho'(t)\| = |\rho'|(t)$ for Lebesgue-a.e. $t \in (a, b)$. Next in [3], this fact has been applied to the metric space of probability measures on a Hilbert space X. This space is equipped with the Wasserstein's metric. It is shown that the class of absolutely continuous curves ρ_t in this metric space coincides with solutions of the continuity equation of physicists. More precisely, given an absolutely continuous curve ρ_t, one can find a Borel time-dependent velocity field $V_t : X \to X$ such that $\|V_t\|_{L^p(\rho_t)} \leq |\rho'|(t)$ for a.e. t and the continuity equation $\frac{\partial\rho}{\partial t} + \nabla.(\rho V) = 0$ holds. Conversely, if ρ_t solves the continuity equation for some Borel velocity field V_t with $\int_a^b \|V_t\|_{L^p(\rho_t)}dt < \infty$, then ρ_t is an absolutely continuous curve and $\|V_t\|_{L^p(\rho_t)} \leq |\rho'|(t)$ for a.e. $t \in (a, b)$. As a consequence, we see that among all velocity fields V_t which produce the same flow ρ_t, there is a unique optimal one with the smallest $L^p(\rho_t, X)$-norm, equal to the metric derivative of ρ_t. One can view this optimal field as the "tangent" vector field to the curve ρ_t. Next, Villani, Lott, and Sturm studied the concept of Ricci curvature for a metric measure space [4, 5]. The references [6, 7, 8, 9, 4] studied Riemannian geometry and dynamics on the Wasserstein space by putting the structure of an informal manifold, first introduced by Otto [10].

In this paper, we will work with the informal differentiable structure on the Wasserstein space. In Section 2, we will define the concept of a weak differentiable function. In Section 3, we will state the weak form of the Euler-Lagrange equation and derive Schrödinger equation.

2. Weakly Differentiable Functions

This paper is based on the following observation which redefines the Newton-Leibniz derivative of a function.

Theorem 2.1. *A curve $x(t) \in M$ over a manifold M is differentiable if and only if there exists a vector field $v(t) \in T_{x(t)}M$ along $x(t)$ such that*

$$\frac{d}{dt} f(x(t)) = df_{x(t)}(v(t)),$$

for all differentiable functions f on M and moreover we have $v(t) = dx(t)/dt$.

In this paper, we call the function $x(t) \in M$ a *strong differentiable curve.*

Definition 2.2. A **weak curve** over an oriented Riemannian manifold M is a curve ρ_t of probability densities over M, i.e. $\rho(t, x) \geq 0, a \leq t \leq b, x \in M$, $\int_M \rho(t, x)dx = 1$ for all $a \leq t \leq b$, and ρ vanishes at infinity with respect to the x-variable, where integral is with respect to the Riemannian volume form of M.

Next, we define the *derivative of a weak curve* by imitating Theorem 2.1.

Definition 2.3. A weak curve ρ over M is called differentiable if there exists a differentiable time-dependent vector field V_t over M such that

$$\frac{d}{dt} \int \rho(t, x)f(x)dx = \int \rho(t, x)df(V(t, x)), \tag{2.1}$$

for all $f \in C_c^\infty(M)$, in the space of all smooth compactly-supported functions on M.

Theorem 2.4. *A weak curve ρ is differentiable if and only if there exists a differentiable time-dependent vector field V_t such that*

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0. \tag{2.2}$$

Proof: We have $\frac{d}{dt}\int \rho f = \int \frac{\partial \rho}{\partial t} f$. On the other hand by the divergence theorem and that f has compact support and ρ vanishes at infinity, we have

$$\int \rho df(V) = \int \nabla.(\rho f V) - \int f \nabla.(\rho V) = -\int f \nabla.(\rho V).$$

Thus (2.1) holds if and only if $\int \left(\frac{\partial \rho}{\partial t} + \nabla.(\rho V)\right) f = 0, \forall f$. The later is equivalent with (2.2). □

Note that the vector field V is not unique. Thus we redefine our concept of differentiability as follows.

Definition 2.5. A **differentiable weak curve** is a pair (ρ_t, V_t) where ρ_t is a curve of probability densities and V_t is a vector field over M such that (2.2) holds.

Next, we define a several variable differentiable weak function.

Definition 2.6. A *differentiable weak function* from an open subset $U \subseteq \mathbb{R}^m$ into M is a system $(\rho, V_1, \cdots, V_m)$ including a differentiable function $\rho : U \times M \to [0, \infty)$, vanishing at infinity with respect to M, together with differentiable vector fields $V_j : U \times M \to TM$, $1 \leq j \leq m$, such that

$$\frac{\partial \rho}{\partial u_j} + \nabla.(\rho V_j) = 0.$$

A weak function $\rho : U \times M \to [0, \infty)$ is a replacement for a strong function $f : U \to M$ and the vector fields V_i are replacements for the partial derivatives $V_i = \frac{\partial f}{\partial u_i}$. For the strong function f we have $\frac{\partial^2 f}{\partial u_i \partial u_j} = \frac{\partial^2 f}{\partial u_j \partial u_i}$. Namely, for $V_i = \frac{\partial f}{\partial u_i}$ we have

$$\frac{\partial V_i}{\partial u_j} = \frac{\partial V_j}{\partial u_i}.$$

For weak functions, we have the following result.

Theorem 2.7. *If $(\rho, V_1, \cdots, V_m)$ is a differentiable weak function from an open region $U \subseteq \mathbb{R}^m$ to M then*

$$\rho\Big(\frac{\partial V_i}{\partial u_j} - \frac{\partial V_j}{\partial u_i} - [V_i, V_j]\Big) = 0.$$

Here $[V_i, V_j]$ means the Lie bracket of vector fields V_i and V_j with respect to M.

Proof: For any compactly-supported function g on M, since ρ vanishes at infinity, by the divergence theorem we have

$$\frac{\partial}{\partial u_i}\int \rho g = \int \frac{\partial \rho}{\partial u_i} g = -\int g \nabla.(\rho V_i) = -\int \nabla(\rho g V_i) + \int \rho V_i g = \int \rho V_i g. \tag{2.3}$$

Thus

$$\begin{aligned} \frac{\partial^2}{\partial u_j \partial u_i}\int \rho g &= \int \frac{\partial \rho}{\partial u_j} V_i g + \int \rho \frac{\partial V_i}{\partial u_j} g = -\int \nabla.(\rho V_j) V_i g + \int \rho \frac{\partial V_i}{\partial u_j} g \\ &= -\int \nabla.(\rho(V_i g)V_j) + \int \rho V_j V_i g + \int \rho \frac{\partial V_i}{\partial u_j} g = \int \rho(\frac{\partial V_i}{\partial u_j} + V_j V_i) g. \end{aligned}$$

Hence

$$0 = \frac{\partial^2}{\partial u_j \partial u_i}\int \rho g - \frac{\partial^2}{\partial u_i \partial u_j}\int \rho g = \int \rho(\frac{\partial V_i}{\partial u_j} + V_j V_i - \frac{\partial V_j}{\partial u_i} - V_i V_j) g = \int \rho(\frac{\partial V_i}{\partial u_j} - \frac{\partial V_j}{\partial u_i} + [V_i, V_j]) g.$$

Since g is arbitrary, the desired identity is obtained. □

Remark 2.8. Without using the divergence theorem, we can prove at most the following identity

$$\nabla.\Big(\rho(\frac{\partial V_i}{\partial u_j}-\frac{\partial V_j}{\partial u_i}-[V_i,V_j])\Big)=0.$$

To prove the later, first, we state an important identity that we discovered recently and we have not seen before elsewhere; one can show by some calculation that for any two vector fields V and W and any function f on M we have

$$\nabla.\Big(\nabla.(fW)V\Big)-\nabla.\Big(\nabla.(fV)W\Big)=\nabla.(f[W,V]).$$

Now using this identity we have

$$\frac{\partial^2\rho}{\partial u_j\partial u_i}=-\frac{\partial}{\partial u_j}\nabla.(\rho V_i)=-\nabla.(\frac{\partial\rho}{\partial u_j}V_i)-\nabla.(\rho\frac{\partial V_i}{\partial u_j})=\nabla.\Big(\nabla.(\rho V_j)V_i\Big)-\nabla.(\rho\frac{\partial V_i}{\partial u_j}).$$

Thus

$$\begin{aligned}0 &= \frac{\partial^2\rho}{\partial u_j\partial u_i}-\frac{\partial^2\rho}{\partial u_i\partial u_j}\\ &= \nabla.\Big(\nabla.(\rho V_j)V_i\Big)-\nabla.\Big(\nabla.(\rho V_i)V_j\Big)-\nabla.(\rho\frac{\partial V_i}{\partial u_j})+\nabla.(\rho\frac{\partial V_j}{\partial u_i})\\ &= -\nabla.(\rho[V_i,V_j])-\nabla.(\rho\frac{\partial V_i}{\partial u_j})+\nabla.(\rho\frac{\partial V_j}{\partial u_i})\\ &= \nabla.\Big(\rho(\frac{\partial V_j}{\partial u_i}-\frac{\partial V_i}{\partial u_j}-[V_i,V_j])\Big).\end{aligned}$$

Example 2.9. Let $A\in M_{n\times m}(\mathbb{R})$ be a matrix of size $n\times m$ with real entries whose columns are $A_i, 1\leq i\leq m$, and let σ be a probability density over $\mathbb{R}^n$. Then the pair $(\rho,V_1,\cdots,V_m)$ given below is a differentiable weak function from $\mathbb{R}^m$ into $\mathbb{R}^n$.

$$\rho(x,y):=\sigma(y-Ax),\qquad V_i(x,y):=A_i.$$

Proof: We have $\frac{\partial\rho}{\partial x_i}=-\nabla\sigma(y-Ax).A_i$ and $\nabla^y.(\rho(x,y)V_i(x,y))=\nabla^y.(\sigma(y-Ax)A_i)=\nabla\sigma(y-Ax).A_i$. □

Definition 2.10. A *differentiable weak function* from a manifold P into an oriented Riemannian manifold M is a pair (ρ,V) including a differentiable function $\rho:P\times M\to[0,\infty)$, together with a differentiable map

$$V:TP\times M\to TM,\qquad V(w,q)\in T_qM,$$

$\forall w\in TP, q\in M$, such that V is linear concerning the first variable and for any coordinate system $u=(u_1,\cdots,u_m)\in U\subseteq\mathbb{R}^m$ for P

$$\frac{\partial\rho}{\partial u_j}+\nabla.(\rho V_j)=0, \tag{2.4}$$

where $V_j:U\times M\to TM, V_j(u,q):=V(\frac{\partial}{\partial u_j}(u),q)\in T_qM$. Here the divergence operator $\nabla.$ is relative to the variable $q\in M$.

Proposition 2.11. *Equations (2.4) do not depend on the coordinate systems of P.*

Proof: Let $W_i(v):=V_p(\frac{\partial}{\partial v_i}(p))$ where $(v_1,\cdots,v_m)$ is another coordinate system for P. Then since $\frac{\partial}{\partial v_i}=\sum_j\frac{\partial u_j}{\partial v_i}\frac{\partial}{\partial u_j}$ we get $W_i(v)=\sum_j\frac{\partial u_j}{\partial v_i}V_j(u)$. Thus

$$\begin{aligned}&\frac{\partial\rho(v,q)}{\partial v_i}+\nabla.(\rho(v,q)W_i(v,q))\\ =\ &\sum_j\frac{\partial u_j}{\partial v_i}\frac{\partial\rho(u,q)}{\partial u_j}+\nabla.(\rho(u,q)\sum_j\frac{\partial u_j}{\partial v_i}V_j(u,q))\\ =\ &\sum_j\frac{\partial u_j}{\partial v_i}[\frac{\partial\rho(u,q)}{\partial u_j}+\nabla.(\rho(u,q)V_j(u,q))].\end{aligned}$$

Thus since the matrix $(\frac{\partial u_j}{\partial v_i})$ is invertible we conclude that the expression $\frac{\partial \rho(v,q)}{\partial v_i} + \nabla.(\rho(v,q)W_i(v,q))$ vanishes if and only if $\frac{\partial \rho(u,q)}{\partial u_i} + \nabla.(\rho(u,q)V_i(u,q))$ vanishes.

□

3. Weak Form of Euler-Lagrange Equation

Classical mechanics over Wasserstein space of probability measures over a manifold has been studied extensively, [6, 7, 11, 12]. In this section, we are going to give another viewpoint on this subject. In classical mechanics over strong states, i.e. points over a finite-dimensional configuration manifold M, we study the following action functional

$$S(\alpha) = \int_a^b L(\alpha(t), \dot{\alpha}(t))dt, \tag{3.1}$$

over all paths $\alpha(t) \in M, a \leq t \leq b$, and $L : TM \to \mathbb{R}$ is a Lagrangian. The Lagrangian can have an arbitrary, i.e. non-restrictive, dependence to both α and its derivative. Now, in this paper we are going to replace the space M with the space $W^\infty(M)$ of smooth probability measures over M, i.e. smooth Wasserstein space. The first natural question would be what is the correct counterpart of the above action functional in the case of the space $W^\infty(M)$? The Lagrangian in the above action is a function of α and its derivative. So the Lagrangian over $W^\infty(M)$ must be a function of a path $d\mu_t = \rho(t)dx \in W^\infty(M)$ and its derivative. In this paper we work with probability measures which have density with respect to the volume form of Riemannian manifold M where its Riemannian volume form is denoted by dx. We know that the time-derivative of $\rho(t)$ is a vector field $V(t)$ over M satisfying the continuity equation in the sense of metric measure space theory. So the Lagrangian must be a function of ρ and V. The first candidate for an action functional over $W^\infty(M)$ is

$$S(\rho, V) = \int_a^b \int_M \rho(t,x)L(x, V(t,x))dxdt,$$

where $L : TM \to \mathbb{R}$ is a classical Lagrangian over M and dx is the Riemannian volume form of M. But this is too restrictive Lagrangian since its dependence to the density ρ is very simple unlike the action (3.1) which has nonrestrictive dependence to α. Thus we must add some extra term to the Lagrangian which depends to ρ and perhaps to its spatial derivatives.

In [7] the action functional is of the form

$$S(\mu, V) = \frac{1}{2}\int_a^b \int_{\mathbb{R}^n} \|V(t,x)\|^2 d\mu_t(x)dt - \int_a^b U(\mu_t)dt,$$

where μ_t is a path of probability measures and V_t is vector field over $\mathbb{R}^n$. Here, we do not assume that the continuity equation between μ_t and V_t holds.

In an unpublished work[a] the action functional is of the following form

$$S(\mu, V) = \int_a^b \int_{\mathbb{R}^n} L(x, V(t,x))d\mu_t(x)dt + \int_a^b F(\mu_t)dt + G(\mu(b)),$$

where μ_t is a curve of probability measures over $\mathbb{R}^n$, V_t is a curve of vector fields over $\mathbb{R}^n$, $L : T\mathbb{R}^n \to \mathbb{R}$, and F as well as G are functionals over the space of probability measures and the continuity equation between μ_t and V_t holds, i.e. $\frac{\partial \mu_t}{\partial t} + \nabla.(\mu_t V_t) = 0$ in the sense of distributions.

In this section, we study the following action functional

$$S(\rho, V) = \int_a^b \int_M \rho(t,x)\Big(L(x, V(t,x)) + F(\rho, \partial_i\rho, \partial^2_{ij}\rho, \partial^3_{ijk}\rho, \cdots)\Big)dxdt, \tag{3.2}$$

for some function $F(y, y_i, y_{ij}, y_{ijk}, \cdots)$ and where $\partial_i = \frac{\partial}{\partial x_i}, \partial^2_{ij} = \frac{\partial}{\partial x_i \partial x_j}$, etc. Here, we assume that (ρ, V) is a weak differentiable function.

[a]P. Cardaliaguet, Paris Dauphine University, April 2019: Analysis in the space of measures. Lecture notes `https://dottorato.math.unipd.it/sites/default/files/Pierre_Cardaliaguet.pdf`

For simplicity, we will work on space $M = \mathbb{R}^n$ but the result can be extended to an arbitrary manifold. In order to be able to apply the least action principle to the action 3.2, we first need to define a perturbation of the differentiable curve (ρ_t, V_t). To do this task, we recall the meaning of a perturbation of a strong differentiable curve $x(t) \in M, a \leq t \leq b$. It is just a two-variable function $x(t,s) \in M, a \leq t \leq b, -\epsilon \leq s \leq \epsilon$, such that $x(t,0) = x(t), x(a,s) = x(a)$ and $x(b,s) = x(b)$. Notice, that we have the following identity

$$\frac{\partial^2 x}{\partial s \partial t} = \frac{\partial^2 x}{\partial t \partial s}.$$

We define a variation of a differentiable weak curve $(\rho_t, V_t), a \leq t \leq b$, over M to be a triple $(\rho_{t,s}(x), V_{t,s}(x), W_{t,s}(x))$ for some parameter s in a small interval around zero, satisfying

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0, \qquad \frac{\partial \rho}{\partial s} + \nabla.(\rho W) = 0,$$

and moreover for $s = 0$ we recover the original pair (ρ_t, V_t) and also $\rho_{a,s}(x) = \rho_a(x), \rho_{b,s}(x) = \rho_b(x), W_{a,s}(x) = W_{b,s}(x) = 0, \forall x$. The notations in the following lemma will be used in our calculus of variation.

Lemma 3.1. *Let A and B be two vectors in $\mathbb{R}^n$ and let X be an $n \times n$ matrix. Denote by AB the inner product and by XA the vector XA^T. Then we have*
i) $(XA)B = (XB)A$,
ii) $(AX)B = A(XB)$.

The proof is obvious.

Theorem 3.2. *By the variation of the action functional (3.2) under the above meaning of variation we arrive to the following equation which we call* weak Euler-Lagrange Equation

$$\rho\Big((\frac{\partial}{\partial t} + V\frac{\partial}{\partial x})\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} - \frac{\partial}{\partial x}F - \frac{\partial}{\partial x}(\rho\frac{\partial F}{\partial y} - \frac{\partial}{\partial x_i}(\rho\frac{\partial F}{\partial y_i}) + \frac{\partial^2}{\partial x_i \partial x_j}(\rho\frac{\partial F}{\partial y_{ij}}) - \cdots)\Big) = 0, \tag{3.3}$$

where ∇ is the gradient operator with respect to the variable $x \in M$ and in general for given vector fields X and function f, by $(X.\nabla)f$ we mean the derivative of function f in the direction of vector field X. Here for repeated indexes we use Einstein summation rule. In particular, if the following identity holds for all function $\rho(x)$

$$\rho\frac{\partial F}{\partial y} - \frac{\partial}{\partial x_i}(\rho\frac{\partial F}{\partial y_i}) + \frac{\partial^2}{\partial x_i \partial x_j}(\rho\frac{\partial F}{\partial y_{ij}}) - \cdots = 0, \tag{3.4}$$

then the weak Euler-Lagrange equation becomes

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0, \quad \rho\Big((\frac{\partial}{\partial t} + V.\nabla)\frac{\partial L}{\partial \dot{x}}(x,V) - \frac{\partial L}{\partial x}(x,V) - \frac{\partial}{\partial x}F\Big) = 0. \tag{3.5}$$

Proof: We set

$$S_1 := \int_a^b \int_M \rho L(x,V) dx dt,$$

and

$$S_2 := \int_a^b \int_M \rho F(\rho, \partial_i \rho, \partial^2_{ij}\rho, \partial^3_{ijk}\rho, \cdots) dx dt.$$

Let $(\rho_{t,s}(x), V_{t,s}(x), W_{t,s}(x))$ be a variation of the pair (ρ_t, V_t). We set $K := L(x, V(t,s,x))$. Based on the notation introduced in the previous Lemma, we compute the derivative

$$\begin{aligned}
\frac{dS_1}{ds} &= \iint \Big(\frac{\partial\rho}{\partial s}K + \rho\frac{\partial L}{\partial\dot{x}}\frac{\partial V}{\partial s}\Big)dxdt \\
&= \iint \rho\Big(\frac{\partial K}{\partial x}W + \frac{\partial L}{\partial\dot{x}}\frac{\partial V}{\partial s}\Big)dxdt \\
&= \iint \rho\Big(\frac{\partial K}{\partial x}W + \frac{\partial L}{\partial\dot{x}}\frac{\partial W}{\partial t} + \frac{\partial L}{\partial\dot{x}}[V,W]\Big)dxdt \\
&= \iint \Big(\rho\frac{\partial K}{\partial x}W + \frac{\partial}{\partial t}(\rho\frac{\partial L}{\partial\dot{x}}W) - \rho(\frac{\partial}{\partial t}\frac{\partial L}{\partial\dot{x}})W - \frac{\partial\rho}{\partial t}\frac{\partial L}{\partial\dot{x}}W + \rho\frac{\partial L}{\partial\dot{x}}[V,W]\Big)dxdt \\
&= \iint \rho\Big(\frac{\partial K}{\partial x}W - (\frac{\partial}{\partial t}\frac{\partial L}{\partial\dot{x}})W - ((\frac{\partial}{\partial x}(\frac{\partial L}{\partial\dot{x}}W))V + \frac{\partial L}{\partial\dot{x}}[V,W]\Big)dxdt \\
&= \iint \rho\Big(\frac{\partial K}{\partial x}W - (\frac{\partial}{\partial t}\frac{\partial L}{\partial\dot{x}})W - ((\frac{\partial}{\partial x}\frac{\partial L}{\partial\dot{x}})W)V - (\frac{\partial L}{\partial\dot{x}}\frac{\partial W}{\partial x})V \\
&+ \frac{\partial L}{\partial\dot{x}}(\frac{\partial W}{\partial x}V - \frac{\partial V}{\partial x}W)\Big)dxdt \\
&= \iint \rho\Big(\frac{\partial K}{\partial x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial\dot{x}} - (\frac{\partial}{\partial x}\frac{\partial L}{\partial\dot{x}})V - \frac{\partial L}{\partial\dot{x}}\frac{\partial V}{\partial x}\Big)W dxdt.
\end{aligned}$$

But $\frac{\partial K}{\partial x} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial\dot{x}}\frac{\partial V}{\partial x}$, and thus

$$\delta S_1 = \frac{dS_1}{ds}|_{s=0} = \iint \rho\Big(\frac{\partial L}{\partial x} - (\frac{\partial}{\partial t} + V\frac{\partial}{\partial x})\frac{\partial L}{\partial\dot{x}}\Big)W dxdt.$$

By applying integration by parts and the fact that ρ and thus its derivatives vanish at infinity, we have

$$\begin{aligned}
\frac{dS_2}{ds} &= \iint \Big(\frac{\partial\rho}{\partial s}F + \rho(\frac{\partial F}{\partial y}\frac{\partial\rho}{\partial s} + \frac{\partial F}{\partial y_i}\frac{\partial^2\rho}{\partial s\partial x_i} + \frac{\partial F}{\partial y_{ij}}\frac{\partial^3\rho}{\partial s\partial x_i\partial x_j} + \cdots)\Big)dxdt \\
&= \iint \Big(\frac{\partial\rho}{\partial s}F + \rho\frac{\partial F}{\partial y}\frac{\partial\rho}{\partial s} - (\frac{\partial}{\partial x_i}(\rho\frac{\partial F}{\partial y_i}))\frac{\partial\rho}{\partial s} + (\frac{\partial^2}{\partial x_i\partial x_j}(\rho\frac{\partial F}{\partial y_{ij}}))\frac{\partial\rho}{\partial s} - \cdots\Big)dxdt \\
&= \iint \frac{\partial\rho}{\partial s}\Big(F + \rho\frac{\partial F}{\partial y} - \frac{\partial}{\partial x_i}(\rho\frac{\partial F}{\partial y_i}) + \frac{\partial^2}{\partial x_i\partial x_j}(\rho\frac{\partial F}{\partial y_{ij}}) - \cdots\Big)dxdt \\
&= \iint \rho\frac{\partial}{\partial x}\Big(F + \rho\frac{\partial F}{\partial y} - \frac{\partial}{\partial x_i}(\rho\frac{\partial F}{\partial y_i}) + \frac{\partial^2}{\partial x_i\partial x_j}(\rho\frac{\partial F}{\partial y_{ij}}) - \cdots\Big)W dxdt.
\end{aligned}$$

Thus

$$\frac{dS}{ds}|_{s=0} = \frac{dS_1}{ds}|_{s=0} + \frac{dS_2}{ds}|_{s=0},$$

for all W implies the desired result. □

The following fact is due to Iranian theoretical physicists Mehdi Golshani, Mahdi Atiq and Mozafar Karamian, [13],[14] which we have adapted with our theory.

Corollary 3.3. *When $L(x,v) = \frac{1}{2m}\|v\|^2 - U(x)$ and*

$$F(y, y_i, y_{ij}) = -\frac{\hbar^2}{2m}(-\frac{\sum_i y_i^2}{4y^2} + \frac{\sum_i y_{ii}}{2y}),$$

then first of all the identity (3.4) holds for all ρ and secondly if in the weak Euler-Lagrange equation the vector field V is of the form $V = \frac{\nabla S}{m}$ for some function S then the weak Euler-Lagrange equations convert to the Schrödinger equation (1.1), under the transformation

$$\psi := \sqrt{\rho}e^{\frac{i}{\hbar}S}. \tag{3.6}$$

In this case we have,

$$F(\rho, \partial_i\rho, \partial^2_{ij}\rho) = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}, \tag{3.7}$$

which is called quantum potential in the literature of Bohmian quantum mechanics.

Proof: We have $\frac{\partial F}{\partial y} = \frac{1}{2}y^{-3}\sum_i y_i^2 - \frac{1}{2}y^{-2}\sum_i y_{ii}$, $\frac{\partial F}{\partial y_i} = \frac{-1}{2}y^{-2}y_i$ and $\frac{\partial F}{\partial y_{ij}} = \frac{1}{2}\delta_{ij}y^{-1}$. Thus $\frac{\partial F}{\partial y}(\rho, \partial_i\rho, \partial^2_{ij}\rho) = \frac{1}{2}\rho^{-3}\sum_i(\partial_i\rho)^2 - \frac{1}{2}\rho^{-1}\sum_i \partial^2_{ii}\rho$, $\frac{\partial F}{\partial y_i}(\rho, \partial_i\rho, \partial^2_{ij}\rho) = \frac{-1}{2}\rho^{-2}\partial_i\rho$ and $\frac{\partial F}{\partial y_{ij}}(\rho, \partial_i\rho, \partial^2_{ij}\rho) = \frac{1}{2}\delta_{ij}\rho^{-1}$. Hence,

$$\begin{aligned}
&\rho\frac{\partial F}{\partial y}(\rho, \partial_i\rho, \partial^2_{ij}\rho) - \sum_i \partial_i(\rho\frac{\partial F}{\partial y_i}(\rho, \partial_i\rho, \partial^2_{ij}\rho)) + \sum_{ij}\partial^2_{ij}(\rho\frac{\partial F}{\partial y_{ij}}(\rho, \partial_i\rho, \partial^2_{ij}\rho)) \\
=\ & \frac{1}{2}\rho^{-2}\sum_i(\partial_i\rho)^2 - \frac{1}{2}\rho^{-1}\sum_i\partial^2_{ii}\rho + \frac{1}{2}\sum_i(\partial_i\rho^{-1}\partial_i\rho) \\
=\ & \frac{1}{2}\rho^{-2}\sum_i(\partial_i\rho)^2 - \frac{1}{2}\rho^{-1}\sum_i\partial^2_{ii}\rho + \frac{1}{2}(-\rho^{-2}\sum_i(\partial_i\rho)^2 + \rho^{-1}\sum_i\partial^2_{ii}\rho) \\
=\ & 0.
\end{aligned}$$

Also, one can easily check (3.7).

Next, equation (3.3) in this case becomes

$$\rho\Big((\frac{\partial}{\partial t} + V.\nabla)V + \nabla U + \frac{\hbar^2}{2m}\nabla\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\Big) = 0.$$

One can easily check that the later together with the equation $\frac{\partial\rho}{\partial t} + \nabla.(\rho V) = 0$ when $V = \frac{\nabla S}{m}$ is equivalent with Schrödinger equation under the transformation (3.6). □

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