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Bounds for a class of two-dimensional integrals involving mixed power-type singularities

Chesneau C.

Abstract. This article investigates two-dimensional integrals involving singularities and multiplicative compositions. The main focus is on deriving lower and upper bounds under various assumptions on the primary function, including monotonicity, convexity, and sub-multiplicativity. The results are presented with complete proofs and are intended to motivate further developments in the study of integral inequalities.

Keywords: Two-dimensional integrals, singular integrals, integral inequalities, mixed power-type singularities, bounds
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1. Introduction

Two-dimensional integrals involving singularities frequently arise in the fields of analysis, probability and partial differential equations. The most famous examples are Hilbert-Hardy-type integrals, which are of the following form:

$$\int_0^{+\infty}\int_0^{+\infty} k(x,y)f(x)g(y)dxdy, \tag{1.1}$$

where $k : [0,+\infty)^2 \to [0,+\infty)$ is a kernel function, and $f, g : [0,+\infty) \to [0,+\infty)$ are the primary functions of interest. A key aim of this framework is to derive inequalities that bound the integral in Equation (1.1) from above by a constant times the product of two (possibly weighted) integral norms of f and g. Comprehensive overviews of this topic can be found in [1, 2, 3, 4]. Both classical and modern approaches, including techniques for sharp estimates and extensions to various functional settings, are discussed in [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

In this article, we focus on integrals of the following specific form:

$$\int_0^{\beta}\int_0^{\alpha} \frac{1}{x^p + y^q} f(xy)dxdy, \tag{1.2}$$

where $\alpha, \beta > 0$ are fixed, $p, q > 0$, and $f : [0, \alpha\beta] \to [0,+\infty)$ is the primary function of interest. The integrand in Equation (1.2) exhibits a mixed singularity structure, governed by the interaction between the terms x^p and y^q, combined with a nonlinear dependence on the product xy through the function f. This article aims to establish effective lower and upper bounds for the integral in Equation (1.2) under various assumptions on f, such as monotonicity, convexity, and sub-multiplicativity. Thus, we analyze how the interplay between the singular kernel $k(x,y) = 1/(x^p + y^q)$ and the composite function $f(xy)$ influences the behavior of the integral. These results have applications in potential theory, weighted integral inequalities, and probabilistic models involving joint multiplicative structures. All findings are presented with complete proofs and are intended to stimulate further research in the field of integral inequalities.

The rest of the article is organized as follows: Section 2 presents the lower bounds under the monotonicity assumption on f. Section 3 is devoted to the upper bounds under various assumptions on f. Section 4 provides a conclusion.

2. Lower bounds

A lower bound for the integral in Equation (1.2) under the monotonicity assumption on f is given below.

Proposition 2.1. *Let $p, q, \alpha, \beta > 0$ and $f : [0, \alpha\beta] \to [0, +\infty)$ be a monotonic function with $f(0) < +\infty$ and $f(\alpha\beta) < +\infty$. Then we have*

$$\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy)dxdy \geq \frac{\alpha\beta}{\alpha^p + \beta^q} \min(f(\alpha\beta), f(0)).$$

Proof: For any $x \in [0, \alpha]$ and $y \in [0, \beta]$, we have $x^p + y^q \leq \alpha^p + \beta^q$ and, since f is monotonic, i.e., non-decreasing or non-increasing,

$$f(xy) \geq \min(f(\alpha\beta), f(0)).$$

These inequalities imply that

$$\begin{aligned}
&\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy)dxdy \geq \int_0^\beta \int_0^\alpha \frac{1}{\alpha^p + \beta^q} \min(f(\alpha\beta), f(0))dxdy \\
&= \frac{1}{\alpha^p + \beta^q} \min(f(\alpha\beta), f(0)) \left(\int_0^\alpha dx\right) \left(\int_0^\beta dy\right) \\
&= \frac{\alpha\beta}{\alpha^p + \beta^q} \min(f(\alpha\beta), f(0)).
\end{aligned}$$

This completes the proof of the proposition. □

It is worth noting that if the double integral diverges to $+\infty$, the inequality holds trivially.

The proposition below establishes a more refined lower bound. Note that it is assumed that $p, q \geq 1$.

Proposition 2.2. *Let $p, q \geq 1$, $\alpha, \beta > 0$ and $f : [0, \alpha\beta] \to [0, +\infty)$ be a monotonic function with $f(0) < +\infty$ and $f(\alpha\beta) < +\infty$. Then we have*

$$\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy)dxdy \geq \min(f(\alpha\beta), f(0)) \max\left(\frac{\alpha}{\beta^{q-1}} \log\left(1 + \frac{\beta^q}{\alpha^p}\right), \frac{\beta}{\alpha^{p-1}} \log\left(1 + \frac{\alpha^p}{\beta^q}\right)\right).$$

Proof: For any $x \in [0, \alpha]$ and $y \in [0, \beta]$, since $p \geq 1$, we have $x^p + y^q \leq \alpha^{p-1}x + \beta^q$ and, since f is monotonic,

$$f(xy) \geq \min(f(\alpha\beta), f(0)).$$

These inequalities imply that

$$\begin{aligned}
&\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy)dxdy \geq \int_0^\beta \int_0^\alpha \frac{1}{\alpha^{p-1}x + \beta^q} \min(f(\alpha\beta), f(0))dxdy \\
&= \min(f(\alpha\beta), f(0)) \left(\int_0^\beta dy\right) \left[\int_0^\alpha \frac{1}{\alpha^{p-1}x + \beta^q} dx\right] \\
&= \min(f(\alpha\beta), f(0))\beta \left[\frac{1}{\alpha^{p-1}} \log(\alpha^{p-1}x + \beta^q)\right]_{x=0}^{x=\alpha} \\
&= \min(f(\alpha\beta), f(0)) \frac{\beta}{\alpha^{p-1}} \log\left(1 + \frac{\alpha^p}{\beta^q}\right). \qquad (2.1)
\end{aligned}$$

Proceeding in a similar way, for any $x \in [0, \alpha]$ and $y \in [0, \beta]$, since $q \geq 1$, we have $x^p + y^q \leq \alpha^p + \beta^{q-1}y$ and, since f is monotonic,

$$f(xy) \geq \min(f(\alpha\beta), f(0)).$$

These inequalities imply that

$$\begin{aligned}
&\int_0^\beta \int_0^\alpha \frac{1}{x^p+y^q} f(xy)dxdy \geq \int_0^\beta \int_0^\alpha \frac{1}{\alpha^p+\beta^{q-1}y} \min(f(\alpha\beta), f(0))dxdy \\
&= \min(f(\alpha\beta), f(0)) \left(\int_0^\alpha dx\right)\left[\int_0^\beta \frac{1}{\alpha^p+\beta^{q-1}y}dy\right] \\
&= \min(f(\alpha\beta), f(0))\alpha \left[\frac{1}{\beta^{q-1}}\log(\alpha^p+\beta^{q-1}y)\right]_{y=0}^{y=\beta} \\
&= \min(f(\alpha\beta), f(0))\frac{\alpha}{\beta^{q-1}}\log\left(1+\frac{\beta^q}{\alpha^p}\right). \qquad (2.2)
\end{aligned}$$

It follows from Equations (2.1) and (2.2) that

$$\int_0^\beta \int_0^\alpha \frac{1}{x^p+y^q} f(xy)dxdy \geq \min(f(\alpha\beta), f(0)) \max\left(\frac{\alpha}{\beta^{q-1}}\log\left(1+\frac{\beta^q}{\alpha^p}\right), \frac{\beta}{\alpha^{p-1}}\log\left(1+\frac{\alpha^p}{\beta^q}\right)\right).$$

This ends the proof of the proposition. □

Using the logarithmic inequality $\log(1+x) \geq x/(1+x)$ with $x > -1$, we have

$$\begin{aligned}
&\int_0^\beta \int_0^\alpha \frac{1}{x^p+y^q} f(xy)dxdy \\
&\geq \min(f(\alpha\beta), f(0)) \max\left(\frac{\alpha}{\beta^{q-1}}\log\left(1+\frac{\beta^q}{\alpha^p}\right), \frac{\beta}{\alpha^{p-1}}\log\left(1+\frac{\alpha^p}{\beta^q}\right)\right) \\
&\geq \min(f(\alpha\beta), f(0)) \max\left(\frac{\alpha}{\beta^{q-1}} \times \frac{\beta^q}{\alpha^p} \times \frac{1}{1+\beta^q/\alpha^p}, \frac{\beta}{\alpha^{p-1}} \times \frac{\alpha^p}{\beta^q} \times \frac{1}{1+\alpha^p/\beta^q}\right) \\
&= \frac{\alpha\beta}{\alpha^p+\beta^q}\min(f(\alpha\beta), f(0)),
\end{aligned}$$

so that Proposition 2.2 implies Proposition 2.1 for the case $p, q \geq 1$.

3. UPPER BOUNDS

3.1. **Under the monotonicity assumption.** An upper bound for the integral in Equation (1.2) under the monotonicity assumption on f is presented below. Note that it is assumed that $p, q \in (0, 2)$.

Proposition 3.1. *Let* $p, q \in (0, 2)$, $\alpha, \beta > 0$ *and* $f : [0, \alpha\beta] \to [0, +\infty)$ *be a monotonic function with* $f(0) < +\infty$ *and* $f(\alpha\beta) < +\infty$. *Then we have*

$$\int_0^\beta \int_0^\alpha \frac{1}{x^p+y^q} f(xy)dxdy \leq 2\max(f(\alpha\beta), f(0))\frac{\alpha^{1-p/2}\beta^{1-q/2}}{(2-p)(2-q)}.$$

Proof: Using the basic inequality $a^2 + b^2 \geq 2ab$ with $a, b \geq 0$, for any $x \in [0, \alpha]$ and $y \in [0, \beta]$, we have

$$x^p + y^q \geq 2x^{p/2}y^{q/2},$$

and, since f is monotonic,

$$f(xy) \leq \max(f(\alpha\beta), f(0)).$$

These inequalities and standard power primitives together with $p, q \in (0, 2)$ give

$$\begin{aligned}
&\int_0^\beta \int_0^\alpha \frac{1}{x^p+y^q} f(xy)dxdy \leq \int_0^\beta \int_0^\alpha \frac{1}{2x^{p/2}y^{q/2}} \max(f(\alpha\beta), f(0))dxdy \\
&= \frac{1}{2}\max(f(\alpha\beta), f(0))\left(\int_0^\alpha \frac{1}{x^{p/2}}dx\right)\left(\int_0^\beta \frac{1}{y^{q/2}}dy\right) \\
&= \frac{1}{2}\max(f(\alpha\beta), f(0))\frac{2}{2-p}\alpha^{1-p/2} \times \frac{2}{2-q}\beta^{1-q/2} \\
&= 2\max(f(\alpha\beta), f(0))\frac{\alpha^{1-p/2}\beta^{1-q/2}}{(2-p)(2-q)}.
\end{aligned}$$

This concludes the proof of the proposition. □

Proposition 3.1 also holds if f is bounded from above, by replacing $\max(f(\alpha\beta), f(0))$ with this upper bound.

The proposition below gives a more refined upper bound under the non-decreasingness assumption on f.

Proposition 3.2. *Let $p, q, \alpha, \beta > 0$ and $f : [0, \alpha\beta] \to [0, +\infty)$ be a non-decreasing function. Then we have*

$$\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy) dx dy \leq \min\left(\alpha^q \int_0^{\beta\alpha} \frac{1}{x^q} f(x) dx, \beta^p \int_0^{\beta\alpha} \frac{1}{x^p} f(x) dx\right),$$

provided that the integrals in the upper bound converge.

Proof: For any $x \in [0, \alpha]$ and $y \in [0, \beta]$, we have $x^p + y^q \geq x^p$ and, since f is non-decreasing,

$$f(xy) \leq f(\beta x).$$

These inequalities and the change of variables $u = \beta x$ give

$$\begin{aligned} &\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy) dx dy \leq \int_0^\beta \int_0^\alpha \frac{1}{x^p} f(\beta x) dx dy = \left(\int_0^\beta dy\right) \left[\int_0^\alpha \frac{1}{x^p} f(\beta x) dx\right] \\ &= \beta \int_0^\alpha \frac{1}{x^p} f(\beta x) dx = \beta^p \int_0^{\beta\alpha} \frac{1}{u^p} f(u) du. \end{aligned} \tag{3.1}$$

Proceeding in a similar way, for any $x \in [0, \alpha]$ and $y \in [0, \beta]$, we have $x^p + y^q \geq y^q$ and, since f is non-decreasing,

$$f(xy) \leq f(\alpha y).$$

These inequalities and the change of variables $v = \alpha y$ gives

$$\begin{aligned} &\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy) dx dy \leq \int_0^\beta \int_0^\alpha \frac{1}{y^q} f(\alpha y) dx dy = \left(\int_0^\alpha dx\right) \left[\int_0^\beta \frac{1}{y^q} f(\alpha y) dy\right] \\ &= \alpha \int_0^\beta \frac{1}{y^q} f(\alpha y) dy = \alpha^q \int_0^{\beta\alpha} \frac{1}{v^q} f(v) dv. \end{aligned} \tag{3.2}$$

Combining Equations (3.1) and (3.2) and uniformizing the notation, we get

$$\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy) dx dy \leq \min\left(\alpha^q \int_0^{\beta\alpha} \frac{1}{x^q} f(x) dx, \beta^p \int_0^{\beta\alpha} \frac{1}{x^p} f(x) dx\right).$$

This completes the proof of the proposition. □

3.2. **Under the convexity assumption.** The proposition below presents an upper bound for the integral in Equation (1.2) under the convexity assumption on f, i.e., for any $x \in [0, \alpha]$, $y \in [0, \beta]$ and $\lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$.

Proposition 3.3. *Let $p > 1$, $q = p/(p - 1)$, $\alpha, \beta > 0$ and $f : [0, \alpha\beta] \to [0, +\infty)$ be a convex function with $f(0) = 0$ and $f(\alpha\beta) < +\infty$. Then we have*

$$\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy) dx dy \leq \max\left(\frac{1}{p}, \frac{1}{q}\right) f(\alpha\beta).$$

Proof: Using the convexity of f and $f(0)=0$, for any $x\in[0,\alpha]$ and $y\in[0,\beta]$, we have

$$f(xy)=f\left(\frac{xy}{\alpha\beta}\alpha\beta+\left(1-\frac{xy}{\alpha\beta}\right)\times 0\right)\leq\frac{xy}{\alpha\beta}f(\alpha\beta)+\left(1-\frac{xy}{\alpha\beta}\right)f(0)=\frac{xy}{\alpha\beta}f(\alpha\beta).$$

Moreover, for any $x\in[0,\alpha]$ and $y\in[0,\beta]$, the Young product inequality gives

$$xy\leq\frac{1}{p}x^p+\frac{1}{q}y^q\leq\max\left(\frac{1}{p},\frac{1}{q}\right)(x^p+y^q).$$

These inequalities imply that

$$\begin{aligned}&\int_0^\beta\int_0^\alpha\frac{1}{x^p+y^q}f(xy)dxdy\leq\int_0^\beta\int_0^\alpha\frac{1}{x^p+y^q}\times\frac{xy}{\alpha\beta}f(\alpha\beta)dxdy\\&=\frac{1}{\alpha\beta}f(\alpha\beta)\int_0^\beta\int_0^\alpha\frac{xy}{x^p+y^q}dxdy\\&\leq\frac{1}{\alpha\beta}f(\alpha\beta)\int_0^\beta\int_0^\alpha\max\left(\frac{1}{p},\frac{1}{q}\right)\frac{x^p+y^q}{x^p+y^q}dxdy\\&=\frac{1}{\alpha\beta}f(\alpha\beta)\max\left(\frac{1}{p},\frac{1}{q}\right)\left(\int_0^\alpha dx\right)\left(\int_0^\beta dy\right)\\&=\frac{1}{\alpha\beta}f(\alpha\beta)\max\left(\frac{1}{p},\frac{1}{q}\right)\alpha\beta=\max\left(\frac{1}{p},\frac{1}{q}\right)f(\alpha\beta).\end{aligned}$$

This ends the proof of the proposition. □

Note that we have

$$\max\left(\frac{1}{p},\frac{1}{q}\right)=\max\left(\frac{1}{p},\frac{p-1}{p}\right)=\begin{cases}\dfrac{p-1}{p}, & \text{if } p\geq 2,\\ \dfrac{1}{p}, & \text{if } p\in(0,2).\end{cases}$$

A more technical upper bound under the decreasingness assumption on f and convexity assumption on $\log(f)$ is given below.

Proposition 3.4. *Let $p>1$, $q=p/(p-1)$, $\alpha,\beta>0$ and $f:[0,\max(\alpha\beta,\alpha^p,\beta^q)]\to[0,+\infty)$ be a non-decreasing function such that $\log(f)$ is convex. Then we have*

$$\begin{aligned}&\int_0^\beta\int_0^\alpha\frac{1}{x^p+y^q}f(xy)dxdy\\&\leq\frac{2}{pq}\sqrt{\int_0^{\alpha^p}x^{2(1/p-1)}\arctan\left[\frac{\beta^{q/2}}{\sqrt{x}}\right]f^{2/p}(x)dx}\sqrt{\int_0^{\beta^q}x^{2(1/q-1)}\arctan\left[\frac{\alpha^{p/2}}{\sqrt{x}}\right]f^{2/q}(x)dx},\end{aligned}$$

provided that the integrals in the upper bound converge.

Proof: First of all, for any $x\in[0,\alpha]$ and $y\in[0,\beta]$, the Young product inequality gives

$$xy\leq\frac{1}{p}x^p+\frac{1}{q}y^q.$$

Using this inequality, the facts that f is non-decreasing, that $\log(f)$ is convex with $1/p+1/q=1$ and that the exponential function is non-decreasing, we have

$$\begin{aligned}f(xy)&\leq f\left(\frac{1}{p}x^p+\frac{1}{q}y^q\right)=e^{\log f(x^p/p+y^q/q)}\\&\leq e^{(1/p)\log f(x^p)+(1/q)\log f(y^q)}=e^{\log[f^{1/p}(x^p)f^{1/q}(y^q)]}=f^{1/p}(x^p)f^{1/q}(y^q).\end{aligned}$$

Using this inequality and the changes of variables $u = x^p$ and $v = y^q$, we obtain

$$\begin{aligned}
&\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy) dx dy \\
&\leq \int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f^{1/p}(x^p) f^{1/q}(y^q) dx dy \\
&= \frac{1}{pq} \int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{u+v} \left[u^{1/p-1} f^{1/p}(u)\right] \left[v^{1/q-1} f^{1/q}(v)\right] du dv \\
&= \frac{1}{pq} \int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{u+v} f_\dagger(u) f_\ddagger(v) du dv, \qquad (3.3)
\end{aligned}$$

where

$$f_\dagger(u) = u^{1/p-1} f^{1/p}(u), \quad f_\ddagger(v) = v^{1/q-1} f^{1/q}(v).$$

Let us now bound this last double integral term. By a suitable decomposition of the integrand and the Cauchy-Schwarz integral inequality, we have

$$\int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{u+v} f_\dagger(u) f_\ddagger(v) du dv = \int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{\sqrt{u+v}} \left(\frac{u}{v}\right)^{1/4} f_\dagger(u) \times \frac{1}{\sqrt{u+v}} \left(\frac{v}{u}\right)^{1/4} f_\ddagger(v) du dv \leq \sqrt{A}\sqrt{B}, \qquad (3.4)$$

where

$$A = \int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{u+v} \sqrt{\frac{u}{v}} f_\dagger^2(u) du dv$$

and

$$B = \int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{u+v} \sqrt{\frac{v}{u}} f_\ddagger^2(v) du dv.$$

Let us now express A and B.

Applying the Fubini-Tonelli integral theorem and using a standard arctangent primitive, we have

$$\begin{aligned}
A &= \int_0^{\alpha^p} \sqrt{u} f_\dagger^2(u) \left[\int_0^{\beta^q} \frac{1}{\sqrt{v}(u+v)} dv\right] du \\
&= \int_0^{\alpha^p} \sqrt{u} f_\dagger^2(u) \left[\frac{2}{\sqrt{u}} \arctan\left[\sqrt{\frac{v}{u}}\right]\right]_{v=0}^{v=\beta^q} du \\
&= 2\int_0^{\alpha^p} f_\dagger^2(u) \arctan\left[\frac{\beta^{q/2}}{\sqrt{u}}\right] du. \qquad (3.5)
\end{aligned}$$

For B, we proceed in a similar way. Applying the Fubini-Tonelli integral theorem and using a standard arctangent primitive, we have

$$\begin{aligned}
B &= \int_0^{\beta^q} \sqrt{v} f_\ddagger^2(v) \left[\int_0^{\alpha^p} \frac{1}{\sqrt{u}(u+v)} du\right] dv \\
&= \int_0^{\beta^q} \sqrt{v} f_\ddagger^2(v) \left[\frac{2}{\sqrt{v}} \arctan\left[\sqrt{\frac{u}{v}}\right]\right]_{u=0}^{u=\alpha^p} dv \\
&= 2\int_0^{\beta^q} f_\ddagger^2(v) \arctan\left[\frac{\alpha^{p/2}}{\sqrt{v}}\right] dv. \qquad (3.6)
\end{aligned}$$

Combining Equations (3.3), (3.4), (3.5) and (3.6), and using the definitions of $f_\dagger$ and $f_\ddagger$, and uniformizing the notation, we get

$$\begin{aligned}&\int_0^\beta\int_0^\alpha \frac{1}{x^p+y^q}f(xy)dxdy\\ &\leq \frac{1}{pq}\sqrt{2\int_0^{\alpha^p} f_\dagger^2(u)\arctan\left[\frac{\beta^{q/2}}{\sqrt{u}}\right]du}\sqrt{2\int_0^{\beta^q} f_\ddagger^2(v)\arctan\left[\frac{\alpha^{p/2}}{\sqrt{v}}\right]dv}\\ &=\frac{2}{pq}\sqrt{\int_0^{\alpha^p} x^{2(1/p-1)}\arctan\left[\frac{\beta^{q/2}}{\sqrt{x}}\right]f^{2/p}(x)dx}\sqrt{\int_0^{\beta^q} x^{2(1/q-1)}\arctan\left[\frac{\alpha^{p/2}}{\sqrt{x}}\right]f^{2/q}(x)dx.\end{aligned}$$

This concludes the proof of the proposition. □

Note that we have

$$\frac{2}{pq}=\frac{2(p-1)}{p^2}.$$

Using the arctangent inequality $\arctan(x)\leq \pi/2$ with $x\in\mathbb{R}$, we have

$$\int_0^\beta\int_0^\alpha \frac{1}{x^p+y^q}f(xy)dxdy\frac{\pi}{pq}\sqrt{\int_0^{\alpha^p} x^{2(1/p-1)}f^{2/p}(x)dx}\sqrt{\int_0^{\beta^q} x^{2(1/q-1)}f^{2/q}(x)dx.}$$

Using the arctangent inequality $\arctan(x)\leq x$ with $x\geq 0$, we also have

$$\int_0^\beta\int_0^\alpha \frac{1}{x^p+y^q}f(xy)dxdy\leq \frac{2}{pq}\alpha^{p/4}\beta^{q/4}\sqrt{\int_0^{\alpha^p} x^{2/p-5/2}f^{2/p}(x)dx}\sqrt{\int_0^{\beta^q} x^{2/q-5/2}f^{2/q}(x)dx.}$$

3.3. **Under the sub-multiplicativity assumption.** The proposition below presents an upper bound for the integral in Equation (1.2) under the sub-multiplicativity assumption on f, i.e., for any $x\in[0,\alpha]$ and $y\in[0,\beta]$, we have $f(xy)\leq f(x)f(y)$.

Proposition 3.5. *Let $p,q,\alpha,\beta>0$ and $f:[0,\alpha\beta]\to[0,+\infty)$ be a sub-multiplicative function. Then we have*

$$\int_0^\beta\int_0^\alpha \frac{1}{x^p+y^q}f(xy)dxdy\leq\frac{1}{2}\left[\int_0^\alpha\frac{1}{x^{p/2}}f(x)dx\right]\left[\int_0^\beta\frac{1}{x^{q/2}}f(x)dx\right],$$

provided that the integrals in the upper bound converge.

Proof: Using the basic inequality $a^2+b^2\geq 2ab$ with $a,b\geq 0$, for any $x\in[0,\alpha]$ and $y\in[0,\beta]$, we have

$$x^p+y^q\geq 2x^{p/2}y^{q/2}.$$

Using this inequality and the fact that f is sub-multiplicative, and uniformizing the notation, we obtain

$$\begin{aligned}&\int_0^\beta\int_0^\alpha \frac{1}{x^p+y^q}f(xy)dxdy\leq\int_0^\beta\int_0^\alpha\frac{1}{2x^{p/2}y^{q/2}}f(x)f(y)dxdy\\ &=\frac{1}{2}\left[\int_0^\alpha\frac{1}{x^{p/2}}f(x)dx\right]\left[\int_0^\beta\frac{1}{y^{q/2}}f(y)dy\right]=\frac{1}{2}\left[\int_0^\alpha\frac{1}{x^{p/2}}f(x)dx\right]\left[\int_0^\beta\frac{1}{x^{q/2}}f(x)dx\right].\end{aligned}$$

This completes the proof of the proposition. □

A more technical upper bound under the sub-multiplicativity assumption on f is developed below.

Proposition 3.6. *Let $p, q > 1$, $\alpha, \beta > 0$ and $f : [0, \alpha\beta] \to [0, +\infty)$ be a sub-multiplicative function. Then we have*

$$\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy)dxdy \leq \frac{2}{\sqrt{p}\sqrt{q}} \sqrt{\int_0^\alpha x^{1-p} \arctan\left[\frac{\beta^{q/2}}{x^{p/2}}\right] f^2(x)dx} \sqrt{\int_0^\beta x^{1-q} \arctan\left[\frac{\alpha^{p/2}}{x^{q/2}}\right] f^2(x)dx},$$

provided that the integrals in the upper bound converge.

Proof: Using the sub-multiplicativity of f and the changes of variables $u = x^p$ and $v = y^q$, we obtain

$$\begin{aligned}
&\int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(xy)dxdy \\
&\leq \int_0^\beta \int_0^\alpha \frac{1}{x^p + y^q} f(x)f(y)dxdy \\
&= \frac{1}{pq} \int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{u+v} \left[u^{1/p-1} f(u^{1/p})\right] \left[v^{1/q-1} f(v^{1/q})\right] dudv \\
&= \frac{1}{pq} \int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{u+v} f_*(u) f_{**}(v) dudv, \qquad (3.7)
\end{aligned}$$

where

$$f_*(u) = u^{1/p-1} f(u^{1/p}), \quad f_{**}(v) = v^{1/q-1} f(v^{1/q}).$$

Let us now bound this last double integral term, proceeding similarly to the proof of Proposition 3.4. By a suitable decomposition of the integrand and the Cauchy-Schwarz integral inequality, we have

$$\begin{aligned}
&\int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{u+v} f_*(u) f_{**}(v) dudv \\
&= \int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{\sqrt{u+v}} \left(\frac{u}{v}\right)^{1/4} f_*(u) \times \frac{1}{\sqrt{u+v}} \left(\frac{v}{u}\right)^{1/4} f_{**}(v) dudv \\
&\leq \sqrt{C}\sqrt{D}, \qquad (3.8)
\end{aligned}$$

where

$$C = \int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{u+v} \sqrt{\frac{u}{v}} f_*^2(u) dudv$$

and

$$D = \int_0^{\beta^q} \int_0^{\alpha^p} \frac{1}{u+v} \sqrt{\frac{v}{u}} f_{**}^2(v) dudv.$$

Let us now express C and D.

Applying the Fubini-Tonelli integral theorem and using a standard arctangent primitive, we have

$$\begin{aligned}
C &= \int_0^{\alpha^p} \sqrt{u} f_*^2(u) \left[\int_0^{\beta^q} \frac{1}{\sqrt{v}(u+v)} dv\right] du \\
&= \int_0^{\alpha^p} \sqrt{u} f_*^2(u) \left[\frac{2}{\sqrt{u}} \arctan\left[\sqrt{\frac{v}{u}}\right]\right]_{v=0}^{v=\beta^q} du \\
&= 2 \int_0^{\alpha^p} f_*^2(u) \arctan\left[\frac{\beta^{q/2}}{\sqrt{u}}\right] du. \qquad (3.9)
\end{aligned}$$

For D, we proceed in a similar way. Applying the Fubini-Tonelli integral theorem and using a standard arctangent primitive, we have

$$\begin{aligned} D &= \int_0^{\beta^q} \sqrt{v} f_{**}^2(v) \left[\int_0^{\alpha^p} \frac{1}{\sqrt{u}(u+v)} du\right] dv \\ &= \int_0^{\beta^q} \sqrt{v} f_{**}^2(v) \left[\frac{2}{\sqrt{v}} \arctan\left[\sqrt{\frac{u}{v}}\right]\right]_{u=0}^{u=\alpha^p} dv \\ &= 2\int_0^{\beta^q} f_{**}^2(v) \arctan\left[\frac{\alpha^{p/2}}{\sqrt{v}}\right] dv. \end{aligned} \tag{3.10}$$

Combining Equations (3.7), (3.8), (3.9) and (3.10), and using the definitions of f_* and f_{**} and the changes of variables $u = x^p$ and $v = x^q$, we get

$$\begin{aligned} &\int_0^{\beta} \int_0^{\alpha} \frac{1}{x^p + y^q} f(xy) dx dy \\ &\leq \frac{1}{pq} \sqrt{2\int_0^{\alpha^p} f_*^2(u) \arctan\left[\frac{\beta^{q/2}}{\sqrt{u}}\right] du} \sqrt{2\int_0^{\beta^q} f_{**}^2(v) \arctan\left[\frac{\alpha^{p/2}}{\sqrt{v}}\right] dv} \\ &= \frac{2}{pq} \sqrt{\int_0^{\alpha^p} u^{2(1/p-1)} \arctan\left[\frac{\beta^{q/2}}{\sqrt{u}}\right] f^2(u^{1/p}) du} \\ &\times \sqrt{\int_0^{\beta^q} v^{2(1/q-1)} \arctan\left[\frac{\alpha^{p/2}}{\sqrt{v}}\right] f^2(v^{1/q}) dv} \\ &= \frac{2}{\sqrt{p}\sqrt{q}} \sqrt{\int_0^{\alpha} x^{1-p} \arctan\left[\frac{\beta^{q/2}}{x^{p/2}}\right] f^2(x) dx} \\ &\times \sqrt{\int_0^{\beta} x^{1-q} \arctan\left[\frac{\alpha^{p/2}}{x^{q/2}}\right] f^2(x) dx}. \end{aligned}$$

This ends the proof of the proposition. □

Note that we have

$$\frac{2}{\sqrt{p}\sqrt{q}} = \frac{2\sqrt{p-1}}{p}.$$

Using the arctangent inequality $\arctan(x) \leq \pi/2$ with $x \in \mathbb{R}$, we have

$$\begin{aligned} &\int_0^{\beta} \int_0^{\alpha} \frac{1}{x^p + y^q} f(xy) dx dy \\ &\leq \frac{\pi}{\sqrt{p}\sqrt{q}} \sqrt{\int_0^{\alpha} x^{1-p} f^2(x) dx} \sqrt{\int_0^{\beta} x^{1-q} f^2(x) dx}. \end{aligned}$$

Using the arctangent inequality $\arctan(x) \leq x$ with $x \geq 0$, we also have

$$\begin{aligned} &\int_0^{\beta} \int_0^{\alpha} \frac{1}{x^p + y^q} f(xy) dx dy \\ &\leq \frac{2}{\sqrt{p}\sqrt{q}} \alpha^{p/4} \beta^{q/4} \sqrt{\int_0^{\alpha} x^{1-3p/2} f^2(x) dx} \sqrt{\int_0^{\beta} x^{1-3q/2} f^2(x) dx}. \end{aligned}$$

4. CONCLUSION

This article provides a detailed analysis of a class of two-dimensional integrals involving mixed singularities and multiplicative compositions. By establishing precise lower and upper bounds on the integrand function under various assumptions, new insights into the behavior of such integrals are revealed. The methods and results presented here could be applied to related areas of analysis, probability, and mathematical physics. Further generalizations, including multidimensional extensions and alternative singular integrands, could be explored in future research.

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Chesneau C.,
Department of Mathematics, LMNO, University of Caen-
Normandie, Caen, France.
e-mail: christophe.chesneau@gmail.com