

## Inverse problem for a system of mixed-type second-order partial differential equations

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**Abstract.** This paper investigates an inverse problem for a system of mixed-type second-order differential equations with a variable time direction. The problem involves finding the unknown right-hand side functions from the solution of the system under given boundary and initial conditions. The method of separation of variables is applied to reduce the problem to a spectral problem, in which eigenvalues and eigenfunctions are determined. The obtained spectral problem is shown to coincide with previously established spectral problems. The orthogonality of the eigenfunctions is used to derive explicit formulas for the coefficients. The existence and uniqueness of the solution are established using the properties of the eigenfunction system and the Hilbert-Schmidt theorem.

**Keywords:** Inverse problem, mixed-type differential equations, separation of variables, spectral problem, eigenvalues, eigenfunctions, orthogonality, Hilbert-Schmidt theorem

**MSC (2020):** 35R30

### 1. INTRODUCTION

Boundary value problems for parabolic equations with variable time direction have been extensively investigated (see, for example, [1]-[2]). Recently, the complexity of formulating inverse problems for these equations has grown, driven by the need to model and control thermal physics and continuum mechanics processes.

The first works on parabolic equations with a changing time direction were by M. Gevrey [1]. Boundary-value problems for such equations and systems were explored by H. Levine [3], K. S. Fayazov, Y. K. Khudayberganov [4], K. S. Fayazov, Y. K. Khudayberganov and S. Pyatkov [5], I. O. Khajiev and Y. K. Khudayberganov [6], K. S. Fayazov, I. O. Khajiev [7], among others. Inverse problems for parabolic equations have been studied quite well by A. I. Kozhanov [8], I. A. Kaliev, M. F. Mugafarov, O. V. Fattahova [2], A. Bubnov [9] and others.

This paper investigates a more general system of equations, with the method used applicable to other systems. Although much progress has been made in inverse problems for parabolic equations [8]-[9], this work's unique contribution is its focus on inverse problems with a time direction that varies.

These equations have widespread applications, including heat transfer in inhomogeneous media, filtration flow interactions, mass transfer near aircraft surfaces, and complex viscous fluid flow. They also have relevance in transport theory, electron scattering [10, 11], plasma theory [12], hydrodynamics (filtration and boundary layer theory) [13, 14, 15], kinetic theory, stochastic processes [16], and diffusion processes [17, 18].

Thus, these problems are important in mathematical modeling, making the task highly relevant as both an applied and mathematical physics problem.

### 2. STATEMENT OF THE PROBLEM

Let  $\Omega = \Omega_- \cup \Omega_+$ , where  $\Omega_- = (-1, 0) \times (0, T)$ ,  $\Omega_+ = (0, 1) \times (0, T)$ .  $u(x, t)$ ,  $v(x, t)$  are solutions in the domain  $\Omega$  of the system of equations

$$\begin{cases} u_t(x, t) - \text{sign}(x)u_{xx}(x, t) + a_1u(x, t) + b_1v(x, t) = f(x), \\ v_t(x, t) - \text{sign}(x)v_{xx}(x, t) + a_2v(x, t) + b_2u(x, t) = g(x). \end{cases} \quad (2.1)$$

Find functions  $v(x, t)$ ,  $u(x, t)$  and  $f(x)$ ,  $g(x)$  satisfying the system of equations (2.1), boundary

$$u_x(\pm 1, t) = 0, \quad v_x(\pm 1, t) = 0, \quad 0 \leq t \leq T \quad (2.2)$$

initial

$$\begin{aligned} u(x, 0) = \varphi(x) &= \begin{cases} \varphi_1(x), & -1 \leq x \leq 0, \\ \varphi_2(x), & 0 \leq x \leq 1, \end{cases} \\ v(x, 0) = \phi(x) &= \begin{cases} \phi_1(x), & -1 \leq x \leq 0, \\ \phi_2(x), & 0 \leq x \leq 1, \end{cases} \end{aligned} \quad (2.3)$$

final

$$\begin{aligned} u(x, T) = \psi(x) &= \begin{cases} \psi_1(x), & -1 \leq x \leq 0, \\ \psi_2(x), & 0 \leq x \leq 1, \end{cases} \\ v(x, T) = \gamma(x) &= \begin{cases} \gamma_1(x), & -1 \leq x \leq 0, \\ \gamma_2(x), & 0 \leq x \leq 1, \end{cases} \end{aligned} \quad (2.4)$$

and gluing

$$\begin{aligned} u(-0, t) = u(+0, t), \quad u_x(-0, t) = -u_x(+0, t), \\ v(-0, t) = v(+0, t), \quad v_x(-0, t) = -v_x(+0, t), \end{aligned} \quad 0 \leq t \leq T. \quad (2.5)$$

conditions, where  $\varphi(x), \psi(x), \phi(x), \gamma(x)$  are sufficiently smooth given functions,  $a_i$  and  $b_i$  are any constants,  $b_2 \neq 0$ ,  $(a_1 - a_2)^2 + 4b_1b_2 > 0$ .

In this formulation,  $f(x), g(x)$  are the unknown right-hand side of the system (2.1). The boundary conditions of the second kind are conditions (2.2) for the corresponding parabolic equations in the domains  $\Omega_-$  and  $\Omega_+$ . The initial conditions are specified using the functions  $\varphi_2(x), \psi_1(x), \phi_2(x)$ , and  $\gamma_1(x)$  the final redefinition conditions necessary to find the functions  $f(x), g(x)$  are specified using the functions  $\varphi_1(x), \psi_2(x), \phi_1(x)$ , and  $\gamma_2(x)$ .

### 3. AUXILIARY STATEMENTS

In order to study problem (2.1) - (2.5) we make the following substitution

$$\begin{aligned} u(x, t) &= \frac{a_1 + k_2}{b_2(k_1 - k_2)}\omega_1(x, t) + \frac{a_1 + k_1}{b_2(k_1 - k_2)}\omega_2(x, t), \\ v(x, t) &= \frac{1}{k_1 - k_2}\omega_1(x, t) + \frac{1}{k_1 - k_2}\omega_2(x, t), \end{aligned} \quad (3.1)$$

where  $k_1$  and  $k_2$  are the roots of the quadratic equation.

$$k^2 + (a_1 + a_2)k + a_1a_2 - b_1b_2 = 0.$$

Then we come to the following problems regarding the functions  $\omega_i(x, t), i = 1, 2$ .

**Problem 1.** Find a function  $\omega_i(x, t)$  satisfying the equation

$$\omega_{it}(x, t) - \text{sign}(x)\omega_{ixx}(x, t) - k_i\omega_i(x, t) = \bar{f}_i(x), \quad (3.2)$$

in the domain  $\Omega = \Omega_- \cup \Omega_+$  and the following conditions:

$$\omega_{ix}(\pm 1, t) = 0, \quad 0 \leq t \leq T, \quad (3.3)$$

$$\omega_i(x, 0) = \bar{\varphi}_i(x) = \begin{cases} \bar{\varphi}_{i1}(x), & -1 \leq x \leq 0, \\ \bar{\varphi}_{i2}(x), & 0 \leq x \leq 1, \end{cases} \quad (3.4)$$

$$\omega_i(x, T) = \bar{\psi}_i(x) = \begin{cases} \bar{\psi}_{i1}(x), & -1 \leq x \leq 0, \\ \bar{\psi}_{i2}(x), & 0 \leq x \leq 1, \end{cases} \quad (3.5)$$

$$\omega_i(-0, t) = \omega_i(+0, t), \quad \omega_{ix}(-0, t) = -\omega_{ix}(+0, t), \quad (3.6)$$

where

$$\begin{aligned} \bar{\varphi}_1(x) &= (a_1 + k_1)\phi(x) - b_2\varphi(x), & \bar{\varphi}_2(x) &= b_2\varphi(x) - (a_1 + k_2)\phi(x), \\ \bar{\psi}_1(x) &= (a_1 + k_1)\gamma(x) - b_2\psi(x), & \bar{\psi}_2(x) &= b_2\psi(x) - (a_1 + k_2)\gamma(x), \\ \bar{f}_1(x) &= (a_1 + k_1)g(x) - b_2f(x), & \bar{f}_2(x) &= b_2f(x) - (a_1 + k_2)g(x). \end{aligned}$$

## 4. APPLICATION OF THE METHOD OF SEPARATION OF VARIABLES

The method of studying the inverse problem in this work is based on the exclusion of the function  $\bar{f}_i(x)$  from equation (3.2) using differentiation with respect to time and the transition to the problem for the function

$$\xi_i(x, t) = \omega_{it}(x, t).$$

From equation (3.2) and conditions (3.3) - (3.6) we obtain the following initial boundary value problem. Find the functions  $\xi_i(x, t)$  and  $\bar{f}_i(x)$  satisfying the equation in the domain  $\Omega$

$$\xi_{it}(x, t) - \text{sign}(x)\xi_{ixx}(x, t) - k_i\xi_i(x, t) = 0, \quad (4.1)$$

boundary conditions

$$\xi_{ix}(\pm 1, t) = 0, \quad 0 \leq t \leq T, \quad (4.2)$$

initial and final conditions

$$\xi_i(x, 0) = \bar{f}_i(x) + \text{sign}(x)\bar{\varphi}_i''(x) + k_i\bar{\varphi}_i, \quad x \in (-1, 0) \cup (0, 1), \quad (4.3)$$

$$\xi_i(x, T) = \bar{f}_i(x) + \text{sign}(x)\bar{\psi}_i''(x) + k_i\bar{\psi}_i, \quad x \in (-1, 0) \cup (0, 1), \quad (4.4)$$

gluing conditions

$$\xi_i(-0, t) = \xi_i(+0, t), \quad \xi_{ix}(-0, t) = -\xi_{ix}(+0, t), \quad 0 \leq t \leq T. \quad (4.5)$$

We also use the separable method to find a solution to the (4.1) - (4.5) problem. As a result, we arrive at the following spectral problem.

**Spectral problem.** Find the values of the parameter  $\lambda$  such that the spectral problem

$$\begin{cases} \text{sign}(x)X''(x) - \lambda X(x) = 0, \\ X'(-1) = X'(1) = 0, \\ X(-0) = X(+0), \\ X'(-0) = -X'(+0) \end{cases} \quad (4.6)$$

has a nontrivial solution.

(4.6) the spectral problem has positive  $\lambda_n^+$ ,  $\lambda_n^-$  negative and zero eigenvalues, which are the roots of the transcendental equation  $\tan \sqrt{\lambda} + \tanh \sqrt{\lambda} = 0$  for  $\forall n \in N$ . The eigenfunctions corresponding to each eigenvalue are described in the form

$$X_n^+(x) = \begin{cases} \frac{\cos \sqrt{\lambda_n^+}(1+x)}{\cos \sqrt{\lambda_n^+}}, & x < 0, \\ \frac{\cosh \sqrt{\lambda_n^+}(1-x)}{\cosh \sqrt{\lambda_n^+}}, & x > 0, \end{cases} \quad X_n^-(x) = \begin{cases} \frac{\cosh \sqrt{-\lambda_n^-}(x+1)}{\cosh \sqrt{-\lambda_n^-}}, & x < 0, \\ \frac{\cos \sqrt{-\lambda_n^-}(1-x)}{\cos \sqrt{-\lambda_n^-}}, & x > 0, \end{cases} \quad (4.7)$$

$$X_0(x) = \begin{cases} \frac{1}{\sqrt{2}}, & x < 0, \\ \frac{1}{\sqrt{2}}, & x > 0. \end{cases}$$

From the results of [18], eigenfunctions of the spectral problem (4.7) are normalized in  $L_2(-1; 0) \cup (0; 1)$  space based on the Hilbert-Schmidt theorem and constitute the Riess basis.

We find the function  $T_i(t)$  from the equation

$$T'_i(t) = (\lambda + k_i)T_i(t).$$

Using the above information we get

$$\xi_i(x, 0) = \bar{f}_i(x) + \text{sign}(x)\bar{\varphi}_i''(x) + k_i\bar{\varphi}_i,$$

$$T_{i0}X_0(x) + \sum_{n=1}^{\infty} T_{in}^-(0)X_n^-(x) + \sum_{n=1}^{\infty} T_{in}^+(0)X_n^+(x) = \bar{f}_i(x) + \text{sign}(x)\bar{\varphi}_i''(x) + k_i\bar{\varphi}_i,$$

$$T_{in}^{\pm}(0) = (\bar{f}_i(x) + \text{sign}(x)\bar{\varphi}''_i(x) + k_i\bar{\varphi}_i, X_n^{\pm}(x)) = A_{in}^{\pm}.$$

Using the information at  $t = T$ , we can write

$$\xi_i(x, T) = \bar{f}_i(x) + \text{sign}(x)\bar{\psi}''_i(x) + k_i\bar{\psi}_i,$$

$$T_{i0}X_0(x) + \sum_{n=1}^{\infty} T_n^-(T)X_n^-(x) + \sum_{n=1}^{\infty} T_{in}^+(T)X_n^+(x) = \bar{f}_i(x) + \text{sign}(x)\bar{\psi}''_i(x) + k_i\bar{\psi}_i,$$

$$T_{in}^{\pm}(T) = (\bar{f}_i(x) + \text{sign}(x)\bar{\psi}''_i(x) + k_i\bar{\psi}_i, X_n^{\pm}(x)) = B_{in}^{\pm}.$$

1) If  $\lambda_n^+ > 0$ , then

$$T_{in}^+(t) = C_{in}e^{(\lambda_n^+ + k_i)t},$$

$$T_{in}^+(0) = A_{in}^+,$$

$$T_{in}^+(T) = B_{in}^+,$$

$$T_{in}^+(t) = B_{in}^+e^{-(\lambda_n^+ + k_i)(T-t)}.$$

2) If  $\lambda_n^- < 0$ , then

$$T_{in}^-(t) = C_{in}e^{(\lambda_n^- + k_i)t},$$

$$T_{in}^-(0) = A_{in}^-,$$

$$T_{in}^-(T) = B_{in}^-,$$

$$T_{in}^-(t) = A_{in}^-e^{(\lambda_n^- + k_i)t}.$$

3) If  $\lambda_0 = 0$ , then

$$T_{i0}(t) = C_i e^{k_i t}.$$

The function  $\xi_i(x, t)$  can be presented in the form

$$\xi_i(x, t) = C_i e^{k_i t} X_0(x) + \sum_{n=1}^{\infty} A_{in}^- e^{(\lambda_n^- + k_i)t} X_n^-(x) + \sum_{n=1}^{\infty} B_{in}^+ e^{-(\lambda_n^+ + k_i)(T-t)} X_n^+(x) \quad (4.8)$$

and we need to find the coefficients  $A_{in}^-, B_{in}^+, C_i$ .

To find the coefficients  $A_{in}^-, B_{in}^+, C_i$  we use conditions (4.3) and (4.4)

$$\begin{aligned} \bar{f}_i(x) = \xi_i(x, 0) - \text{sign}(x)\bar{\varphi}''_i(x) - k_i\bar{\varphi}_i = [C_i X_0(x) + \\ \sum_{n=1}^{\infty} A_{in}^- X_n^-(x) + \sum_{n=1}^{\infty} B_{in}^+ e^{-(\lambda_n^+ + k_i)T} X_n^+(x)] - \text{sign}(x)\bar{\varphi}''_i(x) - k_i\bar{\varphi}_i(x), \end{aligned}$$

$$\begin{aligned} \bar{f}_i(x) = \xi_i(x, T) - \text{sign}(x)\bar{\psi}''_i(x) - k_i\bar{\psi}_i = \left[ C_i e^{k_i T} X_0(x) + \sum_{n=1}^{\infty} A_{in}^- e^{(\lambda_n^- + k_i)T} X_n^-(x) \right. \\ \left. + \sum_{n=1}^{\infty} B_{in}^+ X_n^+(x) \right] - \text{sign}(x)\bar{\psi}''_i(x) - k_i\bar{\psi}_i(x). \end{aligned}$$

From the last two equalities we have

$$\begin{aligned} C_i (1 - e^{k_i T}) X_0(x) + \sum_{n=1}^{\infty} A_{in}^- (1 - e^{(\lambda_n^- + k_i)T}) X_n^-(x) + \\ \sum_{n=1}^{\infty} B_{in}^+ (e^{-(\lambda_n^+ + k_i)T} - 1) X_n^+(x) = \text{sign}(x) (\bar{\varphi}''_i(x) - \bar{\psi}''_i(x)) + k_i (\bar{\varphi}_i(x) - \bar{\psi}_i(x)). \end{aligned} \quad (4.9)$$

We scalar multiply the left and right parts of equality (4.9) by  $X_n^-$  to find the coefficient  $A_{in}$ , by  $X_0$  to find the coefficient  $C_i$  and by  $X_n^+$  to find the coefficient  $B_{in}$ . Taking into account the orthogonality of the system of eigenfunctions, we obtain the following formulas:

$$A_{in}^- = \frac{\bar{\varphi}_{in}^{-(2)} - \bar{\psi}_{in}^{-(2)} + k_i(\bar{\varphi}_{in}^- - \bar{\psi}_{in}^-)}{1 - e^{(\lambda_n^- + k_i)T}}, \quad (4.10)$$

$$B_{in}^+ = \frac{\bar{\varphi}_{in}^{+(2)} - \bar{\psi}_{in}^{+(2)} + k_i(\bar{\varphi}_{in}^+ - \bar{\psi}_{in}^+)}{e^{(\lambda_n^+ + k_i)T} - 1}, \quad (4.11)$$

$$C_i = \frac{\bar{\varphi}_{i0}^{(2)} - \bar{\psi}_{i0}^{(2)} + k_i(\bar{\varphi}_{i0} - \bar{\psi}_{i0})}{1 - e^{k_i T}}, \quad (4.12)$$

where  $\bar{\varphi}_{in}^{\pm(2)} = (\text{sign}x \cdot \bar{\varphi}''_i(x), X_n^{\pm}(x))$ ,  $\bar{\psi}_{in}^{\pm(2)} = (\text{sign}x \cdot \bar{\psi}''_i(x), X_n^{\pm}(x))$ ,  $\bar{\varphi}_{i0}^{(2)} = (\text{sign}x \cdot \bar{\varphi}''_i(x), X_0(x))$ ,  $\bar{\psi}_{i0}^{(2)} = (\text{sign}x \cdot \bar{\psi}''_i(x), X_0(x))$ ,  $\bar{\varphi}_{in}^{\pm} = (\bar{\varphi}_i(x), X_n^{\pm}(x))$ ,  $\bar{\psi}_{in}^{\pm} = (\bar{\psi}_i(x), X_n^{\pm}(x))$ ,  $\bar{\varphi}_{i0} = (\bar{\varphi}_i(x), X_0(x))$ ,  $\bar{\psi}_{i0} = (\bar{\psi}_i(x), X_0(x))$ .

## 5. THE THEOREM OF EXISTENCE AND UNIQUENESS OF A SOLUTION TO THE PROBLEM.

**Theorem 5.1.** *If the functions  $\varphi(x), \psi(x), \phi(x), \gamma(x) \in C^4([-1, 0] \cup (0, 1])$ ,  $\varphi'(\pm 1) = 0$ ,  $\psi'(\pm 1) = 0$ ,  $\phi'(\pm 1) = 0$ ,  $\gamma'(\pm 1) = 0$ ,  $\varphi(-0) = \varphi(+0)$ ,  $\psi(-0) = \psi(+0)$ ,  $\phi(-0) = \phi(+0)$ ,  $\gamma(+0) = \gamma(-0)$ ,  $\varphi'(-0) = -\varphi'(+0)$ ,  $\psi'(-0) = -\psi'(+0)$ ,  $\gamma'(+0) = -\gamma'(-0)$ ,  $\varphi'''(\pm 1) = \psi'''(\pm 1)$ ,  $\phi''(-0) = -\phi''(+0)$ ,  $\varphi''(-0) = -\varphi''(+0)$ ,  $\gamma''(-0) = -\gamma''(+0)$ ,  $\psi''(-0) = -\psi''(+0)$ ,  $\varphi'''(-0) = \varphi'''(+0)$ ,  $\phi'''(-0) = \phi'''(+0)$ ,  $\gamma'''(-0) = \gamma'''(+0)$ ,  $\psi'''(-0) = \psi'''(+0)$ , then there exists a unique classical solution  $u(x, t), v(x, t) \in C_{x,t}^{2,1}(\Omega_- \cup \Omega_+)$ ,  $f(x), g(x) \in C^1([-1, 0] \cup (0, 1])$  of the problem (2.1) - (2.5).*

*Proof:* **1. Uniqueness of the solution.**

Let functions  $(u_1(x, t), v_1(x, t), f_1(x), g_1(x))$  and  $(u_2(x, t), v_2(x, t), f_2(x), g_2(x))$  be solutions of the problem (2.1) - (2.5). We denote  $u = u_1(x, t) - u_2(x, t)$ ,  $f(x) = f_1(x) - f_2(x)$ ,  $v = v_1(x, t) - v_2(x, t)$ ,  $g(x) = g_1(x) - g_2(x)$ . Then functions  $(u, v, f, g)$  satisfy the system of equations

$$\begin{cases} u_t(x, t) - \text{sign}(x)u_{xx}(x, t) + a_1u(x, t) + b_1v(x, t) = f(x), \\ v_t(x, t) - \text{sign}(x)v_{xx}(x, t) + a_2v(x, t) + b_2u(x, t) = g(x) \end{cases} \quad (5.1)$$

with the following homogeneous conditions:

$$u_x(\pm 1, t) = 0, \quad v_x(\pm 1, t) = 0, \quad 0 \leq t \leq T, \quad (5.2)$$

$$u(x, 0) = 0, \quad v(x, 0) = 0, \quad -1 \leq x \leq 1, \quad (5.3)$$

$$u(x, T) = 0, \quad v(x, T) = 0, \quad -1 \leq x \leq 1, \quad (5.4)$$

$$\begin{cases} u(-0, t) = u(+0, t), \quad u_x(-0, t) = -u_x(+0, t), \\ v(-0, t) = v(+0, t), \quad v_x(-0, t) = -v_x(+0, t), \end{cases} \quad 0 \leq t \leq T. \quad (5.5)$$

In relation to this problem, we find the following:

**Problem 1.** Find a function  $\omega_i(x, t)$  that satisfies equation

$$\omega_{it}(x, t) - \text{sign}(x)\omega_{ixx}(x, t) - k_i\omega_i(x, t) = \bar{f}_i(x) \quad (5.6)$$

in the domain  $\Omega = \Omega_- \cup \Omega_+$  and the following conditions

$$\omega_{ix}(\pm 1, t) = 0, \quad 0 \leq t \leq T, \quad (5.7)$$

$$\omega_i(x, 0) = 0, \quad -1 \leq x \leq 1, \quad (5.8)$$

$$\omega_i(x, T) = 0, \quad -1 \leq x \leq 1, \quad (5.9)$$

$$\omega_i(-0, t) = \omega_i(+0, t), \quad \omega_{ix}(-0, t) = -\omega_{ix}(+0, t), \quad 0 \leq t \leq T, \quad (5.10)$$

Let's introduce the notations

$$\omega_{in}^+ = \int_{-1}^1 \omega_i(x, t) X_n^+ dx \quad (5.11)$$

$$\bar{f}_{in}^+ = \int_{-1}^1 \bar{f}_i(x) X_n^+ dx \quad (5.12)$$

Differentiating  $\omega_i(x, t)$  in (5.11) under the integral sign once and taking into account equation (5.6), we obtain

$$\frac{d}{dt} \omega_{in}^+ = \int_{-1}^1 \text{sign}(x) \omega_{ixx}(x, t) X_n^+ dx + k_i \omega_{in}^+ + \bar{f}_{in}^+.$$

Integrating twice by parts the integral in the last equality, taking into account the gluing conditions (5.10) and the characteristic equations (4.6), we obtain an equation that  $\omega_n^+$  must satisfy

$$\frac{d}{dt} \omega_{in}^+ = (\lambda_n^+ + k_i) \omega_{in}^+ + \bar{f}_{in}^+. \quad (5.13)$$

From conditions (5.8) and (5.9), we obtain that  $u_n^+$  must satisfy the conditions

$$\omega_{in}^+(0) = 0, \quad \omega_{in}^+(T) = 0 \quad (5.14)$$

The solution to equation (5.13) has the form

$$\omega_{in}^+ = G_i e^{(\lambda_n^+ + k_i)t} - \frac{\bar{f}_{in}^+}{(\lambda_n^+ + k_i)}, \quad G_i = \text{const.}$$

Due to conditions (5.14) we obtain

$$\begin{cases} G_i - \frac{\bar{f}_{in}^+}{(\lambda_n^+ + k_i)} = 0, \\ G_i e^{(\lambda_n^+ + k_i)T} - \frac{\bar{f}_{in}^+}{(\lambda_n^+ + k_i)} = 0. \end{cases}$$

Let's subtract the second equation of the system from the first one.

$$G_i(1 - e^{(\lambda_n^+ + k_i)T}) = 0$$

Let  $\lambda_n^+ \neq -k_i$ ,  $\forall n = 1, 2, \dots$ , then  $G_i = 0$ . Therefore,  $\bar{f}_{in}^+ = 0$ .

Then we get  $\omega_{in}^+ = 0$ ,  $\forall n = 1, 2, \dots$ . Likewise  $\omega_{in}^- = 0$ ,  $\forall n = 1, 2, \dots$ .

So  $v(x, t) \equiv 0$ ,  $v_1(x, t) \equiv v_2(x, t)$ ,  $u(x, t) \equiv 0$ ,  $u_1(x, t) \equiv u_2(x, t)$ ,  $f \equiv 0$ ,  $f_1(x) \equiv f_2(x)$ ,  $g(x) \equiv 0$ ,  $g_1(x) \equiv g_2(x)$  for  $\forall(x, t) \in L_2((-1; 0) \cup (0; 1))$ . Thus, the uniqueness of the solution to the problem is proven.

## 2. Existence of a solution

We return to functions

$$\omega_i(x, t) = \int_T^t \xi_i(x, \tau) d\tau + \bar{\psi}_{i1}(x) = \bar{\psi}_{i1}(x) + \frac{C_i X_0(x)(e^{k_i t} - e^{k_i T})}{k_i} + \sum_{n=1}^{\infty} \left( A_{in}^- X_n^-(x) \frac{e^{(\lambda_n^- + k_i)t} - e^{(\lambda_n^- + k_i)T}}{\lambda_n^- + k_i} - B_{in}^+ X_n^+(x) \frac{1 - e^{-(\lambda_n^+ + k_i)(T-t)}}{\lambda_n^+ + k_i} \right), \quad (x, t) \in \Omega_- \quad (5.15)$$

$$\omega_i(x, t) = \int_0^t \xi_i(x, \tau) d\tau + \bar{\varphi}_{i2}(x) = \bar{\varphi}_{i2}(x) + \frac{C_i X_0(x) e^{k_i t}}{k_i} + \sum_{n=1}^{\infty} \left( A_{in}^- X_n^-(x) \frac{e^{(\lambda_n^- + k_i)t} - 1}{\lambda_n^- + k_i} + B_{in}^+ X_n^+(x) \frac{e^{-(\lambda_n^+ + k_i)(T-t)} - e^{-(\lambda_n^+ + k_i)T}}{\lambda_n^+ + k_i} \right), \quad (x, t) \in \Omega_+. \quad (5.16)$$

For the coefficients  $A_{in}^-$  from (4.10) we obtain

$$|A_{in}^-| \leq D_i \left( |\bar{\varphi}_{in}^{-(2)}| + |\bar{\psi}_{in}^{-(2)}| + k_i (|\bar{\varphi}_{in}^-| + |\bar{\psi}_{in}^-|) \right), \quad (5.17)$$

where  $D_i = \frac{1}{|1 - e^{(\lambda_1^- + k_i)T}|}$ . Since the functions  $\varphi''(x)$  and  $\psi''(x)$  are continuous on the segment  $[-1, 0) \cup (0, 1]$ , then, as is known from the theory of Fourier series, by virtue of Bessel's inequality the following series in  $L_2(-1, 0) \cup (0, 1)$  converge:

$$\sum_{n=1}^{\infty} \left( \bar{\varphi}_{in}^{-(2)} \right)^2 \leq \|\bar{\varphi}_i''\|^2, \quad (5.18)$$

$$\sum_{n=1}^{\infty} \left( \bar{\psi}_{in}^{-(2)} \right)^2 \leq \|\bar{\psi}_i''\|^2, \quad (5.19)$$

$$\sum_{n=1}^{\infty} \left( \bar{\varphi}_{in}^- \right)^2 \leq \|\bar{\varphi}_i\|^2, \quad (5.20)$$

$$\sum_{n=1}^{\infty} \left( \bar{\psi}_{in}^- \right)^2 \leq \|\bar{\psi}_i\|^2. \quad (5.21)$$

From (5.18)-(5.21) it follows that  $|A_{in}^-|$  is bounded. Similarly, the coefficients  $|B_{in}^+|$  and  $|C_i|$  are also bounded.

Let us estimate the moduli of the common terms from the series (5.15) and (5.16). In the region  $\Omega_+$

$$\begin{aligned} \left| C_i e^{k_i t} X_0(x) \frac{1}{k_i} \right| &\leq \frac{1}{\sqrt{2} |k_i|} |C_i| e^{|k_i|T}, \\ \left| A_{in}^- \left( \frac{e^{(\lambda_n^- + k_i)t} - 1}{\lambda_n^- + k_i} \right) X_n^-(x) \right| &\leq |A_{in}^-| \frac{e^{|k_i|T}}{|k_i + \lambda_n^-| \cos \sqrt{\lambda_n^+}}, \\ \left| B_{in}^+ \left( \frac{e^{-(\lambda_n^+ + k_i)(T-t)} - e^{-(\lambda_n^+ + k_i)T}}{\lambda_n^+ + k_i} \right) X_n^+(x) \right| &\leq \frac{|B_{in}^+| e^{|k_i|T}}{|\lambda_n^+ + k_i|}. \end{aligned}$$

In the region  $\Omega_-$

$$\begin{aligned} \left| \frac{C_i X_0(x) (e^{k_i t} - e^{k_i T})}{k_i} \right| &\leq \frac{1}{\sqrt{2} |k_i|} |C_i| e^{|k_i|T}, \\ \left| A_{in}^- \left( \frac{e^{(\lambda_n^- + k_i)t} - e^{(\lambda_n^- + k_i)T}}{\lambda_n^- + k_i} \right) X_n^-(x) \right| &\leq |A_{in}^-| \frac{e^{|k_i|T}}{|\lambda_n^- + k_i|}, \\ \left| B_{in}^+ \left( \frac{1 - e^{-(\lambda_n^+ + k_i)(T-t)}}{\lambda_n^+ + k_i} \right) X_n^+(x) \right| &\leq |B_{in}^+| \frac{e^{|k_i|T}}{|\lambda_n^+ + k_i| \cos \sqrt{\lambda_n^+}}. \end{aligned}$$

Taking into account that  $\lambda_n^\pm \neq k_i$ , we obtain  $\frac{1}{|\lambda_n^\pm + k_i|} \leq \frac{1}{|\lambda_n^\pm| q}$ , where  $q = \min \left( \left| 1 + \frac{k_i}{\lambda_n^\pm} \right| \right)$ .

If  $n \rightarrow \infty$ , then  $|\lambda_n^\pm| \rightarrow \infty$  and  $\frac{\sqrt{2}}{2} < \cos \sqrt{|\lambda_n^\pm|} < \frac{\sqrt{3}}{2}$  (see [2]). For series (5.15) and (5.16) it is true the following

$$\frac{P_i}{q} \sum_{n=1}^{\infty} \frac{|A_{in}^-| + |B_{in}^+|}{|\lambda_n^\pm|} \leq \frac{P_i}{q} C \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{P_i}{q} C \frac{\pi^2}{6},$$

where  $P_i = 2e^{|k_i|T}$ .

Therefore, the series converges absolutely and uniformly in a closed domain  $\Omega$  by virtue of the Weierstrass criterion. Consequently, the functions  $u(x, t)$ ,  $v(x, t)$  is continuous as the assumption of a uniformly convergent series and a continuous function in the domain  $\Omega$ .

Similar estimates show that the series (5.15) and (5.16) can be differentiated term by term with respect to the variable  $x$  twice, and with respect to the variable  $t$  once. For this, the absolute and uniform convergence of the series obtained by formal differentiation with respect to is proved.

Let's check the conditions of the final redefinition.

Let's consider the domain  $\Omega_+$

$$\begin{aligned} \omega_i(x, T) &= \bar{\varphi}_2(x) + \frac{C_i X_0(x) e^{k_i T}}{k_i} + \sum_{n=1}^{\infty} (A_{in}^- X_n^-(x) \left( \frac{e^{(\lambda_n^- + k_i)T} - 1}{\lambda_n^- + k_i} \right) + \\ B_{in}^+ X_n^+(x) \left( \frac{-e^{-(\lambda_n^+ + k_i)T}}{\lambda_n^+ + k_i} \right)) &= \varphi_2(x) + \frac{\bar{\varphi}_{i0}^{(2)} - \bar{\psi}_{i0}^{(2)} + \bar{k}_i(\bar{\varphi}_{i0} - \bar{\psi}_{i0})}{1 - e^{k_i T}} \cdot \frac{X_0(x) e^{k_i T}}{k_i} \\ &+ \sum_{n=1}^{\infty} X_n^-(x) \left( \frac{-\bar{\varphi}_{in}^{-(2)} + \bar{\psi}_{in}^{-(2)} - k_i(\bar{\varphi}_{in}^- - \bar{\psi}_{in}^-)}{\lambda_n^- + k_i} \right) + \\ \sum_{n=1}^{\infty} X_n^+(x) \left( \frac{\bar{\varphi}_{in}^{+(2)} - \bar{\psi}_{in}^{+(2)} + k_i(\bar{\varphi}_{in}^+ - \bar{\psi}_{in}^+)}{\lambda_n^+ + k_i} \right) &\left( \frac{-e^{-(\lambda_n^+ + k_i)T}}{e^{(\lambda_n^+ + k_i)T} - 1} \right). \end{aligned} \quad (5.22)$$

Integrating the integrals for  $\bar{\varphi}_{in}^{\pm(2)}$ ,  $\bar{\psi}_{in}^{\pm(2)}$ ,  $\bar{\varphi}_{i0}^{(2)}$ , and  $\bar{\psi}_{i0}^{(2)}$  by parts twice, we have  $\bar{\varphi}_{in}^{\pm(2)} = \lambda_n^\pm \bar{\varphi}_{in}$ ,  $\bar{\psi}_{in}^{\pm(2)} = \lambda_n^\pm \bar{\psi}_{in}$ ,  $\bar{\varphi}_{i0}^{(2)} = 0$ , and  $\bar{\psi}_{i0}^{(2)} = 0$ , where  $\bar{\varphi}_{in}^\pm = (\text{sign}(x) \cdot \bar{\varphi}_i(x), X_n^\pm(x))$ ,  $\bar{\psi}_{in}^\pm = (\text{sign}(x) \cdot \bar{\psi}_i(x), X_n^\pm(x))$ ,  $\bar{\varphi}_{i0} = (\text{sign}(x) \cdot \bar{\varphi}_i(x), X_0(x))$ ,  $\bar{\psi}_{i0} = (\text{sign}(x) \cdot \bar{\psi}_i(x), X_0(x))$ .

Substituting the obtained ratios of coefficients into (5.22), we obtain the equalities  $\omega_1(x, T) = \bar{\psi}_{12}(x)$ ,  $\omega_2(x, T) = \bar{\psi}_{22}(x)$ .

Similarly, we check the validity of the equalities  $\omega_1(x, 0) = \bar{\varphi}_{11}(x)$ ,  $\omega_2(x, 0) = \bar{\varphi}_{21}(x)$  in the region  $\Omega_-$ .

Therefore, the obtained solution satisfies all the conditions of the problem.

The existence and uniqueness theorem is proved.  $\square$

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