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On a mixed problem for a multidimensional elliptic equation with singular coefficients

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Abstract. At present the fundamental solutions of the multidimensional singular elliptic equation are known and they are expressed through the well-known Lauricella function $F_A^{(n)}$, the number of variables of which is equal to the number of singular coefficients of the equation under consideration. On the other hand, in applications of any hypergeometric function of many variables, expansion formulas are very important, allowing one to represent this hypergeometric function as an infinite sum of products of one-dimensional hypergeometric Gaussian functions for each variable of the studied function of many variables. In this paper we study one mixed problem for an elliptic equation with many singular coefficients in the first hyperoctant of the unit ball, the uniqueness of its solution is proved by the method of energy integrals, and its existence by the Green's function method. When finding the desired solution, expansion and summation formulas are used, as well as the limit relation for the Lauricella function.

Keywords: multidimensional elliptic equations with singular coefficients; fundamental solution; Gauss hypergeometric function; Lauricella function; mixed problem; Green's function.
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1. INTRODUCTION

Hypergeometric functions play a primary role in both the analysis and the solution of boundary value problems for partial differential equations, because important attributes in solving boundary value problems such as fundamental solutions and Green's functions are written in terms of hypergeometric functions. Numerous applications of hypergeometric functions can be found in mechanics, fluid mechanics, elastic dynamics, electromagnetism, and acoustics [1, 2]. The results of the theory of hypergeometric functions allow one to represent the solution of boundary value problems in explicit forms or in the form of an integral equation with a known kernel. For problems with known Green's functions expressed by hypergeometric functions, the formulation of an integral equation leads to powerful numerical approximation schemes.

Let $\mathbb{R}_m$ be the m-dimensional Euclidean space $(m \geq 2)$, $x := (x_1, ..., x_m)$ be an arbitrary point of it, and n be a positive integer such that $n \leq m$. Define the infinite region R_m^{n+}, which is a 2^{-n}th part of the Euclidean space $\mathbb{R}_m$, as follows:

$$R_m^{n+} = \left\{x \in \mathbb{R}_m : x_i > 0,\, i = 1, ..., n;\, -\infty < x_j < +\infty,\, j = n+1, ..., m\right\}.$$

In R_m^{n+}, consider the generalized singular elliptic equation

$$L_{\alpha,x}^{(m,n)}(u) \equiv \sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} + \sum_{k=1}^{n} \frac{2\alpha_k}{x_k} \frac{\partial u}{\partial x_k} = 0 \tag{1.1}$$

where $\alpha := (\alpha_1, ..., \alpha_n)$ and α_k are real numbers such that $0 < 2\alpha_k < 1$ $(k = \overline{1, n})$.

Note that the fundamental solutions [3] of equation (1.1) and the particular solutions [4] of the more general equation

$$\sum_{j=1}^{k} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=1}^{k} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} = \sum_{j=k+1}^{n} \frac{\partial^2 u}{\partial x_j^2} + \sum_{j=k+1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j}, k = \overline{1, n-1}$$

are explicitly expressed in terms of the Lauricella hypergeometric function F_A^n of n variables. In this paper, using the known fundamental solution of equation (1.1), we find an unique solution of the mixed problem for equation (1.1) in the first hyperoctant of the unit ball in the explicit form.

However, only few works are devoted to the study of the mixed problem for a singular elliptic equation: in [5] was considered a two-dimensional singular elliptic equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{2\alpha}{x_1}\frac{\partial u}{\partial x_1}=0,\ 0<2\alpha<1, \tag{1.2}$$

in the work [6] is discussed the mixed problem for the equation

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}+\frac{2\alpha}{x_1}\frac{\partial u}{\partial x_1}=0,\ \ 0<2\alpha<1,$$

in a domain bounded in a half-space $x>0$, and the work [7] generalizes the results of the works [5, 6] to the $m-$ dimensional equation (1.1) with only one singular coefficient:

$$\sum_{i=1}^{m}\frac{\partial^2 u}{\partial x_j^2}+\frac{2\alpha}{x_1}\frac{\partial u}{\partial x_1}=0\ 0<2\alpha<1,\ m>2. \tag{1.3}$$

In the works [8] and [9], the Dirichlet and Holmgren problems were solved, respectively, by the method of potentials for equation (1.3) in the domain bounded in the subset (half-space) of the Euclidean space.

In the paper [10] for the three-dimensional elliptic equation with two singular coefficients

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}+\frac{2\alpha}{x_1}\frac{\partial u}{\partial x_1}+\frac{2\beta}{x_2}\frac{\partial u}{\partial x_2}=0,\ \ 0<2\alpha,2\beta<1,$$

two mixed problems and Neumann problem in the first quarter of the unit ball are studied.

The paper is organized as follows: First, in Section 2 we give some preliminary information, which will be used in what follows. Section 3 is devoted to the formulation of the problem and the uniqueness theorem. In Section 4 using the Green's functions method we find the solution of the mixed problem for equation (1.1) in the explicit form. Throughout this paper, it is assumed that the dimension of the space $m>2$.

2. Hypergeometric functions of Gauss and Lauricella

A symbol $(\kappa)_\nu$ denotes the general Pochhammer symbol or the shifted factorial, since $(1)_l=l!$ $(l\in N\cup\{0\};\ N:=\{1,2,3,...\})$, which is defined (for $\kappa,\nu\in C$), in terms of the familiar Gamma function, by

$$(\kappa)_\nu:=\frac{\Gamma(\kappa+\nu)}{\Gamma(\kappa)}=\begin{cases}1 & (\nu=0;\kappa\in C\backslash\{0\})\\ \kappa(\kappa+1)...(\kappa+l-1) & (\nu=l\in N;\kappa\in C),\end{cases}$$

it being understood conventionally that $(0)_0:=1$ and assumed tacitly that the $\Gamma-$ quotient exists.

The hypergeometric function of Gauss is defined inside the circle $|z|<1$ as the sum of the hypergeometric series [11] :

$$F(a,b;c;z)\equiv F\left[\begin{matrix}a,b;\\ c;\end{matrix}\ z\right]:=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m m!}z^m,\ \ |z|<1,$$

where a, b, c, are independent of z. We shall call a, b, c the parameters of the hypergeometric function; they are arbitrary complex numbers.

If $Rec>Reb>0$, we have Euler's formula [11]

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}dt. \tag{2.1}$$

Here the right-hand side is a one-valued analytic function of z within the domain $|arg(1-z)|<\pi$; therefore (2.1) gives also the analytic continuation of $F(a,b;c;z)$.

The integral representation (2.1) allows to derive the Boltz formula [11]

$$F(a,b;c;z) = (1-z)^{-b} F\left(c-a, b; c; \frac{z}{z-1}\right) \tag{2.2}$$

and get the value of the Gaussian function in unity (the summation formula) [11]

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ Re(c-a-b) > 0,\ c \neq 0,\ -1,\ -2,\ .... \tag{2.3}$$

The complete Appell hypergeometric function F_2 is defined by the series

$$F_2\left(a,b,b';c,c';x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m}{m!} \frac{y^n}{n!},\ \ |x|+|y|<1. \tag{2.4}$$

Introduce the following notation:

$$\mathbf{a} := (a_1, ..., a_n)\,,\ \ \mathbf{b} := (b_1, ..., b_n)\,,\ \ \mathbf{c} := (c_1, ..., c_n)\,;\ \ \mathbf{x} := (x_1, ..., x_n)\,.$$

In 1893 Lauricella generalized the four known Appell functions and defined his four hypergeometric functions of several variables, the first of which has the form [12]:

$$F_A^{(n)}\left(a, \mathbf{b};\mathbf{c};\mathbf{x}\right) = \sum_{k_1,k_2,...,k_n=0}^{\infty} (a)_{k_1+k_2+...+k_n} \prod_{j=1}^{n} \frac{(b_j)_{k_j}}{(c_j)_{k_j}} \frac{x_j^{k_j}}{k_j!},\ \ \ \sum_{j=1}^{n} |x_j| < 1. \tag{2.5}$$

The numerical parameters and variables of the Lauricella hypergeometric function $F_A^{(n)}$ defined in (2.5) can take complex values. It is easy to see that the function $F_A^{(n)}$ in the two-dimensional case coincides with the Appell function F_2 defined in (2.4).

The following relations hold for the Lauricella function $F_A^{(n)}$:

1) differentiation formula

$$\frac{\partial}{\partial x_k} F_A^{(n)}\left(a,\, \mathbf{b}; \mathbf{c}; \mathbf{x}\right) = \frac{ab_k}{c_k} F_A^{(n)}\left(a+1,\, \mathbf{b}_k+1; \mathbf{c}_k+1; \mathbf{x}\right)$$

2) the relation between contiguous hypergeometric functions [13]

$$\sum_{j=1}^{n} \frac{b_j}{c_j} x_j F_A^{(n)}\left(a+1, \mathbf{b}_j+1; \mathbf{c}_j+1; \mathbf{x}\right) = F_A^{(n)}(a+1, \mathbf{b};\mathbf{c};\mathbf{x}) - F_A^{(n)}(a, \mathbf{b};\mathbf{c};\mathbf{x}),$$

where the vectors $\mathbf{b}_j+1$ and $\mathbf{c}_j+1$ are defined as follows::

$$\mathbf{b}_j + 1 := \left(b_1, ..., b_{j-1}, b_j+1, b_{j+1}, ..., b_n\right),$$

$$\mathbf{c}_j + 1 := \left(c_1, ..., c_{j-1}, c_j+1, c_{j+1}, ..., c_n\right),\ \ j = \overline{1,n}.$$

To study the properties of multiple hypergeometric functions, so-called expansion formulas are used, which allow one to represent a hypergeometric function of many variables through an infinite sum of products of several hypergeometric functions with one variable, and this, in turn, facilitates the process of studying the properties of functions of many variables.

Theorem 2.1. *[14, 15] For any natural* $n \in \mathrm{N}\backslash\{1\}$ *the following expansion formula is valid*

$$F_A^{(n)}\left(a, \mathbf{b};\mathbf{c};\mathbf{x}\right) = \sum_{|\mathbf{m}|=0}^{\infty} \frac{(a)_{A(n,n)}}{M!} \prod_{k=1}^{n} \left\{ \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}} x_k^{B(k,n)} F\left[\begin{matrix} a+A(k,n), b_k+B(k,n); \\ c_k+B(k,n); \end{matrix}\ x_k\right]\right\}, \tag{2.6}$$

where

$$|\mathbf{m}| = \sum_{i=1}^{n}\sum_{j=i}^{n} m_{i,j};\quad M! := \prod_{i=1}^{n}\prod_{j=i}^{n} m_{i,j}!,\ m_{i,j} \geq 0,\ \ 2 \leq i \leq j \leq n,$$

$$A(k,n) = \sum_{i=2}^{k+1}\sum_{j=i}^{n} m_{i,j},\ \ B(k,n) = \sum_{i=2}^{k} m_{i,k} + \sum_{i=k+1}^{n} m_{k+1,i}.$$

Theorem 2.2. *[13] Let* $a, b_1, ..., b_n$ *be real numbers with* $a \neq 0,\ -1,\ -2, ...$ *and* $a > |\mathbf{b}|$. *Then for* $n = 1, 2, ...$ *the following summation formula*

$$\sum_{|\mathbf{m}|=0}^{\infty}\frac{(a)_{A(n,n)}}{M!}\prod_{k=1}^{n}\frac{(b_k)_{B(k,n)}\,(a-b_k)_{A(k,n)-B(k,n)}}{(a)_{A(k,n)}}=\frac{\Gamma\left(a-|\mathbf{b}|\right)\Gamma^{n-1}\left(a\right)}{\prod_{k=1}^{n}\Gamma\left(a-b_k\right)} \tag{2.7}$$

is valid, where $|\mathbf{b}| := b_1 + ... + b_n$.

It is easy to see that formula (2.7) is a natural generalization of the well-known Euler formula (2.3).

Using the expansion (2.6) and the formula (2.2), it is easy to derive an analogue of the Boltz formula for the Lauricella hypergeometric function in the form

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\prod_{k=1}^{n}\left(1-x_k\right)^{-b_k}\sum_{|\mathbf{m}|=0}^{\infty}\frac{(a)_{A(n,n)}}{M!}\prod_{k=1}^{n}\left[\frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}}\times\right.$$
$$\left.\times\left(\frac{x_k}{1-x_k}\right)^{B(k,n)}F\left(\begin{matrix}c_k-a+B(k,n)-A(k,n),b_k+B(k,n); \\ c_k+B(k,n);\end{matrix}\ \frac{x_k}{x_k-1}\right)\right].$$

Theorem 2.3. *[16] Let* a, b_k *and* c_k *be real numbers such that* $c_k \neq 0,\ -1,\ -2, ...,\ a > |\mathbf{b}|$ *and* $c_k > b_k$. *Then the following limit relation is valid for* $n = 1, 2, ...$ *:*

$$\lim_{\varepsilon\to 0}\varepsilon^{-|\mathbf{b}|}F_A^{(n)}\left[a,\mathbf{b};\mathbf{c};1-\frac{z_1(\varepsilon)}{\varepsilon},...,1-\frac{z_n(\varepsilon)}{\varepsilon}\right]=\frac{\Gamma\left(a-|\mathbf{b}|\right)}{\Gamma(a)}\prod_{k=1}^{n}\frac{\Gamma\left(c_k\right)}{\left[z_k(0)\right]^{b_k}\Gamma\left(c_k-b_k\right)}, \tag{2.8}$$

where $z_k(\varepsilon)$ *are arbitrary functions such that* $z_k(0) \neq 0$ $(k = \overline{1,n})$.

3. Formulation of the problem and the uniqueness theorem

Let $x := (x_1, ..., x_m) \in \mathbb{R}_m$ and $\Omega \subset R_m^{n+}$ is a 2^{-n} part of the m-dimensional ball with radius R and origin at the point $O(0, ..., 0)$:

$$\Omega := \left\{x:\ x_1^2+...+x_m^2<R^2,\ x_1>0,\ ...,x_n>0,\right\},$$

and

$$S := \left\{x:\ x_1^2+...+x_m^2=R^2,\ x_1>0,\ ...,x_n>0\right\}$$

is the same part of the sphere corresponding to this ball.

Introduce the following notation:

$$\tilde{x}_p=\left(x_1,...,x_{p-1},x_{p+1},...,x_m\right)\in\mathbb{R}_{m-1};\quad \mathbf{0}=(0,...,0)\in\mathbb{R}_{m-1}$$

$$x^{(2\alpha)}=\prod_{k=1}^{n}x_k^{2\alpha};\ \tilde{x}_p^{(2\alpha)}=\prod_{k=1,\,k\neq p}^{n}x_k^{2\alpha_k};\ dx=\prod_{k=1}^{m}dx_k;\ d\tilde{x}_p=\prod_{k=1,\,k\neq p}^{m}dx_p;$$

$$D_p:=\left\{\begin{matrix}x:\ x_1^2+...+x_{p-1}^2+x_{p+1}^2+...+x_m^2<R^2, \\ x_1>0,...,x_{p-1}>0,x_{p+1}>0,...,x_n>0\end{matrix}\right\};$$

$$S_p:=\left\{\begin{matrix}x:\ x_1^2+...+x_{p-1}^2+x_{p+1}^2+...+x_m^2=R^2, \\ x_1>0,...,x_{p-1}>0,x_{p+1}>0,...,x_n>0\end{matrix}\right\};\ \ p=\overline{1,n}.$$

Mixed problem $\mathbf{D^nM}$. Find a regular solution $u\left(x\right) \in C\left(\overline{\Omega}\right) \cap C^2\left(\Omega\right)$ of Eq. (1.1) satisfying conditions

$$u|_{x_p=0}=\tau_p\left(\tilde{x}_p\right),\ \tilde{x}_p\in\overline{D}_p,\ p=\overline{1,n},\ \left.\frac{\partial u}{\partial\mathbf{N}}\right|_S=\varphi\left(x\right),\ x\in S, \tag{3.1}$$

where $\tau_p \in C\left(\overline{D}_p\right)$ and $\varphi \in C\left(S\right)$ are given smooth enough functions fulfilling the following matching conditions:

$$\tau_p\left(\mathbf{0}\right)=\tau_k\left(\mathbf{0}\right),\ \ \tau_k\left(\tilde{x}_k\right)|_{x_p=0}=\tau_p\left(\tilde{x}_p\right)|_{x_k=0},\ k\neq p\ (k,p=\overline{1,n}).$$

$\mathbf{N}$ is the outer normal to S.

Theorem 3.1. *If the problem $D^n M$ has a solution, then it is unique.*

Proof: One can readily check the validity of the following relation:

$$x^{(2\alpha)}\left[uL_\alpha^{(m,n)}(w)-wL_\alpha^{(m,n)}(u)\right]=\sum_{j=1}^{m}\frac{\partial}{\partial x_j}\left[x^{(2\alpha)}\left(u\frac{\partial w}{\partial x_j}-w\frac{\partial u}{\partial x_j}\right)\right],\ n\le m.$$

Let Ω_ε be a sub-domain of Ω at a distance $\varepsilon>0$ from its boundary $\partial\Omega$ and $\cos(\mathbf{N},x_j)\,dD_p=d\tilde{x}_j$, if $j=p$ and $\cos(\mathbf{N},x_j)\,dD_p=0$ otherwise, where $j=\overline{1,m}$, $p=\overline{1,n}$; $\mathbf{N}$ is the outer normal to $\partial\Omega$.

Integrate both sides of this identity along the domain Ω_ε and use the Gauss-Ostrogradsky formula:

$$\int_{\Omega_\varepsilon}x^{(2\alpha)}\left[uL_\alpha^{(m,n)}(w)-wL_\alpha^{(m,n)}(u)\right]dx=\int_{\partial\Omega_\varepsilon}x^{(2\alpha)}\sum_{j=1}^{m}\left[\left(u\frac{\partial w}{\partial x_j}-w\frac{\partial u}{\partial x_j}\right)\cos(\mathbf{N},x_j)\right]d\sigma. \tag{3.2}$$

Note that Green's formula (3.2) in a more general form is proved in [17].

Using the equality

$$x^{(2\alpha)}\left[uL_\alpha^{(m,n)}(u)+\sum_{j=1}^{m}\left(\frac{\partial u}{\partial x_j}\right)^2\right]=\sum_{j=1}^{m}\frac{\partial}{\partial x_j}\left(x^{(2\alpha)}u\frac{\partial u}{\partial x_j}\right),\ n\le m,$$

we obtain

$$\int_{\Omega_\varepsilon}x^{(2\alpha)}\left[uL_\alpha^{(m,n)}(u)+\sum_{j=1}^{m}\left(\frac{\partial u}{\partial x_j}\right)^2\right]dx=\int_{\Omega_\varepsilon}\sum_{j=1}^{m}\frac{\partial}{\partial x_j}\left(x^{(2\alpha)}u\frac{\partial u}{\partial x_j}\right)dx.$$

Applying again the formula of Gauss-Ostrogradsky to this equality and letting $\varepsilon\to0$, we get

$$\int_{\Omega}x^{(2\alpha)}\sum_{j=1}^{m}\left(\frac{\partial u}{\partial x_j}\right)^2dx=\sum_{p=1}^{n}\int_{D_p}\tilde{x}_p^{(2\alpha)}\tau_p(\tilde{x}_p)\,f_p(\tilde{x}_p)\,d\tilde{x}_p+\int_S x^{(2\alpha)}\varphi(S)\,u\,dS, \tag{3.3}$$

where

$$f_p(\tilde{x}_p)=\lim_{x_p\to0}x_p^{2\alpha_p}\frac{\partial u(x)}{\partial x_p},\ p=\overline{1,n}.$$

To prove the uniqueness of the solution, as usual, we suppose that the problem has two v, w solutions. Denoting $u=v-w$ we have that u satisfies homogeneous mixed problem $D^n M$ ($\tau_p(\tilde{x}_p)=0$, $p=\overline{1,n}$, $\varphi(x)=0$). Further we have to prove that the homogeneous problem has only trivial solution. In this case from (3.3) one can easily get

$$\int_{\Omega}x^{(2\alpha)}\sum_{j=1}^{m}\left(\frac{\partial u}{\partial x_j}\right)^2dx=0.$$

Hence, it follows that $\dfrac{\partial u}{\partial x_j}=0$, $j=\overline{1,m}$, which implies that u is a constant function. Considering homogeneous conditions (3.1), we conclude that $u(x)\equiv0$ in $\overline{\Omega}$. The Theorem 3.1 is proved. □

4. THE EXISTENCE OF THE SOLUTION

The existence of the solution will be proved by method of Green's functions. The mixed problem $D^n M$ on the plane ($m=2$) for an elliptic equation with one ($n=1$) singular coefficient (1.2) was solved by M.M.Smirnov [5], so here we put $m>2$.

Definition 4.1. Green's function of the mixed problem D^nM for equation (1.1) is a function $G_n\left(x;\xi\right)$, satisfying conditions:

1) this function is a regular solution of (1.1) in the domain Ω, except the point ξ;

2) it satisfies boundary conditions

$$G_n|_{x_p=0}=0,\ \ p=\overline{1,n},\ \ \left.\frac{\partial G_n}{\partial \mathbf{N}}\right|_S=0,$$

3) it can be represented as

$$G_n\left(x;\xi\right)=q_n\left(x;\xi\right)+g_n\left(x;\xi\right),$$

where

$$q_n(x;\xi)=\gamma_n r^{-2\beta_n}\prod_{j=1}^{n}\left[\left(x_j\xi_j\right)^{1-2\alpha_j}\right]\cdot F_A^{(n)}\left[\begin{matrix}\beta_n,1-\alpha_1,...,1-\alpha_n;\\ 2-2\alpha_1,...,2-2\alpha_n;\end{matrix}\ \sigma\right]$$

is a fundamental solution [3], and function

$$g_n\left(x;\xi\right)=-\left(\frac{R}{\rho}\right)^{2\beta_n}q_n\left(x;\bar{\xi}\right),$$

is a regular solution of (1.1) in the domain Ω. Here

$$\beta_n=\frac{m-2}{2}+n-\sum_{j=1}^{n}\alpha_j,\ \ \gamma_n=2^{2\beta_n-m}\frac{\Gamma\left(\beta_n\right)}{\pi^{m/2}}\prod_{j=1}^{n}\frac{\Gamma\left(1-\alpha_j\right)}{\Gamma\left(2-2\alpha_j\right)}, \tag{4.1}$$

$$R^2=\sum_{j=1}^{m}x_j^2;\ \rho^2=\sum_{j=1}^{m}\xi_j^2;\ \bar{\xi}=\left(\bar{\xi}_1,...,\bar{\xi}_m\right),\ \bar{\xi}_j=\frac{R^2}{\rho^2}\xi_j,\ \ j=\overline{1,m}.$$

Therefore, for the domain Ω, the Green's function of the mixed problem D^nM for equation (1.1) has the form:

$$G_n\left(x;\xi\right)=q_n\left(x;\xi\right)-\left(\frac{R}{\rho}\right)^{2\beta_n}q_n\left(x;\bar{\xi}\right).$$

Next, repeating the reasoning given in [16, 18] we obtain the solution of the mixed problem in explicit form

$$u\left(\xi\right)=\gamma_n\xi^{(1-2\alpha)}\sum_{k=1}^{n}\left(1-2\alpha_k\right)\int_{S_k}\tilde{x}_k^{(1)}\tau_k\left(\tilde{x}_k\right)\left[\frac{F_A^{(n-1)}\left(\sigma_{k0}\right)}{X_k^{2\beta_n}}-\frac{F_A^{(n-1)}\left(\bar{\sigma}_{k0}\right)}{Y_k^{2\beta_n}}\right]dS_k+$$
$$+\int_S x^{(2\alpha)}G_n(x;\xi)\varphi\left(x\right)dS, \tag{4.2}$$

where

$$\xi^{(1-2\alpha)}=\prod_{j=1}^{n}\xi_j^{1-2\alpha_j},\ \ \tilde{x}_k^{(1)}=\prod_{j=1,j\neq k}^{n}x_j,$$

$$F_A^{(n-1)}\left(z\right)=F_A^{(n-1)}\left[\begin{matrix}\beta_n,1-\alpha_1,...,1-\alpha_{k-1},1-\alpha_{k+1},...,1-\alpha_n;\\ 2-2\alpha_1,...,2-2\alpha_{k-1},2-2\alpha_{k+1},...,2-2\alpha_n;\end{matrix}\ z\right],$$

z is a vector with $n-1$ components and $z=\sigma_{k0}$ or $z=\bar{\sigma}_{k0}$:

$$\sigma_{k0}:=\left(\sigma_1^0,...,\sigma_{k-1}^0,\sigma_{k+1}^0,...,\sigma_n^0\right),\quad \bar{\sigma}_{k0}:=\left(\bar{\sigma}_1^0,...,\bar{\sigma}_{k-1}^0,\bar{\sigma}_{k+1}^0,...,\bar{\sigma}_n^0\right),$$

$$\sigma_p^0=-\frac{4x_p\xi_p}{X_k^2},\ \ \bar{\sigma}_p^0=-\frac{R^2}{\rho^2}\frac{4x_p\xi_p}{Y_k^2};\ \ X_k^2=\xi_k^2+\sum_{i=1,i\neq k}^{m}\left(\xi_i-x_i\right)^2,$$

$$Y_k^2=\sum_{i=1,i\neq k}^{m}\left(R-\frac{x_i\xi_i}{R}\right)^2+\frac{1}{R^2}\sum_{i=1,i\neq k}^{m}\sum_{j=1,j\neq i}^{m}x_i^2\xi_j^2-\left(m-2\right)R^2.$$

Let us prove that the function defined by relation (4.2) really satisfies the mixed problem for the equation (1.1). First, we have to verify that the function

$$I_k\left(\xi,\sigma_{k0}\right)=\gamma_n\xi^{(1-2\alpha)}\left(1-2\alpha_k\right)\int_{S_k}\tilde{x}_k^{(1)}\tau_k\left(\tilde{x}_k\right)X_k^{-2\beta_n}F_A^{(n-1)}\left(\sigma_{k0}\right)dS_k \tag{4.3}$$

satisfies first n conditions of (3.1), i.e.

$$\lim_{\xi_l\to 0}I_k\left(\xi,\sigma_{k0}\right)=\begin{cases}\tau_k\left(\tilde{\xi}_k\right), & l=k\\ 0, & l\neq k\end{cases},$$

where $\tilde{\xi}_k:=\left(\xi_1,...,\xi_{k-1},\xi_{k+1},...,\xi_n,...,\xi_m\right)\in S_k\subset\mathbb{R}_{m-1}$.

Let $l=k$ and $k\in\overline{1,n}$. Putting $x_i=\xi_i+\xi_k t_i,\ i=\overline{1,m},\ i\neq k$, we transform the formula (4.3) to the form

$$\begin{aligned}I_k\left(\xi,\sigma_{k0}\right)=&\left(1-2\alpha_k\right)\gamma_n\xi^{(1-2\alpha)}\int_{T_k}\tau_k\left(\xi_1+\xi_k t_1,...,\xi_{k-1}+\xi_k t_{k-1},\right.\\ &\left.\xi_{k+1}+\xi_k t_{k+1},...,\xi_m+\xi_k t_m\right)\prod_{s=1,\,s\neq k}^{n}\left[\xi_s+\xi_k t_s\right]\tilde{R}_k^{-2\beta}F_A^{(n-1)}(\tilde{\sigma}_{k0})dT_k,\end{aligned} \tag{4.4}$$

where

$$\tilde{\sigma}_{k0}:=\left(1-\frac{\tilde{R}_{1,k}^2}{\tilde{R}_k^2},...,1-\frac{\tilde{R}_{k-1,k}^2}{\tilde{R}_k^2},1-\frac{\tilde{R}_{k+1,k}^2}{\tilde{R}_k^2},...,1-\frac{\tilde{R}_{n,k}^2}{\tilde{R}_k^2},\right)\in\mathbb{R}_{n-1},$$

$$\tilde{R}_k^2=\xi_k^2\left(1+\sum_{i=1,\,i\neq k}^{m}t_i^2\right),\ \tilde{R}_{l,k}^2=\left(2\xi_l+\xi_k t_l\right)^2+\xi_k^2\left(1+\sum_{i=1,\,i\neq l,\,i\neq k}^{m}t_i^2\right).$$

At the right-hand side of (4.4), pass to the limit as $\xi_k\to 0$. Using the limit relation (2.8) and the values of the parameter β_n and coefficient γ_n given by relation (4.1), we obtain

$$\lim_{\xi_k\to 0}I_k\left(\xi,\sigma_{k0}\right)=\frac{\Gamma\left(\frac{m}{2}-\alpha_k\right)\Gamma\left(1-\alpha_k\right)}{4^{\alpha_k}\pi^{m/2}\Gamma\left(1-2\alpha_k\right)}\tau_k\left(\tilde{\xi}_k\right)\underbrace{\int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty}}_{m-1}\frac{dt_1...dt_{k-1}dt_{k+1}...dt_m}{\left(1+\sum\limits_{j=1,j\neq k}^{m}t_j^2\right)^{\frac{m}{2}-\alpha_k}}.$$

Sequentially use the identities

$$\underbrace{\int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty}}_{p}\frac{dz_1...dz_p}{\left[1+z_1^2+...+z_p^2\right]^q}=\frac{\pi^{p/2}\Gamma\left(q-\frac{p}{2}\right)}{\Gamma\left(q\right)}$$

(see [19, p. 637, Eq. 4.638(3)]), and

$$\Gamma\left(2z\right)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma\left(z\right)\Gamma\left(z+\frac{1}{2}\right),$$

(the Legendre's duplication formula; see [11, p. 5, Eq. 1.2(15)]), we obtain

$$\lim_{\xi_k\to 0}I_k\left(\xi,\sigma_{k0}\right)=\tau_k\left(\tilde{\xi}_k\right).$$

In the same way, we prove that

$$\lim_{\xi_l\to 0}I_k\left(\xi,\sigma_{k0}\right)=0,\ k\neq l,\ l=1,...,n;\ \ \lim_{\xi_k\to 0}I_k\left(\xi,\tilde{\sigma}_{k0}\right)=0,\ \ 1\leq k\leq n.$$

The fulfillment of the remaining conditions of the mixed problem is established in a similar manner.

Hence, the main result of the paper is formulated as the following theorem:

Theorem 4.2. *If* $\tau_p\left(\tilde{x}_p\right)\in C^2\left(\overline{D}_p\right),\ p=\overline{1,n},\ \varphi\in C^1(S)$, *then the mixed problem* D^nM *has unique solution represented by formula (4.2).*

5. CONCLUSION

It's known that the potential theory has played a paramount role in both analysis and computation for boundary value problems for elliptic equations. Recently, the potential theories only for the two-dimensional elliptic equation with two singular coefficients [20] and the multidimensional elliptic equation with one singular coefficient [7] are constructed. In the study of potentials, the properties of the fundamental solutions of the given equation are essentially and fruitfully used. At the present time, fundamental solutions of the multidimensional elliptic equation with several singular coefficients are already known (see [3, 21]).

Despite the fact that fundamental solutions, particular solutions and expansion formulas are known even for a more general equation, the construction of potential theory for a singular elliptic equation was limited to the equations with one and two singular coefficients. For the future, it would be interesting to apply potential theory to the solution of boundary value problems for elliptic equations with two or more singular coefficients, for example, in the beginning for the equation

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y + \frac{2\gamma}{z}u_z = 0,\ 0 < 2\alpha, 2\beta, 2\gamma < 1,\ x > 0, y > 0, z > 0.$$

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