Uzbek Mathematical Journal
2026, Volume 70, Issue 2, pp. 115–120
DOI: 10.29229/uzmj.2026-2-14

On the negativity of separable algorithmic representations of fields with additive inversions

Karimova N.

Abstract. It is established that any algorithmic representation of any field in a signature with an additive inverse operation that has a nontrivial enumerable subset is negative. In this case, computably and enumerably generated topologies coincide, and the operations are continuous with respect to the enumerably generated topology.

Keywords: field with additive inversion, algorithmic representation, effectively generated topology, negativity and positivity, continuity.
MSC (2020): 03D45

1. Introduction

Following Yu. L. Ershov ([1]), we give a definition of the enumerated universal algebra of effective signature $\sigma = \langle =, f_0^{m_0}, f_1^{m_1}, \dots \rangle$ (where m_i is the number of arguments of the function into which the functional symbol $f_i^{m_i}$ is interpreted and the mapping $h : n \longmapsto m_n$ is computable).

Definition 1.1. An algorithmic representation (or enumeration) of a universal algebra $\mathfrak{A} = \langle A; g_0, g_1, \dots \rangle$ of signature σ is any mapping $\nu : \omega \longrightarrow A$ of the set of positive integers ω onto the underlying set A of $\mathfrak{A}$ such that the following condition holds:

there exists a two-place computable function G such that for any $n \in \omega$, any $y_1, \dots, y_{m_n}$ the equality $g_n(\nu y_1, \dots, \nu y_{m_n}) = \nu G(n, c^{m_n}(y_1, \dots, y_{m_n}))$ holds.

In other words, given ν-indices of elements from A and the index of the operation g_n, we can effectively find some ν-index of the result of applying this operation to these elements.

If $\nu : \omega \longrightarrow A$ is an enumeration of the algebra $\mathfrak{A}$, then the pair $(\mathfrak{A}, \nu)$ is called an enumerated algebra.

Note that every at most countable algebra of effective signature has an enumeration ([1]).

The kernel of the algorithmic representation ν of the enumerated algebra $(\mathfrak{A}, \nu)$ is the equivalence $\{\langle x, y\rangle | \nu x = \nu y\}$. If ν is a representation, then its kernel is denoted by $ker(\nu)$. An enumeration is called computable (positive, negative) if its kernel is computable (computably enumerable, co-enumerable).

For a fixed system, the classical problem is studying its various algorithmic representations and relations between them (see the work [2] of S.S. Goncharov and Yu.L. Ershov), in particular, the problem of the existence of good representations (primarily computable ones) and their number (including uniqueness, up to computable isomorphism).

On the other hand, within the duality paradigm, one can fix the kernel of a representation and study the general properties of systems that have representations with this kernel. This approach seems appropriate from the point of view of the theory of algorithmic representations of systems in a number of problems, including in theoretical computer science (see the reviews [3, 4]).

Much attention is currently being paid to the study of algorithmic properties of equivalences on the set of positive integers. A rich bibliography within the framework of the above-mentioned paradigm is available in S.S.Goncharov and Yu.L.Ershov in [2]. Closely related to this problematic are questions of the structure of algebras representable over equivalences. If η is an equivalence on the set of positive integers ω, then the universal algebra $\mathfrak{A}$ is called representable over η (or an η-algebra) if there exists an enumeration ν with kernel equal to η (i.e. $\eta = \{\langle x, y\rangle \mid \nu x = \nu y\}$). In other words, an algebra $\mathfrak{A}$ is representable over an equivalence η if there exists a family (not necessarily effective) F of computable functions consistent with η (i.e., for each $f \in F : \forall \overline{x}, \overline{y}[\overline{x} = \overline{y} \pmod{\eta} \Rightarrow f(\overline{x}) = f(\overline{y}) \pmod{\eta})]$) such that $\mathfrak{A}$ is isomorphic to the quotient algebra $\langle \omega/\eta; F\rangle$ of the computable algebra $\langle \omega; F\rangle$ by the congruence η. Moreover, the natural projecting homomorphism $\nu(x) = \{x\}/\eta$ is an enumeration from the computable algebra $\langle \omega; F\rangle$ onto $\mathfrak{A}$. It is appropriate to note that the role and place of computable algebras in the theory of enumerated algebras are similar to the role and place of absolutely free algebras of suitable rank in the theory of universal algebras, since in both cases there exists a homomorphism from a suitable computable (some free) algebra onto $\mathfrak{A}$. Note that any at most countable universal algebra of countable signature has an enumeration (induced by the Gödel enumeration of the absolutely free algebra of terms in a solvable set of generators, [1]). The case of an ineffective signature (of any algorithmic complexity) does not change the situation with the presence of enumeration, i.e. every at most countable algebra of countable signature has an enumeration ([4]).

The word *equivalence* denotes an equivalence relation on the set of positive integers ω. If η is an equivalence, then the set $\alpha \subseteq \omega$ is called *η-closed* if α is the union of suitable η-classes (i.e. $x \in \alpha \wedge x = y \pmod{\eta} \Rightarrow y \in \alpha$).

If η is an equivalence on ω, then the family of η-closed computable (computably enumerable) sets defines a base of the computable (enumerable) topology $\tau(\eta)_C$ (respectively $\tau(\eta)_{CE}$) on the quotient set ω/η.

In complex self-organizing systems, the problem of recognition is apparently of great importance. If a system is specified by its algorithmic representation, then from a mathematical point of view this means that any pair of distinct elements is separated by algorithmically determined neighborhoods. The most important types of separable enumerations of algebras are effectively separable, i.e., those for which there exist computable (in the sense of [5]) families of separating computably enumerable sets. The foundations of the theory of separably enumerated sets and the connections between separability and effective separability were introduced, developed, and investigated by Yu.L.Ershov (see [5]).

Within the framework of the paradigm of the existence of an effective system of algorithmically generated neighborhoods for an enumerated algebra, sufficient for recognizing the separability of any pair of its elements, it must be recognized that it is precisely the most general concept of a separable enumeration studied by Yu.L.Ershov in [5] in the case of enumerations of universal algebras that can be interpreted as one of the mathematical refinements of the concept of a complex developing system, intensionally defined by a family of representing computable functions together with the kernels of homomorphisms (enumerations) and allowing effective recognition of the difference between its constituent elements by separating them with suitable algorithmically determined neighborhoods.

An equivalence η is called computably separated (separable) if for any $x \neq y \pmod{\eta}$ there exists a computable η-closed set containing x and not containing y (there exists a computably enumerable η-closed set α such that either $x \in \alpha \wedge y \notin \alpha$ or $y \in \alpha \wedge x \notin \alpha$).

Let (N, ν) be an enumerated set. A subset N_0 of N is called *ν-computable (ν-enumerable, ν-co-enumerable)* if the set $\nu^{-1}N_0$ is computable (computably enumerable, co-enumerable). If it is clear from the context which enumeration of a set is meant, then we will call its subsets simply *computable (enumerable, co-enumerable)* without the prefix ν.

Each enumeration ν is uniquely associated with its kernel (enumeration equivalence), i.e. the set $ker(\nu) = \{\langle x, y\rangle \mid \nu x = \nu y\}$ and when speaking about certain properties of an enumeration we often imply the presence of these properties for the kernel. An enumeration is called computably separable (separable) if its kernel is computably separable (separable).

An enumeration ν is called effectively separable if its kernel $ker(\nu)$ is so, i.e. there exists a computable family $\mathfrak{S}$ of computably enumerable $ker(\nu)$-closed sets such that for any $x, y \in \omega$ if $\nu x \neq \nu y$ then there exists $S \in \mathfrak{S}$ such that either $\nu x \in \nu S \wedge \nu y \notin \nu S$ or $\nu y \in \nu S \wedge \nu x \notin \nu S$ ([5]). Such families is called complete.

Thus, if separability ensures T_0-separability of the space generated by ν-enumerable subsets, then effective separability guarantees the existence of a computable enumeration for a suitable complete separating family.

If $(A, \mu), (B, \nu)$ are two indexed algebras, then the homomorphism $\varphi : A \to B$ is called a morphism if it is effective on indexes, i.e. there exists a computable function f such that the diagram $\varphi\mu = \nu f$ is commutative, which is completely natural from the intensional point of view. Therefore, all the homomorphisms we consider are morphisms.

The most important ones in the framework of the structural theory of computably separated (separable) algorithmic representations of universal algebras are negative and effectively separable algebras, since we have

Theorem 1.2. *An algorithmic representation of an universal algebra is computably separable (separable) if and only if it is approximated by negative algebras (effectively separable algebras).*

Proof in [3, 6].

This fact emphasizes the exceptional role of negative algebras in the theory of separable enumerations of universal algebras. Note that over every negative equivalence a finitely generated congruence-simple algebra is definable ([7]). Similarly, over every negative equivalence with infinitely many cosets a dense linear order is negatively definable ([6]). Nothing similar holds for positive equivalences ([4]). The property of algorithmic uniformity inherent in negative algebras allows to solve some fundamental problems in Computer Science ([8]).

Next, unless otherwise stated, by a topological space on the quotient set ω/η by the equivalence η we mean the enumerably generated space $\tau(\eta)_{CE}$. The case of a $\tau(\eta)_C$-space generated by η-closed computable subsets will be specified separately. It is obvious that the topology of $\tau(\eta)_{CE}$ is generally stronger than the topology of $\tau(\eta)_C$, since computability is a special case of computable enumerability.

2. Topological spaces over computably separated equivalences

Let us formulate the strongest separation properties for computably separable enumerations ([9]):

Proposition 2.1. *For an arbitrary equivalence η on ω the following conditions are equivalent:*

(1) η is a computably separated equivalence;

(2) $\tau(\eta)_C$-space is perfectly normal and totally disconnected.

First of all, we note the following important fact concerning computably separated enumerated universal algebras.

Proposition 2.2. *The operations of any enumerated algebra are continuous in a computably separately generated space.*

Proof: Let $(\omega/\eta; \Im)$ be an arbitrary η-algebra (i.e., an algebra representable over η). Note that we do not assume computability of the family $\Im$ of computable functions for which η is a congruence. If f is a computable function of one variable, then there is nothing to prove, since the preimage of a computable set is also computable. Suppose that $f \in \Im$ and the number of arguments n of f is at least 2. Fix a set $\overline{x} = \langle x_1, \dots, x_n \rangle$. We need to show that for any η-closed computable set Y containing a number $f(x_1, \dots, x_n)$, there exist η-closed computable sets $X_1 \ni x_1, \dots, X_n \ni x_n$ such that $f(X_1, \dots, X_n) \subseteq Y$. Take the complete f-preimage X of Y, i.e. $X = \{\overline{u} \,|\, f(\overline{u}) \in Y\}$. Clearly, X is a non-empty computable set of tuples of length n if Y is computable. Fix any Gödel enumeration of all tuples in ω^n.

We construct the sets $X_1, \dots, X_n$ and $Z \subseteq X$ in steps:

Step 0. $X_1^0 = \{x_1\}, \dots, X_n^0 = \{x_n\}$ and $Z^0 = \varnothing$.

Step $s+1$. Let $\overline{z} = \langle z_1, \dots, z_n \rangle$ be the first tuple from X that does not belong to the set $X_1^s \times \dots \times X_n^s \cup Z^s$. If $(X_1^s \cup \{z_1\}) \times \dots \times (X_n^s \cup \{z_n\}) \subseteq X$, then we put $X_1^{s+1} = X_1^s \cup \{z_1\}, \dots, X_n^{s+1} = X_n^s \cup \{z_n\}$ and $Z^{s+1} = Z^s$ (in this case, it may turn out that $X_i^{s+1} = X_i^s \cup \{z_i\}$ for some $1 \leqslant i \leqslant n$), otherwise $X_1^{s+1} = X_1^s, \dots, X_n^{s+1} = X_n^s$ and $Z^{s+1} = Z^s \cup \{\overline{z}\}$. End of step $s+1$.

Now define $X_k = \bigcup_{s\in\omega} X_k^s, 1 \leqslant k \leqslant n$ and $Z = \bigcup_{s\in\omega} Z^s$.

By construction, the computable set X splits into two enumerable disjoint parts — $X_1 \times \dots \times X_n$ and Z, which implies the computability of the direct product. Note that both of these sets are η-closed, and if some set $\langle z_1, \dots, z_n \rangle$ is distributed over $X_k, 1 \leqslant k \leqslant n$ at some step (i.e. $z_1 \in X_1, \dots, z_n \in X_n$), then all sets η-equivalent to this set that appear at later steps are distributed in the same way. The correctness of the construction can easily be shown by induction on the steps of the construction. □

The most important open question: *Is it possible to replace computably separated generated spaces with separably generated spaces in the formulation of proposition 2.2?*

Note that the effective separability of an equivalence means that it lies in the class Π_2^0 of the arithmetic hierarchy, although it does not have a clear "coordinatization" in it, since there exist both effectively separable equivalences outside the class Δ_2^0 and Δ_2^0-equivalences that are not effectively separable, but both Σ_1^0-equivalences and Π_1^0-equivalences lie in the class of effectively separable equivalences ([5]).

3. Effectively non-degenerate representations of fields with additive inversions

Based on the above, a mapping $\nu : \omega \longrightarrow F$ onto the underlying set of an at most countable field $\langle F; +, \bullet, - \rangle$ in a ring signature with the operation of additive inversion is called an algorithmic representation of this field if there exist two binary computable functions f, g such that $\nu x + \nu y = \nu f(x, y)$, $\nu x \bullet \nu y = \nu g(x, y)$ for all $x, y \in \omega$ and one unary computable function h with the property $-\nu x = \nu h(x)$ for all $x \in \omega$.

If η is an equivalence on ω, then F is called representable over η (or an η-field) if there exists a computable algebra of ring signature $R = \langle \omega; +, \bullet \rangle$ with computable operations $+, \bullet$ for which η is a congruence (i.e. $x_0 = y_0 \pmod{\eta} \wedge x_1 = y_1 \pmod{\eta} \Rightarrow x_0 + x_1 = y_0 + y_1 \pmod{\eta}$ and $x_0 = y_0 \pmod{\eta} \wedge x_1 = y_1 \pmod{\eta} \Rightarrow x_0 \bullet x_1 = y_0 \bullet y_1 \pmod{\eta}$) such that F is isomorphic to the quotient algebra $\langle \omega/\eta; +, \bullet \rangle$ of the computable algebra $R = \langle \omega; +, \bullet \rangle$ by the congruence η.

Note that the algebra R may not be a ring. In particular, the laws of associativity and commutativity may not hold in R. The only important thing is that there exists a homomorphism from R onto F, which is a morphism that is an algorithmic representation of ν, where $\nu(x) = \{x\}/\eta$ is the natural projective homomorphism.

Definition 3.1. An algorithmic representation ν of a universal algebra A is called effectively non-degenerate if there exists a non-empty ν-enumerable subset $A_0 \subseteq A$.

Recall that a unary termal operation with fixed elements of an algebra as parameters is called a translation.

Definition 3.2. A universal algebra is called translationally complete if every pair of its distinct elements can be translated into any other pair of distinct elements by a suitable translation.

Note that every translationally complete universal algebra is congruence simple. The converse is not true.

Definition 3.3. A universal algebra is called translationally precomplete if there exists a pair of its distinct elements to which any pair of distinct elements can be translated by a suitable translation.

Obviously, every translationally precomplete universal algebra is subdirectly indecomposable. The converse is not true.

Proposition 3.4. *Every T_2-separable representation of a translationally precomplete algebra is negative.*

Proof: Let (A, ν) be a T_2-separable enumerated translationally precomplete algebra and let a pair of distinct elements $\langle a, b \rangle$ of this algebra be translationally reachable from any pair of distinct elements. Then this pair is also contained in any of its nonzero congruences (i.e. A is subdirectly irreducible). Fix ν-indices of these elements, say $\nu m = a, \nu n = b$. By hypothesis, there exist $ker(\nu)$-closed enumerable disjoint sets α and β such that $m \in \alpha$ and $n \in \beta$.

By T_ν we denote the enumerable set of all translations of A given by the corresponding computable representations of the operations of this algebra in the enumeration ν. Then $x \neq y \Leftrightarrow \exists t \in T_\nu(t(x) \in \alpha \wedge t(y) \in \beta)$. □

In [10] it is shown that the following holds:

Proposition 3.5. *There exists a translationally precomplete algebra possessing a T_1-separable non-Hausdorff effectively separable algorithmic representation ν such that the operations of this algebra are continuous in the enumerable $\tau(ker(\nu))_{CE}$-topology.*

Theorem 3.6. *Every skew field with additive inversion is translationally complete.*

Proof: Let two arbitrary distinct elements a, b of the skew field F be given. Let us fix an arbitrary pair of distinct elements c and d. Let us show that there exists a translation $\lambda x[t(x)]$ (the symbol λ serves for the standard λ-notation) of the skew field F that maps the element a to c, and the element b to d. So let us define the following set of translations $T(a)$, which have a very simple form of linear polynomials: $T(a) = \{\lambda x[f(x-a)+c] | f \in F\}$. Obviously, every translation $\lambda x[t(x)]$ from $T(a)$ maps the element a to c. At the same time, for any $b \neq a$ there is a unique translation $\lambda x[f_0(x-a)+c] \in T(a)$ that takes element b to d. Clearly, this translation will be $\lambda x[f_0(x-a)+c]$ with $f_0 = (d-c)(b-a)^{-1}$. □

Corollary 3.7. *Every field with additive inversion is translation complete.*

Corollary 3.8. *Every effectively non-degenerate representation of any field with additive inversion is negative.*

Corollary 3.9. *The operations of any field with additive inversion are continuous with respect to the natural topology over any of its effectively non-degenerate algorithmic representations.*

If η is an equivalence on ω, then the η-closure of α, denoted by $[\alpha]_\eta$, is the intersection of all η-closed extensions of α.

A set is called η-infinite (η-finite) if its η-closure consists of an infinite (finite) number of η-equivalence classes. Obviously, a necessary (but not sufficient) condition for a set to be η-infinite is that it is infinite.

If η is an equivalence and δ is a set, then we say that an index x is η-rejected by δ if $x \notin [\delta]_\eta$. It is easy to see that for any fixed negative equivalence η the relation "a positive integer x is η-rejected by a finite set δ" is uniformly enumerable over x, δ (this implies an explicit specification of all elements of the set δ, for example, by means of its canonical index). If it is clear from the context what negative enumeration (or its core) is meant, then we will say that x is rejected by this finite set.

Note that the maximal difference in the topologies of $\tau(\eta)_C$ and $\tau(\eta)_{CE}$ is observed, for example, for perfect equivalences (i.e. positive ones that do not have nontrivial subsets closed with respect to this equivalence, [5]). In this case, $\tau(\eta)_C$-space is trivial, and $\tau(\eta)_{CE}$ is discrete.

For negative equivalences, the situation is diametrically opposite, as shown by

Theorem 3.10. *If ν is a negative enumeration, then the topologies $\tau(\eta)_C$ and $\tau(\eta)_{CE}$ coincide.*

Proof: In one direction this is obvious, since every computable set is computably enumerable.

Let α be a ν-closed subset of ω that is open in the $\tau(\nu)_{CE}$-topology.

Without loss of generality we can assume that α is a basic (i.e. enumerable) neighborhood. Let $a \in \alpha$.

To prove the theorem it suffices to construct a partition of ω into ν-closed enumerable disjoint sets β and δ such that $a \in \beta \subseteq \alpha$ and $\beta \cup \delta = \omega$. We will construct them using a step-by-step construction. The finite parts of the sets β and δ accumulated before step s+1 will be denoted by β_s and δ_s, respectively.

Note that the case $\omega \setminus \alpha = \emptyset$ is also taken into account by the proposed construction.

Step 0. Let $\beta_0 = \{a\}$, $\delta_0 = \emptyset$.

Step s+1. Let z be the smallest positive integer from $\omega \setminus (\beta_s \cup \delta_s)$. Simultaneously enumerating the sets $\omega^2 \setminus ker(\nu)$ and α, we wait until at least one of the following conditions is confirmed (it will be shown later that this will definitely happen):

(a) (z is rejected by the set δ_s) $\wedge$ ($z \in \alpha$);

(b) z is rejected by the set β_s.

If (a) is satisfied, then add z to the enumeration β, otherwise to the enumeration δ.

Lemma 3.11. *The following statements are true:*

(1) $\beta \subseteq \alpha$;

(2) $\beta \cap \delta = \emptyset$;

(3) any element of β is rejected by all elements of δ and vice versa*;*

(4) $\beta \cup \delta = \omega$.

Proof: Items 1 and 2 are obvious. Item 3 follows from the fact that, as can be easily verified by induction on the steps of the construction, each element of β_s is rejected by all elements of δ_s and vice versa.

Let condition (b) not be satisfied, i.e. z is not rejected by the set β_s. This means that z is ν-equivalent to an element of β_s, and since by virtue of item 1 already proved we have $\beta_s \subseteq \beta \subseteq \alpha$ and α is ν-closed by condition, we obtain $z \in \alpha$. Further, since z is ν-equivalent to some element of β_s, then by virtue of item 3 already proved we obtain that z is rejected by the set δ_s.

To prove item 4, it suffices to show that at each step s one of the conditions (a), (b) in the description of the construction is necessarily satisfied and, as a result, the next z will fall into either β or δ.

Let condition (b) not be satisfied, i.e. z is not rejected by the set β_s. This means that z is ν-equivalent to some element of β_s, and since by virtue of item 1 already proved we have $\beta_s \subseteq \beta \subseteq \alpha$ and α is ν-closed by condition, we obtain $z \in \alpha$. Further, since z is ν-equivalent to some element from β_s, then, by virtue of the already proven item 3, we have the fact of rejection of z by the set δ_s. Thus, in the case of failure to satisfy condition (b), condition (a) will be satisfied. □

The closedness of the sets β and δ with respect to the kernel of the enumeration easily follows from item 3 of the proven lemma 3.11. □

References

[1] Ershov Y. L., Decidability Problems and Constructive Models. Nauka, Moscow, (1980).

[2] Goncharov S. S., Ershov Y. L., Constructive Models. Kluwer Academic/Plenum Publishers, New York, (2000).

[3] Kasymov N. K., Recursively separable enumerated algebras. Russian Mathematical Surveys, (1996) Vol. 51, Iss. 3, P. 509–538.

[4] Kasymov N. K., Dadazhanov R. N., Ibragimov F. N., Separable algorithmic representations of classical systems and their applications. Journal of Mathematical Sciences, (2024) Vol. 278, Iss. 3, P. 476–519.

[5] Ershov Y. L., Theory of Numberings. Nauka, Moscow, (1977).

[6] Kasymov N. K., Homomorphisms onto effectively separable algebras. Siberian Mathematical Journal, (2016) Vol. 57, Iss. 1, P. 36–50.

[7] Kasymov N. K., Algebras over negative equivalences. Algebra and Logic, (1994) Vol. 33, Iss. 1, P. 46–48.

[8] Kasymov N. K., Enumerated algebras with uniformly recursive-separable classes. Siberian Mathematical Journal, (1993) Vol. 34, Iss. 5, P. 869–882.

[9] Kasymov N. K., Separation axioms and partitions of the set of natural numbers. Siberian Mathematical Journal, (1993) Vol. 34, Iss. 3, P. 468–471.

[10] Kasymov N. K., Morozov A. S., Khodzhamuratova I. A., T_1-Separable numberings of subdirectly indecomposable algebras. Algebra and Logic, (2021) Vol. 60, Iss. 4, P. 263–278.

Karimova N. R.,
National University of Uzbekistan, Tashkent, Uzbekistan
e-mail: nodirakarimova@bk.ru