Uzbek Mathematical Journal
2026, Volume 70, Issue 2, pp. 145–150
DOI: 10.29229/uzmj.2026-2-17

Limit theorem for one scheme of particle evolution according to the critical Galton-Watson branching processes.

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Abstract. We consider the critical Galton-Watson branching process a special form and possibly with the infinite variance the number of descendants of one particle. Local limit theorem for such process proved.

Keywords: Critical Galton-Watson branching process, generation function, local limit theorem
MSC (2020): 60J80, 60F05

1. INTRODUCTION

Let $\{Z(n), \ n \geq 0\}$ be a Galton-Watson branching process starting with a single particle $(Z(0) = 1)$ in which particles leave immediate descendants according to probability law having a generating function

$$f(s) = \sum_{k=0}^{\infty} p_k s^k, \ \ 0 \leq s \leq 1,$$

where $p_k = P(Z(1) = k)$. The first work in which a conditional limit theorem for a subcritical Galton-Watson branching process was proved under the condition of non-degeneration of the process belong to A. M. Yaglom. Yaglom [1] proved for subcritical Galton-Watson branching processes that when $n \to \infty$ the conditional probabilities $P(Z(n) = k \,|Z(n) > 0)$ converge for all integers $k \geq 0$ and the limiting quantities form is probability distribution. An analogue of Yaglom's theorem for Markov branching processes with continuous time was proved by B.A. Sevastyanov [2]. The first work in which local limit theorems for Markov branching processes with continuous time were proved belongs to V.M. Zolotarev [3]. In it was found asymptotic behavior of the probability $P(Z(t) = k)$ as $t \to \infty$ for fixed k. Under the assumption of the existence of the fourth moment of the number of immediate descendants of one particle, V.P. Chistyakov [4] found an asymptotic behavior $P(Z(t) = k)$ as $t, k \to \infty$ for Markov branching processes with continuous time. S.V. Nagaev and R. Mukhamedkhanova [5] proved local limit theorem for critical Galton-Watson branching processes under the assumption of the existence of the fourth moment of the number of immediate descendants of one particle.

The local limit theorem for critical Galton-Watson branching processes was proved in the joint work of H.Kesten, P.Ney, F.Spitzer [6] under the condition $\sum\limits_{k=2}^{\infty} k^2 \ln k \cdot p_k < \infty$: if k and n tend to infinity so that their ratio remains bounded, then

$$\lim_{n\to\infty} \frac{\sigma^4 n^2}{4d} \exp\left\{\frac{2kd}{\sigma^2 n}\right\} P(Z(n) = kd) = 1, \tag{1.1}$$

where $\sigma^2 = f''(1), \ d = g.c.d.\{k : p_k > 0\}$.

Despite the assertion of the authors of [6] that for the validity of the local limit theorem for critical Galton-Watson branching processes it is sufficient that the variance of the number of descendants of one particle be finite, no proof of this assertion was known for more than forty years. The final version of the proof of the local limit theorem for critical Galton-Watson branching processes under the condition of finite variance of the number of immediate descendants belongs to S.V. Nagaev and V. Vakhtel [7]: let $\sigma^2 < \infty$. Then for all $k \leq n$

$$\lim_{n\to\infty} \frac{\sigma^4 n^2}{4d} \left(1 + \frac{2d}{\sigma^2 n}\right)^{k+1} P(Z(n) = kd) = 1. \tag{1.2}$$

We also note that the question of the validity of the local limit theorem for critical Bellman-Harris processes was investigated by V.A. Vatutin [8], V.A. Topchiy [9] proved an analogue of (1.1) for critical

Bellman-Harris processes with discrete time. We also note the works of K.B. Athreya, P. Ney [10], S. Dubic and E. Seneta [11], H. Imai [12] in which the validity of the local limit theorem for supercritical Galton-Watson branching processes was investigated.

Let the generating function $f\left(s\right)$ satisfy the condition

$$f\left(s\right)=s+\left(1-s\right)^{1+\alpha}L\left(1-s\right),\ 0\leq s\leq 1, \tag{1.3}$$

where $\alpha\in(0,1]$, $L\left(x\right)$- slowly varying function as $x\rightarrow +0$.

Let $f_0\left(s\right)=s$ and we defined the iteration of the function $f\left(s\right)$ from relation $f_n\left(s\right)=f\left(f_{n-1}\left(s\right)\right),\ n=1,2,...$ It is well known (see [13]) that when the condition (1.3) hold, then

$$\left(1-f_n\left(0\right)\right)^{\alpha}L\left(1-f_n\left(0\right)\right)\sim\frac{1}{\alpha n} \tag{1.4}$$

as $n\rightarrow\infty$ and

$$Q\left(n\right):=P\left(Z\left(n\right)>0\right)=1-f_n\left(0\right)\sim\frac{1}{\left[\alpha nL\left(1-f_n\left(0\right)\right)\right]^{1/\alpha}} \tag{1.5}$$

as $n\rightarrow\infty$. Moreover, for anyone $x\geq 0$

$$F\left(x\right)=\lim_{n\rightarrow\infty}P\left(Q\left(n\right)Z\left(n\right)\leq x|\,Z\left(n\right)>0\right)=$$
$$=\begin{cases}1-e^{-x}, & as\ \alpha=1,\\ 1-\frac{\alpha}{\Gamma(1/\alpha)}\int\limits_0^{\infty}e^{-(x/u)^{\alpha}}p\left(u,\alpha,1\right)du, & as\ \ 0<\alpha<1,\end{cases}$$

where $p\left(u,\alpha,1\right)$ is the density of one-sided stable law with Laplace transform

$$\int_0^{\infty}e^{-\lambda u}p\left(u,\alpha,1\right)du=e^{-\lambda^{\alpha}},\ \lambda\geq 0.$$

It is known [1] that the Laplace transform $\psi\left(\lambda\right)$ corresponding to the distribution $F\left(x\right)$ has the form

$$\psi\left(\lambda\right)=1-\lambda\left(1+\lambda^{\alpha}\right)^{-1/\alpha}.$$

We note that in the case $\alpha=0$ of the well-known result of S.V. Nagaev and V. Vakhtel [7] that the conditional distribution of the correspondingly transformed $Z\left(n\right)$ under the condition of non-degeneration $\left(Z\left(n\right)>0\right)$ by the moment n weakly converges to the exponential law.

Slack [13] proved that if (1.3) hold, then for any (fixed) $i,j\geq 1$ there are limit

$$\lim_{n\rightarrow\infty}L^{1/\alpha}\left(1-f_n\left(0\right)\right)\left(\alpha n\right)^{1+1/\alpha}P_n\left(i,j\right)=i\pi\left(j\right), \tag{1.6}$$

at that

$$\sum_{i=1}^{\infty}\pi\left(i\right)P_1\left(i,j\right)=\pi\left(j\right),\ j\geq 1,\ \sum_{i=1}^{\infty}\pi\left(i\right)p_0^i=1 \tag{1.7}$$

and as $J\rightarrow\infty$

$$\pi\left(1\right)+\pi\left(2\right)+...+\pi\left(J\right)\sim\frac{1}{\alpha\left(1+\alpha\right)}J^{\alpha}\frac{1}{L\left(\frac{1}{J}\right)}, \tag{1.8}$$

where $P_n\left(i,j\right)$ is probability transition of the chain $\left\{Z\left(n\right),\ n\geq 0\right\}$ from state i to state j for n-step.

We note that relations (1.6)-(1.8) are proved in [6] for the case $\alpha=1,\ \sigma^2<\infty$.

However the asymptotic of the probability $P_n\left(i,j\right)$ for $n,j\rightarrow\infty$ in the case when (1.3) and $\sigma^2=\infty$ unknown.

Let condition A be satisfied: the probability generating function $f\left(s\right)$ has the form

$$f\left(s\right)=1-\left[C\alpha+\left(1-s\right)^{-\alpha}\right]^{-1/\alpha},\ s\in[0,1),\ f\left(1\right)=1, \tag{1.9}$$

where $\alpha\in(0,1]$, C is positive constant number.

It is easy to see that

$$f\left(s\right)=s+\left(1-s\right)^{1+\alpha}L^{*}\left(1-s\right)$$

as $s\rightarrow 1$, here $L^{*}\left(x\right)\rightarrow C$ as $x\rightarrow 0$, $f'\left(1\right)=1$ and $f''\left(1\right)=+\infty$ for $\alpha\in(0,1)$.

The purpose of this paper is to establish the asymptotic of the transition probabilities $P_n\left(1,k\right)$ of the critical Galton-Watson branching process $\left\{Z\left(n\right),\ n\geq 0\right\}$under condition A in the case when $n,k\rightarrow\infty,\ \ k=o\left(n^{1/(2-\alpha)}\right)$.

2. FORMULATION OF THE RESULT.

Theorem 2.1. *Let the probability generating function $f(s)$ satisfy condition A. Then*
1. for any fixed integer $k \geq 1$

$$C^{1/\alpha}\left[\alpha n\right]^{1+1/\alpha} P_n(1,k) \to \pi_k := \frac{1}{Ck}\prod_{i=1}^{k-1}\left(1+\frac{\alpha}{i}\right) \tag{2.1}$$

as $n \to \infty$;
2. as $n \to \infty,\ k \to \infty,\ k = o\left(n^{1/(2-\alpha)}\right)$

$$\left(C\alpha n\right)^{1/\alpha+1}\Gamma(\alpha+1)\,k^{1-\alpha}P_n(1,k) \to 1, \tag{2.2}$$

3. as $n \to \infty,\ k \to \infty,\ k = o(n)$

$$\pi_k - \frac{f_n^{(k)}(0)}{Ck!f_n'(0)} = \mathrm{O}\left(\frac{k}{n}\right),\ \ \pi_k \sim \frac{1}{C\Gamma(1+\alpha)\,k^{1-\alpha}}. \tag{2.3}$$

3. AUXILIARY CONCEPTS AND RESULTS.

Let $N = \{1,2,...\}$ and $N_0 = N \cup \{0\}$. For each pair of integers $k \leq J$, we introduce the set

$$D(J,k) := \{i_1 + i_2 + ... + i_{J-k+1} = k,\ i_1 + 2i_2 + ... + (J-k+1)\,i_{J-k+1} = J\}.$$

Bell polynomials of the second type are functions $B_{J,k}(x_1,...,x_{J-k+1})$, $k \leq J$ determined from the relation

$$\frac{1}{k!}\left(\sum_{j=1}^{\infty} x_j\frac{s^j}{j!}\right)^k = \sum_{J=k}^{\infty} B_{J,k}(x_1,...,x_{J-k+1})\,\frac{s^J}{J!} \tag{3.1}$$

and they can be calculated using the formula

$$B_{J,k}(x_1,...,x_{J-k+1}) = \sum_{D(J,k)}\frac{J!}{i_1!\cdot...\cdot i_{J-k+1}!}\prod_{r=1}^{J-k+1}\left(\frac{x_r}{r!}\right)^{i_r}.$$

Note that due to (3.1)

$$B_{J,1}(x_1,...,x_J) = \sum_{D(J,k)}\frac{J!}{i_1!\cdot...\cdot i_J!}\prod_{r=1}^{J}\left(\frac{x_r}{r!}\right)^{i_r} = x_J,$$

$$B_{J,k}(\gamma x_1,...,\gamma x_{J-k+1}) = \gamma^k B_{J,k}(x_1,...,x_{J-k+1}). \tag{3.2}$$

It is known ([14], relations (1.7)) that

$$B_{n,k}(1!,2!,...,(n-k+1)!) = C_{n-1}^{k-1}\frac{n!}{k!}, \tag{3.3}$$

here

$$C_n^k = \frac{n!}{k!\,(n-k)!}.$$

We will use Bell polynomials of the second kind to find derivatives of a superposition function $F(G(x))$ from differentiable functions $F(x)$ and $G(x)$ using the Faa-De Bruno formula:

$$\frac{d^k}{dx^k}F(G(x)) = F'(G(x))\,G^{(k)}(x) + \sum_{J=2}^{k}F^{(J)}(G(x))\,B_{k,J}\left(G'(x),...,G^{(k-J+1)}(x)\right), \tag{3.4}$$

where the second term is considered equal to zero if $J = 1$.

4. Proof of the theorem.

It is easy to see that

$$f_n\left(s\right) = 1 - \left[C\alpha n + \left(1-s\right)^{-\alpha}\right]^{-1/\alpha}.$$

We denote

$$F\left(x\right) = 1 - x^{-1/\alpha},\ G\left(x\right) = C\alpha n + \left(1-x\right)^{-\alpha}.$$

We have

$$F^{(k)}\left(x\right) = \left(-1\right)^{k+1}\frac{1}{\alpha}\left(\frac{1}{\alpha}+1\right)...\left(\frac{1}{\alpha}+k-1\right)x^{-1/\alpha-k},$$

$$G^{(k)}\left(x\right) = \alpha\left(\alpha+1\right)...\left(\alpha+k-1\right)\left(1-x\right)^{-\alpha-k}.$$

According (3.4) we have

$$f_n^{(k)}\left(s\right) = \left[\,\mathrm{F}\left(\mathrm{G}\left(\mathrm{s}\right)\right)\right]^{(k)} = \frac{1}{\alpha}\left[C\alpha n + \left(1-s\right)^{-\alpha}\right]^{-1/\alpha-1}\prod_{i=0}^{k-1}\left(\alpha+i\right)\left(1-s\right)^{-\alpha-k} +$$

$$+\sum_{j=2}^{k}\left(-1\right)^{j+1}\prod_{i=0}^{j-1}\left(\frac{1}{\alpha}+i\right)\left[C\alpha n + \left(1-s\right)^{-\alpha}\right]^{-\frac{1}{\alpha}-j}\times$$

$$\times B_{k,j}\left(\alpha\left(1-s\right)^{-\alpha-1}, \alpha\left(\alpha+1\right)\left(1-s\right)^{-\alpha-2}, ..., \prod_{i=0}^{k-j}\left(\alpha+i\right)\left(1-s\right)^{-\alpha-k+j-1}\right).$$

From here, considering the relation (3.1) and (3.2), we obtain

$$f_n^{(k)}\left(0\right) = \frac{1}{\alpha}\left[C\alpha n+1\right]^{-1/\alpha-1}\prod_{i=0}^{k-1}\left(\alpha+i\right) +$$

$$+\sum_{j=2}^{k}\left(-1\right)^{j+1}\prod_{i=0}^{j-1}\left(\frac{1}{\alpha}+i\right)\left[C\alpha n+1\right]^{-\frac{1}{\alpha}-j}B_{k,j}\left(\alpha, \alpha\left(\alpha+1\right), ..., \alpha\prod_{i=1}^{k-j}\left(\alpha+i\right)\right) =$$

$$= \frac{1}{\alpha}\left[C\alpha n+1\right]^{-1/\alpha-1}\prod_{i=0}^{k-1}\left(\alpha+i\right) + \Delta_{k,n}, \tag{4.1}$$

where

$$\Delta_{k,n} = \sum_{j=2}^{k}\left(-1\right)^{j+1}\prod_{i=0}^{j-1}\left(\frac{1}{\alpha}+i\right)\left[C\alpha n+1\right]^{-\frac{1}{\alpha}-j}\alpha^{j}B_{k,j}\left(1, \left(\alpha+1\right), ..., \prod_{i=1}^{k-j}\left(\alpha+i\right)\right). \tag{4.2}$$

From (3.1) follows

$$B_{k,j}\left(1, \left(\alpha+1\right), ..., \prod_{i=1}^{k-j}\left(\alpha+i\right)\right) \le B_{k,j}\left(1!, 2!, ..., \left(k-j+1\right)!\right). \tag{4.3}$$

Further, relations (4.2)-(4.3) and (3.3) lead us to the estimate

$$\left|\Delta_{k,n}\right| \le \left[C\alpha n+1\right]^{-1/\alpha-2}\sum_{j=2}^{k}\prod_{i=0}^{j-1}\left(\frac{1}{\alpha}+i\right)\alpha^{j}C_{k-1}^{j-1}\frac{k!}{j!}n^{-j+2} \le \tag{4.4}$$

$$\le \left[C\alpha n+1\right]^{-1/\alpha-2}kk!\sum_{j=2}^{k}\frac{1}{j!}\alpha^{j-1}\prod_{i=1}^{j-1}\left(1+\frac{1}{i\alpha}\right).$$

The series $\sum_{j=2}^{\infty} \frac{1}{j!}\alpha^j \prod\limits_{i=1}^{j-1} \left(1+\frac{1}{i\alpha}\right)$ is converges. Now from (4.1) and (4.4) we conclude that

$$P_n(1,k) = \frac{1}{k!} f_n^{(k)}(0) = \frac{1}{k}\left[C\alpha n+1\right]^{-1/\alpha-1} \prod_{i=1}^{k-1}\left(1+\frac{\alpha}{i}\right) + \delta_{k,n}, \tag{4.5}$$

here

$$|\delta_{k,n}| \le \left|\frac{1}{k!}\Delta_{k,n}\right| \le \left[C\alpha n+1\right]^{-1/\alpha-2} k \sum_{j=2}^{\infty} \frac{1}{j!}\alpha^{j-1} \prod_{i=1}^{j-1}\left(1+\frac{1}{i\alpha}\right),$$

from which the validity of (2.1) follows. Now we write (4.5) in the form

$$k\left[C\alpha n+1\right]^{1/\alpha+1} \left[\prod_{i=1}^{k-1}\left(1+\frac{\alpha}{i}\right)\right]^{-1} P_n(1,k) = 1+\tau_{k,n}, \tag{4.6}$$

here

$$|\tau_{k,n}| \le \left[C\alpha n+1\right]^{-1} k^2 \left[\prod_{i=1}^{k}\left(1+\frac{\alpha}{i}\right)\right]^{-1} \sum_{j=2}^{\infty} \frac{1}{j!}\alpha^{j-1} \prod_{i=1}^{j-1}\left(1+\frac{1}{i\alpha}\right)$$

Further, according to the Weierstrass's formula

$$\Gamma(\alpha) = \frac{e^{-\alpha\gamma}}{\alpha} \prod_{i=1}^{\infty}\left(1+\frac{\alpha}{i}\right)^{-1} e^{\alpha/i},$$

here γ is the Euler-Mascheroni's constant. Hence considering the main property Γ-function ($\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$) and Euler's formula for the harmonic series we find for large$k$

$$\prod_{i=1}^{k}\left(1+\frac{\alpha}{i}\right) = \frac{1}{\alpha e^{\alpha\gamma}\Gamma(\alpha)} \prod_{i=1}^{k} e^{\alpha/i}(1+o(1)) = \frac{1}{\alpha e^{\alpha\gamma}\Gamma(\alpha)} e^{\alpha\sum_{i=1}^{k}\frac{1}{i}}(1+o(1)) =$$

$$= \frac{1}{\alpha e^{\alpha\gamma}\Gamma(\alpha)} e^{\alpha(\gamma+\ln k)}(1+o(1)) = \frac{k^\alpha}{\alpha\Gamma(\alpha)}(1+o(1)) = \frac{k^\alpha}{\Gamma(\alpha+1)}(1+o(1)). \tag{4.7}$$

From (4.6)-(4.7) as $n\to\infty,\ k\to\infty,\ k = o\left(n^{\frac{1}{2-\alpha}}\right)$ we conclude

$$(C\alpha n)^{1/\alpha+1}\Gamma(\alpha+1)\, k^{1-\alpha} P_n(1,k) = 1 + \mathrm{O}\left(\frac{k^{2-\alpha}}{n}\right) \to 1,$$

which proves the validity of (2.2). Now we will show the validity of (2.3). We put

$$q_{k,n} = \frac{P_n(1,k)}{CP_n(1,1)} = \frac{f_n^{(k)}(0)}{Ck!f_n'(0)},\ k = 1,2,...$$

From (4.1) we have

$$f_n'(0) = \left[C\alpha n+1\right]^{-1/\alpha-1}. \tag{4.8}$$

From (4.5) and (4.8) we conclude that for any fixed $k \ge 1$ the sequence $q_{k,n},\ n \ge 1$ converges to a finite limit

$$\lim_{n\to\infty} q_{k,n} = \pi_k = \frac{1}{Ck}\prod_{i=1}^{k-1}\left(1+\frac{\alpha}{i}\right). \tag{4.9}$$

Further, by virtue of (4.1), (4.4) and (4.8) we have

$$|q_{k,n+1} - q_{k,n}| \le K\frac{k}{Cn^2}\sum_{j=2}^{k}\prod_{i=1}^{j-1}\left(1+\frac{1}{i\alpha}\right)\cdot\frac{1}{(j-1)!}$$

where K is positive constant depending on α.
Conseq uently,

$$|\pi_k - q_{k,n}| = \left|\sum_{r=n}^{\infty}(q_{k,r+1} - q_{k,r})\right| \le K\frac{k}{Cn}\sum_{j=2}^{\infty}\prod_{i=1}^{j-1}\left(1+\frac{1}{i\alpha}\right)\cdot\frac{1}{(j-1)!}.$$

It follows that for all $k = o(n)$ as $n \to \infty$

$$\sup_{k\le n, k=o(n)} |\pi_k - q_{k,n}| \to 0.$$

From the relation (4.7) and (4.9) we concluded that

$$\pi_k \sim \frac{1}{\Gamma(1+\alpha)\,k^{1-\alpha}}$$

as $k \to \infty$. By (4.9) it is easy to see that the sequence $\{\pi_k, k \in N\}$ is monotonically decreasing.
The proof of the theorem is complete.

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