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Optimization of the Euler-Maclaurin type quadrature formula in the Hilbert space of periodic functions

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Abstract. In this paper, we address the construction of an optimal quadrature formula in the sense of Sard within a Hilbert space composed of periodic, complex-valued functions, specifically for the purpose of numerically approximating Fourier integrals. The quadrature formula is expressed as a linear combination of the function's values taken at nodes of a uniform grid. An upper estimate of the quadrature error is obtained through the norm of the associated error functional, which is evaluated using the Cauchy-Schwarz inequality. To determine this norm explicitly, the method relies on the concept of an extremal function. The extremal function corresponding to the error functional is identified by applying the Riesz representation theorem, which guarantees that the norm of this function coincides with the norm of the error functional in the dual (conjugate) space. Ultimately, the resulting expression for the norm of the error functional takes the form of a multivariate quadratic function dependent on the coefficients of the quadrature formula. In addition, this work determines the optimal coefficients of a derived quadrature formula. Using these optimal coefficients, the exponential-weighted integrals of several functions have been approximately evaluated.

Keywords: Hilbert space, extremal function, error functional, optimal quadrature formula, strongly oscillatory integrals, Fourier transform
MSC (2020): 41A05, 41A15, 65D30, 65D32

1. Introduction and statement of the problem

1.1. **Introduction.** The numerical computation of strongly oscillating integrals is a critical problem in numerical analysis, as such integrals are extensively applied in science and technology. The following types of Fourier integrals also serve as examples of strongly oscillating integrals for sufficiently large values of $\omega \in \mathbb{R}$

$$I(\omega, \varphi) = \int_0^1 e^{2\pi i\omega x}\varphi(x)dx,$$

where φ is non-oscillating functions.

The numerical approximation of these integrals can be particularly challenging when the oscillations are highly localized and rapidly alternating. However, several techniques exist to achieve accurate numerical approximations of such integrals. One such method is the stationary phase method, which involves identifying the stationary points of the integrand and approximating the integral based on the behaviour of the integrand near these points. Another approach is to apply a quadrature method specifically designed for oscillatory integrals, such as the Filon method, Clenshaw-Curtis method, modified Clenshaw-Curtis method, Levin-type methods, Gauss-Laguerre quadrature, and generalized quadrature methods (for comprehensive details, see [1], for example, [2, 3] and the references therein).

In recent years, Kh.M. Shadimetov [4, 5, 6, 7, 8], G.V. Milovanović [2, 1], and A.R. Hayotov [9, 10, 11, 12, 13], N.D. Boltaev [14], I.O. Jalolov [15, 16] and S.S. Babaev [17, 18, 19] have conducted scientific research focused on constructing optimal quadrature formulas for calculating Fourier coefficients and integrals in various spaces, including $L_2^{(m)}$ and $W_2^{(m,m-1)}$. The outcomes of constructing optimal quadrature formulas for the numerical evaluation of Fourier coefficients in $\widetilde{W}_2^{(2,1)}$ space of periodic functions, as well as applying these formulas to reconstruct computed tomography images, were obtained in the studies of A.R. Hayotov [20]. It can also be added that optimal quadrature formulas for the numerical calculation of singular integrals in the spaces $L_2^{(m)}$ and $W_2^{(m,m-1)}$ are constructed in the following works [21, 22, 23].

In this paper, the problem of constructing an optimal quadrature formula of Euler-Maclaurin type in the Hilbert space $\widetilde{W}_2^{(2,1)}$ of periodic functions is studied.

1.2. **Statement of the problem.** This study focuses on the derivation of optimal quadrature formulas under the assumption that the integrand lies within the Hilbert space $W_2^{(2,1)}$. For a detailed description of this function space, the reader is referred to the definition provided, for instance, in the references [14, 24, 25, 26].

The space $W_2^{(2,1)}$ is a Hilbert space consisting of complex-valued functions, and it is defined as follows

$$W_2{}^{(2,1)} = \{\varphi : [0,1] \to \mathbb{C} | \varphi' \text{ is abs. continuous and } \varphi'' \in L_2(0,1)\}.$$

The space is the Hilbert space with the inner product

$$\langle \varphi, \psi \rangle_{W_2^{(2,1)}} = \int_0^1 (\varphi''(x) + \varphi'(x))(\overline{\psi}''(x) + \overline{\psi}'(x))dx, \tag{1.1}$$

and the corresponding norm is

$$\|\varphi\|_{W_2^{(2,1)}} = \left(\langle \varphi, \varphi \rangle_{W_2^{(2,1)}} \right)^{1/2}.$$

Let $\widetilde{W}_2^{(2,1)}(0,1]$ denote the subspace of $W_2^{(2,1)}(0,1)$ that consists of 1-periodic complex-valued functions. Each function belonging to $\widetilde{W}_2^{(2,1)}(0,1]$ satisfies the condition of 1-periodicity

$$\varphi(x + \beta) = \varphi(x) \text{ for } x \in \mathbb{R} \text{ and } \beta \in \mathbb{Z}.$$

We consider the quadrature formula of the following form

$$\int\limits_0^1 e^{2\pi i w x} \varphi(x) dx \cong \sum_{k=1}^{N} C_k \varphi(hk) + \sum_{k=1}^{N} A_k \varphi'(hk), \tag{1.2}$$

where $\omega \in \mathbb{Z} \setminus \{0\}$, $\varphi \in \widetilde{W_2}^{(2,1)}$, C_k coefficients are given in the work [5]

$$C_k = \frac{2}{4\pi^2\omega^2 + 1} \cdot \frac{e^{2h} - 2e^h \cos(2\pi\omega h) + 1}{e^{2h} - 1} \cdot e^{2\pi i \omega h k} \quad \text{for} \quad k = 1, 2, \dots, N,$$

the coefficients A_k are unknowns and need to be determined, while h represents the step size of the equal mesh. The difference between the integral and the quadrature sum is called *the error* of the quadrature formula (1.2)

$$(\ell, \varphi) = \int_0^1 e^{2\pi i w x} \varphi(x) dx - \sum_{k=1}^{N} C_k \varphi(hk) - \sum_{k=1}^{N} A_k \varphi'(hk), \tag{1.3}$$

and the corresponding *error functional* is

$$\ell(x) = \left(e^{2\pi i w x} \varepsilon_{(0,1]}(x) - \sum_{k=1}^{N} C_k \delta(x - hk) + \sum_{k=1}^{N} A_k \delta'(x - hk) \right) * \phi_0(x), \tag{1.4}$$

where $\varepsilon_{(0,1]}(x)$ is the characteristic function of the interval $(0,1]$, δ is the Dirac delta-function and $\phi_0(x) = \sum\limits_{\beta=-\infty}^{\infty} \delta(x - \beta)$. The error functional ℓ is called the periodic error functional of the quadrature formula (1.2), and it belongs to the conjugate space $\widetilde{W}_2^{(2,1)*}(0,1]$.

Since the error functional ℓ is defined on the space $\widetilde{W}_2^{(2,1)}(0,1]$, the following equality is valid as in the work [27]

$$(\ell, 1) = 0. \tag{1.5}$$

The error (1.3) of the quadrature formula (1.3) is a linear functional in $\widetilde{W}_2^{(2,1)*}(0,1]$. The absolute value of the error (1.3) is estimated by the Cauchy-Schwarz inequality as follows

$$|(\ell, \varphi)| \le \|\ell\|_{\widetilde{W_2}^{(2,1)*}} \cdot \|\varphi\|_{\widetilde{W_2}^{(2,1)}},$$

where

$$\|\ell\|_{\widetilde{W_2}^{(2,1)*}} = \sup_{\varphi,\ \|\varphi\|_{\widetilde{W_2}^{(2,1)}}\neq 0} \frac{|(\ell,\varphi)|}{\|\varphi\|}$$

is the norm of the error functional (1.4).

Hence, to get the least upper bound of the error for the quadrature formula (1.2), we solve the following problems.

Problem 1.1. *Find the norm of the error functional* (1.4) *of the quadrature formula* (1.2) *in the space* $\widetilde{W}_2^{(2,1)*}(0,1]$.

It is clear that the norm of the error functional ℓ depends on the coefficients A_k. Minimizing $\|\ell\|$ by adjusting the coefficients A_k is a simple linear problem.

Problem 1.2. *Find the coefficients* $\overset{\circ}{A}_k$ *that minimize the value of the norm of the error functional* (1.4).

The coefficients that give $\|\ell\|_{\widetilde{W_2}^{(2,1)*}}$ the minimum value are called *optimal coefficients* and are denoted by $\overset{\circ}{A}_k$. The quadrature formula (1.2) with these coefficients is *the optimal quadrature formula in the sense of Sard.* Similar problem as Problem 1.1 was first proposed by S.L. Sobolev [27], later the problem was solved in the space $W_2^{(m,m-1)}$ of functions in works [9] and in the space $\widetilde{L_2^{(m)}}$ of periodic functions in works [28] and in works [28, 4] for optimal quadrature formulas of Euler-Maclaurin type.

Furthermore, to solve Problems 1.1 and 1.2 in the next sections, we will do the following:

- first, we find the extremal function of the quadrature formula (1.2) and use it to find the analytic representation of the norm for the error functional (1.4) in section 2;
- in section 3, the process of solving the system of linear equations for the coefficients of the optimal quadrature formula is presented;
- and in the last section, we find the sharp upper bound of the error of the constructed optimal quadrature formula.

2. Finding the norm of the error functional

In this section, we will deal with Problem 1.1. To do this, we use the concept of an extremal function.

2.1. **The extremal function.** To calculate the norm $\|\ell\|_{\widetilde{W_2}^{(2,1)*}}$, we use *the extremal function* ψ_ℓ for the error functional ℓ (see [27]) that satisfies the following equality:

$$(\ell,\psi_\ell) = \|\ell\|_{\widetilde{W_2}^{(2,1)*}} \cdot \|\psi_\ell\|_{\widetilde{W_2}^{(2,1)}}. \tag{2.1}$$

Since $\widetilde{W_2}^{(2,1)}$ is the Hilbert space, using the Riesz representation theorem, we obtain

$$(\ell,\varphi) = \langle\varphi,\psi_\ell\rangle_{\widetilde{W_2}^{(2,1)}}, \tag{2.2}$$

where $\langle\psi_\ell,\varphi\rangle_{\widetilde{W_2}^{(2,1)}}$ is the inner product of the functions ψ_ℓ and φ which is defined by expression (1.1). In addition, the equality $\|\ell\|_{\widetilde{W_2}^{(2,1)*}} = \|\psi_\ell\|_{\widetilde{W_2}^{(2,1)}}$ is satisfied. Thus, taking into account (2.1), we obtain

$$(\ell,\psi_\ell) = \|\ell\|^2_{\widetilde{W_2}^{(2,1)*}}. \tag{2.3}$$

Then from (2.2), integrating by parts, we obtain

$$\int_0^1 \ell(x)\varphi(x)dx = \int_0^1 \left(\overline{\psi_\ell}^{IV}(x) - \overline{\psi_\ell}''(x)\right)\varphi(x)dx.$$

From the last equation, we have

$$\overline{\psi_\ell}^{IV}(x) - \overline{\psi_\ell}''(x) = \ell(x). \tag{2.4}$$

For the solution of the last equation the following holds.

Theorem 2.1. *The solution of differential equation (2.4) is identified as the extremal function ψ_ℓ associated with the error functional ℓ, and can be expressed in the following form:*

$$\psi_\ell(x) = d_0 + e^{-2\pi i\omega x}\cdot\kappa(\omega) - \sum_{k=1}^{N}\overline{C_k}\sum_{\beta\neq 0}\kappa(\beta)e^{2\pi i\beta(x-hk)} - \sum_{k=1}^{N}\overline{A_k}\sum_{\beta\neq 0}\kappa(\beta)\cdot(2\pi i\beta)e^{2\pi i\beta(x-hk)}, \tag{2.5}$$

where d_0 is a complex number, $\overline{C_k}$ and $\overline{A_k}$ are the complex conjugates of C_k, A_k, respectively and

$$\kappa(\omega) = \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2}. \tag{2.6}$$

Proof: We find the periodic solution of equation (2.4) using the following properties of the Fourier transform (see, for instance [14])

$$F[\varphi] = \int_{-\infty}^{\infty}\varphi(x)e^{2\pi ipx}dx, F^{-1}[\varphi] = \int_{-\infty}^{\infty}\varphi(p)e^{-2\pi ipx}dp,$$

$$F[\varphi^{(\alpha)}] = (-2\pi ip)^{\alpha}F[\varphi],\ (\alpha\in\mathbb{N}), F^{-1}[F[\varphi(x)]] = \varphi(x).$$

We apply the Fourier transform to both sides of equation (2.4)

$$F[\overline{\psi_\ell}^{IV} - \overline{\psi_\ell}''] = F[\ell].$$

Since the Fourier transform is a linear operator, we have

$$\left((2\pi ip)^4 - (2\pi ip)^2\right)F[\overline{\psi_\ell}] = F\left[\ell\right]. \tag{2.7}$$

To find the Fourier transform of the error functional ℓ, we simplify it

$$\ell(x) = \left(e^{2\pi iwx}\varepsilon_{(0,1]}(x) - \sum_{k=1}^{N}C_k\delta(x-hk) + \sum_{k=1}^{N}A_k\delta'(x-hk)\right) * \sum_{\beta=-\infty}^{\infty}\delta(x-\beta).$$

Taking into account the convolution of two continuous functions f and g, which is

$$f(x)*g(x) = \int_{-\infty}^{\infty}f(y)\cdot g(x-y)dy$$

and properties of the Dirac delta-function, we get the following

$$\ell(x) = e^{2\pi iwx} - \sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-hk-\beta) + \sum_{k=1}^{N}A_k\sum_{\beta=-\infty}^{\infty}\delta'(x-hk-\beta). \tag{2.8}$$

Using (2.8) we rewrite eqution (2.7)

$$\left((2\pi ip)^4 - (2\pi ip)^2\right)F[\overline{\psi_\ell}] = F\left[e^{2\pi iwx} - \sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-hk-\beta) + \sum_{k=1}^{N}A_k\sum_{\beta=-\infty}^{\infty}\delta'(x-hk-\beta)\right],$$

or

$$\begin{aligned}\left((2\pi ip)^4 - (2\pi ip)^2\right)F[\bar{\psi}_\ell] &= F\left[e^{2\pi i\omega x}\right] - \sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}F\left[\delta(x-hk-\beta)\right]\\ &+ \sum_{k=1}^{N}A_k\sum_{\beta=-\infty}^{\infty}F\left[\delta'(x-hk-\beta)\right].\end{aligned} \tag{2.9}$$

Now, using the following equalities

$$F\left[e^{2\pi i\omega x}\right]=\delta(p+\omega),\quad \sum_{\beta=-\infty}^{\infty}F\left[\delta'(x-hk-\beta)\right]=(-2\pi ip)e^{2\pi iphk}\sum_{\beta=-\infty}^{\infty}\delta(p-\beta),$$

$$\sum_{\beta=-\infty}^{\infty}e^{2\pi ip\beta}=\sum_{\beta=-\infty}^{\infty}\delta(p-\beta),$$

$$\sum_{\beta=-\infty}^{\infty}F\left[\delta(x-hk-\beta)\right]=\sum_{\beta=-\infty}^{\infty}e^{2\pi ip(hk+\beta)}=e^{2\pi iphk}\sum_{\beta=-\infty}^{\infty}e^{2\pi ip\beta}=e^{2\pi iphk}\sum_{\beta=-\infty}^{\infty}\delta(p-\beta)$$

we can rewrite equation (2.9) as follows

$$\left((2\pi ip)^4-(2\pi ip)^2\right)F[\bar{\psi}_\ell]=\delta(p+\omega)-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}e^{2\pi iphk}\delta(p-\beta)-\sum_{k=1}^{N}A_k\sum_{\beta=-\infty}^{\infty}(2\pi ip)e^{2\pi iphk}\delta(p-\beta).$$

We consider that the coefficient on the left-hand side of the last equation is not equal to zero. Consequently, we can divide both sides of the last equation by $\kappa(p)$, which is defined by (2.6). This division is not uniquely defined. From the last equation, the function $F[\bar{\psi}_\ell]$ is defined up to the term of the form $\delta(p)$. Taking into account the above-mentioned and the properties of the delta-function, we get

$$F[\bar{\psi}_\ell]=\delta(p+\omega)\kappa(p)-\sum_{k=1}^{N}C_k\sum_{\beta\neq 0}e^{2\pi iphk}\delta(p-\beta)\kappa(p)-\sum_{k=1}^{N}A_k\sum_{\beta\neq 0}(2\pi ip)e^{2\pi iphk}\kappa(p)+d_0\delta(p),$$

where $\kappa(p)$ is defined by formula (2.6) and d_0 is a constant.

Using the property $f(x)\delta(x-a)=f(a)\delta(x-a)$ of delta-function, we have the following

$$F[\bar{\psi}_\ell]=\delta(p+\omega)\kappa(\omega)-\sum_{k=1}^{N}C_k\sum_{\beta\neq 0}e^{2\pi i\beta hk}\kappa(\beta)\delta(p-\beta)-\sum_{k=1}^{N}A_k\sum_{\beta\neq 0}(2\pi i\beta)e^{2\pi i\beta hk}\kappa(\beta)\delta(p-\beta)+d_0\delta(p).$$

Then, applying the inverse Fourier transform to both sides of the last equation, we have

$$\bar{\psi}_\ell=e^{2\pi i\omega x}\kappa(\omega)-\sum_{k=1}^{N}C_k\sum_{\beta\neq 0}e^{2\pi i\beta hk}\kappa(\beta)e^{-2\pi i\beta x}-\sum_{k=1}^{N}A_k\sum_{\beta\neq 0}(2\pi i\beta)e^{2\pi i\beta hk}\kappa(\beta)e^{-2\pi i\beta x}+d_0.$$

Since $\bar{\bar{\psi}}_\ell$ is equal to ψ_ℓ, we obtain (2.5), that is, Theorem 2.1 is proved. □

2.2. **The norm of the error functional.** The following result corresponds to the solution to Problem 1.1.

Theorem 2.2. *Within the functional space* $\widetilde{W}_2^{(2,1)*}(0,1]$, *the squared norm* $\|\ell\|_{\widetilde{W}_2^{(2,1)*}}$ *corresponding to* $\omega\in Z\backslash\{0\}$ *is represented by the following expression*

$$\begin{aligned}\|\ell\|^2_{\widetilde{W}_2^{(2,1)*}} &= \|\ell_2\|^2_{\widetilde{W}_2^{(2,1)*}}+2\pi i\omega\cdot\kappa(\omega)\left[\sum_{k=1}^{N}A_ke^{-2\pi i\omega hk}+\sum_{k'=1}^{N}\overline{A_{k'}}e^{2\pi i\omega hk'}\right]\\ &+\sum_{k=1}^{N}\sum_{k'=1}^{N}\sum_{\beta\neq 0}2\pi i\beta\cdot\kappa(\beta)e^{2\pi i\beta h(k-k')}\left[A_k\overline{C_{k'}}+C_k\overline{A_{k'}}\right]\\ &+\sum_{k=1}^{N}\sum_{k'=1}^{N}A_k\overline{A_{k'}}\sum_{\beta\neq 0}(2\pi i\beta)^2\cdot\kappa(\beta)e^{2\pi i\beta h(k-k')},\end{aligned}$$

where

$$\|\ell_2\|^2_{\widetilde{W}_2^{(2,1)*}}=\kappa(\omega)-\kappa(\omega)\left[\sum_{k=1}^{N}C_ke^{-2\pi i\omega hk}+\sum_{k'=1}^{N}\overline{C_{k'}}e^{2\pi i\omega hk'}\right]+\sum_{k=1}^{N}\sum_{k'=1}^{N}C_k\overline{C_{k'}}\sum_{\beta\neq 0}\kappa(\beta)e^{2\pi i\beta h(k-k')}$$

and $\kappa(\cdot)$ *is defined by* (2.6).

Proof: To prove Theorem 2.2, we calculate the norm $\|\ell\|_{\widetilde{W}_2^{(2,1)*}}$ and use equalities (2.3), (2.8) and (2.5), respectively. As a result we have the following

$$\begin{aligned}\|\ell\|^2_{\widetilde{W}_2^{(2,1)*}} = (\ell,\psi_\ell) &= \int_0^1 \ell(x)\Bigg(d_0 + e^{-2\pi i\omega x}\cdot\kappa(\omega) - \sum_{k=1}^{N}\overline{C_k}\sum_{\beta\neq 0}\kappa(\beta)e^{2\pi i\beta(x-hk)} \\ &- \sum_{k=1}^{N}\overline{A_k}\sum_{\beta\neq 0}\kappa(\beta)\cdot(2\pi i\beta)e^{2\pi i\beta(x-hk)}\Bigg)dx.\end{aligned}$$

As such, given the algorithm to calculate the norm $\|\ell_0\|_{\widetilde{W}_2^{(2,1)*}}$ in the work [5].

By taking condition (1.5) into account and simplifying the preceding expression, we arrive at the result stated in the previous theorem. Thus, Theorem 2.2 is completely proved. □

3. The optimal coefficients of the quadrature formula

In this section, we determine the optimal coefficients of the quadrature formula (1.2) that minimize the norm of the error functional ℓ. To find the minimum of the norm $\|\ell\|_{\widetilde{W}_2^{(2,1)*}}$with respect to the coefficients A_k, we employ the method of Lagrange multipliers for constrained optimization. Accordingly, we consider the following function:

$$L(A_k,\overline{A_{k'}}) = \|\ell\|^2 \text{ for } k=1,2,\dots,N \quad\text{and}\quad k'=1,2,\dots,N.$$

By setting the partial derivatives of the function $L(A_k,\overline{A_{k'}})$ with respect to all variables A_k and $\overline{A_{k'}}$ equal to zero, we obtain the following system of equations

$$\begin{aligned}\frac{\partial L}{\partial A_k} &= 2\pi i\omega\cdot\kappa(\omega)e^{-2\pi i\omega hk} + \sum_{k'=1}^{N}\overline{C_{k'}}\sum_{\beta\neq 0}2\pi i\beta\kappa(\beta)e^{-2\pi i\beta h(k-k')} \\ &- \sum_{k'=1}^{N}\overline{A_{k'}}\sum_{\beta\neq 0}(2\pi\beta)^2\kappa(\beta)e^{-2\pi i\beta h(k-k')} = 0,\ \ k=1,2,\dots,N, \qquad (3.1)\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial \overline{A_{k'}}} &= 2\pi i\omega\cdot\kappa(\omega)e^{2\pi i\omega hk'} + \sum_{k=1}^{N}C_k\sum_{\beta\neq 0}2\pi i\beta\cdot\kappa(\beta)e^{2\pi i\beta h(k-k')} \\ &- \sum_{k'=1}^{N}A_k\sum_{\beta\neq 0}(2\pi\beta)^2\kappa(\beta)e^{2\pi i\beta h(k-k')} = 0,\ \ k=1,2,\dots,N, \qquad (3.2)\end{aligned}$$

where $\kappa(\cdot)$ is defined by (2.6).

A solution of this system which we denote by $\overset{\circ}{A}_k$, is a minimum point for the function $L(A_k,\overline{A_{k'}})$. The system of equations (3.1) is equivalent to system (3.2). Therefore, it is sufficient to solve the system (3.1).

The following theorem is valid for the solution of the system (3.1).

Theorem 3.1. *Among all quadrature formulas of the form* (1.2), *where* $\omega\in\mathbb{Z}\backslash\{0\}$ *and* $\omega h\notin\mathbb{Z}$, *within the space* $\widetilde{W}_2^{(2,1)}(0,1]$ *of complex-valued periodic functions, there exists a unique quadrature formula with optimal coefficients, and the optimal coefficients of which are a unique solution of the system of equations* (3.1), *which admits the following representation:*

$$\overset{\circ}{A}_k = \frac{2i}{4\pi^2\omega^2+1}\cdot\frac{e^{2h}+1-2e^h\cos 2\pi\omega h}{e^h+1}\cdot\left[\frac{1}{2\pi\omega(e^h-1)} - \frac{\cot\pi\omega h}{e^h+1}\right]\cdot e^{2\pi i\omega hk},\ \ k=1,2,\dots,N. \quad (3.3)$$

The proof of the theorem is the same as the proof of Theorem 3 in [5].

Thus, Problem 1.2 is solved.

4. A sharp upper bound for the error of the optimal quadrature formula

Now, we evaluate the error of the optimal quadrature formula of the form (1.2). To do this, we use the results of Theorems 2.2 and 3.1, that is, using the optimal coefficients $\overset{\circ}{A}_k$ determined by equation (3.3) when calculating the square of the norm for the error functional in Theorem 2.2, we obtain the following theorem

Theorem 4.1. *The norm of the error functional corresponding to the optimal quadrature formula of the form* (1.2)*, that is, the sharp upper bound of the error* (1.3)*, is expressed by the following expression:*

$$\left\|\overset{\circ}{\ell}\right\|^2 = \left\|\overset{\circ}{\ell}_2\right\|^2 - E^2(\omega, h),$$

where

$$\left\|\overset{\circ}{\ell}_2\right\|^2 = \frac{1}{(2\pi\omega)^4 + (2\pi\omega)^2}\left[1 - \frac{2N}{(2\pi\omega)^4 + (2\pi\omega)^2} \times \left[\frac{h}{1-\cos 2\pi\omega h} - \frac{e^{2h}-1}{e^{2h} - 2e^h\cos 2\pi\omega h + 1}\right]^{-1}\right] \tag{4.1}$$

is given in Theorem 4 in [5] and

$$E^2(\omega, h) = \frac{1}{2h}\cdot\left(\frac{e^{2h} + 1 - 2e^h\cos 2\pi\omega h}{(4\pi^2\omega^2+1)(e^h+1)}\right)^2 \cdot \left(\frac{1}{\pi\omega(e^h-1)} - \frac{2\cot\pi\omega h}{e^h+1}\right)^2 \cdot \left(\frac{e^h+1}{e^h-1} - \frac{2}{h}\right). \tag{4.2}$$

To show that the function $E^2(\omega, h)$ is positive for arbitrary values of ω and h, we consider each multiplier in the formula separately. It is evident from formula (4.2) that the first factor $\frac{1}{2h}$ is positive for $h > 0$, and the second and third factors are non-negative as they are squares of real expressions. Therefore, the sign of $E^2(\omega, h)$ is determined by the last factor:

$$D(h) = \frac{e^h+1}{e^h-1} - \frac{2}{h}.$$

Using the identity $\frac{e^h+1}{e^h-1} = \coth(h/2)$, we can rewrite this term as:

$$D(h) = \coth\left(\frac{h}{2}\right) - \frac{1}{h/2}.$$

Using the inequality $\tanh(x) < x$ for $x > 0$, we have $\coth(x) = \frac{1}{\tanh(x)} > \frac{1}{x}$. Letting $x = h/2$, we obtain $\coth(h/2) > \frac{2}{h}$, which implies $D(h) > 0$. Thus, the function $E^2(\omega, h)$ is strictly positive for all $h > 0$ and ω.

Now we show numerically that the function $E^2(\omega, h)$ tends to zero as ω or N tends to infinity in Table 1.

	$\omega = 2$	$\omega = 11$	$\omega = 101$	$\omega = 1001$
$N = 10$	7.45e-8	1.08e-9	1.82e-13	1.92e-17
$N = 100$	9.23e-14	2.59e-12	2.09e-15	2.20e-19
$N = 1000$	9.25e-20	2.76e-18	2.21e-16	2.21e-21
$N = 10000$	9.26e-26	2.77e-24	2.33e-22	2.17e-20

Table 1. Value of the function $E^2(\omega, h)$ is positive and $E^2(\omega, h) \to 0$ as $\omega \to \infty$ or $N \to \infty$.

Theorem 4.1 shows that the absolute value of the error of the optimal quadrature formula constructed in this work is smaller than the absolute value of the error of the optimal quadrature formula in the work [5].

Conclusion

This work addresses the construction of an optimal quadrature formula involving derivatives, in the sense of Sard, for the approximation of strongly oscillatory integrals in the Hilbert space of complex-valued, periodic functions. To derive a sharp upper bound for the absolute error of the quadrature formula, the analytical expression for the norm of the associated error functional is obtained. This derivation begins with the identification of the extremal function corresponding to the error functional. Moreover, in this work, the optimal coefficients of the quadrature formula of the form (1.2) are derived. Additionally, the sharp upper bound for the error of the optimal quadrature formula is established, and it is demonstrated that this bound is smaller than the corresponding error obtained in the work [5].

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