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Optimal approximation techniques for solving Hadamard type integral equations

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Abstract. This paper discusses the frequent occurrence of the Hadamard type integral equation of the first kind in solving problems in aerodynamics, fluid dynamics, wave propagation theory, and ecology. Since the exact solution of this equation is also expressed by an integral, the problem of constructing an optimal quadrature formula for its approximate calculation is considered. The analytical form of the coefficients of the optimal quadrature formula is found.

Keywords: Hadamard type integral eqaution, Cauchy type singular integral, quadrature formula, error functional, extremal function, optimal coefficients.

Mathematics Subject Classification (2020): 65D32

1. Introduction

Hypersingular integral equations are a type of integral equations where the kernel the function inside the integral is highly "singular," meaning it has an infinitely large value at a certain point. This singularity is of a higher order than what's found in more common singular integral equations. These equations are a powerful tool for analyzing complex, three-dimensional problems in various fields, including:

Aerodynamics and Fluid Dynamics: Studying how air and fluids move around objects; Elasticity: Analyzing how materials deform under stress; Wave Theory: Understanding the diffraction of electromagnetic and acoustic waves; Ecology: Modeling certain environmental systems;

Often, hypersingular integral equations are derived from Neumann boundary value problems. These are mathematical problems for equations like the Laplace or Helmholtz equations, where you're given the values of the function's derivative on the boundary of a region, not the function's value itself. The process of converting these boundary value problems into integral equations involves using a concept called the double-layer potential.

We consider the following hypersingular integral equation of the first kind

$$\int_{-1}^{1} \frac{g(x)}{(x-t)^2} dx = \varphi(t), \tag{1.1}$$

here the functions $g(x)$ and $\varphi(x)$ satisfy the Hölder condition (or belongs to the class H), $-1 < t < 1$. We integrate the left side of integral equation (1.1) by parts and obtain the following

$$\frac{g(-1)}{-1-t} + \frac{g(1)}{1-t} + \int_{-1}^{1} \frac{g'(x)}{(x-t)} dx = \varphi(t). \tag{1.2}$$

For integral equation (1.1) or (1.2) to have a unique solution in the given class, the value of the function $g(x)$ at the boundaries of the segment must be equal to zero, i.e.

$$g(1) = g(-1) = 0$$

or

$$\int_{-1}^{1} g'(x) dx = 0. \tag{1.3}$$

We introduce denotation $g'(x) = \rho(x)$, and get

$$\int_{-1}^{1} \frac{\rho(x)}{x-t} dx = \varphi(t), \quad -1 < t < 1. \tag{1.4}$$

The unique exact analytical solution of integral equation (1.4) that satisfies condition (1.3) is equal to the following [16]

$$\rho(t) = -\frac{1}{\pi^2\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sqrt{1-x^2}\varphi(x)}{x-t}dx \tag{1.5}$$

The function under this integral (1.5) is very complex in practical problems. As a result, it is impossible to find the antiderivative of this function. Therefore, to approximate such singular integral, scientists have used the following methods: the discrete vortex method[15, 10], interpolation methods [14, 11, 12, 5], the piecewise interpolation method [6, 7, 9] and other numerical quadrature methods [8, 13]. These methods are not an optimal approximation techniques. In the present paper, we construct an optimal quadrature formula for approximation of the integral (1.5) in the space $L_2^{(2)}(-1,1)$.

2. Statement of the problem

We consider the following quadrature formula

$$\int\limits_{-1}^{1}\frac{\sqrt{1-x^2}\varphi(x)}{x-t}dx \cong \sum_{\beta=0}^{N}\left(C_0[\beta]\varphi(x_\beta)+C_1[\beta]\varphi'(x_\beta)\right), -1<t<1, \tag{2.1}$$

in the Sobolev space $L_2^{(2)}(-1,1)$. This space is a Hilbert space of classes of all real valued functions φ defined in the interval $[-1,1]$ that differ by a linear polynomial of degree second and square integrable with the second order derivative. Here $C_0[\beta]$, $C_1[\beta]$ are the coefficients, x_β are the nodes of the quadrature formula, N is a natural number.

The following difference is called *the error* of quadrature formula (2.1):

$$(\ell,\varphi) = \int\limits_{-1}^{1}\frac{\sqrt{1-x^2}\varphi(x)}{x-t}dx - \sum_{\beta=0}^{N}C_0[\beta]\varphi(x_\beta) - \sum_{\beta=0}^{N}C_1[\beta]\varphi'(x_\beta) = \int\limits_{-\infty}^{\infty}\ell(x)\varphi(x)dx,$$

where

$$\ell_1(x) = \frac{\sqrt{1-x^2}\varepsilon_{[-1,1]}(x)}{x-t} - \sum_{\beta=0}^{N}C_0[\beta]\delta(x-x_\beta) + \sum_{\beta=0}^{N}C_1[\beta]\delta'(x-x_\beta), \tag{2.2}$$

$\varepsilon_{[-1,1]}(x)$ is the indicator of the interval $[-1,1]$, δ is the Dirac delta-function, $\ell(x)$ is the error functional of quadrature formula (2.1).

It is important to note that the coefficients are determined and are given by the following [14, 15, 16].

Theorem 2.1. *Among all quadrature formulas of the form*

$$\int\limits_{-1}^{1}\frac{\sqrt{1-x^2}\varphi(x)}{x-t}\,\mathrm{d}x \cong \sum_{\beta=0}^{N}C_0[\beta]\varphi(x_\beta), -1<t<1,$$

with the error functional

$$\ell(x) = \frac{\sqrt{1-x^2}\varepsilon_{[-1,1]}(x)}{x-t} - \sum_{\beta=0}^{N}C_0[\beta]\delta(x-x_\beta),$$

in the space $L_2^{(1)}(-1,1)$, there is a unique quadrature formula whose coefficients are defined by the equalities

$$C_0[0] = h^{-1}\left[f_0(h)+\frac{\pi}{4}\left(2t^2+2t-1\right)-\frac{h}{2}\pi t\right],$$

$$C_0[\beta] = h^{-1}\left[f_0(h\beta-h)-2f_0(h\beta)+f_0(h\beta+h)\right], \beta=\overline{1,N-1},$$

$$C_0[N]=h^{-1}[f_0(1-h)-\frac{\pi}{4}\left(2t^2-2t-1\right)-\frac{h}{2}\pi t],$$

$$f_0(h\beta)=\int_{-1}^{1}\frac{\sqrt{1-x^2}|x-h\beta+1|}{2(x-t)}\,\mathrm{d}x=\left(\frac{h\beta-1}{2}-t\right)\sqrt{1-(h\beta-1)^2}+\left(t^2-t(h\beta-1)-\frac{1}{2}\right)\mathrm{arcsin(h\beta-1)}$$

$$+(t-(h\beta-1))\sqrt{1-t^2}\ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|. \tag{2.3}$$

Since functional (2.2) defined in the space $L_2^{(2)}(-1,1)$ in [17], then we have

$$(\ell,x)=0. \tag{2.4}$$

The main aim of the present paper is to construct optimal quadrature formulas in the sense of Sard of the form (2.1) in the space $L_2^{(2)}(-1,1)$ by the Sobolev method for approximate integration of the Cauchy type singular integral. This means to find the coefficients $C_1[\beta]$ which satisfy the following variation problem

$$\|\mathring{\ell}|L_2^{(2)*}\|:=\inf_{C_1[\beta]}\|\ell|L_2^{(2)*}\|. \tag{2.5}$$

Thus, in order to construct optimal quadrature formulas of the form (2.1) in the sense of Sard we have to consequently solve the following problems.

Problem 2.1. Find the norm of error functional (2.2) of quadrature formula (2.1) in the space $L_2^{(2)*}(-1,1)$.

Problem 2.2. Find the coefficients $C_1[\beta]$ which satisfy equality (1.4).

In the works [1, 18, 19] for the norm of the error functional the following form was obtained

$$\begin{aligned}\|\ell|L_2^{(2)*}(-1,1)\|^2 \;=\; &\sum_{k=0}^{1}\sum_{\alpha=0}^{1}\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N}(-1)^k C_k[\gamma]C_\alpha[\beta]\frac{(h\beta-h\gamma)^{3-\alpha-k}\mathtt{sign}(h\beta-h\gamma)}{2(3-\alpha-k)!}-\\ &-2\sum_{\alpha=0}^{1}\sum_{\beta=0}^{N}(-1)^\alpha C_\alpha[\beta]\int_{-1}^{1}\frac{\sqrt{1-x^2}(x-(h\beta-1))^{3-\alpha}\mathtt{sign}(x-(h\beta-1))}{2(3-\alpha)!(x-t)}dx+\\ &+\int_{-1}^{1}\int_{-1}^{1}\frac{\sqrt{1-x^2}\sqrt{1-y^2}(x-y)^3\mathtt{sign}(x-y)}{12(x-t)(y-t)}dxdy\end{aligned} \tag{2.6}$$

Thus, Problem 2.1 is solved for quadrature formulas of the form (2.1) in the space $L_2^{(2)}(-1,1)$.

3. The main results

Now we turn to minimizing the norm (2.5) of the error functional for the quadrature formulas with the orthogonality condition (2.4).

Here, we use the $C_0[\beta]$ coefficients of Theorem 2.1 and substitute them into expression (2.5). We then minimize $\|\ell\|^2$ with respect to the $C_1[\beta]$ based on condition (2.4). In order to do this, we apply the Lagrange method.

We denote $\mathbf{C}=(C_1[0],C_1[1],...,C_1[N])$ and λ_1. Consider the function

$$\Phi(C,\lambda)=\|\ell|L_2^{(2)*}(-1,1)\|^2-2\lambda_1(\ell,x).$$

We obtain

$$\sum_{\gamma=0}^{N}C_1[\gamma]\frac{|h\beta-h\gamma|}{2}-\lambda_1=F_1(h\beta),\quad \beta=0,1,2,...,N, \tag{3.1}$$

$$\sum_{\gamma=0}^{N}C_1[\gamma]=g_1-\sum_{\gamma=0}^{N}C_0[\gamma](h\gamma-1), \tag{3.2}$$

where

$$
\begin{aligned}
F_1(h\beta) &= -f_1(h\beta)+\sum_{\gamma=0}^{N} C_0[\gamma]\frac{|h\beta-h\gamma|^2}{4},\\
f_1(h\beta) &= \int_{-1}^{1}\frac{\sqrt{1-x^2}|x-h\beta+1|^2}{4(x-t)}dx\\
&= -\frac{1}{2}\Bigg[\frac{1}{6}\Big(2(h\beta)^2-h\beta(4+9t)+9t+6t^2\Big)\sqrt{1-(h\beta-1)^2}\\
&+\Big(\frac{1}{2}(t-2(h\beta-1))-t(t-(h\beta-1))^2\Big)\mathtt{arcsin}(h\beta-1)-\\
&-(t-(h\beta-1))^2\sqrt{1-t^2}ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|\Bigg],
\end{aligned} \tag{3.3}
$$

$$
g_1=\int_{-1}^{1}\frac{\sqrt{1-x^2}x}{x-t}dx=\frac{\pi}{2}(1-2t^2). \tag{3.4}
$$

The uniqueness of the solution of such type of systems was discussed in [17, 20].

We give the algorithm for solution of system (3.1)-(3.2) when $x_\beta=h\beta-1$, $h=\frac{2}{N}$, N is a natural number. Here we use similar method suggested by S.L. Sobolev [17] for finding the coefficients of optimal quadrature formulas in the Sobolev space $L_2^{(2)}(-1,1)$.

Suppose that $C_1[\beta]=0$ when $\beta<0$ and $\beta>N$. Using the definition of convolution, we rewrite system (3.1)-(3.2) in the following form:

$$
G_1(h\beta)*C_1[\beta]+\lambda_1=F_1(h\beta),\quad \beta=0,1,...,N, \tag{3.5}
$$

$$
\sum_{\beta=0}^{N}C_1[\beta]=g_1-\sum_{\gamma=0}^{N}C_0[\gamma](h\gamma-1), \tag{3.6}
$$

where

$$
G_1(h\beta) = \frac{(h\beta)\mathtt{sgn}(h\beta)}{2},
$$

λ_1 is an arbitrary constant, $\mathtt{sgn}(h\beta)$ is the signum function.

Thus we have the following problem.

Problem 3.3. Find the discrete function $C_1[\beta]$ and constant λ_1 which satisfy the system (3.5)-(3.6). Further we investigate Problem 3.3. Instead of $C_1[\beta]$ we introduce the following functions

$$
v(h\beta) = G_1(h\beta)*C_1[\beta], \tag{3.7}
$$

$$
u(h\beta) = v(h\beta)+\lambda_1. \tag{3.8}
$$

In such statement the coefficients $C_1[\beta]$ are expressed by the function $u(h\beta)$, i.e. taking into account

$$
hD_1(h\beta)*G_1(h\beta)=\delta(h\beta),
$$

where

$$
D_1(h\beta)=\begin{cases} 0, & |\beta|\geq 2,\\ h^{-2}, & |\beta|=1,\\ -2h^{-2}, & \beta=0.\end{cases} \tag{3.9}
$$

There are (3.9) and (3.8) , for the coefficients we have

$$
C_1[\beta]=hD_1(h\beta)*u(h\beta). \tag{3.10}
$$

Thus, if we find the function $u(h\beta)$, then the coefficients $C_1[\beta]$ can be found from equality (3.10). To calculate the convolution (3.10) it is required to find the representation of the function $u(h\beta)$ for all integer values of β. From equality (3.5) we get that $u(h\beta) = F_1(h\beta)$ when $h\beta - 1 \in [-1, 1]$, i.e. $\beta = 0, 1, ..., N$. Now we need to find the representation of the function $u(h\beta)$ when $\beta < 0$ and $\beta > N$.

Since $C_1[\beta] = 0$ when $h\beta \notin [-1, 1]$ then $C[\beta] = hD_1(h\beta) * u(h\beta) = 0, \ \ h\beta \notin [-1, 1]$.

Now we calculate the convolution $v(h\beta) = G_1(h\beta) * C_1[\beta]$ when $\beta \leq 0$ and $\beta \geq N$. Suppose $\beta \leq 0$, then taking into account that $G_1(h\beta) = \frac{|h\beta|}{2}$ and equality (3.6), we have

$$v(h\beta) = -\frac{1}{2}(h\beta)g_1 - p_1 + \lambda_1, \tag{3.11}$$

here $p_1 = \frac{1}{2}\sum\limits_{\gamma=0}^{N} C_0[\gamma](h\gamma)$.

Similarly, in the case $\beta \geq N$ for the convolution $v(h\beta) = G_1(h\beta) * C[\beta]$ we obtain

$$v(h\beta) = \frac{1}{2}(h\beta)g_1 + p_1 + \lambda_1. \tag{3.12}$$

Then we denote

$$a_1^- = -p_1 + \lambda_1 \tag{3.13}$$

$$a_0^+ = p_1 + \lambda_1. \tag{3.14}$$

Taking into account (3.8), (3.11) and (3.12) we get the following problem

Problem 3.4. Find the solution of the equation

$$hD_1(h\beta) * u(h\beta) = 0, \quad\quad h\beta \notin [-1, 1] \tag{3.15}$$

having the form:

$$u(h\beta) = \begin{cases} -\frac{1}{2}(h\beta)g_1 + a_0^-, & \beta < 0, \\ F_1(h\beta), & 0 \leq \beta \leq N, \\ \frac{1}{2}(h\beta)g_1 + a_1^+, & \beta > N. \end{cases} \tag{3.16}$$

Here a_1^- and a_1^+ are unknown constants.

If we find a_1^- and a_1^+ then from (3.13), (3.14) we have

$$\lambda_1 = \frac{1}{2}(a_1^- + a_1^+), \tag{3.17}$$

$$p_1 = \frac{1}{2}(a_1^+ - a_1^-). \tag{3.18}$$

Unknowns a_1^- and a_1^+ can be found from equation (3.15), using the function $D_1(h\beta)$ defined by (3.9). Then we obtain explicit form of the function $u(h\beta)$ and from (3.10) we find the coefficients $C_1[\beta]$. Furthermore, from (3.17) we get λ_1.

Thus, Problem 3.4 and respectively Problems 3.3 will be solved.

Then, using the above algorithm, we obtain explicit formulas for coefficients of the optimal quadrature formula (2.1). It should be noted that the quadrature formula (2.1) is exact for linear function.

The following holds

Theorem 3.1. *Coefficients of the optimal quadrature formulas (2.1), with equally spaced nodes in the space $L_2^{(2)}(-1, 1)$, have the following form*

$$\begin{aligned} C_1[0] &= h^{-1}\Big(f_1(h) + \tfrac{h}{2}f_0(h) - \tfrac{h}{8}\pi(2t^2 + 2t - 1) + \tfrac{\pi}{8}(2t^3 + 4t^2 + t - 2)\Big), \\ C_1[\beta] &= h^{-1}\Big(f_1(h\beta - h) - 2f_1(h\beta) + f_1(h\beta + h) - 2\Big(f_0(h\beta - h) - 2f_0(h\beta) + \\ &\quad + f_0(h\beta + h)\Big) + h^2\sum_{\gamma=0}^{\beta} C_0[\gamma] + \tfrac{h^2}{2}\pi t\Big), \quad \beta = 1, 2, ..., N - 1, \\ C_1[N] &= h^{-1}\Big(f_1(1 - h) - \tfrac{h}{2}f_0(1 - h) - \tfrac{h}{8}\pi(2t^2 - 2t - 1) - \tfrac{\pi}{8}(2t^3 - 4t^2 + t + 2)\Big), \end{aligned}$$

where f_0, f_1 *are defined by* (2.3), (3.3), *respectively.*

Proof. From (3.16) with $\beta = 0$ and $\beta = N$ we immediately obtain (3.16), i.e.

$$a_1^- = f_1(0), \tag{3.19}$$

$$a_1^+ = f_1(2) - g_1. \tag{3.20}$$

This means that we have obtained an explicit form of the function $u(h\beta)$.

Further, using (3.9) and (3.16), from (3.10) calculating the convolution $hD_1(h\beta) * u(h\beta)$ for $\beta = \overline{0, N}$, respectively, we obtain results of the theorem. Theorem (3.1) is proved. □

Remark 3.1. So, the approximate calculation of equality (1.5) is as follows

$$\rho(t) \cong -\frac{1}{\pi^2\sqrt{1-t^2}} \sum_{\beta=0}^{N} \left(C_0[\beta]\varphi(x_\beta) + C_1[\beta]\varphi'(x_\beta) \right).$$

4. Conclusion

In the introduction of the article, several problems in various fields were listed. It was stated that the solutions to these problems lead to first-kind hypersingular integral equations. Since there are no exact analytical methods to solve these types of integral equations, approximate solution methods have been proposed. In this work, we have provided a new method to partially overcome the shortcomings of the proposed methods. That is, an optimal quadrature formula was constructed to approximate singular integrals with high accuracy, and its analytical form was found

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