

The Goursat Problem for a Higher-Order Hyperbolic Equation with a Spectral Parameter

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Abstract. In this article, the Goursat problem for higher-order hyperbolic equations with a spectral parameter has been formulated and studied. The main objective of the article is to find an exact analytical solution to this problem based on the Riemann function. The study considers a higher-order hyperbolic equation defined in a domain bounded by characteristic lines. The conditions of the Goursat problem imply that the function and its higher-order normal derivatives are given on these characteristic lines. To find the solution, the method involves transitioning to characteristic coordinates and determining the corresponding Riemann function. This function is expressed using a special function of the Mittag-Leffler. By applying Green's formula and taking the boundary conditions into account, the solution to the Goursat problem is found in an explicit form at any point and then transformed back to the original variables. The article proves that if the boundary conditions are given by sufficiently smooth functions, the problem has a unique solution. The main novelty lies in the fact that this type of problem is solved for the first time using the Riemann method, and the solution is explicitly expressed.

Keywords: Higher-order hyperbolic equation, Goursat problem, Riemann function, Green's formula, the Four-Parametric Mittag-Leffler function, spectral parameter

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1. INTRODUCTION

Higher-order linear hyperbolic partial differential equations and problems for these equations are one of the fundamental problems of modern mathematical physics and are widely used in areas such as wave propagation, the theory of elasticity, electromagnetic dynamics, and quantum mechanics.

A. V. Bitsadze [1] pointed out the importance of studying higher-order equations of the following form:

$$L^m u = 0, m = 1, 2, \dots, \quad (1.1)$$

where, L is a second-order linear differential operator, and $L^m = L^{m-1}L$ is the m th composition of this operator.

In this direction, the work [2] can be mentioned, where equation (1.1) was studied when the Lavrentyev-Bitsadze operator is of type $m \geq 1$ and $L \equiv \partial^2/\partial x^2 + \operatorname{sign} x (\partial^2/\partial y^2)$. For $m = 2$, various problems for the equation (1.1) with the Lavrentyev-Bitsadze operator have been studied in the works [3] and [4].

An analogue of the Cauchy problem for a differential equation separable into wave multipliers was studied in [5]. In [6], the Cauchy problem for equation (1.1), where L is a wave operator, was solved using the Riesz method, with the solution obtained in the form of a sum of single- and double-layer multidimensional hyperbolic potentials.

In [7], the Cauchy problem for the higher-order equation (1.1) with one and several variables was reduced to a single-variable equation and solved using the method of spherical mean values, where L is a wave operator. Based on the exact solution obtained in this work, exact solutions of analogues of the Cauchy problem for multidimensional polywave equations involving the singular Bessel differential operator with respect to one or more variables were constructed in [8, 9, 10, 11] using the method of transmutation operators. In [12], the construction of the Riemann function for equations involving the singular Bessel differential operator with respect to one or more variables was presented. The monograph [13] provides a review of the literature on this problem.

In this article, we find the solution of the Goursat problem using the Riemann's method. This type of problem has not been previously studied by using the Riemann's method.

2. MAIN PART

Let us denote by Ω the domain of the plane (x, y) bounded by the lines $y = 0$, $x + y = 0$, $x - y = 1$, and consider the following equation in this domain:

$$L_1(u) \equiv \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} \right)^m u(x, y) - \lambda u(x, y) = f(x, y), \quad m \in N, \quad (2.1)$$

where, $u(x, y)$ is the unknown function, $\lambda \in R$ and $f(x, y)$ is a given continuous function.

Problem G. Find a function $u(x, y)$ that belongs to class $C^{2m}(\Omega) \cap C^{2m-1}(\bar{\Omega})$, satisfying the equation (2.1) and the following boundary conditions

$$\begin{aligned} u|_{OK} = \varphi_0(x), \quad \frac{\partial u}{\partial n_1} \Big|_{OK} = \varphi_1(x), \quad \frac{\partial^2 u}{\partial n_1^2} \Big|_{OK} = \varphi_2(x), \quad \dots, \quad \frac{\partial^{m-1} u}{\partial n_1^{m-1}} \Big|_{OK} = \varphi_{m-1}(x), \\ u|_{KN} = \psi_0(x), \quad \frac{\partial u}{\partial n_2} \Big|_{KN} = \psi_1(x), \quad \frac{\partial^2 u}{\partial n_2^2} \Big|_{KN} = \psi_2(x), \quad \dots, \quad \frac{\partial^{m-1} u}{\partial n_2^{m-1}} \Big|_{KN} = \psi_{m-1}(x), \end{aligned} \quad (2.2)$$

where

$$OK = \left\{ (x, y) : x + y = 0, 0 \leq x \leq \frac{1}{2} \right\}, KN = \left\{ (x, y) : x - y = 1, \frac{1}{2} \leq x \leq 1 \right\},$$

(Ω_0 is a domain contained within the interior of domain Ω , consisting of a rectangle $CEKF$, $C(x_0, y_0)$ is an arbitrary point of the inside of the domain Ω , and the segments CE and CF are located parallel to the characteristic lines.)

Here, $\varphi_0(x)$, $\psi_0(x)$, $\varphi_1(x)$, $\psi_1(x)$, $\varphi_2(x)$, $\psi_2(x)$, $\dots$, $\varphi_{m-1}(x)$, $\psi_{m-1}(x)$ are given sufficiently smooth functions such that

$$\begin{aligned} \varphi_0\left(\frac{1}{2}\right) = \psi_0\left(\frac{1}{2}\right), \quad -\varphi_1\left(\frac{1}{2}\right) = \psi_1\left(\frac{1}{2}\right), \quad \varphi_2\left(\frac{1}{2}\right) = \psi_2\left(\frac{1}{2}\right), \quad \dots, \\ (-1)^{m-1} \varphi_{m-1}\left(\frac{1}{2}\right) = \psi_{m-1}\left(\frac{1}{2}\right). \end{aligned} \quad (2.3)$$

n_1 and n_2 are the normals drawn to the characteristic lines, and

$$\begin{aligned} \frac{\partial u}{\partial n_1} \Big|_{OK} = \frac{\partial u}{\partial x} \cos \alpha_1 + \frac{\partial u}{\partial y} \cos \beta_1, \quad \frac{\partial^2 u}{\partial n_1^2} \Big|_{OK} = \frac{\partial^2 u}{\partial x^2} \cos^2 \alpha_1 + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \alpha_1 \cos \beta_1 + \\ + \frac{\partial^2 u}{\partial y^2} \cos^2 \beta_1, \quad \dots, \quad \frac{\partial^{m-1} u}{\partial n_1^{m-1}} \Big|_{OK} = \frac{\partial^{m-1} u}{\partial x^{m-1}} \cos^{m-1} \alpha_1 + 2 \frac{\partial^{m-1} u}{\partial x \partial y} \cos^{m-2} \alpha_1 \cos \beta_1 + \dots + \\ + \frac{\partial^{m-1} u}{\partial y^{m-1}} \cos^{m-1} \beta_1, \\ \frac{\partial u}{\partial n_2} \Big|_{NK} = \frac{\partial u}{\partial x} \cos \alpha_2 + \frac{\partial u}{\partial y} \cos \beta_2, \quad \frac{\partial^2 u}{\partial n_2^2} \Big|_{NK} = \frac{\partial^2 u}{\partial x^2} \cos^2 \alpha_2 + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \alpha_2 \cos \beta_2 + \\ + \frac{\partial^2 u}{\partial y^2} \cos^2 \beta_2, \quad \dots, \quad \frac{\partial^{m-1} u}{\partial n_2^{m-1}} \Big|_{NK} = \frac{\partial^{m-1} u}{\partial x^{m-1}} \cos^{m-1} \alpha_2 + 2 \frac{\partial^{m-1} u}{\partial x \partial y} \cos^{m-2} \alpha_2 \cos \beta_2 + \dots + \\ + \frac{\partial^{m-1} u}{\partial y^{m-1}} \cos^{m-1} \beta_2, \end{aligned}$$

where $\alpha_1 = n_1 \wedge x = \frac{5\pi}{4}$, $\beta_1 = n_1 \wedge y = \frac{3\pi}{4}$, $\alpha_2 = n_2 \wedge x = \frac{7\pi}{4}$, $\beta_2 = n_2 \wedge y = \frac{5\pi}{4}$.

Let us switch to characteristic coordinates by $u(x, y) = w(\xi, \eta)$, $\xi = x + y$, $\eta = x - y$. Let us denote by Ω' the domain bounded by lines $\xi = \eta$, $\xi = 0$ and $\eta = 1$. Then equation (2.1), the boundary conditions (2.2), and the conditions (2.3) take the following forms, respectively.

$$\frac{\partial^{2m}}{\partial \xi^m \partial \eta^m} w(\xi, \eta) - \frac{1}{4^m} \lambda w(\xi, \eta) = \frac{1}{4^m} f\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right), \quad (2.4)$$

$$w|_{\xi=0} = \varphi_0\left(\frac{\eta}{2}\right), \quad \frac{\partial w}{\partial \xi}\bigg|_{\xi=0} = -\frac{\sqrt{2}}{2} \varphi_1\left(\frac{\eta}{2}\right), \quad \frac{\partial^2 w}{\partial \xi^2}\bigg|_{\xi=0} = \frac{1}{2} \varphi_2\left(\frac{\eta}{2}\right), \quad \dots,$$

$$\frac{\partial^{m-1} w}{\partial \xi^{m-1}}\bigg|_{\xi=0} = \left(-\frac{\sqrt{2}}{2}\right)^{m-1} \varphi_{m-1}\left(\frac{\eta}{2}\right), \quad w|_{\eta=1} = \psi_0\left(\frac{\xi+1}{2}\right), \quad \frac{\partial w}{\partial \eta}\bigg|_{\eta=1} = \frac{\sqrt{2}}{2} \psi_1\left(\frac{\xi+1}{2}\right),$$

$$\frac{\partial^2 w}{\partial \eta^2}\bigg|_{\eta=1} = \frac{1}{2} \psi_2\left(\frac{\xi+1}{2}\right), \quad \dots, \quad \frac{\partial^{m-1} w}{\partial \eta^{m-1}}\bigg|_{\eta=1} = \left(\frac{\sqrt{2}}{2}\right)^{m-1} \psi_{m-1}\left(\frac{\xi+1}{2}\right), \quad (2.5)$$

$$\varphi_0(1) = \psi_0(0), \quad -\varphi_1(1) = \psi_1(0), \quad \varphi_2(1) = \psi_2(0), \quad \dots, \quad (-1)^{m-1} \varphi_{m-1}(1) = \psi_{m-1}(0).$$

Let us denote the left-hand side of the equation (2.4) by $M(w)$, that is

$$M(w) = \frac{\partial^{2m}}{\partial \xi^m \partial \eta^m} w - \frac{1}{4^m} \lambda w.$$

Now, let us introduce the operator $M^*(v)$, referred to as the adjoint of this operator:

$$M^*(v) = \frac{\partial^{2m} v}{\partial \xi^m \partial \eta^m} - \frac{1}{4^m} \lambda v.$$

Accordingly, one can write the following equality

$$vM(w) - wM^*(v) = \frac{\partial P}{\partial \xi} + \frac{\partial Q}{\partial \eta}, \quad (2.6)$$

where, $P = \sum_{k=1}^m \frac{\partial^{2k-2} v}{\partial \xi^{k-1} \partial \eta^{k-1}} \times \frac{\partial^{2m-2k+1} w}{\partial \xi^{m-k} \partial \eta^{m+1-k}}, \quad Q = -\sum_{k=1}^m \frac{\partial^{2k-1} v}{\partial \xi^k \partial \eta^{k-1}} \times \frac{\partial^{2m-2k} w}{\partial \xi^{m-k} \partial \eta^{m-k}},$ $v = v(\xi_0, \eta_0; \xi, \eta)$ is the Riemann function and satisfies the following conditions:

$$\frac{\partial^{2m}}{\partial \xi^m \partial \eta^m} v(\xi, \eta; \xi_0, \eta_0) - \frac{1}{4^m} \lambda v(\xi, \eta; \xi_0, \eta_0) = 0, \quad (2.7)$$

$$v|_{\xi=\xi_0} = 0, \quad \frac{\partial v}{\partial \xi}\bigg|_{\xi=\xi_0} = 0, \quad \frac{\partial^2 v}{\partial \xi^2}\bigg|_{\xi=\xi_0} = 0, \dots, \quad \frac{\partial^{m-2} v}{\partial \xi^{m-2}}\bigg|_{\xi=\xi_0} = 0,$$

$$\frac{\partial^{m-1} v}{\partial \xi^{m-1}}\bigg|_{\xi=\xi_0} = \frac{(\eta - \eta_0)^{m-1}}{(m-1)!}, \quad (2.8)$$

$$v|_{\eta=\eta_0} = 0, \quad \frac{\partial v}{\partial \eta}\bigg|_{\eta=\eta_0} = 0, \quad \frac{\partial^2 v}{\partial \eta^2}\bigg|_{\eta=\eta_0} = 0, \dots, \quad \frac{\partial^{m-2} v}{\partial \eta^{m-2}}\bigg|_{\eta=\eta_0} = 0,$$

$$\frac{\partial^{m-1} v}{\partial \eta^{m-1}}\bigg|_{\eta=\eta_0} = \frac{(\xi - \xi_0)^{m-1}}{(m-1)!}. \quad (2.9)$$

Integrating (2.6) over the domain $\Omega'_0 = \{(\xi, \eta); 0 < \xi < \xi_0, \eta_0 < \eta < 1\}$, and applying Green's formula, we obtain

$$\iint_{\Omega'_0} [vM(w) - wM^*(v)] d\xi d\eta = \iint_{\Omega'_0} \left(\frac{\partial P}{\partial \xi} + \frac{\partial Q}{\partial \eta} \right) d\xi d\eta = \int_{\partial \Omega'_0} -Q d\xi + P d\eta =$$

$$= \int_{C' E' K' F'} -Q d\xi + P d\eta = 0. \quad (2.10)$$

In the equation (2.10), the boundary of the integral on the right-hand side consists of four segments; therefore, we evaluate it separately on each segment. Here, $C' E' K' F'$ represents a rectangle and is equal to the boundary of Ω'_0 .

a) On $C' F'$, $\xi = \xi_0$, $d\xi = 0$, $\eta_0 \leq \eta \leq 1$, therefore

$$\begin{aligned} \int_{C' F'} (-Q) d\xi + P d\eta &= \int_{C' F'} P d\eta = \int_{\eta_0}^1 \left[\sum_{k=1}^m \frac{\partial^{2k-2} v}{\partial \xi^{k-1} \partial \eta^{k-1}} \times \frac{\partial^{2m-2k+1} w}{\partial \xi^{m-k} \partial \eta^{m+1-k}} \right]_{\xi=\xi_0} d\eta = \\ &= \psi_0 \left(\frac{\xi_0 + 1}{2} \right) - w(\xi_0, \eta_0). \end{aligned}$$

b) On $F' K'$, $\eta = 1$, $d\eta = 0$, $0 \leq \xi \leq \xi_0$, so

$$\begin{aligned} \int_{F' K'} (-Q) d\xi + P d\eta &= \int_{F' K'} (-Q) d\xi = - \int_0^{\xi_0} \left[\sum_{k=1}^m \frac{\partial^{2k-1} v}{\partial \xi^k \partial \eta^{k-1}} \times \frac{\partial^{2m-2k} w}{\partial \xi^{m-k} \partial \eta^{m-k}} \right]_{\eta=1} d\xi = \\ &= - \int_0^{\xi_0} \left[\sum_{k=1}^m \left(\frac{\sqrt{2}}{2} \right)^{m-k} \left[\frac{\partial^{2k-1} v}{\partial \xi^k \partial \eta^{k-1}} \right]_{\eta=1} \psi_{m-k}^{(m-k)} \left(\frac{\xi+1}{2} \right) \right] d\xi. \end{aligned}$$

c) On $K' E'$, $\xi = 0$, $d\xi = 0$, $\eta_0 \leq \eta \leq 1$, therefore

$$\begin{aligned} \int_{K' E'} (-Q) d\xi + P d\eta &= \int_{K' E'} P d\eta = - \int_{\eta_0}^1 \left[\sum_{k=1}^m \frac{\partial^{2k-2} v}{\partial \xi^{k-1} \partial \eta^{k-1}} \times \frac{\partial^{2m-2k+1} w}{\partial \xi^{m-k} \partial \eta^{m+1-k}} \right]_{\xi=0} d\eta = \\ &= - \int_{\eta_0}^1 \left[\left(-\frac{\sqrt{2}}{2} \right)^{m-k} \left[\frac{\partial^{2k-2} v}{\partial \xi^{k-1} \partial \eta^{k-1}} \right]_{\xi=0} \varphi_{m-k}^{(m+1-k)} \left(\frac{\eta}{2} \right) \right] d\eta. \end{aligned}$$

d) On $E' C'$, $\eta = \eta_0$, $d\eta = 0$, $0 \leq \xi \leq \xi_0$, so

$$\int_{E' C'} (-Q) d\xi + P d\eta = \int_{E' C'} (-Q) d\xi = \int_0^{\xi_0} \left[\sum_{k=1}^m \frac{\partial^{2k-1} v}{\partial \xi^k \partial \eta^{k-1}} \times \frac{\partial^{2m-2k} w}{\partial \xi^{m-k} \partial \eta^{m-k}} \right]_{\eta=\eta_0} d\xi = 0.$$

By substituting the obtained expressions into (2.10), we obtain the solution to the Goursat problem for the equation (2.4), determined at an arbitrary point $C'(\xi_0, \eta_0) \in \Omega'$, as follows

$$\begin{aligned} w(\xi_0, \eta_0) &= \psi_0 \left(\frac{\xi_0 + 1}{2} \right) - \int_0^{\xi_0} \left[\sum_{k=1}^m \left(\frac{\sqrt{2}}{2} \right)^{m-k} \left[\frac{\partial^{2k-1} v(\xi, \eta; \xi_0, \eta_0)}{\partial \xi^k \partial \eta^{k-1}} \right]_{\eta=1} \psi_{m-k}^{(m-k)} \left(\frac{\xi+1}{2} \right) \right] d\xi - \\ &- \int_{\eta_0}^1 \left[\sum_{k=1}^m \left(-\frac{\sqrt{2}}{2} \right)^{m-k} \left[\frac{\partial^{2k-2} v(\xi, \eta; \xi_0, \eta_0)}{\partial \xi^{k-1} \partial \eta^{k-1}} \right]_{\xi=0} \varphi_{m-k}^{(m+1-k)} \left(\frac{\eta}{2} \right) \right] d\eta + \iint_{\Omega_0} \frac{1}{4^m} v f d\xi d\eta. \end{aligned} \quad (2.11)$$

According to equation (2.7), we perform integration with respect to variable ξ over the interval from ξ_0 to ξ , and with respect to variable η over the interval from η_0 to η . Using the conditions (2.8) and (2.9), and the following equality [9],

$$\int_a^{z_0} dz_1 \int_a^{z_1} dz_2 \dots \int_a^{z_{n-1}} \theta(\tau) d\tau = \frac{1}{(n-1)!} \int_a^{z_0} (z_0 - \tau)^{n-1} \theta(\tau) d\tau, \quad n \in N,$$

we derive the integral equation with respect to $\nu(\xi, \eta)$ from equation (2.12),

$$\begin{aligned} v(\xi, \eta) - \frac{1}{4^m} \times \frac{\lambda}{(m-1)!(m-1)!} \int_{\xi_0}^{\xi} \int_{\eta_0}^{\eta} (\xi - s)^{m-1} (\eta - t)^{m-1} v(s, t) dt ds = \\ = \frac{(\xi - \xi_0)^{m-1}}{(m-1)!} \times \frac{(\eta - \eta_0)^{m-1}}{(m-1)!}. \end{aligned} \quad (2.12)$$

Since the equation (2.12) is a Volterra-type integral equation, it has a unique solution. We solve it using the method of successive approximations.

Taking $v_0(\xi, \eta) = \frac{(\xi - \xi_0)^{m-1}}{(m-1)!} \times \frac{(\eta - \eta_0)^{m-1}}{(m-1)!}$ as the initial approximation, the subsequent approximations are constructed as follows:

$$\begin{aligned} v_k = \frac{(\xi - \xi_0)^{m-1}}{(m-1)!} \times \frac{(\eta - \eta_0)^{m-1}}{(m-1)!} - \frac{1}{4^m} \times \frac{\lambda}{(m-1)!(m-1)!} \times \\ \times \int_{\xi_0}^{\xi} \int_{\eta_0}^{\eta} (\xi - s)^{m-1} (\eta - t)^{m-1} v_{k-1}(s, t) dt ds. \end{aligned}$$

One can easily show that from the last equation the functions ν_k can be found as follows:

$$\begin{aligned} v_0 &= \frac{(\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1}}{\Gamma^2(m)}, \\ v_1 &= \frac{(\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1}}{\Gamma^2(m)} + \frac{\lambda}{4^m} \times \frac{(\xi - \xi_0)^{2m-1} (\eta - \eta_0)^{2m-1}}{\Gamma^2(2m)}, \\ v_2 &= \frac{(\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1}}{\Gamma^2(m)} + \frac{\lambda}{4^m} \times \frac{(\xi - \xi_0)^{2m-1} (\eta - \eta_0)^{2m-1}}{\Gamma^2(2m)} + \\ &+ \left(\frac{\lambda}{4^m} \right)^2 \times \frac{(\xi - \xi_0)^{3m-1} (\eta - \eta_0)^{3m-1}}{\Gamma^2(3m)}, \\ &\dots \end{aligned}$$

Continuing this process, for an arbitrary value of $n \in N$, we have

$$v_n = (\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1} \sum_{k=0}^n \left(\frac{\lambda}{4^m} \right)^k \frac{[(\xi - \xi_0) (\eta - \eta_0)]^{kn}}{\Gamma(mn + m) \Gamma(mn + m)}.$$

At this point, taking the limit as $n \rightarrow \infty$, and considering this equality [14],

$$E_{\alpha_1, \beta_1; \alpha_2, \beta_2}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha_1 n + \beta_1) \Gamma(\alpha_2 n + \beta_2)}$$

$\alpha_1, \alpha_2 \in R(\alpha_1^2 + \alpha_2^2 \neq 0)$ and $\beta_1, \beta_2 \in C$, we obtain

$$v = \lim_{n \rightarrow \infty} v_n = (\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1} E_{m, m; m, m} \left[\frac{\lambda}{4^m} ((\xi - \xi_0) (\eta - \eta_0))^m \right].$$

Thus, the Riemann function for equation (2.5) has the form

$$v(\xi, \eta; \xi_0, \eta_0) = (\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1} E_{m,m;m,m} \left[\frac{\lambda}{4^m} ((\xi - \xi_0)(\eta - \eta_0))^m \right].$$

One can verify directly that this function indeed satisfies the conditions (2.8) and (2.9).

Returning to the variables x, y and introducing the notations $\xi_0 = x_0 + y_0$ and $\eta_0 = x_0 - y_0$, we find the Riemann function for the problem G :

$$v(x, y; x_0, y_0) = \left[(x - x_0)^2 - (y - y_0)^2 \right]^{m-1} E_{m,m;m,m} \left[\frac{\lambda}{4^m} \left((x - x_0)^2 - (y - y_0)^2 \right)^m \right]. \quad (2.13)$$

Considering the form (2.13) of the Riemann function the solution (2.11) can be written as follows

$$\begin{aligned} u(x_0, y_0) = & \psi_0 \left(\frac{x_0 + y_0 + 1}{2} \right) - \int_0^{x_0 + y_0} \left[\sum_{k=1}^m \left(\frac{\sqrt{2}}{2} \right)^{m-k} \frac{\partial}{\partial \xi} \left[(\xi - x_0 - y_0)^{m-k} \times \right. \right. \\ & \times (1 - x_0 + y_0)^{m-k} E_{m, m-k+1; m, m-k+1} \left. \left. \left[\frac{\lambda}{4^m} ((\xi - x_0 - y_0)(1 - x_0 + y_0))^m \right] \right] \psi_{m-k}^{(m-k)} \left(\frac{\xi + 1}{2} \right) \right] d\xi - \\ & - \int_{x_0 - y_0}^1 \left[\sum_{k=1}^m \left(-\frac{\sqrt{2}}{2} \right)^{m-k} (-x_0 - y_0)^{m-k} (\eta - x_0 + y_0)^{m-k} E_{m, m-k+1; m, m-k+1} \left[\frac{\lambda}{4^m} ((-x_0 - y_0) \times \right. \right. \\ & \times (\eta - x_0 + y_0))^m \left. \left. \varphi_{m-k}^{(m+1-k)} \left(\frac{\eta}{2} \right) \right] d\eta - \frac{1}{4^m} \iint_{\Omega_0} \left[(x - x_0)^2 - (y - y_0)^2 \right]^{m-1} \times \right. \\ & \times E_{m,m;m,m} \left[\frac{\lambda}{4^m} \left[(x - x_0)^2 - (y - y_0)^2 \right]^m \right] f(x, y) dx dy, \end{aligned} \quad (2.14)$$

where Ω_0 is a domain located inside the domain Ω , consisting of a rectangle $CEKF$ bounded by the lines $x + y = 0$, $x + y = y_0 - x_0$, $x - y = 1$, and $x - y = x_0 - y_0$, and the point $C(x_0, y_0)$ is arbitrarily chosen from within the domain Ω .

Theorem. *Let the conditions $\psi_j \in C^{2m}(0, 1/2)$, $\varphi_j \in C^{2m}(1/2, 1)$ be satisfied for the functions ψ_j , φ_j , $j = \overline{0, m-1}$, $\lambda \in R$ and $f(x, y) \in C(\Omega)$, then the problem G has a unique solution given by the equality (2.14).*

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