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On representations of a given number as the sum of two primes and the square of a third prime in an arithmetic progression

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Abstract. This paper considers the problem of representing a given natural number as combination of two prime numbers and the square of a third prime number taken from an arithmetic progression. It was the first to establish the solvability of the equation under consideration in prime numbers from the arithmetic progression, and prove a lower estimate for the number of solutions to this equation. The results obtained are important in the study of additive problems with prime numbers. The proof of the obtained results uses the Hardy-Littlewood circular method and the Vinogradov method of trigonometric sums.

Keywords: diophantine equation; congruent solution; exceptional zero; Dirichlet L-function; principal characters; Legendre symbol; minor arc; major arc; singular series; singular integral
MSC (2020): 11D61

1. Introduction

Let us consider the following equation:

$$a_1p_1 + a_2p_2 + a_3p_3^2 = b, \tag{1.1}$$

where a_1, a_2, a_3, b are integers and p_1, p_2, p_3 are prime numbers.

If in (1.1) we take $a_1 = a_2 = a_3 = 1$, then we arrive at a classical problem posed in 1938 by Hua Loo-Keng [1]. I. Allakov and N. Muzropova [2] proved that equation (1.1) has solutions in prime numbers under certain conditions and obtained a lower bound for the number of such representations. M. C. Liu and Tao Zhan [3], proved in the linear case, that is, for Goldbach's ternary theorem with prime variables from an arithmetic progression, that if N is a sufficiently large number, then there exists a constant number $\delta > 0$ such that for all positive integers $N \le D^\delta$ and for $\sum\limits_{i=1}^{3} l_i = b\,(\bmod D)$ Diophantine equation

$$\begin{cases} b = p_1 + p_2 + p_3, \\ p_i \equiv l_i\,(\bmod D)\,, i = 1, 2, 3, \end{cases}$$

has a solution, where $(l_i, D) = 1$.

In the works of I. Allakov and O. Sh. Imamov [4]–[5], a lower bound was obtained for the number of representations of a natural number as the sum of the squares of five prime numbers from an arithmetic progression. In [6], we improved the estimate of the exceptional set in the problem of representing a natural number as a sum of squares of four prime numbers, while in [7], the problem of representing a natural number as a sum of squares of four prime numbers from an arithmetic progression was considered.

In this work, we examine the solvability conditions of equation (1.1) in prime numbers p_i from an arithmetic progression, $p_i \equiv l_i (\bmod D)$, $i = 1, 2, 3$, $D \le N^\delta$.

We assume, in the general case, that $a_i \ne 0$, $i = 1, 2, 3$ and

$$\gcd(a_1, a_2, a_3) = 1. \tag{1.2}$$

Furthermore, following the approach in Xua‘s work on the Tarry problem (see [8], p. 162), we consider the solvability condition of equation (1.1) in the sense of congruences. That is, we define the quantity $N(q)$ as follows:

$$N(q) := card\,\{\ (n_1, n_2, n_3)\,|\,1 \le n_j \le q,\ (n_j, q) = 1\,,\ \ a_1n_1 + a_2n_2 + a_3n_3^2 \equiv b\,(\bmod q)\} \tag{1.3}$$

and require that the following condition holds:

$$\text{"for all } q \geq 1,\ N(q) \geq 1\text{"}. \tag{1.4}$$

Let us assume that the integers a_1, a_2, a_3, b satisfy conditions (1.2), (1.3) and define B as

$$B := \max\{2, |a_1|, |a_2|, |a_3|\}. \tag{1.5}$$

In the present paper, by combining the methods of [2],[3] the following result has been proved:

Theorem 1.1. *If a_1, a_2, a_3, b satisfy conditions* (1.2) *and* (1.3)*. Then there exists an effective constant $A > 0$ such that the following assertions hold:*

(a) If all of a_1, a_2, a_3 are positive and $b \geq B^A$, then equation (1.1) *has a solution in prime numbers p_1, p_2, p_3 taken from an arithmetic progression, $p_i \equiv l_i (\mathrm{mod} D)$, $i = 1, 2, 3$, $D \leq N^\delta$.*

(b) If not all of a_1, a_2, a_3 have the same sign, then equation (1.1) *has a solution in primes p_1, p_2, p_3 from an arithmetic progression $p_i \equiv l_i (\mathrm{mod} D)$, $i = 1, 2, 3$, $D \leq N^\delta$, satisfying for p_1, p_2, p_3 not exceeding $3|b| + B^A$.*

Corollary 1.2. *If N is sufficiently large, then the number of solutions of equation* (1.1) *in prime numbers $NB^{-1} < p_1, p_2, p_3^2 \leq N$, $p_i \equiv l_i (\mathrm{mod} D)$, $i = 1, 2, 3$, $D \leq N^\delta$ is at least $c_1 N^{3/2} \left(BQ^{83/420} D \ln^3 N\right)^{-1}$, where c_1 is a positive constant and $Q = N^{21\delta}$.*

2. Integral representation of the problem and minor arcs

From now on, we denote a prime number by p (with or without indices). The constants $c_1, c_2, \ldots$ are effective positive absolute constants. The constant δ is a sufficiently small effective positive number and its value may depend on the values of the constants c_j.

Let

$$Q := N^{21\delta},\ T := Q^{\frac{1}{\sqrt{\delta}}},\quad L := NB^{-1},\ L_1 := \sqrt{L},\ N_1 := \sqrt{N}. \tag{2.1}$$

We choose N so that it satisfies

$$B \leq Q^\delta. \tag{2.2}$$

For any integer y and any positive integer q, we define $e(y) = e^{2\pi i y}$ and $e_q(y) = e^{2\pi i \frac{y}{q}} = e(y/q)$.

We define the sums

$$S_i(y) := \sum_{\substack{L < n_i \leq N \\ n_i \equiv l_i (\bmod D)}} \Lambda(n_i)\, e(n_i y)\ ,\qquad i = 1, 2; \quad S_3(y) = \sum_{\substack{L < n_3^2 \leq N \\ n_3 \equiv l_3 (\bmod D)}} \Lambda(n_3)\, e\left(n_3^2 y\right), \tag{2.3}$$

where $\Lambda(n)$ is the Mangoldt function.

Let

$$\tau = T^{1/4} N^{-1}. \tag{2.4}$$

We divide the interval $[\tau; 1 + \tau]$ into main and additional subintervals in the usual way (see. [2]).

We define

$$I(N, b, D) = I(N) := \sum_{\substack{L < n_1, n_2, n_3^2 \leq N \\ a_1 n_1 + a_2 n_2 + a_3 n_3^2 = b \\ n_i \equiv l_i (\bmod D)}} \Lambda(n_1)\, \Lambda(n_2)\, \Lambda(n_3). \tag{2.5}$$

Using (2.3) and (2.5), we can express $I(N)$ as

$$I(N) = \int_{\tau}^{1+\tau} e(-bx)\, S_1(a_1 x)\, S_2(a_2 x)\, S_3(a_3 x)\, dx. \tag{2.6}$$

Noting that $M \cup M' = [\tau,\ 1 + \tau]$, we rewrite (2.6) as the sum of two integrals:

$$I(N) = \left(\int_M + \int_{M'}\right) e(-bx) \prod_{i=1}^{3} S_i(a_i x)\, dx = I_1(N) + I_2(N)\ . \tag{2.7}$$

In (2.7), the integral over the major arcs is denoted by $I_1(N)$, while the integral over the minor arcs is denoted by $I_2(N)$. From (2.7), it follows that $I(N) > I_1(N) - |I_2(N)|$. If we can show that $I(N) > 0$, then we can conclude that equation (1.1) has a solution in prime numbers in arithmetic progression.

First, we analyze the integral $I_2(N)$, for which we prove the following lemma.

Lemma 2.1. *For any $x \in M'$, the following estimate holds: $S_3(a_3x) \ll d^{1/2}D^{-1}N^{\frac{1+\varepsilon}{2}}Q^{-\frac{1}{4}}B^{\frac{1}{4}}$, where $d = (D, q)$.*

Proof. According to corollary 1.2.1 in [9], if $(h, q) = 1,\ 1 \le h \le q,\ \ |\alpha - hq^{-1}| < q^{-2}$ and $D^2 \le N$, then the estimate $S_3(\alpha) \ll D^{-1}N^{1+\varepsilon}\left(d^2q^{-1} + DN^{-1/2} + qDN^{-2}d^{-1}\right)^{1/4}$ holds. According to Dirichlet's theorem on Diophantine approximations, there exists integers h and q satisfying

$$1 \le q \le \tau^{-1},\ \ (h, q) = 1,\ \ \left|a_ix - hq^{-1}\right| < \tau q^{-1}. \tag{2.8}$$

We devide both sides of the inequality in (2.8) by $\left|x - h(|a_i|\, q)^{-1}\right| < \tau(|a_i|\, q)^{-1}$. From this, we get

$$\left|x - h'(q')^{-1}\right| < \tau(q')^{-1}. \tag{2.9}$$

Here, q' is defined as the positive divisor of a_iq, and $\gcd(h',\ q') = 1$. It is not difficult to see that $q' > Q$. In fact, if $q' \le Q$, then from (2.9) and $x \in [\tau,\ 1+\tau]$ it follows that $1 \le h' \le q' \le Q$, that is, $x \in M$, which contradicts the condition $x \in M'$. Therefore,

$$Q|a_j|^{-1} < q \le \tau^{-1}. \tag{2.10}$$

According to (2.10) we obtain, $S_3(a_3x) \ll D^{-1}N^{\frac{1+\varepsilon}{2}}d^{1/2}B^{1/4}Q^{-1/4}$. □

Lemma 2.2. *If $\varepsilon < 0,05\delta$, then $|I_2(N)| \ll D^{-1}N^{3/2}d^{1/2}B^{1/4}Q^{-1/5}\ln N$ holds.*

Proof. Using Lemma 2.1 and (2.7), we obtain:

$$|I_2(N)| \ll D^{-1}N^{\frac{1}{2}+\varepsilon}d^{1/2}B^{1/4}Q^{-1/4}\int\limits_{\tau}^{1+\tau}\left\{|S_1(a_1x)|^2 + |S_2(a_2x)|^2\right\}dx. \tag{2.11}$$

Here, $\int\limits_{\tau}^{1+\tau}|S_i(a_ix)|^2dx = \sum\limits_{L<n_i\le N}\Lambda^2(n) \ll \ln N \cdot \sum\limits_{n\le N}\Lambda(n) \ll N\ln N$ [10]. Thus, from (2.11) and the lemma conditions, we obtain: $|I_2(N)| \ll D^{-1}N^{3/2}d^{1/2}B^{1/4}Q^{-1/5}\ln N$. □

3. Simplification of the integral over major arcs

Let $\chi(\bmod\, q)$ and $\chi_0(\bmod\, q)$ denote, respectively, an arbitrary character and the principal Dirichlet character modulo q. It is known (see §2, Chapter III of the work [9]) that there exists a constant c_2 such that the L function $L(s, \chi)$ may have at most one real zero $\tilde{\beta}$ for a given real character $\tilde{\chi}\,(\bmod \tilde{r})$ with modulus $\tilde{r} \le T$, in the region $\sigma > 1 - c_2(\ln T)^{-1},\ |t| \le T$. If such a zero exists, it is unique and is called the exceptional zero of the function $L(s, \chi)$, where $s = \sigma + it$. Moreover, this exceptional zero $\tilde{\beta}$ satisfies the inequality

$$c_3\left(\tilde{r}^{\frac{1}{2}}\ln^2\tilde{r}\right)^{-1} \le 1 - \tilde{\beta} \le c_2/\ln T. \tag{3.1}$$

The summation $\sum\limits_{h=1}^{q}{}'$ or $\sum\limits_{(h,q)=1}$ denotes the sum taken over h satisfying the condition $(h, q) = 1$, $1 \le h \le q$. For the character $\chi(\bmod\, Dq/d)$, we define the following functions:

$$S_i(\chi, y) = \sum_{\substack{L<n_i\le N\\ n_i\equiv l_i(\bmod\, D)}}\Lambda(n_i)\,\chi(n_i)\,e(n_iy)\,,\quad S_3(\chi, y) = \sum_{\substack{L<n_3^2\le N\\ n_3\equiv l_3(\bmod\, D)}}\Lambda(n_3)\,\chi(n_3)\,e\left(n_3^2y\right),$$

$$I_i(y)=\int_L^N e(x_iy)\,dx_i,\ I_3(y):=\int_{L_1}^{N_1} e\left(x_3^2y\right)dx_3,\ \tilde{I}_i(y)=\int_L^N x_i^{\tilde{\beta}-1}e(x_iy)\,dx_i,\ \tilde{I}_3(y)=\int_{L_1}^{N_1} x_3^{\tilde{\beta}-1}e\left(x_3^2y\right)dx_3,$$

$$I_i(\chi,y)=\sum_{|\gamma|\le T}{}' \int_L^N x_i^{\rho-1}e(x_iy)\,dx_i,\qquad I_3(\chi,y)=\sum_{|\gamma|\le T}{}' \int_{L_1}^{N_1} x_3^{\rho-1}e\left(x_3^2y\right)dx_3, \tag{3.2}$$

where $i=1,2$ and the summation $\sum\limits_{|\gamma|\le T}{}'$ denotes the sum over zeros $\rho=\beta+i\gamma$ of the function $L(s,\chi)$ in the region $1/2\le\beta\le 1-c_2(\ln T)^{-1},\ |\gamma|\le T$ with $\tilde{\beta}$ being the exceptional zero if it exists.

Lemma 3.1. *For any real number y and characters $\chi(\bmod Dq/d)$ with $Dq/d\le T$, the following equalities hold:*

$$S_i(\chi,y)=\delta_\chi' I_i(y)-\delta_\chi\tilde{I}_i(y)-I_i(\chi,y)+O\left((1+|y|N)NT^{-1}\ln^2 N\right),\quad i=1,2$$

$$S_3(\chi,y)=\delta_\chi' I_3(y)-\delta_\chi\tilde{I}_3(y)-I_3(\chi,y)+O\left((1+|y|N_1)N_1T^{-1}\ln^2 N\right), \tag{3.3}$$

where

$$\delta_\chi'=\begin{cases}1, & if\ \ \chi\equiv\chi_0(\bmod Dq/d)\\ 0, & otherwise\end{cases},\qquad \delta_\chi=\begin{cases}1, & if\ \ \chi\equiv\tilde{\chi}\chi_0(\bmod Dq/d),\\ 0, & otherwise.\end{cases}$$

A proof of this lemma can be found in [9]. In the next step, to transform the exponential sum $S_i(\alpha)$ containing the character χ into the integral form above, we need the following new definitions.

$d(q)$ is defined as follows: $D=p_1^{\alpha_1}\cdots p_s^{\alpha_s}D_0,\ q=p_1^{\beta_1}\cdots p_s^{\beta_s}q_0,\ (D_0,\,q_0)=1,\ d(q)=p_1^{\gamma_1}\cdots p_s^{\gamma_s}$, where $\gamma_i=\min(\alpha_i,\,\beta_i)$, for $i=1,...,s$. We define $d_1(q)$ and $d_2(q)$ as follows:

$$d_1(q):=p_1^{\delta_1}\cdots p_s^{\delta_s},\quad \delta_i=\begin{cases}\alpha_i, & \text{if }\ \beta_i>\alpha_i\\ 0, & otherwise,\end{cases},\quad d_2(q):=d(q)/d_1(q)\,. \tag{3.4}$$

For convenience, we write $d=d(q)$, $d_1=d_1(q)$ va $d_2=d_2(q)$. From the above definitions, it follows that $(d_1,\,d_2)=1$ and $(D/d_1,q/d_2)=1$.

Lemma 3.2. *If $y=a_i\left(hq^{-1}+\lambda\right)$, then the equality*

$$S_i(y)=S_i\left(a_i\left(hq^{-1}+\lambda\right)\right)=\frac{1}{\varphi(\frac{D}{d_1})\varphi(\frac{q}{d_2})}\sum_{\zeta(\bmod D/d_1)}\bar{\zeta}(l_i)\sum_{\eta(\bmod q/d_2)}G_i(h,\bar{\eta},q)S_i(\zeta\eta,a_i\lambda)+O\left(\ln^2N\right),$$

holds for $i=1,2,3$. Here,

$$G_i(h,\bar{\eta},q)=G(D,l_i,h,\bar{\eta},q)=\sum_{\substack{(z_i,q)=1,\\ z_i\equiv l_i(\bmod D)}} e\left(\frac{a_ihz_i}{q}\right)\bar{\eta}(z_i),\quad i=1,2;$$

$$G_3(h,\bar{\eta},q)=G_3(D,l_3,h,\bar{\eta},q)=\sum_{\substack{(z_3,q)=1,\\ z_3\equiv l_3(\bmod D)}} e\left(\frac{a_3hz_3^2}{q}\right)\bar{\eta}(z_3) \tag{3.5}$$

and η, ζ- are characters modulo q/d_2 and D/d_1, respectively.

Proof. By the definition of $S_i(y)$, we have:

$$S_i(y)=\sum_{\substack{L<n_i\le N,\\ n_i\equiv l_i(\bmod D),\ (n,q)=1}}\Lambda(n_i)e(n_iy)+O\left(\sum_{\substack{p^k\le N\\ p|q,\quad k\ge2}}\ln p\, e(p^ky)\right)=$$

$$= \sum_{\substack{(z_i,q)=1 \\ z_i\equiv l_i(\bmod d)}} e\left(\frac{a_ihz_i}{q}\right) \sum_{\substack{L<n_i\le N \\ n_i\equiv l_i(\bmod D),\ n_i\equiv z_i(\bmod q)}} \Lambda(n_i)e(a_in_i\lambda)+O\left(\ln^2N\right).$$

If $z_i \equiv l_i(\bmod d)$, then the inner sum over the main range is empty, and therefore we can restrict the sum over z_i satisfying the condition $z_i \equiv l_i(\bmod d)$. On the other hand, the condition $z_i \equiv l_i(\bmod d)$ is equivalent to the conditions $n_i \equiv l_i(\bmod D)$ and $n_i \equiv z_i(\bmod q)$. These, in turn, are equivalent to $n_i \equiv l_i(\bmod D/d_1)$ and $n_i \equiv c_i(\bmod q/d_2)$, respectively. In this case, due to the orthogonality property of characters [9] and according to (3.2), the following equality holds for $S_i(y)$:

$$S_i(y)=\frac{1}{\varphi(D/d_1)\varphi(q/d_2)} \sum_{\zeta(\bmod D/d_1)} \bar{\zeta}(l_i) \sum_{\eta(\bmod q/d_2)} \sum_{\substack{(z_i,q)=1 \\ z_i\equiv l_i(\bmod d)}} e\left(\frac{a_ihz_i}{q}\right)\bar{\eta}(z_i)\times$$

$$\times \sum_{\substack{L<n_i\le N \\ n_i\equiv l_i(\bmod D)}} \zeta\eta(n_i)\Lambda(n_i)e(a_in_i\lambda)+O(\ln^2N).$$

For $S_3(y)$ we have the following:

$$S_3(y)=\varphi^{-1}(D/d_1)\varphi^{-1}(q/d_2)\sum_{\zeta(\bmod D/d_1)}\bar{\zeta}(l_3)\sum_{\eta(\bmod q/d_2)}G_3(h,\bar{\eta},q)S_3(\zeta\eta,\lambda)+O\left(\ln^2N\right).$$

Thus proving Lemma 3.2. □

Now, using the lemmas above, we simplify $I_1(N)$. For any $y=(hq^{-1}+\lambda)\in m(h,q)$ satisfying $|\lambda|<\tau/q$ and $q\le Q$, equation (3.3) and Lemma 3.2 imply that $S_i(y)$ can be written as follows:

$$S_i(y)=\varphi^{-1}(D/d_1)\varphi^{-1}(q/d_2)\times$$

$$\times\left\{G_i(h,\bar{\eta}_0,q)I(a_i\lambda)-\delta_q\tilde{\zeta}\zeta_0(l_i)\,G_i(h,\bar{\eta}\eta_0,q)\tilde{I}(a_i\lambda)-\sum_{\substack{\zeta\ (\bmod\ D/d_1) \\ \eta\ (\bmod\ q/d_2)}}\tilde{\zeta}(l_i)\,G_i(h,\bar{\eta},q)\tilde{I}(\zeta\eta,a_i\lambda)\right\}+$$

$$+O(\varphi^{-1}(q/d_2)\sum_{\eta\ (\bmod\ q/d_2)}|G_i(h,\bar{\eta},q)|(1+|a_i\lambda|N)NT^{-1}\ln^2N)+O(\ln^2N),$$

where $\tilde{\zeta}\zeta_0\,(\bmod D/d_1)\,\tilde{\eta}\eta_0\,(\bmod q/d_2)=\tilde{\chi}\chi_0\,(\bmod Dq/d)$, $\tilde{\zeta},\tilde{\eta}$ are primitive characters and

$$\delta_q:=\begin{cases}1, & \text{if } \tilde{\chi}\ (\bmod\ \tilde{r})\ \text{exists and}\ \tilde{r}\mid Dq/d,\\ 0, & \textit{otherwise}.\end{cases}$$

From (1.5), (2.4) and (3.5), since $|a_i|\le B$, $|\lambda|<\tau/q$ and $|a_i\lambda|N<BT^{1/4}q^{-1}$, we can trivially estimate the following term as: $\sum_{\eta\ (\bmod\ q/d_2)}|G_i(h,\bar{\eta},q)|\ll\varphi(q/d_2)\varphi(q)$. Using this, we can evaluate the term under the symbol $O:\ \varphi^{-1}(q/d_2)\sum_{\eta\ (\bmod\ q/d_2)}|G_i(h,\bar{\eta},q)|(1+|a_i\lambda|N)NT^{-1}\ln^2N\le NBT^{-3/4}\ln^2N$. Therefore, for $y=h/q+\lambda\in m(h,q)$ we obtain the following:

$$S_i(y)=\varphi^{-1}(D/d_1)\,\varphi^{-1}(q/d_2)\,H_i(h,q,\lambda)+O\left(NBT^{-3/4}\ln^2N\right). \tag{3.6}$$

By similarly reasoning, we estimate $S_3(y)$ as follows:

$$S_3(y)=\varphi^{-1}(D/d_1)\,\varphi^{-1}(q/d_2)\,H_3(h,q,\lambda)+O\left(N^{1/2}BT^{-1}\ln^2N\right). \tag{3.7}$$

The remaining term is estimated as: $\frac{1}{\varphi(\frac{q}{d_2})}\sum\limits_{\eta\ (\bmod\ \frac{q}{d_2})}|G_3(h,\bar\eta,q)|(1+|a_3\lambda|N_1)N_1T^{-1}\ln^2 N \ll \frac{B\ln^2 N}{T}$.

Here

$$\begin{cases} H_i\left(h,q,\lambda\right) := G_i(h,\bar\eta_0,q)I_i(\lambda) - \delta_q\tilde\zeta\zeta_0\left(l_i\right)G_i(h,\bar\eta\eta_0,q)\tilde I_i(\lambda) - F_i\left(h,q,\lambda\right), \\ F_i\left(h,q,\lambda\right) := \sum\limits_{\zeta\ (\bmod\ D/d_1)\eta\ (\bmod\ q/d_2)} \tilde\zeta\left(l_i\right)G_i(h,\bar\eta,q)\tilde I_i(\zeta\eta,\lambda). \end{cases} \tag{3.8}$$

To estimate $H_i\left(h,q,\lambda\right)$ we use Lemma 3.3 from [11] and Lemma 3 from [2]. Then we have: $\varphi^{-1}\left(D/d_1\right)\varphi^{-1}\left(q/d_2\right)H_i\left(h,q,\lambda\right) \ll \varphi\left(q\right)N$, $\varphi^{-1}\left(D/d_1\right)\varphi^{-1}\left(q/d_2\right)H_3\left(h,q,\lambda\right) \ll \varphi\left(q\right)N^{1/2}$.
So, from (3.6) and (3.7), we have: $S_1\left(y\right) = \varphi^{-1}\left(D/d_1\right)\varphi^{-1}\left(q/d_2\right)\{H_1+R\}$,
$S_2\left(y\right) = \varphi^{-1}\left(D/d_1\right)\varphi^{-1}\left(q/d_2\right)\{H_2+R\}$, $S_3\left(y\right) = \varphi^{-1}\left(D/d_1\right)\varphi^{-1}\left(q/d_2\right)\{H_3+R_1\}$.

Thus

$$I_1\left(N\right) = \sum_{q\le Q}\varphi^{-3}\left(\frac{D}{d_1}\right)\varphi^{-3}\left(\frac{q}{d_2}\right)\sum_{(h,q)=1}\int\limits_{-\tau/q}^{\tau/q} e\left(-b\left(\frac{h}{q}+\lambda\right)\right)\prod_{i=1}^{3}H_i\left(h,q,\lambda\right)d\lambda +$$

$$+O\left(\sum_{q\le Q}\sum_{(h,q)=1}\frac{\tau}{q}\varphi^2\left(q\right)N^{\frac{5}{2}}B^3T^{-\frac{3}{4}}\ln^6 N\right).$$

We estimate the remainder term in the final expression as follows: $\ll N^{\frac{3}{2}}T^{-1/2}Q^3B^3\ln^6 N \ll N^{\frac{3}{2}}Q^{-1}$.
As a result, we obtain the following expression for $I_1\left(N\right)$:

$$I_1\left(N\right) = \sum_{q\le Q}\varphi^{-3}\left(D/d_1\right)\varphi^{-3}\left(q/d_2\right)\sum_{(h,q)=1}\int\limits_{-\tau/q}^{\tau/q} e\left(-b\left(h/q+\lambda\right)\right)\prod_{i=1}^{3}H_i\left(h,q,\lambda\right)d\lambda + O\left(N^{\frac{3}{2}}Q^{-1}\right).$$

Now we extend the integration interval $\left[-\frac{\tau}{q},\frac{\tau}{q}\right]$ to $(-\infty;\infty)$. Arguing as at the end of §3 of paper [2], we obtain

$$I_1\left(N\right) = \sum_{q\le Q}\frac{1}{\varphi^3\left(\frac{D}{d_1}\right)\varphi^3\left(\frac{q}{d_2}\right)}\sum_{(h,q)=1}e_q\left(-bh\right)\int\limits_{-\infty}^{\infty}e\left(-b\lambda\right)\prod_{i=1}^{3}H_i\left(h,q,a_i\lambda\right)d\lambda + O\left(N^{3/2}Q^{-1}\right). \tag{3.9}$$

4. Singular Series and Singular Integral of the Problem

Regarding the special series and the special integral, taking into account the specifics of our problem, we formulate several lemmas, the proof of which is, in principle, similar to the proof of analogous results given in the works [7], [12].

Lemma 4.1. *If $\chi\left(\bmod p^\beta/d_2\right)$ is an arbitrary character such that $\beta\ge 0$, $d_2 = d_2\left(p^\beta\right)$ is defined as in (3.4) and α is defined from the relation $p^\alpha\| D$. Then the following holds:*
(a) if $\chi\left(\bmod p^\beta\right)$ is a primitive character and $p|\, h$, $\beta>\alpha$, then $G_i\left(h,\chi,p^\beta\right)=0$, $i=1,2,3$.
(b) if η_0 is a character modulo $p^t/d_2\left(p^t\right)$ such that, $p\nmid h$ and $t\ge\theta+\max\{\theta,\alpha,\beta\}$, then $G_i\left(h,\chi\eta_0,p^t\right)=0$. Here $\theta = 1+[2/p]$, $i=1,2,3$.
(c) if $p\nmid h$, then $G_i\left(h,\chi,p^\beta\right)\le(2,p)\left(h,p^\beta\right)^{1/2}p^{\beta/2}$, $i=1,2$, $G_3\left(h,\chi,p^\beta\right)\le 2\left(2,p\right)\left(h,p^\beta\right)^{1/2}p^{\beta/2}$.

In our subsequent analysis, we encounter the following sums:

$$Z\left(q\right) := Z\left(q,\eta_1,\eta_2,\eta_3\right) := \sum_{(h,q)=1}e_q\left(-bh\right)\prod_{i=1}^{3}G_i\left(a_ih,\eta_i,q\right) \tag{4.1}$$

and

$$Y\left(q\right) := Y\left(q,\eta_1,\eta_2,\eta_3\right) := \sum_{h=1}^{q}e_q\left(-bh\right)\prod_{i=1}^{3}G_i\left(a_ih,\eta_i,q\right), \tag{4.2}$$

where η_i is a character modulo $q/d_2(q)$. The expression $Y(q)$ can be written in the following form:

$$Y(q,\eta_1,\eta_2,\eta_3) = q\sum_{(q)}\eta_1(z_1)\,\eta_2(z_2)\,\eta_3(z_3). \tag{4.3}$$

The sum $\sum\limits_{(q)}$ is taken over all triples $z_1,\ z_2,z_3$ such that (1.3) and (1.4). In expression (4.3) if all the characters η_i are principal, then it can be seen that:

$$Y(q,\eta_0,\eta_0,\eta_0) = qN(q), \tag{4.4}$$

this equality holds. Moreover, we define

$$A(q) := \varphi^{-3}\left(q(D,q)^{\odot}/d\right)Z(q,\eta_0,\eta_0,\eta_0), \tag{4.5}$$

where $(D,q)^{\odot}$ denotes the product of the common prime divisors of D and q. $(D,q)^{\odot}\Big\| D$ indicates that it is the largest divisor of D such that if $p^{\alpha}\|(D,q)^{\odot}$, then necessarily $p^{\alpha}\| D$.

Lemma 4.2. *The functions $Z(q)$ and $Y(q)$ are multiplicative in the following sense: that is, if $q = q_1\cdots q_t$ and $(q_i,\ q_j)=1$ for $i\neq j$ then for each $i=1,2,3$, $\eta_i\,(\mathrm{mod}\, q/d_2(q)) = \prod\limits_{j=1}^{t}\eta_{ij}\,(\mathrm{mod}\, q_j/d_2(q_j))$ is a valid decomposition. In this case, we have: $Z(q,\eta_1,\eta_2,\eta_3) := \prod\limits_{j=1}^{t} Z(q_j,\eta_{1j},\eta_{2j},\eta_{3j})$ and $Y(q,\eta_1,\eta_2,\eta_3) := \prod\limits_{j=1}^{t} Y(q_j,\eta_{1j},\eta_{2j},\eta_{3j})$, also hold. In particular, this implies that $N(q)$ and $A(q)$ are also multiplicative functions with respect to q.*

Lemma 4.3. *For any positive integer q the following estimate holds:*
$\varphi^{-3}\left(\frac{Dq}{d(q)}\right)Z(q) \ll \frac{d^3(q)}{D^3}q^{-1/2}\mathcal{L}^{-3}$, *where* $\mathcal{L} = \left(\ln\ln\frac{Dq}{d(q)}\right)$.

Lemma 4.4. *Suppose that $\chi_i\left(\mathrm{mod} p^{\beta_i}\right)$ for $i = 1,2,3$ are primitive characters or that $\beta = \max\{\beta_1,\ \beta_2,\beta_3\}$ and we choose $\alpha := \alpha(p)$ so that the condition $p^{\alpha(p)}\big| D$ is satisfied. For simplicity, in the notation we write $Z(p^t) = Z(p^t;\ \chi_1\chi_0,\ \chi_2\chi_0,\ \chi_3\chi_0)$, where χ_0, is the principal character modulo p^t. Then the following statements hold:*
(a) If $\beta>\alpha$, then $Z\left(p^{\beta}\right) = Y\left(p^{\beta}\right)$.
(b) If $t\geq\theta+\max\{\theta,\ \beta,\ \alpha\}$, then $Z(p^t)=0$. Where $\theta = 1+[2/p\,]$.
(c) If $\beta>\alpha$, then $\sum\limits_{v=\beta}^{t}\varphi^{-3}(p^v)\,Z(p^v) = \varphi^{-3}(p^t)\,Y(p^t)$ or $\beta=0$ and $t>\alpha$, then
$\sum\limits_{v=0}^{\alpha}\varphi^{-3}(p^a)\,Z(p^v) + \sum\limits_{v=\alpha+1}^{t}\varphi^{-3}(p^v)\,Z(p^v) = \varphi^{-3}(p^t)\,Y(p^t)$.

In Lemma 4.4, let $\chi_1 = \chi_2 = \chi_3 = \chi_0$ and $\beta = 0$ then the following result is obtained.

Corollary 4.5. *Suppose $N(q)$, $A(q)$ and $\alpha = \alpha(p)$ are defined respectively as in* (4.6), (4.5) *and similarly as in Lemma 4.4. Then the following statements hold:*
(a) if $p\geq 3$, $t\geq 1+\alpha$, then $A(p^t)=0$, if $t\geq 2+\max\{2,\alpha\}$, then $A(2^t)=0$.
(b) if $p\geq 3$, $t\geq\alpha$, then $p^t\varphi^{-3}(p^t)\,N(p^t) = p^{\alpha}\varphi^{-3}(p^{\alpha})\,N(p^{\alpha})$.
(c) if $t\geq\alpha'$, where $a' = 1+\max\{2,\alpha\}$, then $2^t\varphi^{-3}(2^t)\,N(2^t) = 2^{\alpha'}\varphi^{-3}\left(p^{\alpha'}\right)N\left(p^{\alpha'}\right)$.

Taking the above result into account, we introduce the following notation:

$$s(p) := \sum_{0\leq t<\theta+\max\{\theta,\alpha(p)\}} A\left(p^t\right) = \varphi^{-3}\left(\sigma\left(p^{\alpha(p)}\right)p^{\alpha(p)}\right)N\left(\sigma\left(p^{\alpha(p)}\right)p^{\alpha(p)}\right)\sigma\left(p^{\alpha(p)}\right)p^{\alpha(p)}. \tag{4.6}$$

Here $\sigma(q) = 1,4,2$ corresponds respectively to $2\nmid q$, $2\,\|q$ and $4\,|q$.

Lemma 4.6. *For $s(p)$ the following assertions hold:*
(a) if $p \neq 2$, $\alpha = \alpha(p) \geq 1$, then $s(p) = \varphi^{-3}(p^{\alpha}) p^{\alpha}$.
(b)

$$s(2) = \begin{cases} 2^3, & if\ \alpha(2) = 1, \\ \varphi^{-3}\left(2^{\alpha(2)}\right) 2^{\alpha(2)+1}, & if\ \alpha(2) \geq 2. \end{cases}$$

Therefore, we also have: $s(2) = \varphi^{-3}(2^{\alpha}) 2^{\alpha}\sigma(D)$.
(c) if $p \neq 2$ and $p \nmid D$, then $s(p) = 1 + A(p)$, and if $2 \nmid D$, then $s(2) = 1 + A(2) + A(2^2) + A(2^3)$.

Lemma 4.7. *The following statements hold:*
(a) if $p \nmid D$, then $|A(p)| < 10p^{-2}$.
(b) The product $\prod\limits_{p} s(p)$ converges absolutely, and $\prod\limits_{p} s(p) \gg \varphi^{-3}(D) D\sigma(D)$.
(c) $\sum\limits_{q=1,\ (q,r)=1}^{\infty} \varphi^{-3}(Dq/d)Z(q;\eta_0,\eta_0,\eta_0) = \prod\limits_{p\nmid r} s(p) = \frac{\sigma(D/(D,r))D/(D,r)}{\varphi^3(D/(D,r))} \prod\limits_{p\nmid r,\ p\nmid D} s(p)$.
(d) $\sum\limits_{q\geq y} \varphi^{-3}(Dq/d)Z(q;\eta_0,\eta_0,\eta_0) \ll y^{-1}D^{-1}\ln^{10}(y+1)$.

Lemma 4.8. *Suppose $r_i|\, Dq/d$, for $i = 1,2,3$ and $\chi_i(\mathrm{mod} r_i) = \zeta_i\left(\mathrm{mod}\left(r_i, \frac{D}{d_1}\right)\right)\eta_i\left(\mathrm{mod}\left(r_i, \frac{q}{d_2}\right)\right)$. All characters are primitive and if $r = [r_1, r_2, r_3]$, then the following estimate holds:*
(a) $\sum\limits_{q\leq Q,\ r|Dq/d} \left|\varphi^{-3}\left(\frac{Dq}{d}\right) Z(q,\eta_1\eta_0,\eta_2\eta_0,\eta_3\eta_0) \prod\limits_{i=1}^{3} \zeta_i\zeta_0(l_i)\right| \ll r^{-1/2}L^{-3}$.
(b) Suppose $\alpha(p)$ is defined as in Lemma 4.4, and furthermore let $r_i = r_i^{(1)}r_i^{(2)}$, $r^{(j)} = \left[r_1^{(j)}, r_2^{(j)}, r_3^{(j)}\right]$, $\chi_i(\mathrm{mod} r_i) = \chi_i^{(1)}\left(\mathrm{mod} r_i^{(1)}\right)\chi_i^{(2)}\left(\mathrm{mod} r_i^{(2)}\right)$, $\left(r_i^{(1)}, r_i^{(2)}\right) = 1$, $i = 1,2,3$, $j = 1,2$, if $p^{\beta}\| r_i^{(1)}$, then $\beta > \alpha(p)$ and if $p^{\beta}\| r_i^{(2)}$, then $\beta \leq \alpha(p)$. If $D = D_1D_2$, $(D_1, D_2) = 1$, then $p^{\beta}\| r$ and if $p|\, D_1$, then $\beta \leq \alpha(p)$, if $p^{\beta}\| r$ and $p|\, D_2$, then $\beta > \alpha(p)$. In this case, we have

$$E := \sum_{q\leq Q,\ r|dq/h} \varphi^{-3}\left(\frac{Dq}{d}\right) Z(q,\eta_1\eta_0,\eta_2\eta_0,\eta_3\eta_0) \prod_{i=1}^{3} \zeta_i\zeta_0(l_i) =$$

$$= \prod_{i=1}^{3} \chi_i^{(2)}(l_i)(\mathrm{mod} r_2) \frac{\sigma(D_1) D_1}{\varphi^3(D_1)} \cdot \frac{Y\left(\sigma\left(r^{(1)}\right) r^{(1)}\right)}{\varphi^3\left(\sigma\left(r^{(1)}\right) r^{(1)}\right)} \prod_{p\nmid D,\ p\nmid r} s(p) + O\left(Q^{-1/2}\ln^4 Q\right).$$

Lemma 4.9. *For arbitrary complex numbers, $0 < \mathrm{Re}\,\rho_i \leq 1$, $i = 1,2,3$ the following equality holds:*

$$\int_{-\infty}^{\infty} e(-n\lambda) \prod_{i=1}^{2} \left(\int_{L}^{N} {x_i}^{\rho_i - 1} e(a_i x_i \lambda)\, dx_i\right) \int_{L^{1/2}}^{N^{1/2}} {x_3}^{\rho_3 - 1} e(a_3 x_3 \lambda)\, dx_3 d\lambda =$$

$$= \frac{N^{3/2}}{2(|a_3|)} \int_{\mathcal{D}} (Nx_1)^{(\rho_1-1)}(Nx_2)^{(\rho_2-1)}(Nx_3)^{(\rho_3-1)/2} x_3^{-1/2} dx_1 dx_2, \tag{4.7}$$

where $x_3 := (bN^{-1} - a_1x_1 - a_2x_2)\, a_3^{-1}$ and

$$\mathcal{D} := \left\{(x_1, x_2) :\ LN^{-1} \leq x_1, x_2 \leq 1,\ \ \left(LN^{-1}\right)^{1/2} \leq x_3 \leq 1\right\}. \tag{4.8}$$

Furthermore

$$\int_{\mathcal{D}} x_3^{-1/2} dx_1 dx_2 \gg 1. \tag{4.9}$$

The proofs of Lemmas 4.1-4.9 are essentially analogous to the arguments presented in [7], [12], [13] and [14]. Therefore, given the limited space of the article, we omit the evidence. These lemmas take into account the specifics of the present problem and for convenience of reference, we have stated them without proofs.

5. Estimation of the Integral $I_1(N)$ and Completion of the Proof of the Theorem 1

We now aim to obtain the required lower bound for $I_1(N)$. From equality (3.8) it follows that the product $\prod_{i=1}^{3} H_i(h,q,\lambda)$ consists of a sum of $3^3 = 27$ terms. We will divide these terms into the following three categories.
(C1): the term $\prod_{i=1}^{3} G_i(h,\bar{\eta}_0,q)I(\lambda)$.
(C2): the 19 terms each of which has at least one $F_i(h,q,\lambda)$as factor.
(C3): the 7 remaining terms.
For convenience, we write, for $i=1,2,3$,

$$M_i = \sum_{q\le Q}\varphi^{-3}(Dq/d)\sum_{(h,q)=1} e_q(-bh)\int_{-\infty}^{\infty} e(-b\lambda)\{sum\ of\ the\ terms\ in\ (C_i)\}\,d\lambda. \tag{5.1}$$

In view of (3.9), we have

$$I_1(N) = M_1 + M_2 + M_3 + O\left(N^{3/2}Q^{-1}\right). \tag{5.2}$$

For distinct integers $m_1, m_2, ...$ taken from the set $\{1,\,2,3\}$, let:

$$P(m_1,m_2,...) := \frac{N^{3/2}}{2(|a_3|)}\int_{\mathcal{D}} (Nx_{m_1})^{(\rho_1-1)}(Nx_{m_2})^{(\rho_2-1)}(Nx_{m_3})^{(\rho_3-1)/2}x_3^{-1/2}dx_1dx_2 \tag{5.3}$$

and

$$\Delta(m_1,m_2,...) := \tilde{\chi}(n_{m_1})\,\tilde{\chi}(n_{m_2})\,..., \tag{5.4}$$

where the region $\mathcal{D}$ is defined in (4.8), $\tilde{\chi}$ and $\tilde{\beta}$ are the exceptional character and exceptional zero respectively. Let

$$P_0 := \frac{N^{3/2}}{2(|a_3|)}\int_{\mathcal{D}} x_3^{-1/2}dx_1dx_2. \tag{5.5}$$

Clearly, from (4.9) we have

$$|P(m_1,m_2,...)| \le P_0 \ll N^{3/2}B^{-1}. \tag{5.6}$$

Lemma 5.1. *The following equality holds.* $M_1 = \frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid d} s(p)\,P_0 + O\left(N^{3/2}B^{-1}D^{-2}Q^{-1}\ln^{10}Q\right)$.

Proof. Based on (5.1)

$$M_1 = \sum_{q\le Q}\varphi^{-3}(Dq/d)\sum_{(h,q)=1} e_q(-bh)G_1(h,\bar{\eta}_0,q)G_2(h,\bar{\eta}_0,q)G_3(h,\bar{\eta}_0,q)\times$$

$$\times\int_{-\infty}^{\infty} e(-b\lambda)\int_L^N e(a_1x_1\lambda)\,dx_1\int_L^N e(a_2x_2\lambda)\,dx_2\int_L^N e\left(a_3x_3^2\lambda\right)dx_3d\lambda.$$

If in (5.3) we set $\rho_1=\rho_2=\rho_3=1$, then the above integral is equals P_0. In view of (4.1), the above double sum is $\sum_{q\le Q}\varphi^{-3}(Dq/d)Z(q)$. By Lemma 4.7 (c), (d), this can be written as:

$$\frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D} s(p) + O\left(\sum_{q>Q}|A(q)|\right) = \frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D} s(p) + O\left(Q^{-1}D^{-2}\ln^{10}Q\right).$$

From this expression and equality (5.6), the proof of Lemma 5.1 follows. Namely,
$M_1 = \left(\frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D} s(p) + O\left(\frac{\ln^{10}Q}{QD^2}\right)\right)P_0 = \frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D} s(p)\,P_0 + O\left(N^{3/2}\{BD^2Q\}^{-1}\ln^{10}Q\right).$ □

Lemma 5.2. *If the exceptional zero $\tilde{\beta}$ exists and $\tilde{r}_1$, d_1 are defined as in Lemma 4.8 (b), by taking $r^{(1)} = \tilde{r}_1$, then*

(a) $$M_3 = \frac{\sigma(D_1)D_1}{\varphi^3(D_1)} \cdot \frac{\sigma(\tilde{r}_1)\tilde{r}_1}{\varphi^3(\sigma(\tilde{r}_1)\tilde{r}_1)} \prod_{p\nmid D,\ p\nmid\tilde{r}_1} s(p) \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \Big\{ -\sum_{i=1}^{3} \Delta(i)\, P(i) + \sum_{1\le i<j\le 3} \Delta(i,j)\, P(i,j) -$$
$$-\Delta(1,2,3)\, P(1,2,3) \Big\} + O\left(N^{3/2}B^{7/2}Q^{-1/2}\ln^4 Q\right).$$

(b) $M_3 \ll d^3 D^{-3} N^{3/2} B \tilde{r}_1^{-1/2} \mathcal{L}^{-3}$.

Proof. Part (a) is proved by reasoning similar to that in the proof of part (a) of Lemma 13 in [2]. The bound in (b) can be deduced directly from Lemma 4.3. □

Define
$$\Omega = \begin{cases} \left(1-\tilde{\beta}\right)\ln T, & if\ \tilde{\beta}\ \text{exists}, \\ 1, & \text{otherwise}. \end{cases} \tag{5.7}$$

In view of Corollary 4.5, Lemma 4.6 and (4.6), we have
$$\prod_{p\nmid D,\ p\nmid \tilde{r}_1} s(p) = \sigma(r'')\, r'' \varphi^{-3}\left(\sigma(r'')\, r''_1\right) N\left(\sigma(r'')\, r''_1\right), \tag{5.8}$$

$$\frac{\sigma(r'_1)\, r'_1}{\varphi^3\left(\sigma(r'_1)\, r'_1\right)} N\left(\sigma(r'_1)\, r'\right) = \frac{\sigma(D_2)\, D_2}{\varphi^3\left(\sigma(D_2)\, D_2\right)} N\left(\sigma(D_2)\, D_2\right) = \frac{\sigma(D_2)\, D_2}{\varphi^3(D_2)}, \tag{5.9}$$

where $r''r' = r$, $(r'', r') = 1$, $(r'', D) = 1$, $r' |\, D^{\odot}$, $r'_1, D\,|(r, D)$ have the same prime factors and the exponent of each prime factor of D_2 is less than in r'_1. Hence we can write M_1 in the form.
$$M_1 = \frac{\sigma(D_1)\, D_1}{\varphi^3(D_1)} \cdot \frac{\sigma(\tilde{r}_1)\, \tilde{r}_1}{\varphi^3\left(\sigma(\tilde{r}_1)\, \tilde{r}_1\right)} \prod_{p\nmid D,\ p\nmid\tilde{r}_1} s(p) \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} P_0 + O\left(N^{3/2}B^{-1}D^{-2}Q^{-1}\ln^{10} Q\right).$$

Comparing it with the form of M_3 in Lemma 5.2 (a), we have
$$M_1 + M_3 = \frac{\sigma(D_1)D_1}{\varphi^3(D_1)} \cdot \frac{\sigma(\tilde{r}_1)\tilde{r}_1}{\varphi^3(\sigma(\tilde{r}_1)\tilde{r}_1)} \prod_{p\nmid D,\ p\nmid\tilde{r}_1} s(p) \Bigg(\sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} P_0 + \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \Big\{ -\sum_{i=1}^{3} \Delta(i)\, P(i) +$$
$$+ \sum_{1\le i<j\le 3} \Delta(i,j)\, P(i,j) - \Delta(1,2,3)\, P(1,2,3) \Big\} \Bigg) + O\left(N^{3/2}B^{7/2}Q^{-1/2}\ln^4 Q\right).$$

Applying formulas (5.3), (5.4) and (5.5) to the right-hand side of the last equality, we obtain:
$$M_1 + M_3 = \frac{\sigma(D_1)\, D_1}{\varphi^3(D_1)} \cdot \frac{\sigma(\tilde{r}_1)\, \tilde{r}_1}{\varphi^3\left(\sigma(\tilde{r}_1)\, \tilde{r}_1\right)} \prod_{p\nmid D,\ p\nmid\tilde{r}_1} s(p) \Bigg\{ \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \frac{N^{3/2}}{2(|a_3|)} \int_{\mathcal{D}} x_3^{-1/2} dx_1 dx_2 -$$
$$-\sum_{i=1}^{2} \left(\sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \tilde{\chi}(n_i) \left(\frac{N^{3/2}}{2(|a_3|)} \int_{\mathcal{D}} \prod_{i=1}^{2} (Nx_i)^{(\rho_i - 1)} dx_1 dx_2 \right) \right) -$$
$$- \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \tilde{\chi}(n_3) \left(\frac{N^{3/2}}{2(|a_3|)} \int_{\mathcal{D}} (Nx_3)^{\frac{(\rho_3 - 1)}{2}} dx_1 dx_2 \right) +$$
$$+ \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \tilde{\chi}(n_1)\, \tilde{\chi}(n_2) \left(\frac{N^{3/2}}{2(|a_3|)} \int_{\mathcal{D}} (Nx_1)^{(\rho_1 - 1)} (Nx_2)^{(\rho_2 - 1)} dx_1 dx_2 \right) +$$

$$+\sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)}\tilde{\chi}(n_2)\tilde{\chi}(n_3)\left(\frac{N^{3/2}}{2(|a_3|)}\int_{\mathcal{D}}(Nx_2)^{(\rho_2-1)}(Nx_3)^{\frac{(\rho_3-1)}{2}}dx_1dx_2\right)+$$

$$+\sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)}\tilde{\chi}(n_1)\tilde{\chi}(n_3)\left(\frac{N^{3/2}}{2(|a_3|)}\int_{\mathcal{D}}(Nx_1)^{(\rho_1-1)}(Nx_3)^{\frac{(\rho_3-1)}{2}}dx_1dx_2\right)-\sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)}\tilde{\chi}(n_1)\tilde{\chi}(n_2)\tilde{\chi}(n_3)\times$$

$$\times\left(\frac{N^{3/2}}{2(|a_3|)}\int_{\mathcal{D}}(Nx_1)^{(\rho_1-1)}(Nx_2)^{(\rho_2-1)}(Nx_3)^{\frac{(\rho_3-1)}{2}}dx_1dx_2\right)\Bigg\}+O\left(N^{3/2}B^{7/2}Q^{-1/2}\ln^4Q\right). \quad (5.10)$$

The expression inside the curly braces is not less than

$$\geq\sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)}\int_{\mathcal{D}}x_3^{-1/2}\prod_{i=1}^{2}\left(1-\tilde{\chi}(n_i)(Nx_i)^{(\rho_i-1)}\right)\left(1-\tilde{\chi}(n_3)(Nx_3)^{\frac{(\rho_3-1)}{2}}\right)dx_1dx_2.$$

It remains to estimate the integral. Since $LN^{-1}\leq x_1,x_2,x_3\leq 1$ and $\left(1-\tilde{\chi}(n_i)(Nx_i)^{(\rho_i-1)}\right)\geq 1-L^{\tilde{\beta}-1}$, $\left(1-\tilde{\chi}(n_3)(Nx_3)^{\frac{(\rho_3-1)}{2}}\right)\geq 1-L^{\tilde{\beta}-1}$ are present in the domain $\mathcal{D}$ we obtain the following: $\prod_{i=1}^{2}\left(1-\tilde{\chi}(n_i)(Nx_i)^{(\rho_i-1)}\right)\left(1-\tilde{\chi}(n_3)(Nx_3)^{\frac{(\rho_3-1)}{2}}\right)\geq\left(1-L^{\tilde{\beta}-1}\right)^3$. From equalities (2.1) and (2.2), it follows that: $L=NB^{-1}\geq NQ^{-\delta}=N^{1-\delta^2}>\sqrt{N}$. Thus, using formulas (2.1) and (5.7), we obtain the following: $1-L^{(\tilde{\beta}-1)}\geq 1-\exp\left(-1/2\left(1-\tilde{\beta}\right)\ln N\right)\geq\min\left\{1/2,\ 1/4\left(\left(1-\tilde{\beta}\right)\ln N\right)\right\}\geq\Omega$. Then the main term in (5.10) is $\gg\Omega^3\frac{\sigma(D_1)D_1}{\varphi^3(D_1)}\cdot\frac{\sigma(\tilde{r}_1)\tilde{r}_1}{\varphi^3(\sigma(\tilde{r}_1)\tilde{r}_1)}\prod_{p\nmid D,\ p\nmid\tilde{r}_1}s(p)\sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)}P_0$. Hence, by (5.8) and (5.9), we have.

Lemma 5.3. $M_1+M_3\geq\Omega^3\frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D}s(p)P_0+O\left(N^{3/2}B^{7/2}Q^{-1/2}\ln^4Q\right)$.

We need to estimate M_2 By the Deuring–Heilbronn phenomenon, we have.

Lemma 5.4. $M_2\ll\Omega^3\exp\left(-c/\sqrt{\delta}\right)\frac{\sigma(D)D}{\varphi^3(D)}P_0\prod_{p\nmid D,\,p\nmid r}s(p)+O\left(N^{3/2}Q^{-1/2}B^{7/2}\ln^4Q\right)$.

Proof. It is proved by reasoning similar to that used in the proof of Lemma 15 in [15]. □

Combining the results obtained above, we can derive the following estimate for $I_1(N)$. To this end, let us consider two cases.

Case 1. If there is no $\tilde{\beta}$ exceptional zero of the Dirichlet L-function, or if it exists and the modulus of the corresponding exceptional character $\tilde{r}>Q^{1/10}$. From Lemma 5.1 and part (b) of Lemma 5.2, as well as from Lemma 5.4 for sufficiently small δ we obtain:

$$I_1(N)\geq\frac{\sigma(D)D}{\varphi^3(D)}P_0\prod_{p\nmid D,\ p\nmid r}s(p)+O\left(\frac{d^3}{D^3}N^{3/2}B^{-1}Q^{-1/20}L^{-3}\right).$$

Then, according to Lemma 4.7 (a), we have: $I_1(N)\gg N^{3/2}\left(BQ^{1/21}D\right)^{-1}$. Here, $D\leq Q^{1/21}$.

Case 2. If there exists a $\tilde{\beta}$ exceptional zero of the Dirichlet L-function and the modulus of the corresponding exceptional character $\tilde{r}\leq Q^{1/10}$. Then, using Lemmas 5.3 and 5.4, and for sufficiently small δ, we obtain:

$$I_1(N)\geq\Omega^3\frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D}s(p)P_0+O\left(N^{3/2}B^{7/2}Q^{-1/2}\ln^4Q\right).$$

Here, taking into account that $\Omega\gg\left(\tilde{r}^{1/2}\ln^2\tilde{r}\right)^{-1}\ln T\gg Q^{-1/20}\ln^{-1}Q$, is as in (3.1), we consider $I_1(N)\gg N^{3/2}\left(BQ^{83/420}D\right)^{-1}$ and finally, by comparing the estimates in both cases with Lemma 2.3, it follows that for sufficiently large N, $I_1(N)>|I_2(N)|$ holds. Thus, our theorem is proved.

5.1. **Proof of Corollary 1.1.** Now let us estimate the number of solutions of equation (1.1) in prime numbers:

$$R(b) = R(b, N, D) = \sum_{\substack{L<p_1,p_2,p_3^2\leq N\\ b=a_1p_1+a_2p_2+a_3p_3^2}} 1$$

Based on (2.8) we have:

$$I(N) = R(b)\ln^3 N + O(N\ln N). \tag{5.11}$$

Since $I(N) > I_1(N) - |I_2(N)|$, it follows from (5.11) that $R(b) \gg (I_1(N) - |I_2(N)|)(\ln N)^{-3} + O\left(N(\ln N)^{-2}\right)$. Hence we obtain an estimate $R(b) \gg N^{3/2}\left(BQ^{83/420}D\ln^3 N\right)^{-1}$.

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