

Application of the method of quadrature's with derivatives for solving problems of the Euler - Bernoulli beam

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Abstract. This paper proposes a method for solving the problem of the Euler - Bernoulli beam, in which the problem is reduced to the solution of the Fredholm integral equation of the second kind. This integral equation is solved by the method of optimal quadrature formulas with derivatives. Then, using the Green function, the analytic solution is determined. Different boundary conditions can be used to determine the numerical and analytical solutions. To determine the suitability of the method, the numerical results are compared with the analytical solution.

Keywords: differential equation, integral equation, optimal quadrature formula, beam deflection value, absolute error

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1. INTRODUCTION

In connection with the development of high-speed rail transportation, the problem of modeling the dynamic effects of a rolling load on bending elements is widely used in the field of design and control of the safety of transport facilities. This issue is increasingly relevant in relation to the increase in traffic speeds, especially in the field of railway transport. The theoretical foundations of the control complex method are laid out in the works [1] - [2].

Mathematical models and the calculation of structures that lie on an elastic base are the subject of many research papers [3] - [4]. In these works, mathematical models of Timoshenko, Rayleigh, and Euler-Bernoulli were used. To solve the problems, the Fourier integral transform method, the finite element method, and the finite difference method were used.

The development of numerical methods, in particular, the widespread use of the optimal quadrature formula method, made it possible to model various structural systems in sufficient detail under the influence of various external influences.

This paper proposes a new approach to solving the problem of the reliability of railway tracks. The essence of this approach lies in the fact that the Euler-Bernoulli model is chosen for the main calculation model, the differential equation is reduced to the Fredholm integral equation of the second kind, which is solved by the method of optimal quadrature formulas with derivatives.

An analytical solution of this model is given. A specific example is considered. The numerical results are compared with the analytical solution. Appropriate conclusions are issued on the method.

2. PROBLEM STATEMENT

We consider the following problem: a beam of length L , lying on a solid elastic base, under the action of a moving load with P a constant velocity V . You must define the deflection of the beam at each point and in each time layer.

To solve this problem, the following mathematical models of beam oscillation can be considered [3]: Tymoshenko's model

$$EI \frac{\partial^4 u}{\partial x^4} - \rho I \frac{\partial^4 u}{\partial x^2 \partial t^2} - \frac{\rho EI}{\gamma G} \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{\rho I}{\gamma G} \frac{\partial^4 u}{\partial t^4} + \rho A \frac{\partial^2 u}{\partial t^2} + c \frac{\partial u}{\partial t} + ku = q(x, t), \quad (1)$$

Rayleigh model

$$EI \frac{\partial^4 u}{\partial x^4} - \rho I \frac{\partial^4 u}{\partial x^2 \partial t^2} + \rho A \frac{\partial^2 u}{\partial t^2} + c \frac{\partial u}{\partial t} + ku = q(x, t), \quad (2)$$

Euler-Bernoulli model

$$EI \frac{\partial^4 u}{\partial x^4} + \rho A \frac{\partial^2 u}{\partial t^2} + c \frac{\partial u}{\partial t} + ku = q(x, t), \quad (3)$$

where E is the modulus of elasticity of the beam; G is modulus of shear; I is the moment of inertia of the beam cross-section; ρ is beam material density; A is cross-sectional area of the beam; k is coefficient of the bed of the elastic base; c is base damping coefficient; γ is the coefficient of influence of the form; $q(x, t)$ is the intensity of the impact on the beam.

Conductive rails mainly hold shear forces and the transverse dimensions of the rail are small compared to its length, so that the rails can be considered within the framework of the Euler-Bernoulli beam model. The intensity of the impact on the beam is of a harmonic type. Therefore, imagine a moving load

$$q(x, t) = P \cos \omega t \delta(x - Vt),$$

where ω is the vibration frequency of the beam.

As the main calculation model, let us consider the differential equation of the Euler-Bernoulli beam (the basis of the Winkler model):

$$EI \frac{\partial^4 u}{\partial x^4} + \rho A \frac{\partial^2 u}{\partial t^2} + ku = P \cos \omega t \delta(x - Vt), x \in [0, L], t \in [0, T], \quad (3)$$

with the corresponding initial conditions:

$$u(x, 0) = 0, \frac{\partial u(x, 0)}{\partial t} = 0, x \in [0, L],$$

where L is the length of the region under consideration, T is the study period,

$$\delta(x - Vt) = \begin{cases} 1, & \text{if } x = Vt, \\ 0, & \text{if } x \neq Vt. \end{cases}$$

As boundary conditions, we consider different models of problems:

Model 1. Beam with fixed ends:

$$u(0, t) = 0, \frac{\partial u(0, t)}{\partial x} = 0, u(L, t) = 0, \frac{\partial u(L, t)}{\partial x} = 0, t \in [0, T].$$

Model 2. The left end is fixed and the right end is hinged:

$$u(0, t) = 0, \frac{\partial u(0, t)}{\partial x} = 0, u(L, t) = 0, \frac{\partial^2 u(L, t)}{\partial x^2} = 0, t \in [0, T].$$

Model 3. The left end is hinged and the right end is fixed:

$$u(0, t) = 0, \frac{\partial^2 u(0, t)}{\partial x^2} = 0, u(L, t) = 0, \frac{\partial u(L, t)}{\partial x} = 0, t \in [0, T].$$

Model 4. The ends of the beam are hinged:

$$u(0, t) = 0, \frac{\partial^2 u(0, t)}{\partial x^2} = 0, u(L, t) = 0, \frac{\partial^2 u(L, t)}{\partial x^2} = 0, t \in [0, T].$$

It is necessary to find an approximate and analytical solution to the above problems and to evaluate an approximate solution using an analytical solution. This estimate allows us to apply this approximate method to more complex Euler-Bernoulli beam problems. More precisely, to problems for which it is impossible to find an analytical solution.

Before the numerical solution of mechanical problems, equations, initial and boundary conditions are converted to dimensionless variables:

$$\xi = \frac{x}{L}, \tau = \frac{t}{t_0}, w(\xi, \tau) = \frac{u(x, t)}{u_0},$$

$t_0 = \sqrt{\frac{\rho AL^4}{EI}}$ is characteristic frequency,

$u_0 = \frac{PL^3}{EI}$ is characteristic scale of deflection.

Let us put all the variables in their places, get the equation in dimensionless variables:

$$\frac{\partial^4 w}{\partial \xi^4} + \frac{\partial^2 w}{\tau^2} + \lambda w = \cos \omega_0 \tau \delta(\xi - v\tau), \xi \in [0, 1], \tau \in [0, T_0]. \quad (4)$$

Here

$\omega_0 = \omega t_0$ is dimensionless frequency,

$\lambda = \frac{KL^4}{EI}$ is dimensionless Base factor,

$v = \frac{Vt_0}{L}$ is dimensionless speed,

$T_0 = \frac{T}{t_0}$ is the dimensionless duration of the period under consideration.

Dimensionless initial conditions:

$$w(\xi, 0) = 0, \frac{\partial w(\xi, 0)}{\partial \tau} = 0 = 0, \xi \in [0, 1]. \quad (5)$$

Dimensionless boundary conditions:

Model 1:

$$w(0, \tau) = 0, \frac{\partial w(0, \tau)}{\partial \xi} = 0, w(1, \tau) = 0, \frac{\partial w(1, \tau)}{\partial \xi} = 0, \tau \in [0, T_0], \quad (6)$$

Model 2:

$$w(0, \tau) = 0, \frac{\partial^2 w(0, \tau)}{\partial \xi^2} = 0, w(1, \tau) = 0, \frac{\partial w(1, \tau)}{\partial \xi} = 0, \tau \in [0, T_0], \quad (7)$$

Model 3:

$$w(0, \tau) = 0, \frac{\partial w(0, \tau)}{\partial \xi} = 0, w(1, \tau) = 0, \frac{\partial^2 w(1, \tau)}{\partial \xi^2} = 0, \tau \in [0, T_0], \quad (8)$$

Model 4:

$$w(0, \tau) = 0, \frac{\partial^2 w(0, \tau)}{\partial \xi^2} = 0, w(1, \tau) = 0, \frac{\partial^2 w(1, \tau)}{\partial \xi^2} = 0, \tau \in [0, T_0], \quad (9)$$

Thus, we have four problems that need to be solved in an approximate and analytical ways.

3. PROBLEM-SOLVING METHODS

3.1. Method of quadratures with derivatives. To solve the above problems, we apply the following approach. Since the intensity of the action on the beam is a harmonic function, and the desired function that determines the forced oscillations of the same frequency ω_0 is represented as follows: $w(\xi, \tau) = y(\xi) \cos \omega_0 \tau$, where $y(\xi)$ is the unknown function.

Since the function $\delta(\xi - v\tau)$ depends on time, it will be necessary to find $y(\xi)$ for each time layer. Therefore, moving to finite differences in time and substituting the above expressions into equation (4), we have a boundary value problem in ordinary differential equations:

$$y_j^{(4)}(\xi) = \mu y_j(\xi) + \delta(\xi - v\tau_j), \quad (10)$$

where $\mu = \omega_0^2 - \lambda$.

Boundary Conditions:

Model 1:

$$y_j(0) = \frac{dy_j}{dx}(0) = y_j(1) = \frac{dy_j}{dx}(1) = 0, j = \overline{0, M}, \quad (11)$$

Model 2:

$$y_j(0) = \frac{dy_j}{dx}(0) = y_j(1) = \frac{d^2 y_j}{dx^2}(1) = 0, j = \overline{0, M}, \quad (12)$$

Model 3:

$$y_j(0) = \frac{d^2 y_j}{dx^2}(0) = y_j(1) = \frac{dy_j}{dx}(1) = 0, j = \overline{0, M}, \quad (13)$$

Model 4:

$$y_j(0) = \frac{d^2 y_j}{dx^2}(0) = y_j(1) = \frac{d^2 y_j}{dx^2}(1) = 0, j = \overline{0, M}, \quad (14)$$

where $y_j(\xi)$ is the desired function is in the j time layer, $\tau_j = \Delta\tau j, j = \overline{0, M}, \Delta\tau$ is the step in time.

To solve boundary value problems (10) – (14), equation (10) is reduced to the Fredholm integral equation of the second kind [5]:

$$y_j(\xi) = \mu \int_0^1 G(\xi, s)y_j(s)ds + \int_0^1 G(\xi, s)\delta(s - v\tau_j)ds, \quad (15)$$

where $G(x, s)$ is the Green function. Under the conditions for determining the Green function, all boundary conditions (11) - (14) are taken into account. The second integral in equations (15)

$$\int_0^1 G(\xi, s)\delta(s - v\tau_j)ds = G(\xi, v\tau_j).$$

Hence, equation (15) will take the following form

$$y_j(\xi) = \mu \int_0^1 G(\xi, s)y_j(s)ds + G(\xi, v\tau_j).$$

To solve the integral equation (16), we apply the method of optimal quadrature formulas with derivatives in space $L_2^{(m)}(0, 1)$ [6]. In this method, the integral of (16) is replaced by the sum, i.e.

$$\int_0^1 G(\xi_i, s)y_j(s)ds \cong \sum_{k=0}^m \sum_{\beta=0}^N C_{i\beta}^{(k)} y_{\beta j}^k, i = \overline{0, N}, j = \overline{0, M}, \quad (17)$$

Here ξ_i is the nodes of the grid, $\xi_i = ih, i = \overline{0, N}, N = h^{-1}, h$ is the grid step, $y_{\beta j}^{(k)} = y_j^k(\xi_\beta), k = \overline{0, m-1}, C_{i\beta}^{(k)}$ are the optimal coefficients of the quadrature formula (17).

The optimal coefficients of the quadrature formula (17) are determined using the following formulas:

$$\begin{aligned} C_{i0}^{(k)} &= \frac{p_{ik}}{2} + h^{-1}(-1)^k [F_{ik1} - F_{ik0}], \\ C_{i\beta}^{(k)} &= h^{-1}(-1)^k [F_{ik\beta-1} - 2F_{ik\beta} + F_{ik\beta+1}], \beta = \overline{1, N-1}, \\ C_{iN}^{(k)} &= \frac{p_{ik}}{2} + h^{-1}(-1)^k [F_{ikN-1} - F_{ikN}], \\ i &= \overline{0, N}, k = \overline{0, m-1}, m = 1, 2, \dots, \end{aligned}$$

where

$$\begin{aligned} F_{ik\beta} &= f_{ik\beta} - \sum_{\nu=0}^{k-1} \sum_{\gamma=0}^N (-1)^\nu C_{i\gamma}^{(\nu)} \frac{(h\beta - h\gamma)^{k-\nu+1} \text{sign}(h\beta - h\gamma)}{2(k-\nu+1)!}, \\ f_{ik\beta} &= (-1)^k \int_0^1 G^{(k)}(\xi_i, s) \frac{(s - h\beta)^{k+1} \text{sign}(s - h\beta)}{2(k+1)!} ds, \\ g_{ik} &= \int_0^1 G^{(k)}(\xi_i, s) s^k ds, G^{(k)}(\xi_i, s) = \frac{\partial^k}{\partial \xi^k} G(\xi, s)|_{\xi=\xi_i}, \\ p_{ik} &= \frac{g_{ik}}{k!} - \sum_{\nu=0}^{k-1} \sum_{\gamma=0}^N C_{i\gamma}^{(k)} \frac{(h\gamma)^{k-\nu}}{(k-\nu)!}, \\ i &= \overline{0, N}, k = \overline{0, m-1}, m = 1, 2, \dots \end{aligned}$$

Applying the optimal quadrature formula (17) to the integral equation (16) and moving on to the difference equations, we get:

$$y_{ij} - \mu \sum_{k=0}^{m-1} \sum_{\beta=0}^N C_{i\beta}^{(k)} y_{\beta j}^{(k)} = G_{ij}, i = \overline{0, N}, j = \overline{0, M}, m = 1, 2, \dots \tag{18}$$

Where $G_{ij} = G(\xi_i, v\tau_j)$.

In the system of linear algebraic equations (18) at each time layer, the number of equations is $N + 1$, and the number of unknowns $m(N + 1)$. To obtain the remaining equations, we differentiate the integral equation (16) $m - 1$ times with respect to the variable ξ :

$$\begin{aligned} y'_j(\xi) - \mu \int_0^1 G'_\xi(\xi, s)y_j(s)ds &= G'_\xi(\xi, v\tau_j), \\ y''_j(\xi) - \mu \int_0^1 G''_\xi(\xi, s)y_j(s)ds &= G''_\xi(\xi, v\tau_j), \\ &\dots \dots \dots \\ y_j^{(m-1)}(\xi) - \mu \int_0^1 G_\xi^{(m-1)}(\xi, s)y_j(s)ds &= G_\xi^{(m-1)}(\xi, v\tau_j), \\ &j = \overline{0, M}, m = 1, 2, \dots \end{aligned} \tag{19}$$

Furthermore, applying the optimal quadrature formula (17) for each equation (19) and moving on to the difference equations, we obtain a system of linear algebraic equations to determine the values of the desired function $y_j(\xi_i)$ and the values of the derivatives $y'_j(\xi_i), y''_j(\xi_i), \dots, y_j^{(m-1)}(\xi_i)$ in the nodes of the grid $\xi_i (i = \overline{0, N})$:

$$\begin{aligned} y_{ij} - \mu \sum_{k=0}^{m-1} \sum_{\beta=0}^N C_{i\beta}^{(k)} y_{\beta j}^{(k)} &= G_{ij}, \\ y'_{ij} - \mu \sum_{k=0}^{m-1} \sum_{\beta=0}^N C_{i\beta}^{(k)} y_{\beta j}^{(k)} &= G'_{ij}, \\ &\dots \dots \dots \\ y_{ij}^{(m-1)} - \mu \sum_{k=0}^{m-1} \sum_{\beta=0}^N C_{i\beta}^{(k)} y_{\beta j}^{(k)} &= G_{ij}^{(m-1)}. \end{aligned} \tag{20}$$

From the solution on each time layer j of the linear system of algebraic equations (20), the first $N + 1$ terms, i.e. $y_{ij}(i = \overline{0, N})$ will be the numerical solution of the integral equation (16).

Now let's move on to the function. Then $w_{ij} = w(\xi_i, \tau_j) = y_{ij} \cos \omega_0 \tau_j, i = \overline{0, N}, j = \overline{0, M}$.

3.2. Analytical solution. To find an analytical solution to our problems, we define the Green function for a homogeneous equation [5]:

$$y^{(4)}(\xi) + \mu y(\xi) = 0, \xi \in [0, 1]$$

taking into account the boundary conditions of each model (11) - (14). Each model will have its own Green function. After that, you can write an analytical solution to the inhomogeneous equation

$$y(\xi, \tau) = \int_0^1 G(\xi, s)\delta(s - v\tau)ds = G(\xi, v\tau).$$

Hence $w(\xi, \tau) = G(\xi, v\tau) \cos \omega_0 \tau$.

TABLE 1. Initial information for beam deflection calculation

№	Parameter names	Unit	Parameter values
1	Coefficient of elasticity	N/m ²	2.0 × 10 ¹¹
2	Moment of inertia of the section	m ⁴	0.0001
3	Density of Beam Material	kg/m ³	7850
4	Cross-sectional Area of Beam	m ²	0.03
5	Elastic Base Bed Coefficient	N/m ²	1.0E+7
6	Beam Length	m	10
7	Beam Load	N	50000
8	Load velocity	m/s	50
9	Oscillation frequency	-	0
10	Duration of the study period	s	10

4. NUMERICAL RESULTS

According to the above algorithm, a program is compiled in Matlab. Thanks to the Matlab language capability, the Green function is obtained in analytical form. All operations are performed using the program. For the numerical experiment, the initial information given in Table 1 is adopted.

For the integral equations and their right-hand sides in each model, the Green function is obtained: Model 1:

$$G(x, \xi) = \begin{cases} -x^2(\xi - 1)^2(x - 3\xi + 2x\xi)/6, & \text{if } x \leq \xi, \\ -\xi^2(x - 1)^2(\xi - 3x + 2x\xi)/6, & \text{if } x > \xi, \end{cases}$$

Model 2:

$$G(x, \xi) = \begin{cases} -x(\xi - 1)(\xi^2 - 2\xi + x^2)/6, & \text{if } x \leq \xi, \\ -\xi(x - 1)(\xi^2 - 2x + x^2)/6, & \text{if } x > \xi, \end{cases}$$

Model 3:

$$G(x, \xi) = \begin{cases} x^2((-\xi^3 + 3\xi^2 - 2)x + (3\xi^3 - 9\xi^2 + 6\xi))/12, & \text{if } x \leq \xi, \\ \xi^2(x - 1)(2\xi - 6x + 2\xi x - \xi x^2 + 3x^2)/12, & \text{if } x > \xi, \end{cases}$$

Model 4:

$$G(x, \xi) = \begin{cases} -x(\xi - 1)^2(\xi x^2 - 3\xi + 2x^2)/12, & \text{if } x \leq \xi, \\ -\xi(x - 1)^2(\xi^2 x - 3x + 2\xi^2)/12, & \text{if } x > \xi, \end{cases}$$

Next, for the analytical solution, the Green function for each model is also obtained: Model 4:

$$G(x, \xi) = \begin{cases} G_1(x, \xi), & \text{if } x \leq \xi, \\ G_2(x, \xi), & \text{if } x > \xi, \end{cases}$$

Model 1:

$$\begin{aligned} G_1(x, \xi) = & (\sin(\beta^* \xi - \beta x) - \sinh(\beta^* \xi - \beta x) - \cos(\beta^*) \sinh(\beta^* - \beta^* \xi + \beta x) + \\ & + \cosh(\beta^*) \sin(\beta^* - \beta^* \xi + \beta x) + \cos(\beta^* \xi) \sinh(\beta x) - \cosh(\beta^* \xi) \sin(\beta x) - \sin(\beta^* \xi) \cosh(\beta x) + \\ & + \sinh(\beta^* \xi) \cos(\beta x) - \cos(\beta^* \xi - \beta^* + \beta x) \sinh(\beta^*) + \cosh(\beta^* \xi - \beta^* + \beta x) \sin(\beta^*) - \\ & - \cos(\beta^* - \beta x) \sinh(\beta^* \xi - 1)) + \cosh(\beta^* - \beta x) \sin(\beta^* \xi - 1)) + \cos(\beta^* (\xi - 1)) \sinh(\beta^* - \beta x) - \\ & - \cosh(\beta^* \xi - 1)) \sin(\beta^* - \beta x)) / (4\beta^{*3} (\cos(\beta^*) \cosh(\beta^*) - 1)); \\ G_2(x, \xi) = & -(\sin(\beta^* \xi - \beta x) - \sinh(\beta^* \xi - \beta x) + \cos(\beta^*) \sinh(\beta^* + \beta^* \xi - \beta x) - \\ & - \cosh(\beta^*) \sin(\beta^* + \beta^* \xi - \beta x) - \cos(\beta^* \xi) \sinh(\beta x) + \cosh(\beta^* \xi) \sin(\beta x) + \\ & + \sin(\beta^* \xi) \cosh(\beta x) - \sinh(\beta^* \xi) \cos(\beta x) + \cos(\beta^* \xi - \beta^* + \beta x) \sinh(\beta^*) - \end{aligned}$$

$$\begin{aligned}
 & - \cosh(\beta^* \xi - \beta^* + \beta x) \sin(\beta^*) + \cos(\beta^* - \beta x) \sinh(\beta^* \xi - 1) - \cosh(\beta^* - \beta x) \sin(\beta^* \xi - 1) - \\
 & - \cos(\beta^* (\xi - 1)) \sinh(\beta^* - \beta x) + \cosh(\beta^* (\xi - 1)) \sin(\beta^* - \beta x) / (4\beta^{*3} (\cos(\beta^*) \cosh(\beta^*) - 1));
 \end{aligned}$$

Model 2:

$$\begin{aligned}
 G_1(x, \xi) &= (\cos(\beta^* \xi) \sin(\beta x)) / (2\beta^{*3}) - (\cosh(\beta^* \xi) \sinh(\beta x)) / (2\beta^{*3}) - \\
 & - (\cot(\beta^*) \sin(\beta^* \xi) \sin(\beta x)) / (2\beta^{*3}) + (\cosh(\beta^*) \sinh(\beta^* \xi) \sinh(\beta x)) / (2 \sinh(\beta^*) \beta^{*3}); \\
 G_2(x, \xi) &= (\sin(\beta^* \xi) \cos(\beta x)) / (\beta^{*3}) - (\sinh(\beta^* \xi) \cosh(\beta x)) / (2\beta^{*3}) - \\
 & - (\cot(\beta^*) \sin(\beta^* \xi) \sin(\beta x)) / (2\beta^{*3}) + (\cosh(\beta^*) \sinh(\beta^* \xi) \sinh(\beta x)) / (2 \sinh(\beta^*) \beta^{*3});
 \end{aligned}$$

Model 3:

$$\begin{aligned}
 G_1(x, \xi) &= (\cosh(\beta^*) \cos(\beta^* - \beta^* \xi + \beta x) + \sinh(\beta^*) \sin(\beta^* - \beta^* \xi + \beta x) - \cos(\beta^* \xi - \beta^* + \beta x) \cosh(\beta^*) - \\
 & - \sin(\beta^* \xi - \beta^* + \beta x) \sinh(\beta^*) - 2 \cosh(\beta^*) \sinh(\beta x) \sin(\beta^* (\xi - 1)) + 2 \sinh(\beta^*) \cosh(\beta x) \sin(\beta^* (\xi - 1)) + \\
 & + 2 \cosh(\beta^*) \sinh(\beta^* \xi) \sin(\beta^* - \beta x) - 2 \sinh(\beta^*) \cosh(\beta^* \xi) \sin(\beta^* - \beta x) + 2 \cos(\beta^*) \cosh(\beta^*) \sinh(\beta^* \xi) \\
 & \sinh(\beta x) - 2 \cos(\beta^*) \sinh(\beta^*) \cosh(\beta^* \xi) \sinh(\beta x) - 2 \cosh(\beta^*) \sin(\beta^*) \sinh(\beta^* \xi) \cosh(\beta x) + \\
 & + 2 \sin(\beta^*) \sinh(\beta^*) \cosh(\beta^* \xi) \cosh(\beta x)) / (4\beta^{*3} (\cos(\beta^*) \sinh(\beta^*) - \cosh(\beta^*) \sin(\beta^*))); \\
 G_2(x, \xi) &= (\cosh(\beta^*) \cos(\beta^* + \beta^* \xi - \beta x) + \sinh(\beta^*) \sin(\beta^* + \beta^* \xi - \beta x) - \cos(\beta^* \xi - \beta^* + \\
 & + \beta x) \cosh(\beta^*) - \sin(\beta^* \xi - \beta^* + \beta x) \sinh(\beta^*) - 2 \cosh(\beta^*) \sinh(\beta x) \sin(\beta^* (\xi - 1)) + \\
 & + 2 \sinh(\beta^*) \cosh(\beta x) \sin(\beta^* (\xi - 1)) + 2 \cosh(\beta^*) \sinh(\beta^* \xi) \sin(\beta^* - \beta x) - \\
 & - 2 \sinh(\beta^*) \cosh(\beta^* \xi) \sin(\beta^* - \beta x) + 2 \cos(\beta^*) \cosh(\beta^*) \sinh(\beta^* \xi) \sinh(\beta x) - \\
 & - 2 \cos(\beta^*) \sinh(\beta^*) \sinh(\beta^* \xi) \cosh(\beta x) - 2 \cosh(\beta^*) \sin(\beta^*) \cosh(\beta^* \xi) \sinh(\beta x) + \\
 & + 2 \sin(\beta^*) \sinh(\beta^*) \cosh(\beta^* \xi) \cosh(\beta x)) / (4\beta^{*3} (\cos(\beta^*) \sinh(\beta^*) - \cosh(\beta^*) \sin(\beta^*)));
 \end{aligned}$$

Model 4:

$$\begin{aligned}
 G_1(x, \xi) &= -(\sin(\beta^* \xi) \sinh(\beta x) + \sinh(\beta^* \xi) \sin(\beta x) - \cos(\beta^*) \cosh(\beta^*) \sin(\beta^* \xi) \sin(\beta x) - \\
 & - \cos(\beta^*) \sinh(\beta^*) \cos(\beta^* \xi) \sin(\beta x) + \cosh(\beta^*) \sin(\beta^*) \cos(\beta^* \xi) \sin(\beta x) - \\
 & - \cos(\beta^*) \cosh(\beta^*) \sinh(\beta^* \xi) \sinh(\beta x) + \cos(\beta^*) \sinh(\beta^*) \cosh(\beta^* \xi) \sinh(\beta x) - \\
 & - \cosh(\beta^*) \sin(\beta^*) \cosh(\beta^* \xi) \sinh(\beta x) - \sin(\beta^*) \sinh(\beta^*) \sin(\beta^* \xi) \sin(\beta x) + \\
 & + \sin(\beta^*) \sinh(\beta^*) \sinh(\beta^* \xi) \sinh(\beta x)) / (2\beta^{*3} (\cos(\beta^*) \sinh(\beta^*) - \cosh(\beta^*) \sin(\beta^*))); \\
 G_2(x, \xi) &= -(\sin(\beta^* \xi) \sinh(\beta x) + \sinh(\beta^* \xi) \sin(\beta x) - \cos(\beta^*) \cosh(\beta^*) \sin(\beta^* \xi) \sin(\beta x) - \\
 & - \cos(\beta^*) \sinh(\beta^*) \sin(\beta^* \xi) \cos(\beta x) + \cosh(\beta^*) \sin(\beta^*) \sin(\beta^* \xi) \cos(\beta x) - \\
 & - \cos(\beta^*) \cosh(\beta^*) \sinh(\beta^* \xi) \sinh(\beta x) + \cos(\beta^*) \sinh(\beta^*) \sinh(\beta^* \xi) \cosh(\beta x) - \\
 & - \cosh(\beta^*) \sin(\beta^*) \sinh(\beta^* \xi) \cosh(\beta x) - \sin(\beta^*) \sinh(\beta^*) \sin(\beta^* \xi) \sin(\beta x) + \\
 & + \sin(\beta^*) \sinh(\beta^*) \sinh(\beta^* \xi) \sinh(\beta x)) / (2\beta^{*3} (\cos(\beta^*) \sinh(\beta^*) - \cosh(\beta^*) \sin(\beta^*))),
 \end{aligned}$$

where β is the solution of the characteristic equation $\beta^4 + \mu = 0$, $\beta^* -$ is a conjugate complex number.

On the basis of these calculation formulas, deflection calculations were carried out for all models approximately by the method of quadrature formulas with derivatives and the values of the analytical solution. The results of the calculations are given in Table 2 and Table 3.

TABLE 2. Initial information for beam deflection calculation

Time	Model 1	Model 2	Model 3	Model 4
0,0	0.000E + 00	0.000E - 00	0.000E + 00	0.000E + 00
0,1	5.278E - 08	1.330E - 07	3.086E - 08	1.072E - 07
0,2	4.088E - 07	3.065E - 07	1.886E - 07	3.150E - 07
0.3	3.071E - 07	2.965E - 07	1.638E - 07	2.626E - 07
0.4	7.154E - 07	3.002E - 07	3.248E - 07	3.534E - 07
0.5	5.271E - 07	2.638E - 07	2.669E - 07	2.904E - 07
0.6	6.925E - 07	2.139E - 07	3.609E - 07	2.249E - 07
0.7	5.013E - 07	1.798E - 07	2.955E - 07	1.792E - 07
0.8	3.088E - 07	2.567E - 07	2.438E - 07	2.158E - 07
0.9	2.172E - 07	2.453E - 07	1.978E - 07	1.908E - 07
1.0	2.685E - 07	2.516E - 07	2.162E - 07	2.383E - 07
Average	3.636E - 07	2.225E - 07	2.081E - 07	2.161E - 07

TABLE 3. Average values of absolute error for models corresponding to the method parameter m.

m	Model 1	Model 2	Model 3	Model 4
1	9.263E-05	2.125E-05	1.990E-05	2.869E-05
2	2.235E-06	4.894E-07	4.959E-07	9.053E-07
3	1.142E-06	2.997E-07	3.132E-07	4.674E-07
4	3.636E-07	2.081E-07	2.161E-07	2.225E-07
5	3.636E-07	2.081E-07	2.161E-07	2.225E-07
6	3.636E-07	2.081E-07	2.161E-07	2.225E-07

5. CONCLUSION

Based on the results obtained, the following conclusions can be drawn:

- In the calculation process, the optimal value of the parameter of the method of derivative quadrature formulas was determined ($m = 4$);
- An analytical solution has been obtained using the Green function;
- Green's function is obtained in symbolic form, which is used to solve the integral equation;
- a high order of accuracy of the approximate solution was obtained, which is proved by the values of absolute error;
- the possibility of considering another model, i.e. other boundary conditions must be taken into account in the construction of the Green function.

The proposed method can be generalized to solve more complex problems of Euler-Bernoulli beams.

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