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Geometric description of the unit ball of a neutral strongly facially symmetric space

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Abstract. This paper is devoted to the study of the geometric structure of neutral strongly facially symmetric (SFS) spaces, which provide a natural framework for modeling aspects of quantum mechanics in geometric terms. We investigate the representation of symmetric faces associated with complete and maximal geometric tripotents and establish their connections with split faces. Based on these results, we obtain a geometric description of the unit ball in neutral SFS-spaces. Furthermore, we give a characterization of real neutral SFS-spaces, highlighting their structural properties and the role of geometric tripotents in determining the organization of the unit ball.

Keywords: strongly facially symmetric spaces, neutral spaces, geometric tripotents, symmetric faces, split faces, unit ball, convex geometry
MSC (2020): 4B20

1. Introduction

Mathematical models of quantum mechanics provide a powerful and versatile framework for describing the behavior of quantum systems. These models comprise a wide spectrum of methods and approaches, reflecting both algebraic and geometric perspectives.

In the works of Sh.A. Ayupov, B. Iochum, and N. Yadgorov, a so-called "convex" model was developed, whose central concept is the study of convex sets of states of physical systems. Within this framework, the authors investigated geometric characterizations of projective convex sets [1, 2, 3, 4]. Further contributions by E. Alfsen, F. Schultz, H. Hanche-Olsen, and B. Iochum [5, 6] established geometric conditions on convex sets that ensure their affine isomorphism to the state space of a C^*-algebra, a von Neumann algebra, or a JB-algebra. The geometric properties of convex sets presented in the works of E. Alfsen and F. Schultz demonstrate the existence of unordered analogues of JB^*-triples.

A significant geometric approach to quantum mechanics is provided by the theory of strongly facially symmetric (SFS) spaces, introduced by J. Friedman and B. Russo in [7, 8]. These spaces supply a natural framework for analyzing the unit ball of the predual of a JBW^*-triple, enabling one to describe the associated convex sets entirely in geometric terms.

In [8, Proposition 4.5] it was proved that for any fixed geometric tripotent ω in a neutral strongly facially symmetric space Z, the set

$$L_\omega := \{v \in G\mathfrak{U} : v \leq \omega\} \cup \{0\}$$

is a complete orthomodular lattice with the least element 0, the greatest element ω, and orthocomplementation given by $v \mapsto v^\perp = \omega - v$, where $G\mathfrak{U}$ denotes the set of all geometric tripotents of the unit ball of the dual space Z^*. It should be emphasized that a detailed description and representation of the symmetric face F_ω corresponding to the geometric tripotent ω makes it possible to obtain a geometric characterization of the unit ball of a facially symmetric space and to describe its internal structure, since symmetric faces of the unit ball play a fundamental role in the theory of facially symmetric spaces.

In [9] the notion of a strongly split face of the unit ball of a neutral strongly facially symmetric space was introduced, and it was shown that if for every element $u \in L_\omega$ the symmetric face F_u is a strongly split face, then the lattice L_ω acquires the structure of a Boolean algebra.

In the present paper, relying on the indicated structural role of symmetric and strongly split faces, we describe a representation of the symmetric face F_ω corresponding to a complete (see Theorem 3.1) and maximal (see Theorem 3.4) geometric tripotent ω, by means of a split face F_u, where $u \in L_\omega$.

Such a representation makes it possible to explicitly relate the lattice structure of geometric tripotents belonging to L_ω to the geometry of the corresponding symmetric face of the unit ball of a facially symmetric space. It should be stressed that in the theory of facially symmetric spaces the unit ball is the principal geometric object, since its facial structure completely determines the geometry and analytic properties of the space. In this context, Theorems 3.1 and 3.4 play a key role, as they provide a precise description of the structure of symmetric faces corresponding to complete and maximal geometric tripotents and thereby form the basis for a geometric characterization of the unit ball. As direct consequences of these results, Corollary3.8 gives a representation of the unit ball of a neutral strongly facially symmetric space, and Corollary 3.9 provides a description of a real neutral SFS space.

Throughout this work, we adhere to the terminology and notation established in [7, 8, 10, 11, 12, 13].

2. Preliminaries

We start by reviewing certain notions from the theory of JBW^*-triples that are essential for formulating and understanding the corresponding concepts in facially symmetric spaces.

Definition 2.1. A Banach space E over $\mathbb{C}$ is called a *JB^*-triple* if it is equipped with a triple product $(a, b, c) \mapsto \{a, b, c\}$ mapping $E \times E \times E$ into E, such that

(i) $\{a, b, c\}$ is linear in a and c, and conjugate linear in b;

(ii) $\{a, b, c\}$ is symmetric in the outer variables, i.e. $\{a, b, c\} = \{c, b, a\}$;

(iii) for every $x \in E$, the operator $L(x, x) : E \to E$ defined by $L(x, x)y = \{x, x, y\}$, where $y \in E$, is Hermitian with nonnegative spectrum;

(iv) the triple product satisfies the following identity, called the *main identity*:

$$L(x, x)\{a, b, c\} = \{L(x, x)a, b, c\} - \{a, L(x, x)b, c\} + \{a, b, L(x, x)c\};$$

(v) for every $x \in E$, the following equality holds:

$$\|\{x, x, x\}\| = \|x\|^3.$$

A nonzero element u of a JB^*-triple E is called a *tripotent* if $u = \{u, u, u\}$. Two elements a and b are said to be *orthogonal* ($a \perp b$) if $\{a, b, x\} = 0$ for every $x \in E$. The *quadratic operator* Q on E is defined (see [13, §1]) by

$$Q(x)y = \{x, y, x\}, \quad x, y \in E.$$

The *Peirce projections* $P_k(u)$ ($k = 0, 1, 2$) corresponding to a tripotent u are defined as

$$P_2(u) = Q(u)^2, \quad P_1(u) = 2\big(L(u, u) - Q(u)^2\big), \quad P_0(u) = I - 2L(u, u) + Q(u)^2.$$

It is clear that $\sum_{j=0}^{2} P_j(u) = I$ and $L(u, u) = P_2(u) + \frac{1}{2}P_1(u)$. Moreover, in [13, Corollary 1.2] it was shown that the operators $P_k(u)$ ($k = 0, 1, 2$) are contractive projections.

We denote by $E_k(u)$ the range of the projection $P_k(u)$: $E_k(u) = P_k(u)E$, where $k = 0, 1, 2$. It is known (see [14, Fact 4.2.14]) that

$$E = E_2(u) \oplus E_1(u) \oplus E_0(u), \quad E_j(u) = \{x \in E : \{u, u, x\} = \tfrac{j}{2}x\}, \quad j = 0, 1, 2.$$

A JB^*-triple E whose predual E_* is a Banach space is called a *JBW^*-triple*.

We now introduce the key concepts from the theory of facially symmetric spaces, illustrate them with examples, and explain the motivation behind their definitions. For clarity, we compare these notions with their analogues in operator algebra theory, particularly in the theory of JBW^*-triples. This approach makes it possible to trace the logical steps leading to the definition of facially symmetric spaces and to describe the structure of their unit balls.

Let Z be a real or complex normed space. A central notion in the construction of facially symmetric spaces is the definition of orthogonality of elements of Z: elements $f, g \in Z$ are called *orthogonal* ($f \diamondsuit g$) if $\|f \pm g\| = \|f\| + \|g\|$.

Example 2.2. [15, Lemma 2]. Let f and g be normal functionals on a von Neumann algebra A, and let u and v be partial isometries corresponding to their polar decompositions, respectively. Then u and v are orthogonal if and only if $f\diamondsuit g$.

Example 2.3. Let E_* be the predual of a JBW^*-triple E. Then for every nonzero element $f \in E_*$ there exists a nonzero tripotent $u \in E$ which is the least tripotent such that

$$u(f) = \|f\|.$$

This u is called the *support tripotent* for f and is denoted by $v(f)$ (see [16, 13]).

In [17, Theorem 5.4] it was proved that for $f, g \in E_*$,

$$f\diamondsuit g \quad \Leftrightarrow \quad v(f)\perp v(g).$$

Recall that a convex subset F of the unit ball $Z_1 = \{f \in Z : \|f\| \leq 1\}$ is called a *face* if for $f, g \in Z_1$ and $t \in (0,1)$ the condition $tf + (1-t)g \in F$ implies $f, g \in F$. A face F is called a *norm-exposed face* if $F = F_u = \{f \in Z_1 : f(u) = 1\}$ for some $u \in Z^*$ with $\|u\| = 1$.

Subsets $S, T \subset Z$ are called *orthogonal* ($S\diamondsuit T$) if $f\diamondsuit g$ for all $(f, g) \in S \times T$. For a subset S of Z we set

$$S^\diamondsuit = \{f \in Z : \, f\diamondsuit g, \ \forall g \in S\}$$

and call $S^\diamondsuit$ the *orthogonal complement* of S.

An element $u \in Z^*$ is called a *projective unit* if $\|u\| = 1$ and $\langle g, u\rangle = 0$ for all $g \in F_u^\diamondsuit$. Note that if $F_u \neq \emptyset$, then F_u is a norm-exposed face of the unit ball Z_1 parallel to $F_u^\diamondsuit$, i.e. $\langle u, F_u\rangle = 1$ and $\langle u, F_u^\diamondsuit\rangle = 0$.

We denote by $\mathfrak{F}$ and $\mathfrak{U}$ the sets of all norm-exposed faces of Z_1 and all projective units in Z_1^*, respectively.

Definition 2.4. A norm-exposed face F is called a *symmetric face* if there exists a linear isometry $S_F : Z \to Z$ with $S_F^2 = I$ such that the fixed-point set of S_F is $\overline{sp}F \oplus F^\diamondsuit$. In particular, $F^\diamondsuit$ is a closed linear subspace. We call S_F the *symmetry* associated with F.

If a symmetric face F of the unit ball Z_1 consists of a single point, it is called a *norm-exposed point.* We denote by $\exp Z_1$ the set of all norm-exposed points of Z_1.

Using symmetric faces, we now define the notion of a geometric tripotent, which plays the role of the analogue of a tripotent in a JBW^*-triple.

Definition 2.5. A projective unit $u \in Z^*$ is called a *geometric tripotent* if F_u is a symmetric face and $S_{F_u}^* u = u$ for the symmetry S_{F_u}.

We denote by $S\mathfrak{F}$ and $G\mathfrak{U}$ the sets of all symmetric faces of Z_1 and all geometric tripotents in Z_1^*, respectively.

Definition 2.6. A real or complex normed space Z is called a *weakly facially symmetric space* (WFS-*space*) if every norm-exposed face of Z_1 is a symmetric face.

One of the fundamental examples of a WFS-space is the predual of a von Neumann algebra (see [12, Lemma 2.8 and Lemma 2.10]). It should also be noted that in [8, Proposition 1.6] the bijectivity of the mapping

$$G\mathfrak{U} \ni u \;\longmapsto\; F_u \in S\mathfrak{F}$$

was established in the case where Z is a WFS-space.

Earlier, we introduced the definitions of Peirce projections in a JBW^*-triple. A significant property of JBW^*-triples is the existence of a Peirce decomposition associated with a given norm-exposed face, where each component is itself a JBW^*-triple.

On a WFS-space the projections $P_k(F_u)$ $\;(k \in \{0, 1, 2\})$, which are analogues of the Peirce projections in a JBW^*-triple, are defined for each symmetric face F_u ($u \in G\mathfrak{U}$). They are called *geometric* (or *generalized*) Peirce projections:

$$P_1(F_u) = \tfrac{1}{2}(I - S_{F_u}), \quad P_1(F_u)(Z) = \{f \in Z : \, S_{F_u} f = -f\}.$$

Here $P_0(F_u)$ and $P_2(F_u)$ project Z onto $F_u^{\diamondsuit}$ and $\overline{sp}F_u$, respectively.

For convenience, we introduce the notation:

$$Z^* = U,\ \ P_k(F_u) = P_k(u),\ \ P_k(u)Z = Z_k(u),\ \ Z_k^*(u) = U_k(u),\ \ S_{F_u} = S_u,$$

$$T = \{\lambda \in K:\ |\lambda| = 1\},\quad k \in \{1,2,3\},\ \ K = \mathbb{R} \text{ or } K = \mathbb{C}.$$

It is clear that

$$P_0(u) + P_2(u) = \tfrac{1}{2}(I + S_u),\quad P_0(u) + P_1(u) + P_2(u) = I,\quad P_0(u) - P_1(u) + P_2(u) = S_u.$$

Recall that a contractive projection Q on a normed space Z is said to be *neutral* if, for every $f \in Z$, the equality $\|Qf\| = \|f\|$ implies $Qf = f$. A normed space Z is called *neutral* if, for each symmetric face F_u, the corresponding projection $P_2(u)$ is neutral.

Definition 2.7. A WFS-space Z is called a *strongly facially symmetric space* (*SFS-space*) if for each symmetric face F_u of Z_1 and for every $v \in Z^*$ with $\|v\| = 1$ and $F_u \subset F_v$, we have $S_u^* v = v$.

Important examples of SFS-spaces are the predual spaces of von Neumann algebras [12, Theorem 2.11] and the predual spaces of JBW^*-triples [12, Theorem 3.1]. In these cases, the SFS-spaces are neutral.

Definition 2.8. Let Z be a strongly facially symmetric space. A geometric tripotent u is called

- *maximal* if $U_0(u) = \{0\}$;
- *complete* if $U_2(u) = U$;
- *unitary* if $Z_1 = \mathrm{co}(F_u \cup -F_u)$.

For geometric tripotents u, v we write $u \leq v$ if $F_u \subset F_v$. If $u \leq v$ and $u \neq v$, we write $u < v$.

Definition 2.9. (see [7, §2]) Let Z be a normed space. Elements $u, v \in Z^*$ are called *orthogonal* if there exists a symmetric face $F \subset Z_1$ such that

(i) $u \in \mathrm{im}\, P_2(F)^*$ and $v \in \mathrm{im}\, P_0(F)^*$;

or

(ii) $u \in \mathrm{im}\, P_0(F)^*$ and $v \in \mathrm{im}\, P_2(F)^*$.

In this case we write $u \diamondsuit v$ and $v \diamondsuit u$.

Definition 2.10. A neutral SFS-space Z is said to satisfy the *FE-condition* if every norm-closed face of Z_1 (different from Z_1 itself) is a norm-exposed face.

3. Geometric description of the unit ball of a neutral strongly facially symmetric space

Recall that a face F of a convex set K is called a *split face* if there exists a face G, called the *complementary face* to F, such that $F \cap G = \emptyset$ and K is the direct convex sum $F \oplus_c G$, i.e., every element $f \in K$ can be uniquely represented in the form

$$f = tg + (1-t)h,\quad t \in [0,1],\ g \in F,\ h \in G.$$

Let Z be a strongly facially symmetric space and let $u, \omega \in G\mathfrak{U}$. If $u < \omega$, then by [8, Lemma 4.2] it follows that $(\omega - u) \in G\mathfrak{U}$ and $(\omega - u)\diamondsuit u$. In this case the geometric tripotent $\omega - u$ is called the *orthogonal complement* of u and is denoted by $u^{\perp}$.

Let Z be a neutral SFS-space and let ω be a fixed geometric tripotent in the space Z^*. A symmetric face $F_u \subset F_\omega$ is called a *strongly split face* if the following equality holds:

$$F_\omega = F_u \oplus_c F_{u^\perp}.$$

Theorem 3.1. *Let Z be a neutral SFS-space with the (FE)-condition and let ω be a complete geometric tripotent. If a symmetric face $F_u \subset F_\omega$ corresponding to a geometric tripotent u is a split face, then F_ω admits the following representation:*

$$F_\omega = \left\{ f \in Z_1 :\ f = \|P_2(u)f\|g + \|P_2(u^\perp)f\|h,\ g \in F_u,\ h \in F_{u^\perp} \right\}. \tag{3.1}$$

We first prove the following auxiliary lemmas.

Lemma 3.2. *Let Z be a neutral SFS-space. If ω is a complete geometric tripotent, then for every $f \in F_\omega$ and for every $u < \omega$ we have:*

(i) $\|P_2(u)f\| + \|P_2(u^\perp)f\| = 1$;

(ii) $\|P_2(u)f\|g + \|P_2(u^\perp)f\|h \in F_{u+u^\perp}, \quad g \in F_u,\ h \in F_{u^\perp}$.

Proof: (i) From [18, Corollary 3.4(iii)] it follows that $P_2(u^\perp) = P_0(u)$. Hence, by [8, Proposition 1.5] we obtain:

$$\|P_2(u)f\| + \|P_2(u^\perp)f\| = \|P_2(u)f\| + \|P_0(u)f\| = \|P_2(u)f + P_0(u)f\| = \frac{\|f + S_u f\|}{2}.$$

According to [19, Lemma 1] we have $S_u f \in F_\omega$, hence $(f + S_u f)(\omega) = 2$, and therefore

$$\|P_2(u)f\| + \|P_2(u^\perp)f\| = \frac{\|f + S_u f\|}{2} = 1.$$

(ii) Since $g \diamondsuit h$, by [20, p. 5.127] we have

$$\|P_2(u)f\|g \diamondsuit \|P_2(u^\perp)f\|h.$$

Then, by the definition of orthogonality and property (i), it follows that

$$\big\|\|P_2(u)f\|g + \|P_2(u^\perp)f\|h\big\| = \|P_2(u)f\| + \|P_2(u^\perp)f\| = 1.$$

Moreover, by [7, Proposition 1.1(d)] we have

$$\left(\|P_2(u)f\|g + \|P_2(u^\perp)f\|h\right)(u + u^\perp) = \|P_2(u)f\| + \|P_2(u^\perp)f\| = 1.$$

Thus,

$$\|P_2(u)f\|g + \|P_2(u^\perp)f\|h \in F_{u+u^\perp}.$$

□

Lemma 3.3. *Let Z be a neutral SFS-space. If ω is a complete geometric tripotent, then for any $u < \omega$ we have:*

$$F_u \oplus_c F_{u^\perp} = \left\{f \in F_{u+u^\perp} : f = \|P_2(u)f\|\, g + \left\|P_2(u^\perp)f\right\| h,\ g \in F_u,\ h \in F_{u^\perp}\right\}. \tag{3.2}$$

Proof: Denote the expression on the right-hand side of equality (3.2) by A. Let $\forall f \in F_u \oplus F_{u^\perp}$, i.e. $f = \alpha g + \beta h$, where $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, $g \in F_u$ and $h \in F_{u^\perp}$. Since $g \diamondsuit h$, by [7, Proposition 1.1] and [11, (16)] we have

$$\alpha = f(u) = \|P_2(u)f\|, \quad \beta = f(u^\perp) = \left\|P_2(u^\perp)f\right\|,$$

hence $f \in A$, and therefore $F_u \oplus F_{u^\perp} \subset A$.

Conversely, let $\forall f \in A$, i.e.

$$f = \|P_2(u)f\|\, g + \left\|P_2(u^\perp)f\right\| h, \quad g \in F_u,\ h \in F_{u^\perp}. \tag{3.3}$$

From Lemma 3.2(i) it follows that $f \in co(F_u \cup F_{u^\perp})$. It remains to prove the uniqueness of representation (3.3).

First, consider the case when $\|P_2(u)f\| \neq 0$ and $\left\|P_2(u^\perp)f\right\| \neq 0$. Suppose

$$f = \|P_2(u)f\|\, g + \left\|P_2(u^\perp)f\right\| h = \|P_2(u)f\|\, g' + \left\|P_2(u^\perp)f\right\| h',$$

where $g, g' \in F_u$ and $h, h' \in F_{u^\perp}$. Since $Z_0(u), Z_2(u)$ are subspaces of Z, and $g, g' \in Z_2(u)$, $h, h' \in Z_0(u)$, then by [8, Proposition 1.5] we obtain

$$\|P_2(u)f\|\,(g - g') \diamondsuit \left\|P_2(u^\perp)f\right\|(h - h').$$

Hence,

$$\|P_2(u)f\|\,\|g-g'\|+\left\|P_2(u^{\perp})f\right\|\,\|h-h'\|=0.$$

Since $\|P_2(u)f\|\neq 0$ and $\left\|P_2(u^{\perp})f\right\|\neq 0$, it follows that $\|g-g'\|=0$ and $\|h-h'\|=0$, i.e. $g=g'$ and $h=h'$.

Now consider the cases when $\|P_2(u)f\|=0$ or $\left\|P_2(u^{\perp})f\right\|=0$. Then $f=\left\|P_2(u^{\perp})f\right\|h$ or $f=\|P_2(u)f\|\,g$, respectively. Uniqueness in these cases is obvious. Note that $\|P_2(u)f\|$ and $\left\|P_2(u^{\perp})f\right\|$ cannot both be zero simultaneously, since $\|f\|=1$.

Thus, $F_u\oplus_c F_{u^{\perp}}=A$. □

Proof of theorem 3.1. From [9, Theorem 2.2] it follows that F_u is a strongly split face, i.e. $F_u\oplus_c F_{u^{\perp}}=F_\omega$. Hence, from (3.2) it follows (3.1). □

Theorem 3.4. *Let Z be a real neutral SFS-space with a unitary geometric tripotent. Then any symmetric face F_ω corresponding to a maximal geometric tripotent ω admits the following representation:*

$$F_\omega=\left\{f\in Z_1:\ f=\|P_2(u)f\|\,g+\left\|P_2(u^{\perp})f\right\|h,\ g\in F_u,\ h\in F_{u^{\perp}}\right\},\tag{3.4}$$

where $u\in G\mathfrak{U}$ and $u<\omega$.

We first prove the following auxiliary lemmas.

Lemma 3.5. *If u is not a maximal geometric tripotent, then there exists a geometric tripotent e such that $u<e$.*

Proof: Since $P_0(u)\neq 0$, there exists a nonzero element $f\in Z_0(u)\cap\partial Z_1$. Then $v(f)\in U_0(u)$ (see [11, §2]), and hence, by [7, Lemma 2.5], it follows that $v(f)\diamondsuit u$ and $v(f)+u\in G\mathfrak{U}$. Therefore, $u<v(f)+u$. □

Lemma 3.6. *Let u,v be mutually orthogonal geometric tripotents. Then $u+v$ is maximal if and only if $u-v$ is maximal.*

Proof: Suppose $u+v$ is maximal. Assume that $u-v$ is not a maximal geometric tripotent. Then, by Lemma 3.5, there exists a geometric tripotent e such that $u-v<e$. Set $\omega=e-(u-v)$. Then, according to [8, Lemma 4.2], we have $\omega\diamondsuit(u-v)$. Consequently, from [7, Corollary 2.6] it follows that $\omega\diamondsuit u$ and $\omega\diamondsuit v$. Hence, by [21, Lemma 3.2(e)] we obtain $\omega\diamondsuit(u+v)$. Therefore, $u+v\leq u+v+\omega$. This contradicts the maximality of $u+v$. □

Lemma 3.7. *If e is a maximal geometric tripotent, then any symmetric face $F_u\subset F_e$ is strongly split.*

Proof: First, we show that $F_e=co\{F_u\cup F_{u^{\perp}}\}$. For this, it suffices to establish the inclusion $F_e\subseteq co\{F_u\cup F_{u^{\perp}}\}$. Since e is maximal and $e=u+u^{\perp}$, it follows from Lemma 3.6 that the geometric tripotent $u-u^{\perp}$ is maximal, and therefore unitary (see [22, Lemma 4.1]). Hence,

$$Z_1=co\{F_{u-u^{\perp}}\cup F_{u^{\perp}-u}\}.$$

Thus, every element $f\in F_e$ can be written in the form

$$f=tg+(1-t)h$$

for some $g,h\in F_{u-u^{\perp}},F_{u^{\perp}-u}$ respectively, with $0\leq t\leq 1$.

We now consider the following cases.

Case 1. If $t=0$, then $f\in F_e\cap F_{u^{\perp}-u}=F_{u+u^{\perp}}\cap F_{u^{\perp}-u}=F_{u^{\perp}}$.

Case 2. If $t=1$, then $f\in F_e\cap F_{u-u^{\perp}}=F_{u+u^{\perp}}\cap F_{u-u^{\perp}}=F_u$.

Case 3. If $0<t<1$, then applying the geometric tripotent $e=u+u^{\perp}$ to $f=tg+(1-t)h$, we obtain

$$1=\langle e,f\rangle=\langle u+u^{\perp},tg+(1-t)h\rangle=t\langle u,g\rangle+t\langle u^{\perp},g\rangle+(1-t)\langle u,h\rangle+(1-t)\langle u^{\perp},h\rangle.\tag{3.5}$$

Since $g \in F_{u-u^\perp}$ and $h \in F_{u^\perp - u}$, we have

$$1 = \langle u - u^\perp, g\rangle = \langle u, g\rangle - \langle u^\perp, g\rangle, \quad 1 = \langle u^\perp - u, h\rangle = \langle u^\perp, h\rangle - \langle u, h\rangle.$$

Thus,

$$\begin{aligned} 1 = t + 1 - t = t(\langle u, g\rangle - \langle u^\perp, g\rangle) + (1-t)(\langle u^\perp, h\rangle - \langle u, h\rangle) = \\ = t\langle u, g\rangle - t\langle u^\perp, g\rangle + (1-t)\langle u^\perp, h\rangle - (1-t)\langle u, h\rangle. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6) it follows that

$$t\langle u, g\rangle + (1-t)\langle u^\perp, h\rangle = 1.$$

Since $\left|\langle u, g\rangle\right| \leq 1$ and $\left|\langle u^\perp, h\rangle\right| \leq 1$, this equality implies that

$$\langle u, g\rangle = \langle u^\perp, h\rangle = 1.$$

Hence, $g \in F_u$ and $h \in F_{u^\perp}$. Therefore,

$$f = tg + (1-t)h \in co\{F_u \cup F_{u^\perp}\},$$

which proves that $F_e \subseteq co\{F_u \cup F_{u^\perp}\}$.

Thus, $F_e = co\{F_u \cup F_{u^\perp}\}$. Taking into account that $F_u \diamondsuit F_{u^\perp}$, we obtain $F_e = F_u \oplus_c F_{u^\perp}$, i.e. the symmetric face F_u is strongly split. □

Proof of theorem 3.4. Since ω is a maximal geometric tripotent, it follows from Lemma 3.7 that any symmetric face $F_u \subset F_\omega$ is strongly split. Then, from Theorem 3.1 it follows that F_ω admits the representation (3.4). □

From Theorem 3.4 the following statements follow.

Corollary 3.8. *The unit ball admits the following representation:*

$$Z_1 = \left\{f \in Z : \ f = \|P_2(u) f\| \, g + \left\|P_2\left(u^\perp\right) f\right\| h, \, g \in co\left(F_u \cup F_{-u}\right), \, h \in co\left(F_{u^\perp} \cup F_{-u^\perp}\right)\right\}.$$

Corollary 3.9. *Let ω be a maximal geometric tripotent.*

(i) *For any geometric tripotent $u < \omega$ we have*

$$Z = Z_2(u) \oplus Z_2\left(u^\perp\right), \quad \textit{and} \quad P_1(u) = 0.$$

(ii) *Let u_1 and u_2 be mutually orthogonal geometric tripotents such that $u_1, u_2 < \omega$. Then the following equality holds:*

$$P_2(u_1 + u_2) = P_2(u_1) + P_2(u_2).$$

Proof: (i) From the unitarity of ω (see [22, Lemma 4.1]) and the strong splitness of F_u (see Lemma 3.7) we obtain

$$Z = sp\, Z_1 = sp\left\{co\left(F_e \cup F_{-e}\right)\right\} = sp\, F_e = sp\left(F_u \oplus_c F_{u^\perp}\right) = sp\, F_u \oplus sp\, F_{u^\perp}.$$

Since $sp\, F_u \diamondsuit sp\, F_{u^\perp}$, it follows that $\overline{sp}\, F_u \diamondsuit \overline{sp}\, F_{u^\perp}$, and hence

$$Z = \overline{sp} F_u \oplus \overline{sp} F_{u^\perp} = Z_2(u) \oplus Z_2\left(u^\perp\right).$$

This implies that $P_2(u) + P_2\left(u^\perp\right) = I$. Since $P_1(u)\, P_0(u) = 0$ and $P_2\left(u^\perp\right) = P_0(u)\, P_2\left(u^\perp\right)$ (see [8, Corollary 3.4]), we obtain $P_1(u)\, P_2\left(u^\perp\right) = 0$. Therefore,

$$P_1(u) = P_1(u)\, I = P_1(u)\left[P_2(u) + P_2\left(u^\perp\right)\right] = 0.$$

(ii) From [8, Lemma 1.8] we have

$$P_0(u_1 + u_2) = P_0(u_1)\, P_0(u_2).$$

Using this equality and taking into account $P_1(u_1) = P_1(u_2) = P_1(u_1 + u_2) = 0$, as well as [8, Corollary 3.4], we obtain:

$$\begin{aligned} P_2(u_1 + u_2) &= I - P_0(u_1 + u_2) = I^2 - P_0(u_1 + u_2) \\ &= (P_2(u_1) + P_0(u_1))(P_2(u_2) + P_0(u_2)) - P_0(u_1) P_0(u_2) \\ &= P_2(u_1) + P_2(u_2) + P_0(u_1) P_0(u_2) - P_0(u_1) P_0(u_2) \\ &= P_2(u_1) + P_2(u_2). \end{aligned}$$

□

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