

Recovering the time dependent coefficient in a multidimensional multi-term fractional diffusion equation

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Abstract. We consider the inverse problem of determining the time-dependent coefficient for a diffusion equation involving a fractional Laplacian operator in space and Hilfer multi-term fractional derivatives in time with Dirichlet zero boundary conditions. The solution of the initial-boundary value problem was investigated using the Fourier method. First, we will study the spectral problem related to the fractional Laplacian in a spatial variable. In addition, the existence of a solution to the Cauchy problem for a fractional time differential equation is considered. Using the generalized Gronwall inequality, we obtain an estimate of the solution in terms of the unknown coefficient. The inverse problem is to recover a time-dependent coefficient with an integral type over-determination condition. We consider the analytical solution of the inverse problem and prove the existence and uniqueness of the analytical solution. The inverse problem reduces to an equivalent Volterra-type integral equation. The local existence and global uniqueness of the solution of the inverse problem are proven according to the Banach fixed-point theorem.

Keywords: Hilfer (generalized Riemann-Liouville) fractional derivative, fractional Laplacian, Mittag-Leffler function, inverse problem, integral equation, Fourier series, principle of contraction mapping

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1. INTRODUCTION

In recent years, both linear and nonlinear fractional as well as integer order partial differential equations have gained significant interest in theoretical and applied mathematics. The increased attention to this field in recent years is primarily due to the fact that these equations play an important role in modeling many phenomena in various scientific and engineering sciences [1, 2, 3]. Fractional differential equations have been widely used in engineering, physics, chemistry, biology, and other fields. Many applications can be found in the work [3] and references therein.

Most of the research published to date is related to fractional differential equations. Initial and initial-boundary value problems for fractional-order time diffusion and diffusion-wave equations have been sufficiently studied, for example, see [3, 4, 5]. In the works [5, 6], the authors investigated the unique solvability of fractional initial-boundary value problems for diffusion-wave equations involving second-order elliptic differential operators. In these studies, various initial and boundary conditions are taken into account, and different solution approaches are applied.

In the midst of the inverse problems for the diffusion-wave equation involving fractional derivatives, the most common are inverse source problems with various overdetermination conditions [7, 8, 9, 10, 11, 12, 13]. In particular, in the article [14], the authors investigated direct and inverse problems for subdiffusion equations involving Hilfer fractional derivatives. An arbitrary positive self-adjoint operator A is taken as the elliptic part of the equation. In [15], the inverse problem of determining the coefficient in the lower term of the fractional-order diffusion equation is considered. A direct problem is an initial-boundary value problem for an equation with nonlocal initial and homogeneous Dirichlet conditions. The condition of overdetermination of the integral form with respect to the solution of the direct problem for determining the unknown coefficient is shown. In [16], the author proposed new formulas for representing the solution of the first (Dirichlet), second (Neumann) and third (Robin) initial-boundary value problems for one-dimensional fractional diffusion equations with the time-fractional Caputo derivative. These formulas are obtained by the continuation method used in the theory of differential equations with integer derivatives.

In [17], the correctness and long-term asymptotic behavior of the initial-boundary value problem for fractional-order diffusion equations were investigated, where the time differentiation consists of a finite summation of Caputo derivatives with decreasing orders in $(0, 1)$ and positive constant coefficients.

In [18], the well-posedness and the long-time behavior of the initial-boundary value problems for the multi-term Caputo time-fractional wave equations are considered. The multi-term time-fractional wave equation involves Caputo fractional derivatives in time of different orders between 0 and 2.

In [19], the inverse problem of determining the time-dependent source term along with the diffusion/temperature concentration for the space-time fractional-order diffusion equation is considered. The space-time fractional diffusion equation involves a Caputo fractional derivative in space and Hilfer fractional derivatives in time of different orders between 0 and 1. In [20], the inverse problem of determining the time-dependent source term along with the diffusion concentration for the multi-parameters space-time fractional diffusion equation with nonlocal boundary conditions is considered. The space-time fractional diffusion equation involves several Caputo fractional derivatives in time of different orders between 0 and 1. In [21], the inverse problem of determining the time-dependent source term along with the state variable for a non-homogeneous time-space fractional diffusion equation is considered. The time-space fractional diffusion equation involves a Caputo fractional derivative in time of order $\alpha \in (0, 1)$ and a fractional power of a symmetric uniformly elliptic operator in space.

In this article, we consider the inverse problem of determining the time-dependent coefficient with a state variable for a multidimensional multi-term time-fractional diffusion equation with zero Dirichlet boundary conditions. The multi-term time-fractional diffusion equation involves a multi-term Hilfer fractional derivative in time of different orders between 0 and 1, and a fractional Laplacian operator in space of order $1 < \sigma \leq 2$, where the unknown coefficient is recovered under an integral-type over-determination condition.

Next, we consider the full fractional diffusion-wave equation with the multi-term time-fractional Hilfer operator and a fractional Laplacian operator

$$\begin{aligned} & \left(D_{0+,t}^{\alpha_0,\beta_0} u \right) (x, t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j,\beta_j} u \right) (x, t) \\ & + (-\Delta)^{\frac{\sigma}{2}} u(x, t) + q(t)u(x, t) = f(x, t), \quad (x, t) \in Q_T, \end{aligned} \quad (1.1)$$

with initial and boundary conditions

$$I_{0+,t}^{(1-\alpha_j)(1-\beta_j)} u(x, t) \big|_{t=0} = \varphi_j(x), \quad j = 0, 1, \dots, m, \quad m \in \mathbb{N}, \quad x \in \Omega, \quad (1.2)$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \quad (1.3)$$

where $Q_T = \Omega \times (0, T)$, Ω is an open and bounded domain in $\mathbb{R}^d$, featuring a smooth boundary $\partial\Omega$, and T is a fixed final time ($T = \text{const} > 0$), $\mu_j > 0$, $j = 1, 2, \dots, m$, $(-\Delta)^{\frac{\sigma}{2}}$ is the fractional Laplacian operator of order $1 < \sigma \leq 2$, $D_{0+,t}^{\alpha_j,\beta_j}$, $j = 0, 1, \dots, m$, are Hilfer fractional derivatives in time variable of orders α_j , $j = 0, 1, \dots, m$, $0 < \alpha_n < \dots < \alpha_2 < \alpha_1 < \alpha_0 < 1$ and types $0 \leq \beta_j \leq 1$, $I_{0+,t}^{(1-\alpha_j)(1-\beta_j)}$ denotes the Riemann-Liouville fractional integrals in time variable of orders $(1 - \alpha_j)(1 - \beta_j)$, $j = 0, 1, \dots, m$, $m \in \mathbb{N}$ respectively.

Note that if σ tends to 2, the fractional Laplacian operator $(-\Delta)^{\frac{\sigma}{2}}$ tends to the Laplacian operator $-\Delta$, and thus model (1.1) reproduces the multi-term time-fractional diffusion-wave equation.

Given functions $q(t)$, $f(x, t)$, $\varphi_j(x)$, $j = 0, 1, \dots, m$ and parameters $0 < \alpha_n < \dots < \alpha_2 < \alpha_1 < \alpha_0 < 1$, $0 \leq \beta_j \leq 1$, $j = 0, 1, \dots, m$, $1 < \sigma \leq 2$, the problem of finding the solution to the initial boundary problem (1.1)-(1.3) is called the direct problem.

Inverse problem. For the given functions $f(x, t)$, $\varphi_j(x)$, $j = 0, 1, \dots, m$ and numbers $0 < \alpha_n < \dots < \alpha_2 < \alpha_1 < \alpha_0 < 1$, $0 \leq \beta_j \leq 1$, $j = 0, 1, \dots, m$, $1 < \sigma \leq 2$, it is required to determine the function $q(t)$, $t \in [0, T]$ in equation (1.1), using additional information about the solution to the direct problem (1.1)-(1.3)

$$\int_{\Omega} w(x)u(x, t)dx = h(t), \quad 0 \leq t \leq T, \quad (1.4)$$

where $w(x)$, $h(t)$ are given functions. In heat propagation in a thin rod in which the law of variation $h(t)$ of the total quantity of heat in the rod is given in [22]. This integral condition in parabolic problems is also called heat moments which are analyzed in [23].

Definition 1.1. A function $u(x, t)$ is called a classical solution to initial boundary value problem (1.1)-(1.3), for $q(t) \in C([0, T])$ if:

(i) $t^\gamma u(x, t) \in C(\overline{Q}_T)$, and $t^\gamma (-\Delta)^{\frac{\alpha}{2}} u(\cdot, t) \in C(\overline{Q}_T)$;

(ii) $t^\gamma \left(D_{0+,t}^{\alpha_0,\beta_0} + \sum_{i=1}^m \mu_i D_{0+,t}^{\alpha_i,\beta_i} \right) u(x, t) \in C(\overline{Q}_T)$;

(iii) $u(x, t)$ satisfies (1.1)-(1.3),

where $1 > \gamma > \max \left\{ (1 - \beta_0)(1 - \alpha_0), \max_{0 \leq j \leq m} [1 - \alpha_0 + \beta_j(1 - \alpha_j)] \right\}$.

2. PRELIMINARIES

In this section, we present some useful definitions and results, which will be used later.

Definition 2.1. (see [1]) Let $g \in L_{loc}^1(a, b)$, $-\infty < a < t < b < \infty$ be a locally integrable real-valued function. The left and right sided Riemann-Liouville fractional integral of order $\alpha > 0$ are defined as

$$I_{a+,t}^\alpha g(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} g(\tau) d\tau$$

and

$$I_{b-,t}^\alpha g(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} g(\tau) d\tau$$

respectively.

Definition 2.2. (see [2], pp. 113-116) Let $g_1 \in L^1(a, b)$, $-\infty < a < t < b < \infty$ and $g_1 * g_{(1-\alpha)(1-\beta)} \in AC([a, b])$. The Hilfer fractional derivative of order α , $0 < \alpha < 1$ and type β , $0 \leq \beta \leq 1$ is defined as

$$D_{a+,t}^{\alpha,\beta} g_1(t) := \left(I_{a+,t}^{\beta(1-\alpha)} \frac{d}{dt} I_{a+,t}^{(1-\alpha)(1-\beta)} g_1 \right) (t)$$

where $g_\alpha = t^{\alpha-1}/\Gamma(\alpha)$ and $''*''$ indicates integral convolution stated as

$$(g_1 * g)(t) = \int_0^t g_1(\tau) g(t - \tau) d\tau$$

The Hilfer fractional derivative interpolates both the Riemann-Liouville fractional derivative and the Caputo fractional derivative:

- For $\beta = 0$, the HFD becomes the RLFD, i.e.,

$$D_{a+,t}^{\alpha,0} g(z) = \frac{d}{dt} I_{a+,t}^{(1-\alpha)} g(t) := {}^{RL}D_{a+,t}^\alpha g(t).$$

- For $\beta = 1$, the HFD becomes the Caputo fractional derivative, i.e.,

$$D_{a+,t}^{\alpha,1} g(t) = I_{a+,t}^{1-\alpha} \frac{d}{dt} g(t) := {}^cD_{a+,t}^\alpha g(t).$$

Lemma 2.1. [1] Let g_i be a sequence of functions defined on $(a, b]$ for each $i \in \mathbb{N}$, such that

- $D_{a+,t}^{\alpha,\beta} g_i(t)$ exists $\forall i \in \mathbb{N}$, $t \in (a, b]$,
- both series $\sum_{i=1}^\infty g_i(t)$ and $\sum_{i=1}^\infty D_{a+,t}^{\alpha,\beta} g_i(t)$ are uniformly convergent on the interval $[a + \epsilon, b]$ for any $\epsilon > 0$.

Then,

$$D_{a+,t}^{\alpha,\beta} \sum_{i=1}^\infty g_i(t) = \sum_{i=1}^\infty D_{a+,t}^{\alpha,\beta} g_i(t), \quad 0 < \alpha \leq \beta < 1, \quad a < t < b.$$

We consider the weighted spaces of continuous functions

$$C_\gamma[a, b] := \{g : (a, b] \rightarrow \mathbb{R} : (t - a)^\gamma g(t) \in C[a, b], 0 \leq \gamma < 1, \},$$

$$C_\gamma^{\alpha, \beta, \sigma}(Q_T) = \left\{ u(x, t) \in C_\gamma(Q_T) : D_{0+, t}^{\alpha, \beta} u(x, t) \in C_\gamma(Q_T), 0 < \alpha \leq 1, 0 \leq \beta \leq 1, \right. \\ \left. (-\Delta)^{\frac{\sigma}{2}} u(x, t) \in C_\gamma(Q_T); x \in [0, l], 1 < \sigma \leq 2 \right\}.$$

2.1. Mittag-Leffler type functions. In this subsection, we describe the Mittag-Leffler and multinomial Mittag-Leffler functions with some important estimates.

Two parameter Mittag-Leffler (M-L) function. The two parameter M-L function $E_{\alpha, \beta}(z)$ is defined by the following series:

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\alpha)$ denotes the real part of the complex number α , $\Gamma(\cdot)$ is Euler's Gamma function. The Mittag-Leffler function has been studied by many authors who have proposed and studied various generalizations and applications. A very interesting work that meets many results about this function is due to Kilbas et al. (see [3], pp. 42-44).

For $\beta = 1$, β reduces to the Mittag-Leffler function of single parameter, i.e.

$$E_{\alpha, 1}(z) = E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Proposition 2.1. ([3], pp. 40-45) Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that

$$|E_{\alpha, \beta}(z)| \leq \frac{C}{1 + |z|}, \quad \kappa \leq |\arg(z)| \leq \pi.$$

Multinomial Mittag-Leffler functions

Definition 2.3. (see [24]) For $\beta > 0$, $\alpha_i > 0$, $z_i \in \mathbb{C}$, $i = \overline{1, n}$, $n \in \mathbb{N}$, the multinomial Mittag-Leffler functions is stated as

$$E_{(\alpha_1, \alpha_2, \dots, \alpha_n), \beta}(z_1, z_2, \dots, z_n) := \sum_{k=0}^{\infty} \sum_{\substack{l_1 + l_2 + \dots + l_n = k \\ l_1 \geq 0, \dots, l_n \geq 0}} \frac{(k; l_1, \dots, l_n) \prod_{i=1}^n z_i^{l_i}}{\Gamma\left(\beta + \sum_{i=1}^n \alpha_i l_i\right)},$$

where $(k; l_1, \dots, l_n) = \frac{k!}{l_1! \times \dots \times l_n!}$.

Remark 2.1. For $n = 2$, the multinomial Mittag-Leffler function takes the following form:

$$E_{(\alpha_1, \alpha_2), \beta}(z_1; z_2) = \sum_{k=0}^{\infty} \sum_{\substack{l_1 + l_2 = k \\ l_1 \geq 0, l_2 \geq 0}} \frac{k!}{l_1! l_2!} \frac{z_1^{l_1} z_2^{l_2}}{\Gamma(\beta + \alpha_1 l_1 + \alpha_2 l_2)} \\ = \sum_{k=0}^{\infty} \sum_{i=0}^k \frac{k!}{i!(k-i)!} \frac{z_1^i z_2^{k-i}}{\Gamma(\beta + \alpha_1 l_1 + \alpha_2 l_2)}.$$

Moreover, note that

$$E_{(\alpha_1, \alpha_2, \dots, \alpha_n), \beta}(z_1; z_2; \dots; z_n) = E_{(\alpha_n, \dots, \alpha_2, \alpha_1), \beta}(z_n; \dots; z_2; z_1)$$

Remark 2.2. For $z_1 \neq 0$ and $z_2 = z_3 = \dots = z_n = 0$, the multinomial Mittag-Leffler function takes the following form

$$E_{\alpha_1, \beta}(z_1) = \sum_{k=0}^{\infty} \frac{z_1^k}{\Gamma(\beta + \alpha_1 k)}.$$

Lemma 2.2. [19] For $\alpha_i, \beta, \tau, \sigma_i > 0, i = 1, 2, \dots, m, m \in \mathbb{N}$ the Laplace transform of the multinomial Mittag-Leffler function is given by

$$\mathcal{L}\{\tau^{\beta-1} E_{(\alpha_1, \alpha_2, \dots, \alpha_m), \beta}(-\sigma_1 \tau^{\alpha_1}, -\sigma_2 \tau^{\alpha_2}, \dots, -\sigma_m \tau^{\alpha_m})\} = \frac{s^{-\beta}}{1 + \sum_{i=1}^m \sigma_i s^{-\alpha_i}},$$

$$\text{if } \left| \sum_{i=1}^m \sigma_i s^{-\alpha_i} \right| < 1.$$

The Laplace transform for $m = 1$ is as follows

$$\mathcal{L}[t^{\beta-1} E_{\alpha, \beta}(\pm at^{\alpha})] = \int_0^{\infty} e^{-st} t^{\beta-1} E_{\alpha, \beta}(\pm at^{\alpha}) dt = \frac{s^{-\beta}}{1 \mp as^{-\alpha}}, \quad \Re(s) > |a|^{1/\alpha}.$$

Lemma 2.3. [25] For $0 < \beta < 1$ and $0 < \alpha_n < \dots < \alpha_2 < \alpha_1 < 1$ be given. Assume that $\frac{\alpha_1 \pi}{2} < \kappa < \alpha_1 \pi, \kappa \leq |\arg(\sigma_1 \tau^{\alpha_1})| \leq \pi$ and $\sigma_i \tau > 0$. Then there exists a constant C_0 depending only on $\kappa, \alpha_i, i = 1, 2$ such that

$$|E_{(\alpha_1 - \alpha_n, \dots, \alpha_1 - \alpha_2, \alpha_1), \beta}(-\sigma_n \tau^{\alpha_1 - \alpha_n}, \dots, -\sigma_2 \tau^{\alpha_1 - \alpha_2}, -\sigma_1 \tau^{\alpha_1})| \leq \frac{C_0}{1 + |\sigma_1 \tau^{\alpha_1}|}.$$

We give some definitions concerning the eigenvalues and eigenvectors of the fractional Laplacian.

Definition 2.4. [26] Suppose $\{\lambda_k, e_k\}$ be the eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in Ω with Dirichlet boundary condition on $\partial\Omega$:

$$\begin{aligned} -\Delta e_k(x) &= \lambda_k e_k(x), & \text{in } \Omega \\ e_k(x) &= 0, & \text{on } \partial\Omega. \end{aligned} \quad (2.1)$$

Counting according to the multiplicities, we can assume

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = +\infty.$$

Let

$$H_0^{\sigma}(\Omega) := \left\{ u = \sum_{k=1}^{\infty} a_k e_k : \|u\|_{H_0^{\sigma}(\Omega)}^2 = \sum_{k=1}^{\infty} a_k^2 \lambda_k^{\sigma} < \infty \right\}$$

we define the operator $(-\Delta)^{\frac{\sigma}{2}}$ by

$$(-\Delta)^{\frac{\sigma}{2}} u = \sum_{k=1}^{\infty} a_k \lambda_k^{\frac{\sigma}{2}} e_k$$

which maps $H_0^{\frac{\sigma}{2}}(\Omega)$ onto $L^2(\Omega)$, with the following equivalence:

$$\|u\|_{H_0^{\sigma}} = \|(-\Delta)^{\frac{\sigma}{2}} u\|_{L^2(\Omega)}.$$

Let us present some assertions about eigenfunctions $v_k(x)$ and eigenvalues λ_k proved by V.A.Ilyin [27].

Lemma 2.4. The series $\sum_{k=1}^{\infty} \lambda_k^{-([\frac{d}{2}] + 1)} e_k^2(x)$ converges uniformly in a closed domain $\overline{\Omega}$.

Definition 2.5. Let κ be a fixed number. We will say that $g(x) \in D_{\kappa}(\Omega)$ if

- satisfy conditions
 - 1) $g(x) \in C^p(\overline{\Omega})$, $\frac{\partial^{p+1}g(x)}{\partial x_1^{p_1} \dots \partial x_n^{p_n}} \in L_2(\Omega)$, $p+1 = p_1 + p_2 + \dots + p_n$, $p \geq 1$
 - 2) $g(x)|_{\partial\Omega} = \Delta g(x)|_{\partial\Omega} = \dots = \Delta^{[\frac{p}{2}]}g(x)|_{\partial\Omega} = 0$
with the exponent $p = [\frac{d}{2}]$
- $\left\| A^{\frac{1}{2}([\frac{d}{2}]+1+\sigma)}g \right\|_{L_2(\Omega)} \leq \kappa$.

where $A := -\Delta$.

Lemma 2.5. ([28], p.189) Suppose $b \geq 0$, $\alpha > 0$, $\gamma > 0$, $\alpha + \gamma > 1$ and $a(t)$ nonnegative function locally integrable on $0 \leq t < T$ and suppose $t^{\gamma-1}u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\alpha-1} s^{\gamma-1} u(s) ds$$

a.e. in $(0, T)$; then

$$u(t) \leq a(t) Z_{\alpha, \gamma} \left((b\Gamma(\alpha))^{\frac{1}{\alpha+\gamma-1}} t \right),$$

where $Z_{\alpha, \gamma}(t) = \sum_{m=0}^{\infty} c_m t^{m(\alpha+\gamma-1)}$, $c_0 = 1$, $\frac{c_{m+1}}{c_m} = \frac{\Gamma(m(\alpha+\gamma-1)+\gamma)}{\Gamma(m(\alpha+\gamma-1)+\alpha+\gamma)}$ for $m \geq 0$. As $t \rightarrow +\infty$

$$Z_{\alpha, \gamma}(t) = O \left(t^{\frac{1}{2} \frac{\alpha+\gamma-1}{\alpha} - \gamma} \exp \left(\frac{\alpha + \gamma - 1}{\alpha} t^{\frac{\alpha+\gamma-1}{\alpha}} \right) \right).$$

Theorem 2.1. [29] Let (X, d) be a complete metric space, and assume $A : X \rightarrow X$. Furthermore, assume there exists an $a \in X$ and an $r > 0$ such that

(i) The ball $B(a, r) = \{x \in X : d(x, a) < r\}$ is an invariant set for A , i.e.

$$A : B(a, r) \rightarrow B(a, r)$$

(ii) the map A is a contraction on $B(a, r)$, t.e. there exists a constant $\gamma \in (0, 1)$ such that.

$$d(A(x), A(y)) \leq \gamma d(x, y)$$

for all $x, y \in B(a, r)$.

Then there exists a $x_0 \in B(a, r)$ such that $A(x_0) = x_0$ and, furthermore, this is the unique fixed point of A inside the ball $B(a, r)$.

3. INVESTIGATION OF DIRECT PROBLEM (1.1)–(1.3)

We investigate the solution of the direct problem (1.1)–(1.3) using the Fourier method. By applying the Fourier method, the solution $u(x, t)$ of the problem (1.1)–(1.3) can be expanded in a uniformly convergent series in term of eigenfunctions of the form

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) e_k(x), \quad (3.1)$$

where $u_k(t)$ are the unknowns and satisfy the following fractional differential equation

$$\left(D_{0+,t}^{\alpha_0, \beta_0} u_k \right) (t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} u_k \right) (t) + \lambda_k^{\frac{\sigma}{2}} u_k(t) + q(t) u_k(t) = f_k(t), \quad (3.2)$$

where λ_k and $\{e_n(x)\}_{n \geq 1}$ are eigenvalues and eigenfunctions of the classical Laplace operator $-\Delta$ defined in equation (2.1) and $(f(x, t), e_k(x))_{L_2(\Omega)} =: f_k(t)$.

From the initial conditions (1.2) and the equality (3.1) we obtain the following initial conditions:

$$I_{0+,t}^{(1-\alpha_j)(1-\beta_j)} u_k(t) \Big|_{t=0} = \varphi_{jk}, \quad j = 0, 1, \dots, m, \quad (3.3)$$

where $\varphi_{jk} = (\varphi_j, e_k)_{L_2(\Omega)}$.

Let us investigate the Cauchy problem (3.2) - (3.3). Applying the Laplace transform method to both sides of equation (3.2), we obtain:

$$\begin{aligned} & s^{\alpha_0} \mathcal{L}[u_k(t)](s) - s^{\beta_0(\alpha_0-1)} \left(I_{0+}^{(1-\beta_0)(1-\alpha_0)} u_k \right) (0+) + \sum_{j=1}^m \mu_j s^{\alpha_j} \mathcal{L}[u_k(t)](s) \\ & - \sum_{j=1}^m \mu_j s^{\beta_j(\alpha_j-1)} \left(I_{0+}^{(1-\beta_j)(1-\alpha_j)} u_k \right) (0+) + \lambda_k^{\frac{\sigma}{2}} \mathcal{L}[u_k(t)](s) = \mathcal{L}[f_k(t) - q(t)u_k(t)](s). \end{aligned}$$

Using the above equality and conditions (3.3), we find $\mathcal{L}[u_k(t)](s)$:

$$\begin{aligned} \mathcal{L}[u_k(t)](s) &= \frac{\varphi_{0k} s^{\beta_0(\alpha_0-1)}}{s^{\alpha_0} + \sum_{j=1}^m \mu_j s^{\alpha_j} + \lambda_k^{\frac{\sigma}{2}}} + \frac{\sum_{j=1}^m \varphi_{jk} \mu_j s^{\beta_j(\alpha_j-1)}}{s^{\alpha_0} + \sum_{j=1}^m \mu_j s^{\alpha_j} + \lambda_k^{\frac{\sigma}{2}}} \\ &+ \frac{1}{s^{\alpha_0} + \sum_{j=1}^m \mu_j s^{\alpha_j} + \lambda_k^{\frac{\sigma}{2}}} \mathcal{L}[f_k(t) - q(t)u_k(t)](s). \end{aligned}$$

According to Lemma 2.2 and the method of applying the inverse Laplace transform, we have:

$$\begin{aligned} u_k(t) &= \varphi_{0k} t^{(1-\alpha_0)(\beta_0-1)} E_{\alpha', 1-(1-\alpha_0)(1-\beta_0)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &+ \sum_{j=1}^m \mu_j \varphi_{jk} t^{\alpha_0-1+\beta_j(1-\alpha_j)} E_{\alpha', \alpha_0+\beta_j(1-\alpha_j)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &+ \int_0^t f_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau \\ &- \int_0^t q(\tau) u_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau, \quad (3.4) \end{aligned}$$

where $\alpha' = (\alpha_0, \alpha_0 - \alpha_1, \dots, \alpha_0 - \alpha_m)$.

Lemma 3.1. Let $q(t) \in C[0, T]$, $f_n(t) \in C[0, T]$. For any $n \in \mathbb{N}$ and $t \in [0, T]$, we have

$$\begin{aligned} t^\gamma |u_k(t)| &\leq \left(C |\varphi_{0k}| t^{\gamma+(1-\alpha_0)(\beta_0-1)} + C \sum_{j=1}^m \mu_j |\varphi_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \right. \\ &\quad \left. + C \|f_k\|_{C_\gamma[0, T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) \right) \\ &\quad \times Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0, T]} t^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} t \right), \quad t \in [0, T]; \quad (3.5) \end{aligned}$$

$$\begin{aligned} & \left| t^\gamma \left(D_{0+, t}^{\alpha_0, \beta_0} u_k \right) (t) \right| + \left| t^\gamma \sum_{j=1}^m \mu_j \left(D_{0+, t}^{\alpha_j, \beta_j} u_k \right) (t) \right| \leq \|f_k\|_{C_\gamma[0, T]} \\ & + \left(\lambda_k^{\frac{\sigma}{2}} + \|q\|_{C[0, T]} \right) \left(C |\varphi_{0k}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \right. \\ & \left. + C \sum_{j=1}^m \mu_j |\varphi_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + C \|f_k\|_{C_\gamma[0, T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) \right) \\ & \times Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0, T]} t^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} t \right), \quad t \in [0, T], \quad (3.6) \end{aligned}$$

where C is a constant depending on α_j , β_j , $j = 0, 1, \dots, m$, σ .

Proof: Multiplying equation (3.4) by t^γ and using Lemma 2.3, then we can estimate $t^\gamma |u_k(t)|$ as follows:

$$\begin{aligned} t^\gamma |u_k(t)| &\leq C |\varphi_{0k}| t^{\gamma+(1-\alpha_0)(\beta_0-1)} + C \sum_{j=1}^m \mu_j |\varphi_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \\ &\quad + C \|f_k\|_{C_\gamma[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) + C \|q\|_{C[0,T]} \int_0^t (t-\tau)^{\alpha_0-1} |u_k(\tau)| d\tau. \end{aligned}$$

Then, according to lemma 2.5, we arrive at the first estimate of Lemma 3.1.

Next, let's take the second estimate of Lemma 3.1. For this, we obtain the following from equation (3.2):

$$\left(D_{0+,t}^{\alpha_0,\beta_0} u_k \right) (t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j,\beta_j} u_k \right) (t) = f_k(t) - \lambda_k^{\frac{\sigma}{2}} u_k(t) - q(t) u_k(t). \quad (3.7)$$

Multiply equality (13) by t^γ on both sides and we get estimate

$$\begin{aligned} &\left| t^\gamma \left(D_{0+,t}^{\alpha_0,\beta_0} u_k \right) (t) + t^\gamma \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j,\beta_j} u_k \right) (t) \right| \\ &\leq \|f_k\|_{C_\gamma[0,T]} + \left(\lambda_k^{\frac{\sigma}{2}} + \|q\|_{C[0,T]} \right) t^\gamma |u_k(t)|. \end{aligned}$$

According to the last inequality and (3.5) we obtain the second estimate of Lemma 3.1. Lemma 3.1 is proven. $\square$

The functions $\varphi_j(x)$, $j = 0, 1, \dots, m$, $w(x)$, $f(x, t)$ and $h(t)$ satisfy the following assumptions

A1) $\varphi_j(x) \in D_\kappa(\Omega)$, $j = 0, 1, \dots, m$;

A2) $f(x, t) \in C_\gamma(D_\kappa(\Omega), [0, T])$;

A3) $w(x) \in D_\kappa(\Omega)$;

A4) $D_{0+,t}^{\alpha,\beta} h(t) \in C_\gamma[0, T]$, $D_{0+,t}^{\alpha_j,\beta_j} h(t) \in C_\gamma[0, T]$ and $|t^\gamma h(t)| \geq h_{00} > 0$, h_{00} is a given number,

$$\int_\Omega w(x) \varphi_j(x) dx = I_{0+,t}^{(1-\alpha_j)(1-\beta_j)} h(t)|_{t=0}, \quad j = 0, 1, \dots, m, \quad m \in \mathbb{N}.$$

For the solution of the direct problem (1.1)–(1.3), the following theorem holds.

Theorem 3.1. *Let $q(t) \in C[0, T]$ and conditions A1) -A2) be satisfied. Then there exists a unique solution of the direct problem (1.1)–(1.3), $u(x, t) \in C_\gamma^{\alpha,\beta,\sigma}(\overline{Q}_T)$, where $\overline{Q}_T = \overline{\Omega} \times [0, T]$.*

Proof: First, we prove that the solution $u(x, t)$, given by formula (3.1), is a classical solution, that is $t^\gamma u(x, t) \in C(\overline{Q}_T)$, $t^\gamma (-\Delta)^{\frac{\sigma}{2}} u(\cdot, t) \in C(\overline{Q}_T)$ and $t^\gamma \left(D_{0+,t}^{\alpha_0,\beta_0} + \sum_{i=1}^m \mu_i D_{0+,t}^{\alpha_i,\beta_i} \right) u(x, t) \in C(\overline{Q}_T)$, $j = 0, 1, \dots, m$ represent continuous functions.

Formally, by term-by-term differentiation of the functional series (3.1), we construct the following series

$$(-\Delta)^{\frac{\sigma}{2}} u(x, t) = \sum_{k=1}^{\infty} \lambda_k^{\frac{\sigma}{2}} u_k(t) e_k(x), \quad (3.8)$$

$$\begin{aligned} &\left(D_{0+,t}^{\alpha_0,\beta_0} u \right) (x, t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j,\beta_j} u \right) (x, t) \\ &= \sum_{k=1}^{\infty} \left(D_{0+,t}^{\alpha_0,\beta_0} u_k \right) (t) e_k(x) + \sum_{j=1}^m \mu_j \sum_{k=1}^{\infty} \left(D_{0+,t}^{\alpha_j,\beta_j} u_k \right) (t) e_k(x). \end{aligned} \quad (3.9)$$

Let us prove the uniform convergence of the series (3.1), (3.8), and (3.9) in the domain $\overline{Q}_T$. To do this show the uniform convergence of series (3.1), (3.8), and (3.9) multiplied by t^γ in this domain. These series, for any $(x, t) \in \overline{Q}_T$, are respectively majorized by the following series

$$\begin{aligned} & \sum_{k=1}^{\infty} |e_k(x)| \left(|\varphi_{0k}| C_\epsilon \lambda_k^{-\frac{\sigma}{2} \frac{\gamma+(1-\alpha_0)(\beta_0-1)}{\alpha_0}} + C \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \right. \\ & \left. + C \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \right) Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right), \end{aligned} \quad (3.10)$$

$$\begin{aligned} & \sum_{k=1}^{\infty} |e_k(x)| \lambda_k^{\frac{\sigma}{2}} \left(|\varphi_{0k}| C_\epsilon \lambda_k^{-\frac{\sigma}{2} \frac{\gamma+(1-\alpha_0)(\beta_0-1)}{\alpha_0}} + C \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \right. \\ & \left. + C \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \right) Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right), \end{aligned} \quad (3.11)$$

$$\begin{aligned} & \sum_{k=1}^{\infty} |e_k(x)| \left[\|f_k\|_{C_\gamma[0,T]} + \left(\lambda_k^{\frac{\sigma}{2}} + \|q\|_{C[0,T]} \right) \left(|\varphi_{0k}| C_\epsilon \lambda_k^{-\frac{\sigma}{2} \frac{\gamma+(1-\alpha_0)(\beta_0-1)}{\alpha_0}} \right. \right. \\ & \left. \left. + C \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + C \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \right) \right] \\ & \times Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right). \end{aligned} \quad (3.12)$$

It is not difficult to notice that the sums of the series (3.10), (3.11), and (3.12) do not exceed the following expression:

$$\sum_{k=1}^{\infty} \lambda_k^{\frac{\sigma}{2}} \left(|\varphi_{0k}| + \sum_{j=1}^m \mu_j |\varphi_{jk}| + \|f_k\|_{C_\gamma[0,T]} \right) |e_k(x)|. \quad (3.13)$$

Applying the Cauchy-Bunyakovsky inequality we get

$$\begin{aligned} & \sum_{k=1}^{\infty} \lambda_k^{\frac{\sigma}{2}} \left(|\varphi_{0k}| + \sum_{j=1}^m \mu_j |\varphi_{jk}| + \|f_k\|_{C_\gamma[0,T]} \right) |e_k(x)| \\ & = \sum_{k=1}^{\infty} |\varphi_{0k}| \left(\sqrt{\lambda_k} \right)^{([\frac{\sigma}{2}]+1+\sigma)} \left(\sqrt{\lambda_k} \right)^{-([\frac{\sigma}{2}]+1)} |e_k(x)| \\ & + \sum_{j=1}^m \mu_j \sum_{k=1}^{\infty} |\varphi_{jk}| \left(\sqrt{\lambda_k} \right)^{([\frac{\sigma}{2}]+1+\sigma)} \left(\sqrt{\lambda_k} \right)^{-([\frac{\sigma}{2}]+1)} |e_k(x)| \\ & + \sum_{k=1}^{\infty} \|f_k\|_{C_\gamma[0,T]} \left(\sqrt{\lambda_k} \right)^{([\frac{\sigma}{2}]+1+\sigma)} \left(\sqrt{\lambda_k} \right)^{-([\frac{\sigma}{2}]+1)} |e_k(x)| \\ & \leq \left(\sum_{k=1}^{\infty} |\varphi_{0k}| \lambda_k^{([\frac{\sigma}{2}]+1+\sigma)} \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \lambda_k^{-([\frac{\sigma}{2}]+1)} |e_k^2(x)| \right)^{\frac{1}{2}} \\ & + \sum_{j=1}^m \mu_j \left(\sum_{k=1}^{\infty} |\varphi_{jk}| \lambda_k^{([\frac{\sigma}{2}]+1+\sigma)} \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \lambda_k^{-([\frac{\sigma}{2}]+1)} |e_k^2(x)| \right)^{\frac{1}{2}} \\ & + \left(\sum_{k=1}^{\infty} \|f_k\|_{C_\gamma[0,T]} \lambda_k^{([\frac{\sigma}{2}]+1+\sigma)} \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \lambda_k^{-([\frac{\sigma}{2}]+1)} |e_k^2(x)| \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq \Theta \left(\|\varphi_0\|_{D\left(A\left(\left[\frac{d}{2}\right]+1+\sigma\right)\right)} + \sum_{j=1}^m \mu_j \|\varphi_j\|_{D\left(A\left(\left[\frac{d}{2}\right]+1+\sigma\right)\right)} + \|f\|_{C_\gamma\left(D\left(A\left(\left[\frac{d}{2}\right]+1+\sigma\right)\right), [0, T]\right)} \right).$$

Furthermore, according to Lemma 2.4 the series $\Theta = \left(\sum_{k=1}^{\infty} \lambda_k^{-\left(\left[\frac{d}{2}\right]+1\right)} |e_k^2(x)| \right)^{\frac{1}{2}}$ converges uniformly in a closed domain $\overline{\Omega}$.

If the functions $\varphi_j(x)$, $j = 0, 1, \dots, m$ and $f(x, t)$ satisfy the conditions A1), A2), then, the series (3.1), (3.8), and (3.9) are absolutely and uniformly convergent in $\overline{Q}_T$. $\square$

INVESTIGATION OF THE INVERSE PROBLEM

Let us now investigate the inverse problem (1.1)-(1.4).

Multiplying equation (1.1) by the function $w(x)$ on both sides, we integrate in the domain Ω with respect to the variable x .

$$\begin{aligned} & \int_{\Omega} w(x) \left\{ \left(D_{0+,t}^{\alpha_0, \beta_0} u \right) (x, t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} u \right) (x, t) \right. \\ & \left. + (-\Delta)^{\frac{\sigma}{2}} u(x, t) + q(t) u(x, t) \right\} dx = \int_{\Omega} w(x) f(x, t) dx. \end{aligned}$$

Using condition (1.4), we get

$$\begin{aligned} & \left(D_{0+,t}^{\alpha_0, \beta_0} h \right) (t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) + \int_{\Omega} w(x) (-\Delta)^{\frac{\sigma}{2}} u(x, t) dx \\ & + q(t) h(t) = \int_{\Omega} w(x) f(x, t) dx. \end{aligned}$$

From the above relation, we obtain the following equation for $q(t)$

$$\begin{aligned} q(t) = & \frac{1}{h(t)} \left(\int_{\Omega} w(x) f(x, t) dx - \left(D_{0+,t}^{\alpha_0, \beta_0} h \right) (t) - \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) \right) \\ & - \frac{1}{h(t)} \int_{\Omega} w(x) (-\Delta)^{\frac{\sigma}{2}} u(x, t) dx. \end{aligned} \quad (3.14)$$

According to definition 2.3, we can rewrite the integral $\int_{\Omega} w(x) (-\Delta)^{\frac{\sigma}{2}} u(x, t) dx$ as follows

$$\int_{\Omega} w(x) (-\Delta)^{\frac{\sigma}{2}} u(x, t) dx = \sum_{k=1}^{\infty} \lambda_k^{\frac{\sigma}{2}} u_k(t) \int_{\Omega} w(x) e_k(x) dx.$$

Using the above equality, we write (3.13) for $t \in (0, T]$ as follows

$$\begin{aligned} q(t) = & \frac{1}{h(t)} \left(\int_{\Omega} w(x) f(x, t) dx - \left(D_{0+,t}^{\alpha_0, \beta_0} h \right) (t) - \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) \right) \\ & - \frac{1}{h(t)} \sum_{k=1}^{\infty} \lambda_k^{\frac{\sigma}{2}} u_k(t) w_k. \end{aligned}$$

where $w_n = \int_{\Omega} w(x) e_n(x) dx$.

The functions $u_n(t)$ depend on $q(t)$, i.e. $u_n(t; q)$. After simple conversion, for $t \in [0, T]$ we get the following integral equation for determining $q(t)$:

$$q(t) = q_0(t) - \frac{1}{h_0(t)} \sum_{k=1}^{\infty} \lambda_k^{\frac{\alpha}{2}} t^{\gamma} u_k(t) w_k, \quad (3.15)$$

where $h_0(t) = t^{\gamma} h(t)$,

$$q_0(t) = \frac{1}{h_0(t)} \left(\int_{\Omega} w(x) t^{\gamma} f(x, t) dx - t^{\gamma} \left(D_{0+,t}^{\alpha_0, \beta_0} h \right) (t) - t^{\gamma} \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) \right).$$

We introduce an operator $\mathcal{G}$ defining it by the right hand side of (3.15):

$$\mathcal{G}[q](t) = q_0(t) + \frac{1}{h_0(t)} \sum_{n=1}^{\infty} w_n t^{\gamma} u_n(t; q). \quad (3.16)$$

Then, the equation (3.16) is written in more convenient form as

$$\mathcal{G}[q](t) = q(t). \quad (3.17)$$

Let

$$\begin{aligned} \|q_0\|_{C[0,T]} &:= \max_{t \in [0,T]} |q_0(t)| = \left\| \frac{1}{h_0(t)} \left(\int_{\Omega} w(x) t^{\gamma} f(x, t) dx \right. \right. \\ &\quad \left. \left. - t^{\gamma} \left(D_{0+,t}^{\alpha_0, \beta_0} h \right) (t) - t^{\gamma} \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) \right) \right\|_{C[0,T]} \\ &\leq \frac{1}{h_{00}} \left(w_0 \|f\|_{C_{\gamma}(D_{\kappa}(\Omega), [0,T])} + \|h\|_{C_{\gamma}^{\alpha, \beta}[0,T]} \sum_{j=1}^m \mu_j \right), \end{aligned}$$

where $w_0 = \max_{x \in [0,l]} |w(x)|$, $|h_0(t)| = |t^{\gamma} h(t)| \geq h_{00} > 0$, h_{00} is a given number,

$$\|h\|_{C_{\gamma}^{\alpha, \beta}[0,T]} := \max_{0 \leq j \leq m} \left\{ \max_{t \in [0,T]} \left| \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) \right| \right\}.$$

Fix a number $r > 0$ and consider the ball

$$B_r(q_0) := \{q(t) : q(t) \in C[0, T], \|q - q_0\|_{C[0,T]} \leq r\}.$$

Theorem 3.2. *Let A1)-A3) be satisfied. Then there exists a number $T^* \in (0, T)$, and $r > 0$, such that there exists a unique solution $q(t) \in C[0, T^*]$ of the inverse problem (1.1)-(1.4), $q(t) \in B_r(q_0)$.*

Proof: Let us first prove that for an enough small $T > 0$ the operator $\mathcal{G}$ maps the ball $B_r(q_0)$ implies that $\mathcal{G}[q](t) \in B_r(q_0)$. Indeed, for any continuous function $q(t)$, the function $\mathcal{G}[q](t)$ calculated using formula (3.15) will be continuous. Moreover, estimating the norm of the differences, we find that

$$\begin{aligned} \|\mathcal{G}[q](t) - q_0(t)\| &\leq \frac{w_0}{h_{00}} \sum_{k=1}^{\infty} \left(C |\varphi_{0k}| T^{\gamma + (1 - \alpha_0)(\beta_0 - 1)} \right. \\ &\quad \left. + C \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma + \alpha_0 - 1 + \beta_j(1 - \alpha_j)} + C \|f_k\|_{C_{\gamma}[0,T]} T^{\alpha_0} B(\alpha_0, 1 - \gamma) \right) \end{aligned}$$

$$\times Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0,T]} t^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} t \right).$$

Here we have used the estimate (3.5). In view of Lemmas 2.4 and conditions A1)-A3) last series is convergent series. Note that the function occurring on the right-hand side in this inequality is monotone increasing with T , and the fact that the function $q(t)$ belongs to the ball $B_r(q_0)$ implies the inequality $\|q\|_{C[0,T]} \leq \|q_0\|_{C[0,T]} + r$.

Therefore, we only strengthen the inequality if we replace $\|q\|_{C[0,T]}$ in this inequality with the expression $\|q_0\|_{C[0,T]} + r$. Performing these replacements, we obtain the estimate

$$\begin{aligned} \|\mathcal{G}[q](t) - q_0(t)\| &\leq \frac{w_0}{h_{00}} \sum_{k=1}^{\infty} \left(C |\varphi_{0k}| T^{\gamma+(1-\alpha_0)(\beta_0-1)} \right. \\ &\quad \left. + C \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + C \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \right) \\ &\quad \times Z_{\alpha, 1-\gamma} \left((C (\|q_0\|_{C[0,T]} + r) T^\gamma \Gamma(\alpha))^{\frac{1}{\alpha-\gamma}} T \right) =: m_1(T). \end{aligned} \quad (3.18)$$

Before applying the second condition of the Banach fixed-point theorem, we need the estimate for the difference between the solution of the original integral equation (3.4) and the solution of this equation with perturbed functions $\tilde{q}$, $\tilde{\varphi}_{jk}$, $j = 0, 1, 2, \dots, m$ and $\tilde{f}_k(t)$. Let $\tilde{u}_k(t)$ be solution of the integral equation (3.4) corresponding to the functions $\tilde{\varphi}_{jk}$, $j = 0, 1, 2, \dots, m$, $\tilde{f}_k(t)$ i.e.,

$$\begin{aligned} \tilde{u}_k(t) &= \tilde{\varphi}_{0k} t^{(1-\alpha_0)(\beta_0-1)} E_{\alpha', 1-(1-\alpha_0)(1-\beta_0)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &\quad + \sum_{j=1}^m \mu_j \tilde{\varphi}_{jk} t^{\alpha_0-1+\beta_j(1-\alpha_j)} E_{\alpha', \alpha_0+\beta_j(1-\alpha_j)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &\quad + \int_0^t \tilde{f}_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau \\ &\quad - \int_0^t \tilde{q}(\tau) \tilde{u}_k(\tau) (t-\tau)^{\alpha_0-1} \\ &\quad \times E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau. \end{aligned} \quad (3.19)$$

Composing the difference $u_n(t) - \tilde{u}_n(t)$ with the help of the equations (3.4), (3.19) and introducing the notations $\hat{u}_n(t) = u_n(t) - \tilde{u}_n(t)$, $\hat{q} = q(t) - \tilde{q}(t)$.

The following lemma holds for $|\hat{u}_n(t)|$:

$$\begin{aligned} \hat{u}_k(t) &= \hat{\varphi}_{0k} t^{(1-\alpha_0)(\beta_0-1)} E_{\alpha', 1-(1-\alpha_0)(1-\beta_0)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &\quad + \sum_{j=1}^m \mu_j \hat{\varphi}_{jk} t^{\alpha_0-1+\beta_j(1-\alpha_j)} E_{\alpha', \alpha_0+\beta_j(1-\alpha_j)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &\quad + \int_0^t \hat{f}_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau \\ &\quad - \int_0^t \hat{q}(\tau) u_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau \\ &\quad - \int_0^t \hat{q}(\tau) \hat{u}_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau. \end{aligned} \quad (3.20)$$

Lemma 3.2. Let $q(t), \tilde{q}(t) \in C[0, T]$, $f_n(t) \in C[0, T]$. For any $n \in \mathbb{N}$ and $t \in [0, T]$ we have the estimates:

$$\begin{aligned} t^\gamma |\hat{u}_k(t)| &\leq C \left(|\hat{\varphi}_{0k}| t^{\gamma+(1-\alpha_0)(\beta_0-1)} + \sum_{j=1}^m \mu_j |\hat{\varphi}_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \right. \\ &+ \|\hat{f}_k\|_{C_\gamma[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) + \|\hat{q}\|_{C[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) \left(|\varphi_{0k}| t^{\gamma+(1-\alpha_0)(\beta_0-1)} \right. \\ &+ \sum_{j=1}^m \mu_j |\varphi_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + \|f_k\|_{C_\gamma[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) \Big) \\ &\times Z_{\alpha_0, 1-\gamma} \left((C\|q\|_{C[0,T]} t^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} t \right) Z_{\alpha_0, 1-\gamma} \left((C\|\tilde{q}\|_{C[0,T]} t^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} t \right). \end{aligned} \quad (3.21)$$

Proof: To prove this, we multiply the integral equation (3.20) by t^γ on both sides, then estimate

$$\begin{aligned} t^\gamma |\hat{u}_k(t)| &\leq C |\hat{\varphi}_{0k}| t^{\gamma+(1-\alpha_0)(\beta_0-1)} + C \sum_{j=1}^m \mu_j |\hat{\varphi}_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \\ &+ C \|\hat{f}_k\|_{C_\gamma[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) + C \|\hat{q}\|_{C[0,T]} \|u_k\|_{C_\gamma[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) \\ &+ C \|\tilde{q}\|_{C[0,T]} \int_0^t (t-\tau)^{\alpha_0-1} |\hat{u}_k(\tau)| d\tau. \end{aligned}$$

Using lemma 2.5 and estimate (3.5) in the above inequality, we get the estimate in lemma 3.1. Then according to lemma 2.5, we get the estimate in lemma 3.1. $\square$

Now let us take any functions $q(t), \tilde{q}(t) \in B_r(q_0)$ and estimate the distance between their images $\mathcal{G}[q](t)$ and $\mathcal{G}[\tilde{q}](t)$ in the space $C[0, T]$. Composing the difference $\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)$ with the help of equations (3.4), (3.20) and then estimating its norm, we obtain

$$\|\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)\| \leq \frac{w_0}{h_{00}} \sum_{k=1}^{\infty} \|(t^\gamma u_k(t, q) - t^\gamma \tilde{u}_k(t, \tilde{q}))\|.$$

In the above inequality, using estimate (3.21), we obtain the following inequality

$$\begin{aligned} \|\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)\| &\leq \frac{Cw_0}{h_{00}} \sum_{k=1}^{\infty} \left(|\hat{\varphi}_{0k}| T^{\gamma+(1-\alpha_0)(\beta_0-1)} \right. \\ &+ \sum_{j=1}^m \mu_j |\hat{\varphi}_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + \|\hat{f}_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \\ &+ \|\hat{q}\|_{C[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \left(|\varphi_{0k}| T^{\gamma+(1-\alpha_0)(\beta_0-1)} \right. \\ &+ \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \Big) \\ &\times Z_{\alpha_0, 1-\gamma} \left((C\|q\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right) Z_{\alpha_0, 1-\gamma} \left((C\|\tilde{q}\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right). \end{aligned} \quad (3.22)$$

The function $\tilde{u}_n(t)$ corresponding to $\tilde{q}(t)$ satisfies the integral equation (3.20) with $\varphi_{jk} = \tilde{\varphi}_{jk}$, $j = 0, 1, 2, \dots, m$, $f_k(t) = \tilde{f}_k(t)$. Considering $\varphi_{jk} = \tilde{\varphi}_{jk}$, $j = 0, 1, 2, \dots, m$, $f_k(t) = \tilde{f}_k(t)$ in inequality (3.22), we obtain the following result.

$$\|\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)\| \leq \frac{CT^{\alpha_0} B(\alpha_0, 1-\gamma) w_0}{h_{00}} \sum_{k=1}^{\infty} \left(|\varphi_{0k}| T^{\gamma+(1-\alpha_0)(\beta_0-1)} \right.$$

$$\begin{aligned}
& + \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \Big) \\
& \quad \times Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right) \\
& \quad \times Z_{\alpha_0, 1-\gamma} \left((C \|\tilde{q}\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right) \|\tilde{q}\|_{C[0,T]}. \tag{3.23}
\end{aligned}$$

The functions $q(t)$ and $\tilde{q}(t)$ belong to the ball $B_r(q_0)$, and hence for each of these functions one has inequality $\|q\|_{C[0,T]} \leq \|q_0\|_{C[0,T]} + r$. Note that the function on the right-hand side in inequality (3.23) at the factor $\|q - \tilde{q}\|_{C[0,T]}$ is monotone increasing with $\|q\|_{C[0,T]}$, $\|\tilde{q}\|_{C[0,T]}$ and T . Consequently, replacing $\|q\|_{C[0,T]}$ and $\|\tilde{q}\|_{C[0,T]}$ in inequality (3.23) with $\|q_0\|_{C[0,T]} + r$ will only strengthen the inequality. Thus, we have

$$\begin{aligned}
\|\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)\| & \leq \frac{CT^{\alpha_0} B(\alpha_0, 1-\gamma) w_0}{h_{00}} \sum_{k=1}^{\infty} \left(|\varphi_{0k}| T^{\gamma+(1-\alpha_0)(\beta_0-1)} \right. \\
& + \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \Big) \\
& \quad \times Z_{\alpha_0, 1-\gamma} \left((C (\|q_0\|_{C[0,T]} + r) T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right) \\
& \quad \times Z_{\alpha_0, 1-\gamma} \left((C (\|q_0\|_{C[0,T]} + r) T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right) \|\tilde{q}\|_{C[0,T]}. \tag{3.24}
\end{aligned}$$

In view of these facts, we obtain

$$\begin{aligned}
\|\mathcal{G}[q](t) - q_0(t)\| & \leq \mu_1(T), \\
\|\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)\| & \leq \mu_2(T) \|\tilde{q}\|_{C[0,T]},
\end{aligned}$$

where $\mu_1(T)$ denotes the right-hand side of inequality (3.18) and $\mu_2(T)$ is the multiplier at $\|q - \tilde{q}\|_{C[0,T]} = \|\tilde{q}\|_{C[0,T]}$ on the right-hand side of (3.24) with the only difference that $q(t)$, $\tilde{q}(t)$ are replaced by $\|q_0\|_{C[0,T]} + r$ in both cases.

Note that $\mu_i(T)$, $i = 1, 2$, are positive monotonically increasing functions of T , and $\mu_1(0) = 0$, $\mu_2(0) = 0$. It follows that the equations $\mu_1(T) = r$ and $\mu_2(T) = 1$ have unique positive roots. Denote these roots by T_1 and T_2 , respectively. Then, it is clear that if we choose $T^* < \min(T_1, T_2)$, then the operator $\mathcal{G}$ is a contraction in the ball $B_r(q_0)$. However, in accordance with the Banach theorem 2.1, the operator $\mathcal{G}$ has a unique fixed point in the ball $B_r(q_0)$; i.e., there exists a unique solution of the equation (3.17). Theorem 2 is proven. $\square$

CONCLUSIONS

An inverse problem of determining the time-dependent coefficient for a space-time fractional diffusion equation, with multi-term Hilfer fractional derivatives in time of orders $0 < \alpha_n < \dots < \alpha_2 < \alpha_1 < \alpha_0 < 1$ and types $0 \leq \beta_j \leq 1$ and fractional Laplacian operator in space variable of order $1 < \sigma \leq 2$ with homogeneous boundary condition is investigated. Initially, the initial boundary value problem was investigated. At the same time, the spectral problem related to the fractional Laplacian in a spatial variable was studied. The Cauchy problem for the multi-term fractional order ordinary differential equations obtained by eigenfunction, was investigated using the Laplace transform method. A series representation involving multinomial Mittag-Leffler functions is obtained for $u(x, t)$. The functional series solution $u(x, t)$ is proved to be classical solution. The solution of the inverse problem is proved to be locally well-posed. For this, the inverse problem is reduced to an equivalent integral equation of Volterra type. The local existence and global uniqueness of the solution of the inverse problem were proven by the Banach theorem of fixed points.

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