

On two problems with displacement at three points of boundary characteristics for a degenerate hyperbolic equation

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Abstract. This paper is devoted to proving the well-posedness of the boundary characteristic problem with three displacements for the equation $-(-y)^m u_{xx} + u_{yy} - \frac{m}{2y} u_y = 0$ in the characteristic triangle. The proof of the existence of a solution to the problem, based on integral representations of solutions to the modified Cauchy problem, is reduced to solving a functional equation with two shifts $p(x)$ and $q(x)$. A combined method of successive approximations and iterations is used to solve the functional equation. A counterexample is given indicating the importance of the class of functions where the solution to the functional equation is sought.

Keywords: singular coefficient, modified Cauchy problem, functional equation with two shifts, recurrence relation, combined method of iterations and successive approximations, functional series, majorant series, uniform convergence

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1. INTRODUCTION

Over the past period of time, for degenerate hyperbolic equations, foreign and local scientists: G. Darboux, V.I. Zhegalov [1], A.M. Nakhushev [2] and M.S. Salakhitdinov, M. Mirsaburov [3],[4], A.K. Urinov, A.T. Abdukodirov [5], G.M. Mirsaburova [6], U.M. Mirsaburova [7] and others conducted scientific research on the study of boundary value and nonlocal problems and obtained significant scientific results.

The foundations of the theory of boundary value problems with displacement are laid in the works of V.I. Zhegalov [1], for the Lavrentiev-Bitsadze equation and A.M. Nakhushev [2], for the Tricomi equation. Unlike the Tricomi problem, where the boundary condition is specified only on one boundary characteristic, in problems with displacement [1], [2] both boundary characteristics are covered by a boundary condition where a nonlocal condition (displacement condition) is specified establishing pointwise connections between the function values on both characteristics [1] or pointwise connections are established using the fractional differentiation operator [2] in the Riemann-Liouville sense.

Problems with displacement were intensively developed in the works of S.K. Kumykova [8], V.A. Eleev [9], O.A. Repin [10], and M.M. Smirnov [11]. In these problems, it was characteristic that the displacement condition was specified at two points on the boundary characteristics.

2. STATEMENT OF THE PROBLEM A

Let D_- be a finite simply connected domain of the half-plane $y < 0$, bounded by the segment AB of the axis $y = 0$, characteristics AC and BC of the equation

$$-(-y)^m u_{xx} + u_{yy} - \frac{m}{2y} u_y = 0, \quad (2.1)$$

where the constant $m > 0$, $A = A(-1, 0)$, and $B = B(1, 0)$.

In problems with displacement [1, 2, 3, 4], [8, 9, 10, 11], the displacement condition is specified at two points of the boundary characteristics. In this paper, in the characteristic triangle, for a degenerate hyperbolic equation with a singular coefficient, the correctness of the problem with the displacement condition at three points of the boundary characteristics is investigated.

On the segment AB , we consider the point $E = E(c, 0)$ where $c \in I = (-1, 1)$ – the interval of the axis $y = 0$, and introduce the linear functions $p(x) = ax - b$ and $q(x) = a - bx$ – mapping the segment $[-1, 1]$ onto the segments $[-1, c]$ and $[c, 1]$ respectively, where $p(-1) = -1$, $p(1) = c$, $q(-1) = 1$, $q(1) = c$, i.e. $a = (1 + c)/2$, $b = (1 - c)/2$.

A Problem . In the domain D find a regular solution $u(x, y) \in C(\bar{D}) \cap C^2(\bar{D})$ of equation (2.1), that satisfies the conditions:

$$u(x, 0) = \tau(x), \quad x \in [-1, 1], \quad (2.2)$$

$$u[\theta_0(x)] = \mu_2 u[\theta_1(q(x))] - \mu_1 u[\theta_1(p(x))] + \rho(x), \quad x \in [-1, 1], \quad (2.3)$$

where μ_1, μ_2 – are some positive constants, and $\tau(x), \rho(x) \in C^2(\bar{I})$ – are given functions,

$$\theta_0(x_0) = \frac{x_0 - 1}{2} - i \left[\frac{m+2}{4} (1 + x_0) \right]^{2/m+2}$$

– is the affix of the intersection point of the boundary characteristic AC with the characteristic emanating from the point $x_0 \in [-1, 1]$, and

$$\theta_1(p(x_0)) = \frac{1 + p(x_0)}{2} - i \left[\frac{m+2}{4} (1 - p(x_0)) \right]^{2/m+2},$$

$$\theta_1(q(x_0)) = \frac{1 + p(x_0)}{2} - i \left[\frac{m+2}{4} (1 - q(x_0)) \right]^{2/m+2}$$

– affixes of the intersection points of the boundary characteristic BC with the characteristics emanating from the points $(p(x_0), 0)$ and $(q(x_0), 0)$ respectively.

3. STUDY OF PROBLEM A.

Theorem 3.1. *Problem A is uniquely solvable under the condition*

$$\delta_1 + \delta_2 < 1, \quad (3.1)$$

where $\delta_1 = a\mu_1$, $\delta_2 = b\mu_2$

Proof. D'Alembert formula [12], which gives in the domain D for equation (2.1) a solution to the modified Cauchy problem with initial data

$$u(x, 0) = \tau(x), \quad x \in \bar{I}; \quad \lim_{y \rightarrow -0} (-y)^{-m/2} \frac{\partial u}{\partial y} = \nu(x), \quad x \in I;$$

has the form

$$\begin{aligned} u(x, y) = & \frac{1}{2} \left[\tau \left(x + \frac{2}{m+2} (-y)^{\frac{m+2}{2}} \right) + \tau \left(x - \frac{2}{m+2} (-y)^{\frac{m+2}{2}} \right) \right] - \\ & - \frac{(-y)^{\frac{m+2}{2}}}{m+2} \int_{-1}^1 \nu \left(x + \frac{2t}{m+2} (-y)^{\frac{m+2}{2}} \right) dt. \end{aligned} \quad (3.2)$$

Using the D'Alembert formula (3.2), it is easy to calculate the values

$$u[\theta_0(x)] = \frac{1}{2} [\tau(-1) + \tau(x)] - \frac{1}{2} \int_{-1}^x \nu(t) dt, \quad (3.3)$$

$$u[\theta_1(p(x))] = \frac{1}{2} [\tau(1) + \tau(p(x))] - \frac{1}{2} \int_{p(x)}^1 \nu(t) dt, \quad (3.4)$$

$$u[\theta_1(q(x))] = \frac{1}{2} [\tau(1) + \tau(q(x))] - \frac{1}{2} \int_{q(x)}^1 \nu(t) dt. \quad (3.5)$$

Now substituting (3.3), (3.4), and (3.5) into the boundary condition (2.3), and calculating the derivative with respect to x , taking into account equality (2.2), we have the functional equation:

$$\nu(x) = \delta_1 \nu(p(x)) + \delta_2 \nu(q(x)) + F(x), \quad (3.6)$$

where

$$F(x) = \tau'(x) - a\mu_1\tau'(p(x)) + b\mu_2\tau'(q(x)) - 2\rho(x),$$

is a known function. Equation (3.6) is a functional equation with two displacements, $p(x)$ and $q(x)$.

To solve equation (3.6), we apply a combined method of successive approximations and iterations [6].

To construct the terms of the sequence $\nu_0(x), \nu_1(x), \dots, \nu_n(x), \dots$ we use the recurrence relation

$$\nu_n(x) = \delta_1\nu_n(p(x)) + \delta_2\nu_{n-1}(q(x)) + F(x), x \in \bar{I}, n = 1, 2, \dots, \quad (3.7)$$

As the zero approximation of $\nu_0(x)$ we take the solution of the functional equation

$$\nu_0(x) = \delta_1\nu_0(p(x)) + F(x), x \in \bar{I}, \quad (3.8)$$

We will look for a solution to the functional equation (3.8) in the class of functions bounded at the point $x = -1$. If we abandon this requirement, then the corresponding homogeneous functional equation (3.8) (with $F(x) = 0$) may have a nontrivial solution.

Example. Let $F(x) = 0$, in equation (3.8), then it takes the form

$$\nu_0(x) = \delta_1\nu_0(p(x)), x \in \bar{I} \quad (3.9)$$

Consider the function $\nu_0(x) = (1+x)^\alpha$, choose α such that this function satisfies equation (3.9). It is easy to verify that

$$\nu_0(p(x)) = (1+p(x))^\alpha = a^\alpha(1+x)^\alpha \quad (3.10)$$

By virtue of (3.10), from the homogeneous functional equation (3.9), to find α we obtain the equation

$$\delta_1 a^\alpha - 1 = 0. \quad (3.11)$$

The solution of equation (3.11) has the form

$$\alpha = -\log_a \delta_1,$$

and according to (3.1), where ($\delta_1 < 1$) we conclude that the found $\alpha < 0$. Consequently, the solution $\nu(x) = (1+x)^\alpha$ of the homogeneous functional equation (3.9) is not bounded in the neighborhood of the point $x = -1$. Thus, the class of solutions where the solution of the functional equation (3.9) is sought is essential.

We will solve the functional equation (3.8) by the iteration method. In (11), replacing x with $p(x)$, we obtain

$$\nu_0(p(x)) = \delta_1\nu_0(p(p(x))) + F(p(x)), x \in \bar{I}. \quad (3.12)$$

Substituting now $\nu_0(p(x))$ from (3.12) into (3.8), we have

$$\nu_0(x) = \delta_1^2\nu_0(p_2(x)) + \delta_1 F(p_1(x)) + F(x), x \in \bar{I}, \quad (3.13)$$

here $p_1(x) = p(x), p_2(x) = p_1(p_1(x))$. Relation (3.13) is the first iteration of equation (3.8). Continuing this process, for the n -th iteration we have

$$\nu_0(x) = \delta_1^{n+1}\nu_0(p_{n+1}(x)) + \sum_{k=0}^n \delta_1^k F(p_k(x)), x \in \bar{I}, \quad (3.14)$$

here $p_0(x) = x, p_{n+1}(x) = p_n(p_1(x)) = p_1(p_n(x)), p_k(1) = 1, k \in \overline{1, n}$.

Note that, by (3.1) (where $\delta_1 < 1$),

$$\lim_{n \rightarrow \infty} \delta_1^n = 0. \quad (3.15)$$

Since $p(x) < x, p(-1) = -1$, for $x \in [-1, 1]$ and $p(-1) = -1$, then $p_n(x) = p_{n-1}(p_1(x)) = p_1(p_{n-1}(x)) < p_{n-1}(x)$ i.e., the functional sequence $\{p_n(x)\}$ monotonically decreases and is bounded

below by -1 , i.e. $p_n(x) \geq -1, \forall x \in \bar{I}$. Therefore, by the theorem on the limit of a monotonically decreasing and bounded sequence, the sequence $\{p_n(x)\}$ converges to some function $p^0(x)$

$$\lim_{n \rightarrow \infty} p_n(x) = p^0(x), \quad x \in \bar{I}. \quad (3.16)$$

Now we find $p^0(x)$. Passing to the limit as $n \rightarrow \infty$ in the equality $p_n(x) = p_1(p_{n-1}(x))$ we obtain $p^0(x) = p_1(p^0(x))$ and from this we conclude that $p^0(x) = -1, \forall x \in \bar{I}$, since $p(x) = p_1(x)$, it has a unique fixed point at $x = -1$. Thus,

$$\lim_{n \rightarrow \infty} p_n(x) = p^0(x) = -1, \quad \forall x \in \bar{I}. \quad (3.17)$$

In relation (3.14) passing to the limit as $n \rightarrow \infty$, taking into account (3.15)-(3.17) and the boundedness of the value $\nu_0(-1)$, we obtain

$$\nu_0(x) = \sum_{k=0}^{\infty} \delta_1^k F(p_k(x)) \in C(\bar{I}) \quad (3.18)$$

- the solution of the functional equation (3.8).

It is easy to verify that

$$|\nu_0(x)| \leq \sum_{k=0}^{\infty} \delta_1^k |F(p_k(x))| \leq \sum_{k=0}^{\infty} M \delta_1^k = \frac{M}{1 - \delta_1}, \quad x \in \bar{I}, \quad (3.19)$$

where $M = \max_{x \in \bar{I}} |F(x)|$. Consequently, by virtue of (3.19), the functional series on the right-hand side of (3.18) converges uniformly, and, by virtue of the continuity of $F(x)$ on $\bar{I}$, we conclude that $\nu_0(x) \in C(\bar{I})$. It is also easy to verify that $\nu_0(x) \in C^2(\bar{I})$

Now, in the recurrence relation (3.7) setting $n = 1$, we obtain a functional equation for determining $\nu_1(x)$:

$$\nu_1(x) = \delta_1 \nu_1(p(x)) + \delta_2 \nu_0(q(x)) + F(x) \quad x \in \bar{I}, \quad (3.20)$$

here $\nu_0(x)$ — is a known function defined by equality (3.18).

Applying the iteration method to this functional equation, as above, we obtain a solution to equation (3.20) in the form

$$\nu_1(x) = \sum_{k=0}^{\infty} \delta_1^k \delta_2 \nu_0(q(p_k(x))) + \sum_{k=0}^{\infty} \delta_1^k F(p_k(x)) \in C(\bar{I}). \quad (3.21)$$

where $\nu_0(q(p_k(x)))$ — is a known quantity determined by formula (3.18).

It is easy to verify that the functional series of the right-hand side of (3.21) converge uniformly.

Continuing this process successively, it is easy to find that

$$\nu_n(x) = \sum_{k=0}^{\infty} \delta_1^k \delta_2 \nu_{n-1}(q(p_k(x))) + \sum_{k=0}^{\infty} \delta_1^k F(p_k(x)) \in C(\bar{I}). \quad (3.22)$$

In equation (3.22), $\nu_{n-1}(q(p_k(x)))$ — is a known quantity.

Thus, we have constructed a functional sequence

$$\nu_0(x), \nu_1(x), \dots, \nu_n(x), \dots \quad (3.23)$$

Now, we will prove that the functional sequence (3.23) converges uniformly. To this consider the functional series

$$\nu_0(x) + [\nu_1(x) - \nu_0(x)] + [\nu_2(x) - \nu_1(x)] + \dots + [\nu_n(x) - \nu_{n-1}(x)] + \dots \quad (3.24)$$

Let us find the majorant series for the functional series (3.24) [6],[12]. Subtracting equality (3.18) from (3.21), we obtain

$$|\nu_1(x) - \nu_0(x)| \leq \sum_{k=0}^{\infty} \delta_1^k \delta_2 |\nu_0(q((p_k(x))))| \leq \frac{\delta_2}{1 - \delta_1} \frac{M}{1 - \delta_1} = \frac{M}{\delta_2} \left(\frac{\delta_2}{1 - \delta_1} \right)^2.$$

Continuing this process as in works [6],[12], and taking into account (3.22), it is easy to find an estimate for the members of the functional sequence

$$\begin{aligned} |\nu_n(x) - \nu_{n-1}(x)| &\leq \sum_{k=0}^{\infty} \delta_1^k \delta_2 |\nu_{n-1}(q((p_k(x)))) - \nu_{n-2}(q((p_k(x))))| \leq \\ &\leq \frac{\delta_2}{1 - \delta_1} \frac{M}{\delta_2} \left(\frac{\delta_2}{1 - \delta_1} \right)^n = \frac{M}{\delta_2} \left(\frac{\delta_2}{1 - \delta_1} \right)^{n+1}. \end{aligned} \quad (3.25)$$

By (3.25), the functional series (3.24) is majorized by the numerical series

$$\sum_{k=0}^{\infty} \frac{M}{\delta_2} \left(\frac{\delta_2}{1 - \delta_1} \right)^{k+1},$$

and by (3.1), this series is an infinitely decreasing geometric progression, therefore, the series (3.24) converges uniformly on the segment $\bar{I}$.

Consequently, the functional sequence $\{\nu_n(x)\}$ converges uniformly, i.e. there exists

$$\lim_{n \rightarrow \infty} \nu_n(x) = \nu(x), \quad \forall x \in \bar{I},$$

and $\nu(x) \in C(\bar{I})$.

Thus, we have found $\nu(x)$. Now, we find $\nu'(x)$, for this, differentiating equation (3.6) with respect to the variable x , we have

$$\nu'(x) = a\delta_1 \nu'(p(x)) - b\delta_2 \nu'(q(x)) + F'(x), \quad x \in \bar{I}, \quad (3.26)$$

i.e. we have obtained a functional equation with two displacements relative to the unknown function $\nu'(x)$. Since $0 < a < 1$, then, by (3.1), $a\delta_1 - b\delta_2 < 1$, therefore, equations (3.26) are investigated in the same way as equation (3.6), and we can find $\nu'(x) \in C(\bar{I})$. Similarly, $\nu''(x) \in C(\bar{I})$ is found. Thus, we have found $\nu(x) \in C^2(\bar{I})$. Next, the solution to problem A is reconstructed using d'Alembert's formula (3.2).

Theorem 1 is proven.

4. PROBLEM B.

Problem B. In the domain D find a regular solution $u(x, y) \in C(\bar{D}) \cap C^2(\bar{D})$ of equation (2.1), that satisfies the conditions:

$$\lim_{y \rightarrow -0} (-y)^{-m/2} \frac{\partial u}{\partial y} = \nu(x), \quad x \in (-1, 1), \quad (4.1)$$

$$u[\theta(x)] = \mu_2(x)u[\theta_1(q(x))] - \mu_1(x)u[\theta_1(p(x))] + \rho(x), \quad x \in [-1, 1], \quad (4.2)$$

where $\mu_1(x)$, $\mu_2(x)$, $\nu(x)$, $\rho(x) \in C^2(\bar{I})$ — are given functions.

Theorem 4.1. *Problem B under the condition*

$$\delta_1 + \delta_2 < 1, \quad (4.3)$$

is uniquely solvable, where $\delta_1 = \max_{x \in I} |\mu_1(x)|$, $\delta_2 = \max_{x \in I} |\mu_2(x)|$.

Substituting (3.3), (3.4), and (3.5) into the boundary condition (4.2), taking into account the equality (4.1), we obtain the functional equation:

$$\tau(x) = \mu_1(x)\tau(p(x)) + \mu_2(x)\tau(q(x)) + F_1(x), \quad (4.4)$$

where

$$F_1(x) = \int_{-1}^x \nu(t)dt - \mu_1(x) \int_{p(x)}^1 \nu(t)dt - \mu_2(x) \int_{q(x)}^1 \nu(t)dt + 2\rho(x),$$

is a known function. Equation (4.4) is a functional equation with two displacements $p(x)$ and $q(x)$ relative to the unknown function $\tau(x)$.

Equation (4.4) is investigated in the same way as equation (3.6).

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