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Geometry of orbits of Hamiltonian vector fields

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Abstract. Hamilton systems play a fundamental role in mechanics and mathematics, which usually generated by Hamiltonian vector fields.The geometry and topology of vector field orbits is not only an important branch of modern geometry, but also has important applications in many fields of science and engineering. This article studies the geometric characteristics of the leaves of the foliation generated by the orbits of Hamiltonian vector fields.

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1. INTRODUCTION

Let M be a smooth Riemannian manifold of dimension n with the Riemannian metric g, ∇ is the Levi-Civita connection, $<\cdot,\cdot>$ is inner product defined by the Riemannian metric g.

We denote by $V(M)$ the set of all smooth vector fields defined on M, through $[X,Y]$ Lie bracket of vector fields $X, Y \in V(M)$. The set $V(M)$ is a Lie algebra with Lie bracket.

Let's consider some set $D \subset V(M)$, which contains finite or infinite number of smooth vector fields. For a point $x \in M$ through $t \to X^t(x)$ we will denote the integral curve of a vector field X passing through a point x at $t = 0$. Map $t \to X^t(x)$ is defined in some domain $I(x) \subset R$, which generally depends on field X and point x.

Definition 1.1. The orbit $L(x)$ of a system D of vector fields through a point x is the set of points y in M such that there exist $t_1, t_2, ..., t_k \,\epsilon\, R$ and vector fields $X_1, X_2, ..., X_k \,\epsilon\, D$ such that

$$y = X_k^{t_k}(X_{k-1}^{t_k-1}(...(X_1^{t_1}(x)))),$$

where k is an arbitrary positive integer.

There exist many papers devoted to topology and geometry of orbits [1],[2],[3],[4].

In this paper we study the geometry of orbits of Hamiltonian vector fields.

Let us recall a notion of Hamiltonian vector field. In order to define notion of Hamiltonian vector field we need to define Poisson bracket of functions.

Definition 1.2. A *Poisson bracket* on a smooth manifold M is an operation that assigns a smooth real-valued function $\{F, H\}$ on M to each pair F, H of smooth, real-valued functions, with the basic properties:

(a)*Bilineality:*

$$\{\lambda F + \mu P, H\} = \lambda\{F, H\} + \mu\{P, H\},$$

$$\{F, \lambda H + \mu P\} = \lambda\{F, H\} + \mu\{F, P\}, \ \lambda, \ \mu \ \in \mathbb{R};$$

(b)*Skew-Symmetry:*

$$\{F, H\} = -\{H, F\};$$

(c)*Jacobi Identity:*

$$\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{\{H, P\}, F\} = 0;$$

(c)*Leibniz' Rule:*

$$\{F, H \cdot P\} = \{F, H\} \cdot P + H \cdot \{F, P\}.$$

(Here $\cdot$ denotes the ordinary multiplication of real-valued functions.)

A manifold M with a Poisson bracket is called a Poisson manifold, the bracket defining a Poisson structure on M The notion of a Poisson manifold is slightly more general than that of a simplistic manifold, or manifold with Hamiltonian structure; in particular, the underlying manifold M need not be even-dimensional. This is borne out by the standard examples from classical mechanics.

Let M be the Euclidean space R^m, with coordinates

$$(p, q, z) = (p^1, ..., p^n, q^1, ..., q^n, z^1, ..., z^l),$$

where $m = 2n + l$.

If $F(p, q, z)$ and $H(p, q, z)$ are smooth functions, we define their Poisson bracket to be the function:

$$\{F, H\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} \right).$$

This bracket is clearly bilinear and skew-symmetric; the verifications of the Jacobi identity and the Leibniz rule are simple exercises. We note the particular bracket identities:

$$\{p^i, p^j\} = 0,\ \{q^i, q^j\} = 0,\ \{p^i, q^j\} = \delta^i_j,$$

$$\{p^i, z^k\} = \{q^i, z^k\} = \{z^t, z^k\} = 0.$$

in which i and j run from 1 to n, when t and k run from 1 to l. Here δ^i_j is the Kronecker symbol, which is 1 if $i = j$ and 0 otherwise.

Definition 1.3. Let M be a Poisson manifold and $H : M \to R$ a smooth function. The *Hamiltonian vector field* associated with H is the unique smooth vector field $sgradH$ on M satisfying

$$sgradH(F) = \{F, H\} = -\{H, F\}$$

for every smooth function $F : M \to R$.

The equations governing the flow of $sgradH$ are referred to as *Hamilton's equations* for the *Hamiltonian function H*.

In the case of the canonical Poisson bracket on R^m $(m = 2n + l)$, the Hamiltonian vector field to any $H(p, q, z)$, as clearly, corresponds

$$sgradH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \right)$$

The corresponding flow is obtained by integrating the system of ordinary differential equations

$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i},\ \frac{dq^i}{dt} = \frac{\partial H}{\partial p^i},\ i = 1, ..., n,$$

$$\frac{dz^l}{dt} = 0,\ l = 1, ..., l.$$

There is a fundamental connection between the Poisson bracket of two functions and the Lie bracket of their associated *Hamiltonian vector fields*, which forms the basis of much of the theory of Hamiltonian systems. It is well known following theorem [5]:

Theorem 1.4. *Let M be a Poisson manifold and $\{F, H\} : M \to R$ are smooth functions with corresponding Hamiltonian vector fields $sgradF$, $sgradH$. The Hamiltonian vector field associated with the Poisson bracket of F and H is, up to sign, the Lie bracket of the two Hamiltonian vector fields:*

$$sgrad\{F, H\} = [sgradH, sgardF].$$

The Hamiltonian vector field associated with $H(x)$ has the form

$$sgradH = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\{x^i, x^j\}\frac{\partial H}{\partial x^j}\frac{\partial}{\partial x^i}\right).$$

Let $F(x)$ be a second smooth function. We obtain the basic formula

$$\{F, H\} = \sum_{i=1}^{n}\sum_{j=1}^{n}\{x^i, x^j\}\frac{\partial F}{\partial x^i}\frac{\partial H}{\partial x^j}$$

for the Poisson bracket.

This basic brackets $A^{ij}(x) = \{x^i, x^j\}$ $i, j = 1, ..., m$ are called the *structure functions* of the Poisson manifold M with respect to the given local coordinates.

A skew-symmetric $m \times m$ matrix $A(x) = \{A^{ij}(x)\}$ called the *structure matrix* of M. If the Poisson bracket is non-degenerate ($det(A^{ij}) \neq 0$ everywhere on M), then the Poisson manifold is called a *symplectic manifold.* The symplectic structure in this case has the form $\omega = A_{ij}dx^i \wedge dx^j$ where A_{ij} are components of the matrix inverse to (A^{ij}).

2. Main part

I.Let us consider functions $H_1, H_2 : \mathbb{R}^4 \to \mathbb{R}$ on the Euclidean four dimensional space $\mathbb{R}^4$ with cartezian coordinates p_1, p_2, q_1, q_2 which are given by the formulas

$$H_1 = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)$$

$$H_2 = \frac{1}{2}(p_1^2 + p_2^2 - q_1^2 + q_2^2).$$

The Hamiltonian vector fields corresponding to H_1 and H_2 have following forms:

$$sgradH_1 = -q_1\frac{\partial}{\partial p^1} - q_2\frac{\partial}{\partial p^2} + p_1\frac{\partial}{\partial q^1} + p_2\frac{\partial}{\partial q^2}, sgradH_2 = q_1\frac{\partial}{\partial p^1} - q_2\frac{\partial}{\partial p^2} + p_1\frac{\partial}{\partial q^1} + p_2\frac{\partial}{\partial q^2}. \tag{2.1}$$

Theorem 2.1. *The orbits of vector fields (2.1) generate singular foliation regular leaf of which is three dimensional surface with zero Gauss curvature.*

Proof. We need invariant function from conditions

$$sgradH_1(F) = 0, sgradH_2(F) = 0 \tag{2.2}$$

and we get the function $F(p_1, p_2, q_1, q_2) = p_2^2 + q_2^2$ is a invariant function. For the Puasson bracket of functions H_1 va H_2 we have

$$\{H_1, H_2\} = 2p_1q_1.$$

From theorem 2.1 it follows that

$$[sgradH_2, sgradH_1] = sgrad\{H_1, H_2\} = -2p_1\frac{\partial}{\partial p^1} + 2q_1\frac{\partial}{\partial q^1}$$

If we denote by H_3 the function $\{H_1, H_2\}$ then we have

$$sgradH_3 = [sgradH_2, sgradH_1].$$

And also we have $sgradH_3(F) = 0$. It follows from [6],[5] vector fields $sgradH_1, sgradH_2, sgradH_3$ are tangent vector fields for every orbit of the system D. Let denote by $A(D)$ minimal Lie subalgebra of the algebra $V(R^4)$ containing the set D, and define subspace

$$A_x(D) = \{X(x) : X \in A(D)\}, x = (p_1, p_2, q_1, q_2) \in R^4\}$$

of tangent space at the point $x=(p_1,p_2,q_1,q_2)$. It is known, that

$$dimA_x(D)=\{X(x):X\in A(D)\}\geq dimL(x)$$

where $L(x)$ is a orbit containing x [7]. If $p_1^2+q_1^2>0$ and $p_2^2+q_2^2>0$ then

$$dimA_x(D)=dimL(x)=3.$$

It follows from here in this case the orbit $L(x)$ is three dimensional surface.

Let us consider connected component Φ of the surface defined by equation $p_2^2+q_2^2=c$. Then if $x\in\Phi$ the orbit $L(x)$ is a open subset of the surface Φ. Since the different orbits do not intersect and the set Φ is connected, we obtain that $L(x)=\Phi$.

Let us Φ parameterize the by following equations

$$\begin{cases} p_1=u\\ p_2=\sqrt{c}\cos s\\ q_1=v\\ q_2=\sqrt{c}\sin s\end{cases}$$

or by vector equation

$$r=r(u,v,s)=\{u,\ \sqrt{c}\cos s,\ v,\ \sqrt{c}\sin s\}.$$

We find matrixs of first and second quatric forms:

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&c\end{pmatrix}$$

and

$$B=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-\sqrt{c}\end{pmatrix}.$$

In order to find principal curvatures we have following equation $det|B-\lambda A|=0$. We have following values of principal curvatures

$$\lambda_1=0,\ \lambda_2=0,\ \lambda_3=-\frac{1}{\sqrt{c}}$$

and Gauss curvature $\lambda_1\cdot\lambda_2\cdot\lambda_3=0$.

Following sets are singular leaves:

$L_1=\{0,0,0,0\}-$ zero dimensional orbit;

$L_2=\{(0,p_2,0,q_2):\ p_2^2+q_2^2=c,\ c>0\}-$ one dimensional orbit.

We also have following leaf: $L_3=\{(p_1,0,q_1,0):\ p_1^2+q_1^2>0\}-$ two dimensional plane (p_1,q_1) without the point $(0,0)$.

II. Now let us consider functions

$$H_1=p_1q_1+p_2q_2$$

$$H_2=p_1q_2+p_2q_1$$

and hamiltonian vector fields

$$sgradH_1=-p_1\frac{\partial}{\partial p^1}-p_2\frac{\partial}{\partial p^2}+q_1\frac{\partial}{\partial q^1}+q_2\frac{\partial}{\partial q^2},\ sgradH_2=-p_2\frac{\partial}{\partial p^1}-p_1\frac{\partial}{\partial p^2}+q_2\frac{\partial}{\partial q^1}+q_1\frac{\partial}{\partial q^2}. \quad (2.3)$$

Theorem 2.2. *The orbits of vector fields (2.3) generate singular foliation regular leaf of which is two dimensional surface zero Gauss curvature and zero Gauss torsion.*

Proof. From the equality $\{H_1, H_2\} = 0$ we get

$$[sgradH_2, sgradH_1] = 0.$$

From the following conditions

$$sgradH_1(F_1) = 0,\ sgradH_2(F_1) = 0,\ sgradH_1(F_2) = 0,\ sgradH_2(F_2) = 0$$

we find following invariant functions

$$F_1(p_1, p_2, q_1, q_2) = p_1q_2 + p_2q_1, F_2(p_1, p_2, q_1, q_2) = p_1q_1 + p_2q_2.$$

Let us consider connected component Φ of the surface defined by equation

$$\begin{cases} p_1q_2 + p_2q_1 = c_1, c_1 \neq 0 \\ p_1q_1 + p_2q_2 = c_2, c_2 \neq 0 \end{cases}$$

If $x \in \Phi$ then

$$dimA_x(D) = dimL(x) = 2.$$

It follows from here in this case the orbit $L(x)$ is two dimensional surface. Since the surface Φ is also two-dimensional, the orbit $L(x)$ is an open subset of the surface Φ. Since the different orbits do not intersect and the set Φ is connected, we obtain that $L(x) = \Phi$.

We parameterize this surface by following equations

$$\begin{cases} p_1 = u \\ p_2 = v \\ q_1 = \dfrac{c_2v - c_1u}{v^2 - u^2} \\ q_2 = \dfrac{c_1v - c_2u}{v^2 - u^2} \end{cases} \tag{2.4}$$

or in vector form

$$r = r(u, v) = \{u,\ v,\ \frac{c_2v - c_1u}{v^2 - u^2},\ \frac{c_1v - c_2u}{v^2 - u^2}\}.$$

Using this equations we find coefficients of first quadric form

$$g_{11} = 1 + \frac{(c_1^2 + c_2^2)(u^4 + 6u^2v^2 + v^4) - 8c_1c_2uv(u^2 + v^2)}{(v^2 - u^2)^4}$$

$$g_{12} = g_{21} = \frac{c_1c_2(u^2 + 6u^2v^2 + v^4) - 2(c_1^2 + c_2^2)uv(u^2 + v^2)}{(v^2 - u^2)^4}$$

$$g_{22} = 1 + \frac{(c_1^2 + c_2^2)(u^4 + 6u^2v^2 + v^4) - 8c_1c_2uv(u^2 + v^2)}{(v^2 - u^2)^4},$$

we use denotation $c_1c_2 = \tilde{c}$, $c_1^2 + c_2^2 = \tilde{c}^2$ and also we find normal vectors

$$n_1 = \left\{ \frac{-(c_1 + c_2)}{\sqrt{2(c_1 + c_2)^2 + 2(u + v)^4}}, \frac{-(c_1 + c_2)}{\sqrt{2(c_1 + c_2)^2 + 2(u + v)^4}}, \right.$$

$$\left. \frac{(u + v)^2}{\sqrt{2(c_1 + c_2)^2 + 2(u + v)^4}}, \frac{(u + v)^2}{\sqrt{2(c_1 + c_2)^2 + 2(u + v)^4}} \right\}$$

$$n_2 = \left\{ \frac{c_1 - c_2}{\sqrt{2(c_1 - c_2)^2 + 2(v - u)^4}}, \frac{c_2 - c_1}{\sqrt{2(c_1 - c_2)^2 + 2(v - u)^4}}, \right.$$

$$\left.\frac{(v-u)^2}{\sqrt{2(c_1-c_2)^2+2(v-u)^4}}, \frac{-(v-u)^2}{\sqrt{2(c_1-c_2)+2(v-u)^4}}\right\}.$$

Now we find two second quadric forms

$$II_1 = b_{ij}du^i du^j, II_2 = c_{ij}du^i du^j,$$

where

$$b_{11} = b_{12} = b_{22} = \frac{\sqrt{2}(c_1+c_2)}{(u+v)\sqrt{(c_1+c_2)^2+(u+v)^4}}$$

and

$$c_{11} = -c_{12} = c_{22} = \frac{\sqrt{2}(c_2-c_1)}{(v-u)\sqrt{(c_1-c_2)^2+(v-u)^4}}$$

Let us recall some notions on the geometry of two dimensional submanifolds of four dimensional Euclidean space [8],[9],[10], [11],[12].

Through a point $x_0 \in F^n$ in some tangent direction τ we draw a curve γ lying on F^n. Let s be the length of the arc on γ. Consider the curvature vector k of the curve γ: $k = r_{ss}$. *The normal curvature vector* k_N of a surface F_n in the direction τ at a point x_0 is called the projection of the curvature vector k of a curve γ onto the normal space N_{x_0}.

Let

$$u^i = u^i(s),$$

$i = 1, \ldots, n$,– be the equation of γ. Then

$$r_{ss} = r_{u^i u^j}\frac{du^i}{ds}\frac{du^j}{ds} + r_{u^i}\frac{d^2u^i}{ds^2} \cdot .$$

The projection of the vector k onto the normal space N_{x_0} has the form

$$k_N = \sum_{\sigma=1}^{p}(kn_\sigma)n_\sigma = \sum_{\sigma=1}^{p}(r_{ss}n_\sigma)n_\sigma.$$

Using the expression for r_{ss}, we can write

$$k_N = \sum_{\sigma=1}^{p}(r_{u^i u^j}n_\sigma)\frac{du^i}{ds}\frac{du^j}{ds}n_\sigma = \sum_{\sigma=1}^{p}\frac{II^\sigma n_\sigma}{I}.$$

It follows from here normal curvature vector does not depend from curve γ, it depends only from directionτ.

The *indicatrix of the normal curvature* of a submanifold at a point x_0 is the set of points N_{x_0} of the endpoints of the normal curvature vectors $k_N(\tau)$, taken for each tangent direction τ and plotted from the point x_0.

The indicatrix of the normal curvature is an ellips. Let us denote by a, b the half axis of the *indicatrix of the normal curvature.* We define the Gaussian torsion χ_G of a two-dimensional surface in four-dimensional space as

$$\sigma_G = 2ab.$$

The *Gaussian torsion* of two dimensional surface F in $\mathbb{R}^4$ is a function that assigns a value to each point of the surface, calculated by the following formula

$$\sigma_G = \frac{\langle n_{1u}, n_{2v}\rangle - \langle n_{2u}, n_{1v}\rangle}{\sqrt{g_{11}g_{22} - g_{12}^2}}$$

In our case, the normal vectors n_1, n_2 are non- constant vectors.

Let $F^2 \subset \mathbb{R}^4$ be a 2-dimensional smooth surface embedded in 4-dimensional Euclidean space. At each point $p \in F$, the tangent space T_pF is 2-dimensional, and the normal space N_pF is also 2-dimensional.

Let $\{n_1, n_2\}$ be an orthonormal basis of the normal space N_pF. For each normal vector n_i, we can define: the second fundamental form h^{n_i}.

Then the **Gaussian curvature** $K(p)$ of the surface F at the point p is given by:

$$K(p) = \det(h^{\nu_1}) + \det(h^{\nu_2})$$

where: h^{n_i} is the matrix of the second fundamental form in the direction of n_i. It follows Gaussian curvature of two dimensional surface F is given by the formula

$$K(p) = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} + \frac{c_{11}c_{22} - c_{12}^2}{g_{11}g_{22} - g_{12}^2}.$$

Calculations with formulas gives that

$$K(p) = 0, \sigma_G = 0.$$

Remark: Unlike the case in $\mathbb{R}^3$, where the Gaussian curvature is an intrinsic invariant depending only on the metric of the surface, in $\mathbb{R}^4$ the Gaussian curvature also depends on the extrinsic geometry that is, on how the surface is embedded into the ambient space.

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