

Hypo-EP elements in rings

Semmami R.

Abstract. In this paper, we introduce and investigate a natural extension of the notion of hypo-EP elements previously defined in the framework of C^* -algebras to the broader context of rings with involution. We provide several equivalent characterizations, factorization properties, and study the behavior of these elements under addition and multiplication. Examples and counterexamples are also given to illustrate the scope of the theory.

Keywords: EP element, hypo-EP element, Moore-Penrose inverse, group inverse, ring with involution
MSC (2020): 16B99, 16W10, 46L05.

1. INTRODUCTION

Let R be an associative ring with unity 1. An *involution* $a \mapsto a^*$ in R is an anti-isomorphism of degree 2, that is, $(a^*)^* = a$, $(a + b)^* = a^* + b^*$, $(ab)^* = b^*a^*$, for all $a, b \in R$.

The *Moore-Penrose inverse* $a^\dagger$ of $a \in R$ is defined as the unique solution of the following four equations:

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

There is at most one $a^\dagger$ such that the above conditions hold (see [1]). The set of all Moore-Penrose invertible elements of R will be denoted by $R^\dagger$.

Let $a \in R$. It can be shown that the set of elements $x \in R$ satisfying the conditions

$$axa = a, \quad xax = x, \quad \text{and} \quad ax = xa$$

is either empty or contains exactly one element. If the set contains a single element, this element is called the *group inverse* of a and is denoted by $a^\#$. The set of all group invertible elements will be denoted by $R^\#$.

An element $b \in R$ is **-cancellable* if for all $x \in R$,

$$b^*bx = 0 \implies bx = 0 \quad \text{and} \quad xbb^* = 0 \implies xb = 0. \quad (1.1)$$

Applying the involution to (1.1), we observe that b is *-cancellable if and only if b^* is *-cancellable. In C^* -algebras all elements are *-cancellable. It is demonstrated in [2] that $a \in R^\dagger$ if and only if a is *-cancellable and a^*a is group invertible.

We will also use the following notations: $aR = \{ax : x \in R\}$, $Ra = \{xa : x \in R\}$, $^\circ a = \{x \in R : xa = 0\}$ and $a^\circ = \{x \in R : ax = 0\}$. The usual notation for the commutator of a and b is used: $[a, b] = ab - ba$.

An element $a \in R$ is said to be an EP element if $a \in R^\dagger \cap R^\#$ and $a^\dagger = a^\#$ [2]. The set of all EP elements will be denoted by (EP). Mosić et al. [3, Theorem 2.1] gave several equivalent conditions for an element in R to be an EP element. Patrício and Puystjens in [4, Proposition 2] proved that for a Moore-Penrose invertible element $a \in R$, $a \in (\text{EP})$ if and only if $aR = a^*R$. As for a Moore-Penrose invertible element $a \in R$, $a \in (\text{EP})$ if and only if $[a, a^\dagger] = 0$, thus we deduce that $[a, a^\dagger] = 0$ if and only if $aR = a^*R$. In [5, Theorem 3.1], Rakić et al. investigated some other equivalent conditions. We observe that a is EP if and only if a^* is EP.

The concept of hypo-EP operators was originally introduced in the setting of bounded linear operators on Hilbert spaces (see [6, 7, 8, 9]), and was later extended to elements in C^* -algebras by S. Veeramani in [10] as follows: An element a in a C^* -algebra A is said to be a hypo-EP element if a is regular (i.e., there exists $x \in A$ such that $axa = a$) and $aA \subseteq a^*A$. It is therefore natural to explore a further generalization of this notion within the broader and purely algebraic framework of rings with involution. Such a generalization not only enables the transfer of several known results

into ring theory, but also opens up new directions for the study of generalized inverses in algebraic structures. Motivated by this perspective, the present work introduces and investigates the class of hypo-EP elements in rings with involution, provides several equivalent characterizations, and examines their behavior under algebraic operations such as sums and products. This study represents a first step toward a more complete theory of hypo-EP elements in rings.

2. HYPO-EP ELEMENTS

The class of hypo-EP operators in Hilbert spaces and hypo-EP elements in C^* -algebras in the presence of an involution is naturally extended to rings with involution. Furthermore, we can define hypo-EP elements in a ring with involution using the definition of hypo-EP in C^* -algebras as follows:

Definition 2.1. An element a of a ring R with involution $*$ is said to be **hypo-EP** if $a \in R^\dagger$ and $aR \subseteq a^*R$.

It is clear that if a and a^* are hypo-EP, then a is EP, and the same holds if a and $a^\dagger$ are hypo-EP. Moreover, it is easily seen that every left invertible element is hypo-EP. In the setting of complex matrices, If A is hypo-EP, then A is EP, because $R(A) \subseteq R(A^*)$ and $\text{rank}(A) = \text{rank}(A^*)$ imply $R(A) = R(A^*)$. Thus, in the matrix case, hypo-EP coincides with EP. The following example shows that in a general ring with involution, a hypo-EP element need not be EP.

Example 2.1. Let R denote the ring of all bi-finite real matrices (i.e., matrices that are both row-finite and column-finite) with the transpose acting as involution. Let $e_{i,j}$ be the matrix in R that has 1 in the (i,j) -position and 0 elsewhere. Define $a = \sum_{i=1}^{\infty} e_{i+1,i}$ and $b = a^*$. Then, we have:

$$ab = \sum_{i=2}^{\infty} e_{i,i}, \quad ba = 1.$$

Thus, $a^\dagger = b$, and it is straightforward to verify that b^2 is not left-invertible and that a is not EP (since $ab \neq ba$).

This example also demonstrates that the equality $aR = a^*R$ in the equivalence $(a \in (\text{EP})) \Leftrightarrow (a \in R^\dagger, aR = a^*R)$ cannot be substituted by the inclusions $aR \subseteq a^*R$ or $a^*R \subseteq aR$.

Since every self-adjoint element in $R^\dagger$ is hypo-EP, we then have the following result:

Proposition 2.1. Let $a \in R$ be a self-adjoint element and let $b \in R$ such that $b^*ab \in R^\dagger$; then b^*ab is hypo-EP.

Before characterizing hypo-EP elements, we recall some fundamental properties of the Moore-Penrose inverse in rings, which will be used throughout the paper.

Lemma 2.1. Let $a \in R^\dagger$. We have

- (1) $a^\dagger R = a^\dagger aR = a^*R = a^*aR$;
- (2) $aa^\dagger R = a^*a^\dagger a^*R = a^*a^\dagger R = aR$;
- (3) $(aa^*)^\circ = (a^*)^\circ = (a^\dagger)^\circ$;
- (4) $(1 - a^\dagger a)R = a^\circ$ and $(1 - aa^\dagger)R = (a^*)^\circ$.

Proof: (1) It is clear because $a^* = a^\dagger aa^* = a^*aa^\dagger$ and $a^\dagger = a^\dagger aa^\dagger$.

(2) We have $aa^\dagger R = (aa^\dagger)^*R = a^*a^\dagger a^*R$. From (1), we obtain $aa^\dagger R = a^*a^\dagger a^*R = a^*a^\dagger R = aR$.

(3) Since $a^*aa^\dagger = a^*$, it follows that if $a^\dagger x = 0$, then $a^*x = 0$. Hence, we obtain $(a^\dagger)^\circ \subseteq (a^*)^\circ$. Now, let $x \in (a^*)^\circ$, so that $a^*x = 0$. Multiplying on the left by $a^\dagger a^*$, we get

$$a^\dagger a^* a^\dagger a^* x = a^\dagger (aa^\dagger)^* x = a^\dagger x = 0.$$

Thus, $(a^*)^\circ = (a^\dagger)^\circ$.

Next, we show that $(aa^*)^\circ = (a^*)^\circ$. It is clear that $(a^*)^\circ \subseteq (aa^*)^\circ$. Now, let $x \in (aa^*)^\circ$, so that $a^*x = a^\dagger aa^*x = 0$. Thus, $(aa^*)^\circ \subseteq (a^*)^\circ$.

Therefore, we conclude that $(aa^*)^\circ = (a^*)^\circ = (a^\dagger)^\circ$.

(4) As $a((1 - a^\dagger a)y) = 0$, we have $(1 - a^\dagger a)y \in a^\circ$. Conversely, if $ax = 0$, then $(1 - a^\dagger a)x = x$ which implies $x \in (1 - a^\dagger a)R$. In the same manner, we show that $(1 - aa^\dagger)R = (a^*)^\circ$. $\square$

The following theorem provides several equivalent conditions for an element to be hypo-EP. These characterizations will serve as the foundation for the rest of our study.

Theorem 2.1. *Let $a \in R^\dagger$. The following conditions are equivalent:*

- (i) a is hypo-EP;
- (ii) $aR \subseteq a^\dagger R$;
- (iii) $[a, a^\dagger a] = 0$;
- (iv) $[a^*, a^\dagger a] = 0$;
- (v) $a^\circ \subseteq (a^\dagger)^\circ = (a^*)^\circ$;
- (vi) $[a^\dagger, a^\dagger a] = 0$;
- (vii) $aa^\dagger = a(a^\dagger)^2 a$;
- (viii) $(a^\dagger a)^\circ \subseteq (aa^\dagger)^\circ$;
- (ix) $[a^\dagger a, (a + a^\dagger)] = 0$;
- (x) $aa^\dagger = a^\dagger a^2 a^\dagger$.

Proof: (i) $\iff$ (ii) It is clear by $a^* R = a^\dagger R$.

(ii) $\implies$ (iii) Suppose $aR \subseteq a^\dagger R$; then $a = a^\dagger b$ for some $b \in R$. We get

$$a - a^\dagger a^2 = a^\dagger b - a^\dagger aa^\dagger b = 0.$$

Hence $a = a^\dagger a^2$.

(iii) $\implies$ (iv) Suppose $a = a^\dagger a^2$. By the definition of the Moore-Penrose invertibility of a , we have

$$a^* = ((a^\dagger a)a)^* = a^*(a^\dagger a)^* = a^* a^\dagger a.$$

(iv) $\implies$ (v) Assume $a^* = a^* a^\dagger a$. If $y \in a^\circ$, then $a^* y = a^* a^\dagger a y = 0$. Hence $y \in (a^*)^\circ = (a^\dagger)^\circ$. Thus $a^\circ \subseteq (a^\dagger)^\circ$.

(v) $\implies$ (vi) Suppose that $a^\circ \subseteq (a^*)^\circ$. By Lemma 2.1 (4), we have $(1 - a^\dagger a)R \subseteq (1 - aa^\dagger)R$. Thus $(1 - a^\dagger a) = (1 - aa^\dagger)b$ for some $b \in R$. Multiplying on the left by $a^\dagger$, we get

$$a^\dagger = (a^\dagger)^2 a.$$

(vi) $\implies$ (ii) Assume that $a^\dagger = (a^\dagger)^2 a$. We have $(a^\dagger)^* = a^\dagger a (a^\dagger)^*$ and by Lemma 2.1(2), $(a^\dagger)^* R = aR$, so we get $aR \subseteq a^\dagger R$.

(vi) $\iff$ (vii) Obvious.

(vii) $\implies$ (viii) Suppose $aa^\dagger = a(a^\dagger)^2 a$. Let $x \in (a^\dagger a)^\circ$; then $a^\dagger a x = 0$. Multiplying on the left by $aa^\dagger$, we get $aa^\dagger a^\dagger a x = 0$. Thus by hypothesis $x \in (aa^\dagger)^\circ$.

(viii) $\implies$ (v) Suppose $(a^\dagger a)^\circ \subseteq (aa^\dagger)^\circ$. Let $y \in a^\circ$; then $a^\dagger a y = 0$. By hypothesis $aa^\dagger y = 0$. Multiplying on the left by $a^\dagger$, we get $a^\dagger y = 0$. Hence $y \in (a^\dagger)^\circ$.

(ix) $\implies$ (vii) Assume $(a^\dagger a)(a + a^\dagger) = (a + a^\dagger)(a^\dagger a)$; then $a^\dagger a^2 + a^\dagger = a + (a^\dagger)^2 a$. Multiplying on the left by a , we get $a^2 + aa^\dagger = a^2 + a(a^\dagger)^2 a$. Therefore,

$$aa^\dagger = a(a^\dagger)^2 a.$$

(iii) $\implies$ (ix) By the equivalence of (iii) and (vi) we have

$$a^\dagger a(a + a^\dagger) = a^\dagger a^2 + a^\dagger = a + (a^\dagger)^2 a = (a + a^\dagger)a^\dagger a.$$

From the last two equations, we get the desired result.

(x) $\implies$ (iii) Right-multiply the assumption $aa^\dagger = a^\dagger a^2 a^\dagger$ by a to obtain $a = a^\dagger a^2$.

(iii) $\implies$ (x) Obvious. □

Using the characterizations above and the theorems in [11], we now relate the class of hypo-EP elements to the well-known class of EP elements.

Corollary 2.1. *Let $a \in R^\dagger$. Then the following are equivalent:*

- (i) a is EP;
- (ii) a is hypo-EP and $aR = a^2R$;
- (iii) a is hypo-EP and a is group invertible;
- (iv) $a^\dagger$ is hypo-EP and $Ra = Ra^2$;
- (v) $Ra \subseteq Ra^\dagger$ and $Ra = Ra^2$;
- (vi) $Ra \subseteq Ra^*$ and $Ra = Ra^2$.

In the following proposition we give some conditions for a hypo-EP element to be EP. This result is proved for operators in [12].

Proposition 2.2. *Let $a \in R$ be a hypo-EP element. If a satisfies one of the following statements,*

- (i) $[aa^\dagger, a + a^\dagger] = 0$;
- (ii) $[aa^\dagger, a + a^*] = 0$;
- (iii) $[a, aa^\dagger] = 0$,

then a is an EP element.

Proof: (i) It suffices to show that $a^\dagger$ is hypo-EP. We have

$$0 = [aa^\dagger, a + a^\dagger] = a + a(a^\dagger)^2 - a^2a^\dagger - a^\dagger.$$

Multiplying on the right by a , we get

$$a^\dagger a = a(a^\dagger)^2 a.$$

Thus, according to Theorem 2.1(x), $a^\dagger$ is hypo-EP, which implies that a^* is also hypo-EP. Therefore, a is EP.

(ii) As before,

$$0 = [aa^\dagger, a + a^*] = a + aa^\dagger a^* - a^2a^\dagger - a^*.$$

Multiplying on the left by $aa^\dagger$, we get $a = a^2a^\dagger$ and by Theorem 2.1(vi), $a^\dagger$ is hypo-EP. We conclude that a is EP.

(iii) Again we have $0 = [a, aa^\dagger] = a^2a^\dagger - a$, and by Theorem 2.1(vi) we derive that a is EP. $\square$

Remark 2.1. Proposition 2.2 can be seen as an analogue of the fact that for EP elements, the Moore-Penrose inverse and the group inverse coincide. Here, additional commutation conditions force a hypo-EP element to be EP.

For EP elements, the projection $p = 1 - aa^\dagger$ satisfies $pa = 0$ and $p + a$ is invertible (see [13]). The following proposition extends this characterization to hypo-EP elements.

Proposition 2.3. *Let $a \in R^\dagger$ be a hypo-EP element. Then the projection $p = 1 - aa^\dagger$ satisfies $pa = 0$ and $p + a$ is left-invertible.*

Proof: Let a be a hypo-EP element and set $p = 1 - aa^\dagger$. Then $p^2 = p$ and $pa = 0$. We prove that $p + a$ is left-invertible.

Compute

$$(a(a^\dagger)^2 + p)(p + a) = a(a^\dagger)^2 p + a(a^\dagger)^2 a + p.$$

Since

$$a(a^\dagger)^2 p = a(a^\dagger)^2 (1 - aa^\dagger) = a(a^\dagger)^2 - a(a^\dagger)^2 aa^\dagger = 0,$$

it follows that

$$(a(a^\dagger)^2 + p)(p + a) = a(a^\dagger)^2 a + p.$$

By Theorem 2.1(vii), $aa^\dagger = a(a^\dagger)^2 a$, and hence

$$(a(a^\dagger)^2 + p)(p + a) = aa^\dagger + (1 - aa^\dagger) = 1.$$

Therefore, $a(a^\dagger)^2 + p$ is a left inverse of $p + a$. $\square$

3. FACTORIZATIONS OF HYPO-EP ELEMENTS

Factorizations of hypo-EP elements provide useful structural insights and extend results already established in the setting of C^* -algebras (see [14, 15, 10]). They highlight connections with projection techniques and offer algebraic characterizations that remain valid in rings with involution.

Theorem 3.1. *Let $a \in R^\dagger$. The following statements are equivalent:*

- (i) a is hypo-EP;
- (ii) $\exists b_1 \in R$ such that $a = a^*b_1$;
- (iii) $\exists b_2 \in R$ such that $a^\dagger = b_2a$;
- (iv) $\exists c_1 \in R$ such that $aa^* = c_1a$;
- (v) $\exists b_3, u \in R$ such that $(b_3)^\circ \subseteq ((b_3)^*)^\circ$, $u^\circ = \{0\}$ and $a = ub_3u^*$;
- (vi) $\exists b_4, c_2, u, v, w \in R$ such that $(b_4)^\circ \subseteq (c_2)^\circ$, $u^\circ = v^\circ = \{0\}$ and $a = ub_4w = w^*c_2^*v^*$;
- (vii) $\exists b_5, c_3, u_1, v_1, w_1 \in R$ such that $(b_5)^\circ \subseteq (c_3)^\circ$, $(u_1)^\circ = (v_1)^\circ = \{0\}$, and $a^*a = u_1b_5w_1$, $aa^* = v_1c_3w_1$;
- (viii) $\exists b_6, c_4, u_2, v_2, w_2 \in R$ such that $(b_6)^\circ \subseteq (c_4)^\circ$, $(u_2)^\circ = (v_2)^\circ = \{0\}$, and $a = u_2b_6w_2$, $a^\dagger = v_2c_4w_2$.

Proof: (i) $\implies$ (ii) Since $aR \subseteq a^*R$, we have $a \in a^*R$. Thus we get the desired result.

(ii) $\implies$ (iii) Assume there exists $b_1 \in R$ such that $a = a^*b_1$, which implies $a^* = (b_1)^*a$. Consequently, we obtain

$$a^\dagger = a^\dagger(aa^\dagger)^* = a^\dagger(a^\dagger)^*a^* = b_2a,$$

where $b_2 = a^\dagger(a^\dagger)^*(b_1)^*$.

(iii) $\implies$ (i) Assume $a^\dagger = b_2a$. Let $y \in a^\circ$. Then

$$a^\dagger y = 0.$$

This implies $y \in (a^\dagger)^\circ$. Since $(a^\dagger)^\circ = (a^*)^\circ$, it follows that $a^\circ \subseteq (a^*)^\circ$. Therefore, by Theorem 2.1(v), a is hypo-EP.

(ii) $\implies$ (iv) Let $a = a^*b_1$. Thus $a^* = (b_1^*)a$. Now we have

$$aa^* = a(b_1^*)a = c_1a,$$

where $c_1 = ab_1^*$.

(iv) $\implies$ (i) Suppose $aa^* = c_1a$ for some $c_1 \in R$. Let $x \in a^\circ$; then

$$aa^*x = 0.$$

Hence $x \in (aa^*)^\circ$. By Lemma 2.1(3) and Theorem 2.1(v), a is hypo-EP.

(i) $\implies$ (v) By Theorem 2.1(v), it is enough to choose $b_3 = a$ and $u = 1$.

(v) $\implies$ (vi) Assume $a = ub_3u^*$ with $b_3, u \in R$ such that $(b_3)^\circ \subseteq ((b_3)^*)^\circ$ and $u^\circ = 0$. Let $w = u^*$ and $b_4 = b_3$, so $a = ub_4u$. We observe that

$$a = (u^*)^*((b_4)^*)^*u^*.$$

Now set $w = u^*$, $v = u$, and $c_2 = (b_4)^*$. Then $a = w^*c_2^*v^*$. Since $(b_4)^\circ \subseteq ((b_4)^*)^\circ$ and $u^\circ = 0$, it follows that $(b_4)^\circ \subseteq (c_2)^\circ$ and $v^\circ = 0$.

(vi) $\implies$ (i) By Theorem 2.1(v), it is sufficient to prove $a^\circ \subseteq (a^*)^\circ$. Let $x \in a^\circ$, meaning $ub_4wx = 0$. Since $u^\circ = 0$, we have $b_4wx = 0$, so $wx \in (b_4)^\circ$. Since $(b_4)^\circ \subseteq (c_2)^\circ$, we deduce $c_2wx = 0$, which further leads to $vc_2vx = 0$. Consequently $a^*x = 0$, so $x \in (a^*)^\circ$.

(vii) $\implies$ (i) Assume $a^*a = u_1b_5w_1$ and $aa^* = v_1c_3w_1$ for some $b_5, c_3, u_1, v_1 \in R$ such that $(b_5)^\circ \subseteq (c_3)^\circ$ and $(u_1)^\circ = (v_1)^\circ = 0$. We show $a^\circ \subseteq (a^*)^\circ$. Let $y \in a^\circ$; then $a^*ay = 0$. Substituting the given factorization,

$$u_1b_5wy = 0.$$

Thus $wy \in (b_5)^\circ$, and since $(b_5)^\circ \subseteq (c_3)^\circ$, we get $c_3w_1y = 0$. Then $w_1y \in (c_3)^\circ$, and $v_1c_3wy = 0$ implies $y \in (aa^*)^\circ$. By Lemma 2.1(3) and Theorem 2.1(v), the desired result holds.

(i) $\implies$ (vii) By Lemma 2.1(3) and Theorem 2.1(v), choose $w = 1$, $v_1 = 1$, $u_1 = 1$, $b_5 = a^*a$ and $c_3 = aa^*$.

(viii) $\implies$ (i) Assume $a = u_2b_6w_2$ and $a^\dagger = v_2c_4w_2$ for some $b_6, c_4, u_2, v_2, w_2 \in R$ such that $(b_6)^\circ \subseteq (c_4)^\circ$ and $(u_2)^\circ = (v_2)^\circ = 0$. Let $x \in a^\circ$. Then $u_2b_6w_2x = 0$, so $w_2x \in (b_6)^\circ$ and since $(b_6)^\circ \subseteq (c_4)^\circ$, we have $c_4w_2x = 0$. Hence $a^\dagger x = v_2c_4w_2x = 0$, so $x \in (a^*)^\circ$. By Theorem 2.1(v), the required result holds.

(i) $\implies$ (viii) Choose $u_2 = 1$, $v_2 = 1$, $w_2 = 1$, $b_6 = a$, and $c_4 = a^\dagger$. By Theorem 2.1(v), the desired result follows. $\square$

4. SUM OF HYPO-EP ELEMENTS

In general, the sum of two hypo-EP elements is not necessarily hypo-EP. The following example shows that $a + b$ is not necessarily hypo-EP even though a and b are hypo-EP.

Example 4.1. Let R be the ring of all complex 2×2 matrices. Consider

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a and b are invertible, hence they are hypo-EP. However, $a + b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not hypo-EP because $(a + b)^2 = 0$ and $a + b \neq 0$, which fails condition (iii) of Theorem 2.1.

It is important to understand whether the class of hypo-EP elements is closed under addition. The following result provides a sufficient condition for the sum to be hypo-EP.

Proposition 4.1. *Let a, b be hypo-EP elements such that $a + b \in R^\dagger$. If $(a + b)^\circ \subseteq a^\circ$, then $a + b$ is hypo-EP.*

Proof: By Theorem 2.1(v), it is enough to show $(a + b)^\circ \subseteq ((a + b)^*)^\circ$. Let $x \in (a + b)^\circ$; then $(a + b)x = 0$. By hypothesis, $ax = 0$, thus $bx = 0$. Since both a and b are hypo-EP, by Theorem 2.1 we have $a^*x = 0$ and $b^*x = 0$, so $(a + b)^*x = 0$. Therefore, $a + b$ is hypo-EP. $\square$

We now examine the converse of the previous result under additional assumptions. This helps to better understand the additive structure of hypo-EP elements. Suppose a and b are hypo-EP; then by (ii) of Theorem 3.1, $a^* = c_1a$ and $b^* = c_2b$, where $c_1, c_2 \in R$.

Theorem 4.1. *Let a, b be hypo-EP elements such that $a + b \in R^\dagger$ and $c_1 - c_2$ is invertible, where c_1, c_2 are as defined above. Then $a + b$ is hypo-EP if and only if $(a + b)^\circ \subseteq a^\circ$.*

Proof: Suppose $a + b$ is hypo-EP. Then we have

$$c_1a + c_2b = a^* + b^* = (a + b)^* = d(a + b)$$

for some $d \in R$. This implies

$$(c_1 - d)a = (d - c_2)b.$$

Set $e = c_1 - d$ and $f = d - c_2$; then $ea = fb$ and $(e + f)a = f(a + b)$, so

$$a = (e + f)^{-1}f(a + b).$$

Hence $(a + b)^\circ \subseteq a^\circ$. $\square$

Remark 4.1. The converse of Proposition 4.1 is not true in general if the condition that $c_1 - c_2$ is invertible is omitted. The following example illustrates this point. Let R be the ring of all complex 2×2 matrices. Consider

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then a , b and $a + b$ are hypo-EP (in fact, they are self-adjoint). However, $(a + b)^\circ \not\subseteq a^\circ$ because $a + b = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ has a larger nullspace. Notice that in this example $c_1 = c_2 = I$, hence $c_1 - c_2 = 0$, which is not invertible. This explains why the converse fails when the invertibility condition is dropped.

5. PRODUCT OF HYPO-EP ELEMENTS

This section deals with the product of two hypo-EP elements. We derive some equivalent conditions for the product of two hypo-EP elements to be hypo-EP. We first give an example to show that the product of two hypo-EP elements in a ring is not necessarily hypo-EP.

Example 5.1. Let $R = M_2(\mathbb{C})$ (complex 2×2 matrices) with the transpose as involution. Suppose

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then a is invertible (hence hypo-EP) and b is self-adjoint idempotent up to scalar (hence hypo-EP). Their product

$$ab = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

is not hypo-EP because one can verify that $(ab)^\dagger(ab)^2 \neq (ab)$.

It is natural to ask whether the product of two hypo-EP elements is again hypo-EP. The next theorem provides sufficient conditions for this to hold.

Theorem 5.1. *Let $a, b \in R$ such that $ab \in R^\dagger$. If a, b are hypo-EP, $bR \subseteq aR$ and ${}^\circ b \subseteq {}^\circ a$, then ab is hypo-EP.*

Proof: We begin by proving $aR \subseteq b^*R$. Since b is hypo-EP, it satisfies

$$b - b^\dagger b^2 = 0.$$

Rearranging, $(1 - b^\dagger b)b = 0$, so $1 - b^\dagger b \in {}^\circ b$. By assumption, $1 - b^\dagger b \in {}^\circ a$, leading to

$$a = b^\dagger ba.$$

Applying Lemma 2.1(1), which states $b^\dagger bR = b^*R$, we conclude $a \in b^*R$, thus $aR \subseteq b^*R$. Now, using Lemma 2.1(1) again,

$$\begin{aligned} abR &\subseteq aR \subseteq b^*R = b^*bR \\ &\subseteq b^*aR \subseteq b^*a^*R \\ &= (ab)^*R. \end{aligned}$$

This establishes that ab is hypo-EP. □

The following example demonstrates that the converse of Theorem 5.1 is not necessarily true.

Example 5.2. Let $R = M_2(\mathbb{C})$ and consider

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then a , b , and $ab = 0$ are all hypo-EP (they are self-adjoint projections). However, $aR \not\subseteq bR$ and ${}^\circ a \not\subseteq {}^\circ b$.

The following result connects the class of EP elements to hypo-EP elements via multiplication, giving a characterization in terms of range inclusion.

Theorem 5.2. *Let $a, b \in R$ be EP elements and $ab \in R^\dagger$. Then ab is hypo-EP if and only if $ab^*R \subseteq b^*aR$.*

Proof: By our assumption, we have

$$abR = a(bR) = ab^*R \quad \text{and} \quad b^*aR = b^*(aR) = b^*a^*R.$$

Thus $abR \subseteq (ab)^*R$ if and only if $ab^*R \subseteq b^*aR$. Therefore, ab is hypo-EP if and only if $ab^*R \subseteq b^*aR$. □

Corollary 5.1. *Let a, b be hypo-EP elements in R such that $ab \in R^\dagger$. If $ab^*R \subseteq b^*aR$, then ab is hypo-EP.*

We conclude this section with an alternative condition under which the product of two elements is hypo-EP, further illustrating the flexibility of the definition.

Theorem 5.3. *Let $a, b \in R$ such that $ab \in R^\dagger$. If a is hypo-EP and $(ab)^\circ \subseteq a^\circ$, then ab is hypo-EP.*

Proof: We have $(ab)^\circ \subseteq a^\circ \subseteq (a^*)^\circ \subseteq (b^*a^*)^\circ = [(ab)^*]^\circ$. Thus by Theorem 2.1(v), it follows that ab is hypo-EP. $\square$

Remark 5.1. The condition $(ab)^\circ \subseteq a^\circ$ is not necessary for ab to be hypo-EP. For

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

we have $(ab)^\circ \not\subseteq a^\circ$. But a, b and ab are all hypo-EP elements (over $\mathbb{C}$).

REFERENCES

- [1] Koliha J. J., Djordjević D., Cvetković D., Moore-Penrose inverse in rings with involution. *Linear Algebra Appl.* (2007) Vol. 426, P. 371–381.
- [2] Koliha J. J., Patrício P., Elements of rings with equal spectral idempotents. *J. Australian Math. Soc.* (2002) Vol. 72, P. 137–152.
- [3] Mosić D., Djordjević D. S., Koliha J. J., EP elements in rings. *Linear Algebra Appl.* (2009) Vol. 431, P. 527–535.
- [4] Patrício P., Puystjens R., Drazin-Moore-Penrose invertibility in rings. *Linear Algebra Appl.* (2004) Vol. 389, P. 159–173.
- [5] Rakić D. S., Dinčić N. Č., Djordjević D. S., Group, Moore-Penrose, core and dual core inverse in rings with involution. *Linear Algebra Appl.* (2014) Vol. 463, P. 115–133.
- [6] Itoh M., On some EP operators. *Nihonkai Math. J.* (2005) Vol. 16, P. 49–56.
- [7] Johnson P. S., Vinoth A., Product and factorization of hypo-EP operators. *Spec. Matrices*, (2018) Vol. 6, P. 376–382.
- [8] Patel A. B., Shekhawat M. P., Hypo-EP operators. *Indian J. Pure Appl. Math.* (2016) Vol. 47, P. 73–84.
- [9] Vinoth A., Sam Johnson P., On sum and restriction of hypo-EP operators. *Funct. Anal. Approx. Comput.* (2017) Vol. 9, P. 37–41.
- [10] Veeramani S., Hypo-EP element in C^* -algebra. *Rend. Circ. Mat. Palermo (2)*, (2023) Vol. 72, P. 1115–1123.
- [11] Xu S., Chen J., Benítez J., EP elements in rings with involution. *Bull. Malays. Math. Sci. Soc.* (2019) Vol. 42, P. 3409–3426.
- [12] Semmami R., Ezzahraoui H., Zerouali E. H., On some properties of n-EP and n-hypo-EP operators. *arXiv:2410.20920*, (2024),
- [13] Benítez J., Moore-Penrose inverses and commuting elements of C^* -algebras. *J. Math. Anal. Appl.* (2008) Vol. 345, P. 766–770.
- [14] Chen J., Zhang X., *Algebraic Theory of Generalized Inverses*. Springer, (2023).
- [15] Djordjević D. S., Koliha J. J., Straškraba I., Factorization of EP elements in C^* -algebras. *Linear Multilinear Algebra*, (2009) Vol. 57, P. 587–594.

SEMMAMI Rachid,

Faculty of Sciences, Mohammed V University in Rabat,

Rabat, Morocco.

e-mail: semmami.rachi@gmail.com; rachid_semmami@um5.ac.ma