

Sequential and computational approach for obtaining the quadruples of a Diophantine System

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Abstract. This manuscript is concerned with obtaining non-negative and non-zero sequential and computational solutions to the Diophantine problem, which states that the sum of four non-negative, non-zero integers, the sum of the squares of these integers, and their cubic sum are perfect squares. Hitherto, no sequential pattern has been discovered on this particular Diophantine form of four unknowns under specific conditions. An open question is further addressed at the end.

Keywords: Diophantine problem, Sequential solution, Cubic sum

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1. INTRODUCTION

The paper is primarily concerned with the problem of obtaining four unknowns, provided these unknowns are non-negative, non-zero integers. In detail, the unknowns are associated with the condition that the sum of these unknowns, their squares, and cubes is a square of a non-negative non-zero integer. Two different conditions are presumed for the outcomes. These conditions have not been explored yet in the prevailing scientific literature.

Let us consider a Diophantine system consisting of four unknowns $\alpha, \beta, \gamma, \delta$ associated with the following three concurrent equations, where there are some non-zero, non-negative integers a, b, c .

$$\alpha + \beta + \gamma + \delta = a^2 \quad (1.1)$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = b^2 \quad (1.2)$$

$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = c^2 \quad (1.3)$$

Our prior intention is to find the sequential pattern of the desired solution to the aforementioned problems. Back in 1795, a gentleman named *J. Saul* obtained an integral solution to a simultaneous system that states the sum of two variables, their squares, and their cubes are a perfect square. Based on his remarkable study, *A. Bremner* [1] exhibits all the integral solutions to the problems stated by *J. Saul*. In the problem stated above, when it is considered, the problem turns out to be fully solved by *A. Bremner* [1]. Recently, in the year 2020, *A. Choudhry* [2] extended the problem for three positive integers and found infinitely many parametric solutions. As surveyed till now, no studies have been conducted to attain the solution by addressing one more unknown in the simultaneous system of equations considered by *A. Choudhry* [2].

2. QUADRUPLES FOR THE PRESCRIBED DIOPHANTINE SYSTEM

We will now solve the simultaneous Diophantine system of equations represented by equations (1.1), (1.2), and (1.3), assuming two unknowns are equal.

Let us consider, $\beta = \gamma = \kappa$, where κ is a non-negative non-zero integer. Then (1.1), (1.2), and (1.3) can be rewritten as follows.

$$\alpha + 2\kappa + \delta = a^2 \quad (2.1)$$

$$\alpha^2 + 2\kappa^2 + \delta^2 = b^2 \quad (2.2)$$

$$\alpha^3 + 2\kappa^3 + \delta^3 = c^2 \quad (2.3)$$

Since a, b, c are non-zero non-negative integers, thus there exist some non-zero non-negative integers θ, ψ, p, q, r such that

$$a = \theta + \psi - p \quad (2.4)$$

$$b = \theta^2 - 2p\theta + \psi^2 + q \quad (2.5)$$

$$c = \theta^3 + \psi^3 - 3\theta p(\theta - p) - r \quad (2.6)$$

Our concern is to show whether the values depicted through equations (2.4), (2.5) and (2.6) justify the desired demand of the sequential pattern of the solution.

On solving equations (2.1) and (2.2) by substituting the values of a, b and $q = p^2$, we have obtained a quadratic form of the equation in terms of δ and κ .

$$\begin{aligned} &\delta^2 + (2\kappa - \theta^2 - p^2 + 2\theta p - \psi^2 - 2\theta\psi + 2p\psi)\delta + 2\theta^2\psi^2 + 2p^2\psi^2 - 4\theta p\psi^2 + 3\kappa^2 - 4\theta\psi\kappa \\ &+ 4p\psi\kappa + 2\psi^3\theta - 2\psi^3p - 2\psi^2\kappa + 2\theta^3\psi - 2p^3\psi - 6\theta^2p\psi + 6\theta p^2\psi - 2\kappa\theta^2 - 2\kappa p^2 + 4\kappa\theta p = 0 \end{aligned} \quad (2.7)$$

Without loss of generality, on further substituting $\kappa = \theta p - \psi p$ in equation (2.7), we have

$$\begin{aligned} &\delta^2 + (4\theta p - \theta^2 - p^2 - \psi^2 - 2\theta\psi)\delta + (2\theta^2\psi^2 + p^2\psi^2 + 7\theta^2p^2 - 8\theta^2p\psi + 2\psi^3\theta - 2\theta p\psi^2 \\ &+ 2\theta^3\psi - 2\theta^3p - 2\theta p^3) = 0 \end{aligned} \quad (2.8)$$

It is clearly observed that $(\theta - p)^2$ and ψ^2 satisfies the equation (2.8) while $p = \psi$ is considered.

From equation (2.1), we can also obtain the value of α by substituting the assumed value of $\kappa = \theta p - \psi p$ and $\delta = (\theta - p)^2$. Equation (2.1) leads to $\alpha = (\theta + \psi - p)^2 - 2(\theta p - \psi p) - (\theta - p)^2$ and finally $\alpha = \psi^2$ can be obtained.

While the values of κ, δ and α are substituted in L.H.S. of equation (2.2), it is found to be equal to b^2 that implies that our assumption for κ is correct.

Now, to find the condition under which equation (2.6) is valid to justify the modified Diophantine system of equations portrayed through equation (2.1), (2.2) and (2.3). Putting the values of α, κ, δ and c in equation (2.3), we have

$$\psi^6 + 2(\theta p - \psi p)^3 + (\theta - p)^6 = (\theta^3 + \psi^3 - 3\theta p(\theta - p) - r)^2 \quad (2.9)$$

On solving equation (2.9), we have obtained that $r = \psi^3$.

It shows cases the clarity of our assumption. Hence, we can arrive at a sequential pattern of solution as $\langle \alpha, \beta, \gamma, \delta, a, b, c \rangle = \langle \psi^2, \theta p - \psi p, \theta p - \psi p, (\theta - p)^2, \theta + \psi - p, \theta^2 - 2\theta p + \psi^2 + p^2, \theta^3 + \psi^3 - 3\theta p(\theta - p) - p^3 \rangle$ provided $p = \psi$ i.e.

$$\langle \alpha, \beta, \gamma, \delta, a, b, c \rangle = \langle p^2, \theta p - p^2, \theta p - p^2, (\theta - p)^2, \theta, \theta^2 - 2\theta p + 2p^2, \theta^3 + p^3 - 3\theta p(\theta - p) - p^3 \rangle \quad (2.10)$$

It is the desired sequential pattern, and the required quadruple is

$$\langle \alpha, \beta, \gamma, \delta \rangle = \langle p^2, \theta p - p^2, \theta p - p^2, (\theta - p)^2 \rangle.$$

2.1. Particular Cases.

2.1.1. When $\theta = 2p$ is considered, a special pattern of sequence is acquired as follows. $\langle \alpha, \beta, \gamma, \delta, a, b, c \rangle = \langle p^2, p^2, p^2, p^2, 2p, 2p^2, 2p^3 \rangle$. This is valid for all positive non-trivial integral values of p .

Illustration: For instance, $(1, 1, 1, 1, 2, 2, 2)$, $(4, 4, 4, 4, 8, 16)$, $(9, 9, 9, 9, 18, 54)$ and $(16, 16, 16, 16, 32, 128)$ lie under this pattern. Thus, when $\alpha = \beta = \gamma = \delta$, an infinite symmetric pattern of non-trivial solutions is obtained.

2.1.2. When $p = 1$ and $\theta = n$, (n is a Natural number) is considered, in that case, another sequential pattern has been obtained $\langle \alpha, \beta, \gamma, \delta, a, b, c \rangle = \langle 1, n - 1, n - 1, (n - 1)^2, n, n^2 - 2n + 2, n^3 - 3n(n - 1) \rangle$ This is valid for all integral values of n greater than 1.

Illustration: Examples include $(1, 1, 1, 1, 2, 2, 2)$, $(1, 2, 2, 4, 3, 5, 9)$, $(1, 3, 3, 9, 4, 10, 28)$ and $(1, 4, 4, 16, 5, 17, 65)$. This gives infinitely many non-symmetric solutions with $\alpha \leq \beta = \gamma \leq \delta$.

2.1.3. When $p = n^3, \theta = n^4$, the sequential pattern is reduce to a new pattern such as $\langle \alpha, \beta, \gamma, \delta, a, b, c \rangle = \langle n^6, n^7 - n^6, n^7 - n^6, (n^4 - n^3)^2, n^4, n^8 - 2n^7 + 2n^6, n^{12} - 3n^{11} + 3n^{10} \rangle$. This is valid for all non-zero non-negative values of $n \geq 2$.

Illustration: Examples include (64, 64, 64, 64, 16, 128, 1024) and (729, 1458, 1458, 2916, 81, 3645, 177147). Hence this pattern yields infinitely many non-symmetric solutions of the prescribed system.

3. WHETHER SOLUTIONS EXIST FOR $\alpha \neq \beta \neq \gamma \neq \delta$

We have also found some special solutions that do not hold the sequential pattern of the solution of the equation represented by (2.10). However, through an analytical approach, these solutions cannot be acquired. A computational approach has been employed to obtain such a type of solution, where $\alpha < \beta < \gamma < \delta$. The whole set of solutions ranging from 1 to 500 is as follows.

α	β	γ	δ	a	b	c	α	β	γ	δ	a	b	c	α	β	γ	δ	a	b	c
9	10	60	90	13	109	973	30	57	168	474	27	507	10557	60	241	282	378	31	533	9521
10	44	57	58	13	93	683	33	205	441	477	34	682	14246	62	73	266	440	29	523	10229
10	74	98	142	18	188	2052	34	82	98	110	18	172	1692	67	209	233	275	28	422	6548
14	58	122	130	18	188	2052	36	40	240	360	26	436	7784	88	96	232	260	26	372	5624
17	91	299	377	28	490	9004	40	176	228	232	26	372	5464	98	212	305	346	31	517	8959
18	84	129	210	21	261	3465	42	66	108	145	19	197	2161	125	241	305	485	34	634	12586
22	24	58	65	13	93	703	46	170	202	258	26	372	5516	186	234	258	478	34	620	12068
22	73	130	136	19	203	2261	49	98	170	212	23	293	3937	222	354	370	498	38	748	15148
22	161	266	280	27	419	6705	52	190	233	254	27	397	6003	240	306	313	366	35	619	11053
29	109	301	461	30	562	11250	57	102	222	348	27	429	7371	253	295	341	407	36	658	12204

Thus, the solution exists, but there may not exist sequential pattern for such a type of solution.

The computational search was performed by running through all ordered quadruples $1 \leq \alpha < \beta < \gamma < \delta \leq 500$ and retaining only those for which $\alpha + \beta + \gamma + \delta$, $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$, and $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$ are perfect squares. The corresponding values of a , b , and c are the square roots of these three sums.

4. SOME SIGNIFICANT REMARKS BASED ON THE PRESCRIBED DIOPHANTINE SYSTEM

4.1. **Remark.** The following Diophantine system of equations, having three non-trivial

$$\begin{aligned}\alpha + n\beta + \delta &= a^2 \\ \alpha^2 + n\beta^2 + \delta^2 &= b^2 \\ \alpha^3 + n\beta^3 + \delta^3 &= c^2\end{aligned}$$

unknowns α, β, δ with positive integers a, b, c, n has a sequential pattern of solutions if n is a positive even number.

4.2. **Remark.** For all symmetric and non-symmetric solutions of the prescribed Diophantine system exhibited by equations (1.1), (1.2) and (1.3); the following specific relations have been acquired:

- (1) b greater than or equal to half of the square of a .
- (2) c greater than or equal to one-fourth of the cube of a .
- (3) Square of b is equal to the fourth power of a minus twice the sum of the product of all integers $\alpha, \beta, \gamma, \delta$ twice at a time.
- (4) Sixth power of a is equal to the square of c , plus thrice the product of the square of a and the square of b , minus three times the sum of the products of two terms of $\alpha, \beta, \gamma, \delta$ and sums of two same terms selected.

4.3. **Remark.** In the sequential pattern of solution $\langle \alpha, \beta, \gamma, \delta, z_1, z_2, z_3, z_4, \dots, z_n \rangle$ of the following Diophantine system

$$\begin{aligned}\alpha + \beta + \gamma + \delta &= z_1^2 \\ \alpha^2 + \beta^2 + \gamma^2 + \delta^2 &= z_2^2 \\ &\dots \\ \alpha^n + \beta^n + \gamma^n + \delta^n &= z_n^2\end{aligned}$$

there always exists a term of the form $2p^{2n}$ in z_{2n} , where p is a non-zero non-negative integer and n is a natural number.

4.4. **Remark.** The following system of m -Diophantine equations with $2k$ non-zero non-negative variables $\alpha, \beta_1, \beta_2, \dots, \beta_n, \gamma$ and non-zero non-negative integers $z_1, z_2, z_3, z_4, \dots, z_m$

$$\begin{aligned}\alpha + \beta_1 + \beta_2 + \dots + \beta_n + \gamma &= z_1^2 \\ \alpha^2 + \beta_1^2 + \beta_2^2 + \dots + \beta_n^2 + \gamma^2 &= z_2^2 \\ &\dots \\ \alpha^m + \beta_1^m + \beta_2^m + \dots + \beta_n^m + \gamma^m &= z_m^2\end{aligned}$$

have the generalized sequential pattern of a non-symmetric solution

$$\langle \alpha, \underbrace{\beta_1, \beta_2, \beta_3, \beta_4, \dots, \beta_n}_{(2k-2) \text{ times}}, \gamma \rangle = \langle p^2, \underbrace{\theta p - p^2, \theta p - p^2, \theta p - p^2, \theta p - p^2, \dots, \theta p - p^2}_{(2k-2) \text{ times}}, (\theta - p)^2 \rangle,$$

where k is greater than or equal to 1 and θ, p be any non-zero non-negative integers.

Open Questions It would be challenging and interesting to determine whether there exist any non-integral rational or irrational solutions or any sequential non-integral patterns. Further, it would be useful for comparison if the solutions to the prescribed problem portrayed by (1.1), (1.2), and (1.3) with four distinct unknowns can be acquired by employing an elliptic curve.

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