

Uzbek Mathematical Journal
 2026, Volume 70, Issue 3, pp. 181–189
 DOI: 10.29229/uzmj.2026-3-19

A Free Boundary Problem for a Diffusion-Convection Equation with Nonlocal Condition

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Abstract. In this paper, we consider a free boundary problem for a diffusion-convection equation with a nonlocal boundary condition. Methods for obtaining a priori estimates in the norms of Hölder spaces for the solution are proposed. On the basis of these estimates, the existence and uniqueness of the solution are proved.

Keywords: free boundary, diffusion-convection, parabolic equation, a priori bounds, existence and uniqueness

MSC (2020): 35K20, 35K59, 35R35

1. INTRODUCTION

Free boundary problems play a crucial role in physical, biological, and mathematical studies. They involve boundary value problems for partial differential equations, where determining an unknown moving boundary is essential [1, 2, 3, 4, 5, 6, 7, 8, 9]. In this paper, we formulate a free boundary problem for a nonlinear diffusion-convection equation, namely the Rosen-Fokas-Yorstsos equation [10, 11, 12]. This equation models fluid diffusion with convective effects in porous media and has numerous applications, including groundwater hydrology, oil reservoir engineering, and various biological contexts, such as drug propagation in arterial tissues.

In [10] a one-phase Stefan problem for the diffusion-convection equation with a particular temperature-dependent thermal conductivity and a particular unidirectional speed was studied. Further articles involving nonlinear convective terms in parabolic equations include [13, 14, 15, 16, 17].

We will study a one-dimensional free boundary problem for the diffusion-convection equation with a nonlocal boundary condition at the fixed face $x = 0$ and a Stefan condition on the free boundary which has a convective term:

$$u_t = u^2 (du_{xx} - u_x), \quad 0 < t \leq T, \quad 0 < x < s(t), \quad (1.1)$$

$$u(0, x) = u_0(x) > \beta, \quad 0 \leq x \leq s_0, \quad (1.2)$$

$$u(t, 0) = mu(t, x_0), \quad 0 \leq t \leq T, \quad (1.3)$$

$$u(t, s(t)) = \beta, \quad 0 \leq t \leq T, \quad (1.4)$$

$$s'(t) = -du_x(t, s(t)) + u(t, s(t)), \quad 0 \leq t \leq T, \quad (1.5)$$

$$s(0) = s_0, \quad 0 < m < 1, \quad 0 < x_0 < s_0, \quad \beta > 0, \quad (1.6)$$

where d is the diffusivity, u_0 is the initial concentration.

Problem (1.1)-(1.6) was posed and studied in [10, 12] for a variable Dirichlet and Neumann boundary conditions instead of the nonlocal condition (1.3). In [10, 12] authors through several transformations the problem reduced to a free boundary problem for a linear heat-diffusion equation. They obtained the existence and uniqueness of solution. Problems with nonlocal conditions can also be found in the works [18, 19, 20].

The goal of this study is to derive a priori estimates for solutions to the one-phase Stefan problem with a nonlocal boundary condition on a fixed boundary, ensuring solvability over any finite time interval and proving the uniqueness of the solution.

The present paper is organized as follows. In Section 2, we use several transformations to reformulate the free boundary problem for the Burgers equation. Section 3 focuses on deriving a priori bounds that will be used later in the analysis. Finally, in Section 4, we establish the local-in-time existence and uniqueness of the solution.

1.1. Reduction of the problem to the problem for the Burgers equation. As in [10], we will transform to problem (1.1)–(1.6) in the one which is governed by the Burgers equation. We have:

$$v(t, z) = u(t, x).$$

where

$$\begin{aligned} z(t, x) &= g_0 + \int_0^t (u(\tau, 0) - du_x(\tau, 0)) d\tau + \int_0^x \frac{1}{u(t, \xi)} d\xi, \\ z_0 = z(t, x_0) &= g_0 + \int_0^t \left(mv(\tau, z_0) - d \frac{v_z(\tau, z_0)}{mv(\tau, z_0)} \right) d\tau + \int_0^{x_0} \frac{1}{u(t, \xi)} d\xi, \\ g(t) = z(t, 0) &= g_0 + \int_0^t \left(mv(\tau, z_0) - d \frac{v_z(\tau, z_0)}{mv(\tau, z_0)} \right) d\tau, \\ h(t) = z(t, s(t)) &= h_0 + (\beta + 1)t - \frac{(\beta + 1)d}{\beta^2} \int_0^t v_z(h(\tau), \tau) d\tau, \end{aligned}$$

with g_0 and h_0 are arbitrary constants, is a solution to the problem given by the Burgers equation

$$v_t = dv_{zz} - 2vv_z, \quad (t, z) \in D, \quad (1.7)$$

$$v(0, z) = v_0(z), \quad g_0 \leq z \leq h_0, \quad (1.8)$$

$$v(t, g(t)) = mv(t, z_0), \quad 0 \leq t \leq T, \quad (1.9)$$

$$v(t, h(t)) = \beta, \quad 0 \leq t \leq T, \quad (1.10)$$

$$h'(t) = -\frac{(\beta + 1)d}{\beta^2} v_z(t, h(t)) + \beta + 1, \quad 0 \leq t \leq T, \quad (1.11)$$

$$g(0) = g_0, \quad h(0) = h_0, \quad (1.12)$$

where $D = \{(t, z) : 0 < t \leq T, g(t) < z < h(t)\}$, $h(t)$ is a free boundary,

$$v_0(z) = u_0(\mu^{-1}(z)), \quad \mu(x) = g_0 + \int_0^x \frac{1}{u_0(\eta)} d\eta, \quad h_0 = g_0 + \int_0^{s_0} \frac{1}{u_0(\eta)} d\eta,$$

and the constants s_0 , g_0 and h_0 satisfy the following relation $s_0 = \int_{g_0}^{h_0} v_0(z) dz$.

Throughout this we assume that:

$$v_0(z) \in C^2[g_0, h_0], \quad v_0(0) = mv(0, z_0), \quad v_0(h_0) = \beta. \quad (1.13)$$

2. A PRIORI ESTIMATES

In this section, we establish several a priori bounds that will be used later. First of all, let us establish the boundedness of the functions $v(t, z)$, $h'(t)$.

Theorem 2.1. *Let $v(t, z)$, $h(t)$ be a solution of (1.7)–(1.12). Then there hold estimates:*

$$0 < v(t, z) \leq M_1, \quad (t, z) \in \overline{D}, \quad (2.1)$$

$$0 < h'(t) \leq M_2, \quad 0 \leq t \leq T. \quad (2.2)$$

Proof: We now turn to prove (2.1). The nonnegativity of the trace of $v(t, 0)$ is established as follows: Assume that, at a certain boundary point $(t_0, g(t_0))$, the function $v(t, z)$ attains its negative minimum $v(t_0, g(t_0)) = -M$. Then, by virtue of condition (1.9), we get

$$-M = v(t_0, g(t_0)) = mv(t_0, z_0) > v(t_0, z_0).$$

Thus, we arrive at a contradiction. So, we show that the function $v(t, x)$ does not have negative values on the left boundary.

Next, applying the strong maximum principle to $v(t, z)$, we get

$$0 < v(t, z) \leq \max v_0(z) \equiv M_1, \quad (t, z) \in \overline{D}.$$

Since the conditions (1.8)-(1.10) and the positivity of the function $v(t, z)$ in the domain D , we find $v_z(t, h(t)) < 0$. It then follows from the free boundary condition in (1.11) that $h'(t) > 0$ for $t \in (0, T)$.

Next we prove an upper bound of $h'(t)$. To do so, we define

$$\Omega = \left\{ (t, z) : 0 < t < T, h(t) - \frac{1}{N} < z < h(t) \right\},$$

where N is a positive constant to be determined later. And we construct an auxiliary function on Ω as follows $w(t, z) = M_1 (2N(h(t) - z) - N^2(h(t) - z)^2)$.

We will choose N so that $w(t, z) \geq v(t, z)$ holds over Ω . Direct calculations show that, for $(t, z) \in \Omega$,

$$\begin{aligned} w_t &= 2M_1 N h'(t) (1 - N(h(t) - z)), \\ -w_z &= 2M_1 N (1 - N(h(t) - z)), \quad -w_{zz} = 2M_1 N^2. \end{aligned}$$

then $ww_z = M_1^2 N^2 (-4(h(t) - z) + 6N(h(t) - z)^2 - 2N^2(h(t) - z)^3)$.

Let $h(t) - z = p$, then $0 < p < \frac{1}{N}$ holds over Ω . By $ww_z = 0$, we obtain $p = 0$ or $p = \frac{-6M_1 \pm \sqrt{36N^2 - 32N^2}}{-4N^2}$, in other words, $p = \frac{2}{N}$ or $p = \frac{1}{N}$. Denote $ww_z = P(p)$, then

$$P'(p) = M_1^2 N^2 (-4 + 12Np - 6N^2 p^2).$$

By solving $P'(p) = 0$, we have $p = \left(1 \pm \frac{\sqrt{3}}{3}\right) \frac{1}{N}$. Since $P(p) = 0$ when $p = 0, \frac{1}{N}, \frac{2}{N}$. Using the properties of cubic function, we know that $P(p)$ reaches the minimum at $p = \left(1 - \frac{\sqrt{3}}{3}\right) \frac{1}{N}$ for $0 < p < \frac{1}{N}$, and $P\left(1 - \frac{\sqrt{3}}{3}\right) = -\frac{4}{9}\sqrt{3}M_1^2 N$.

It follows from $h'(t) > 0$ and (2.1) that

$$w_t - dw_{zz} + ww_z \geq \left(2dN^2 - \frac{4}{9}\sqrt{3}M_1 N\right) M_1 > 0 = v_t - dv_{zz} + vv_z.$$

on Ω , provided that $N \geq \frac{2\sqrt{3}}{9d}M_1$. Also note that

$$\begin{aligned} w\left(t, h(t) - \frac{1}{N}\right) &= M_1 \geq v\left(t, h(t) - \frac{1}{N}\right), \\ w(t, h(t)) &= \beta = v(t, h(t)). \end{aligned}$$

Thus, if we can choose N such that $v_0(z) \leq w(0, z)$ for $z \in [h_0 - \frac{1}{N}, h_0]$, then we can apply the maximum principle to $w - v$ over Ω to deduce that $v(t, z) \leq w(t, z)$ for $(t, z) \in \Omega$. It would then follow that

$$\begin{aligned} v_z(t, h(t)) &\geq w_z(t, h(t)) = -2NM_1, \\ h'(t) &= -\frac{(\beta + 1)d}{\beta^2} v_z(t, h(t)) + \beta + 1 \leq M_2 \equiv \frac{2(\beta + 1)NM_1 d}{\beta^2} + \beta + 1. \end{aligned}$$

To complete the proof, we need only find some N independent of T such that

$$v_0(z) \leq w(0, z) \quad \text{for } z \in \left[h_0 - \frac{1}{N}, h_0\right],$$

then we can apply the maximum principle to $w - v$.

By the definition of w , we have $w(0, z) = M_1 [2N(h_0 - z) - N^2(h_0 - z)^2]$ if $z \in [h_0 - \frac{1}{N}, h_0 - \frac{1}{2N}]$, then $w(0, z) \geq \frac{3}{4}M_1$; if $z \in [h_0 - \frac{1}{2N}, h_0]$, then $w_z(0, z) = -2M_1 N [1 - N(h_0 - z)] \leq -M_1 N$. Therefore upon choosing

$$N = \max \left\{ \frac{2\sqrt{3}}{9d}M_1, \frac{4\|v_0\|_{C^1([0, h_0])}}{3M_1} \right\}$$

we will have $w_z(0, z) \leq v'_0(z)$ for $z \in [h_0 - \frac{1}{2N}, h_0]$.

Since $w(0, h_0) = v_0(h_0) = 0$, the above inequality implies $w(0, z) \geq v_0(z)$ for $z \in [h_0 - \frac{1}{2N}, h_0]$.

Moreover, for $z \in [h_0 - \frac{1}{N}, h_0 - \frac{1}{2N}]$, we have $w(0, z) \geq \frac{3}{4}M_1$, $v_0(z) \leq \frac{\|v_0\|_{C^1([0, h_0])}}{N} \leq \frac{3}{4}M_1$. Therefore $v_0(z) \leq w(0, z)$ for $z \in [h_0 - \frac{1}{N}, h_0]$.

This completes the proof. $\square$

Initial estimates have been received. Now, using the results of the work [21], we establish a priori estimates for v_z and higher derivatives.

To transform the varying domain D into a fixed domain, we introduce the transformation:

$$t = t, \quad y = \frac{z - g(t)}{h(t) - g(t)}.$$

Then the problem (1.7)–(1.12) takes the following form in $Q = \{(t, y) : 0 < t \leq T, 0 < y < 1\}$:

$$V_t - A(t)V_{yy} = B(t, y, V, V_y), \quad (t, y) \in Q, \quad (2.3)$$

$$V(0, y) = V_0(y), \quad 0 \leq y \leq 1, \quad (2.4)$$

$$V(t, 0) = mV(t, y_0), \quad 0 \leq t \leq T, \quad (2.5)$$

$$V(t, 1) = \beta, \quad 0 \leq t \leq T, \quad (2.6)$$

where

$$\begin{aligned} A(t) &= \frac{d}{(h(t) - g(t))^2}, \\ B(t, y, V, V_y) &= \left(\frac{2}{h(t) - g(t)} V + \frac{1 - y(h'(t) - g'(t))}{h(t) - g(t)} \right) V_y, \\ h'(t) &= -\frac{1}{h(t) - g(t)} \frac{d^2(\beta + 1)}{\beta^2} V_y(t, h(t)) + \beta + 1. \end{aligned}$$

Without loss of generality, we may assume that $V(0, y) = 0$.

Theorem 2.2. *Suppose that a function $V(t, y)$ continuous in $\bar{Q}$ satisfies the conditions of problem (2.3)–(2.6). Assume that, for $(t, y) \in \bar{Q}$, $|V| \leq M_1$ and any V_y , continuous functions A and B satisfy the condition*

$$\begin{aligned} A &\geq A_0 > 0, \\ \frac{|B|}{A} &\leq K(V_y^2 + 1), \quad K > 0. \end{aligned} \quad (2.7)$$

Under the conditions (2.7), the following estimate is valid

$$|V_y(t, y)| \leq M_3(M_1, M_2, A_0, K, \delta), \quad (t, y) \in Q^\delta, \quad (2.8)$$

where $Q^\delta = \{(t, y) : 0 < \delta \leq y \leq 1 - \delta, 0 < \delta \leq t \leq T\}$.

In addition, in the domain $\{(t, y) \in \bar{Q}, |V| \leq M_1, |V_y| \leq M_3\}$ we have the estimate

$$|V|_{\frac{2}{3}}^{Q^{2\delta}} \leq M_4(M_1, K, \delta).$$

And if it is also known that $V(t, y)$ possesses in $\bar{Q}$ summable with a square generalized derivatives V_{ty} , V_{yy} , then

$$|V|_{1+\alpha}^{Q^{2\delta}} \leq M_5(M_1, A_0, M_3, K, \delta), \quad 0 < \alpha < 1, \quad (2.9)$$

$$|V|_{2+\alpha}^{Q^{4\delta}} \leq M_6(M_1, A_0, M_3, K, \delta). \quad (2.10)$$

If $0 < x_0 < 4\delta$ and $V(t, 1) = 0$, then the estimates (2.8)–(2.10) are also valid in $\bar{Q}$.

Proof: Since estimates $|v| \leq M_1$, $|h'(t)| \leq M_2$ are established, by virtue of Theorem 3 and Theorem 6 in [21], the inner estimates (2.8)-(2.10) are true. In order to establish estimates up to the left boundary, we act as follows: Choosing δ such that $4\delta \leq y_0$ and using the nonlocal condition (2.5) taking into account (2.10), we show that the bounded function $V_t(t, l)$ satisfies the Hölder condition. In the domain $Q_+ = \{(t, y) : 0 \leq t \leq T, 0 \leq y \leq 4\delta\}$ we now introduce new functions

$$W(t, y) = V(t, y) - g_1(t, y), \quad (2.11)$$

$$g_1(t, y) = \frac{4\delta - y}{4\delta} V(t, 0) + \frac{y}{4\delta} V(t, 4\delta).$$

From (2.11) we get

$$W_y(t, y) = V_y(t, y) - g_2(t), \quad W_{yy}(t, y) = V_{yy}(t, y), \quad W_t(t, y) = V_t(t, y) - g_3(t, y),$$

where

$$g_2(t) = \frac{1}{4\delta} (V(t, 4\delta) - V(t, 0)), \quad g_3(t) = g_{1t}(t, y).$$

Let us refine the smoothness of the functions g_i , $i = \overline{1, 3}$. By virtue of (2.10) and the fact that $V_t(t, l - 4\delta)$ and $V_t(t, l)$ are Hölder, we can assert that the functions $g_i(t, y)$, $i = 1, 2$, have a derivative with respect to t satisfying the Hölder condition, and with respect to the variable y they are linear. And the function $g_3(t, y)$ is bounded, satisfies the Hölder condition in t , and is linear in y .

For the function $W(t, y)$ in the domain Q_+ we get the problem

$$\begin{cases} W_t = dW_{yy} + B_2(W, W_y), & (t, y) \in Q_+, \\ W(0, y) = 0, & 0 \leq y \leq 4\delta, \\ W(t, 0) = 0, & W(t, 4\delta) = 0, \quad 0 \leq t \leq T. \end{cases}$$

Hence, applying Theorem 4 in [21], we find

$$|W|_{1+\alpha}^{Q_+} \leq M_6.$$

Therefore,

$$|V|_{1+\alpha}^{Q_+} \leq M_6. \quad (2.12)$$

Using the method of even continuation across the right boundary [21], we obtain estimates for $|V_y|$, $|V|_{1+\alpha}$, $|V|_{1+\alpha}$ in the domain $Q_- = \{(t, y) : 0 \leq t \leq T, 1 - 4\delta \leq y \leq 1\}$.

Estimates in the domains Q_- , Q_+ , $Q^{4\delta}$ give a general estimate in the closed domain $\bar{Q}$. $\square$

3. UNIQUENESS AND EXISTENCE OF A SOLUTION

Theorem 3.1. *Suppose that the conditions of Theorem 2.2 is satisfied. Then the solution of problem (1.7)–(1.12) is unique.*

Proof: To prove the uniqueness of the solution, we derive an integral representation equivalent to (1.11). Multiply equation (1.7) by z and for any $t \in (0, T)$ integrate over the domain D , and we have

$$\begin{aligned} h^2(t) &= h_0^2 + \frac{2(\beta+1)}{\beta} \int_{g_0}^{h_0} \xi v_0^2(\xi) d\xi - 2 \left(\frac{(\beta+1)^2 d}{\beta} + \beta + 1 \right) \int_0^t h(\tau) d\tau - 2(\beta+1)d \cdot t \\ &+ \frac{2(\beta+1)dm}{\beta} \int_0^t v(\tau, z_0) d\tau + \frac{2(\beta+1)}{\beta} \int_{g(t)}^{h(t)} \xi v^2(t, \xi) d\xi + \frac{2(\beta+1)}{\beta} \int_0^t d\tau \int_0^{h(\tau)} v^2(\tau, \xi) d\xi. \end{aligned} \quad (3.1)$$

The theorem is proved first for small $t \in [0, t_0]$, and then it is established that it is valid for any $0 < t < \infty$. Let $h_1(t), v_1(t, z)$ and $h_2(t), v_2(t, z)$ be solution of (1.7)–(1.12) and, in addition, let $y(t) = \min(h_1(t), h_2(t))$ and $\chi(t) = \max(h_1(t), h_2(t))$. Due to (3.1) we get

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq k_1 \int_0^t |h_1(\tau) - h_2(\tau)| d\tau + k_2 \int_{g(t)}^{y(t)} \xi |v_1^2(t, \xi) - v_2^2(t, \xi)| d\xi \\ &\quad + dm k_2 \int_0^t |v_1(\tau, z_0) - v_2(\tau, z_0)| d\tau + k_2 \int_{y(t)}^{\chi(t)} \xi |v_i^2(t, \xi)| d\xi \\ &\quad + k_2 \int_0^t d\tau \int_{g(\tau)}^{y(\tau)} |v_1^2(\tau, \xi) - v_2^2(\tau, \xi)| d\xi + k_2 \int_0^t d\tau \int_{y(\tau)}^{\chi(\tau)} v_i^2(\tau, \xi) d\xi. \end{aligned} \quad (3.2)$$

where $k_1 = \frac{1}{h_0} \left(\frac{d(\beta+1)^2}{\beta} + \beta + 1 \right)$, $k_2 = \frac{\beta+1}{h_0\beta}$; v_i ($i=1,2$) is a solution between $y(t)$ and $\chi(t)$, i.e.,

$$v_i(t, z) = \begin{cases} v_1(t, z) & \text{if } h_2(t) < h_1(t), \\ v_2(t, z) & \text{if } h_2(t) > h_1(t). \end{cases}$$

By virtue of the established estimates for $h_i(t), v_i(t, z)$, we get

$$0 < v_i(t, z) \leq N(y(t) - z), \quad i = 1, 2,$$

$$|v_1(t, y(t)) - v_2(t, y(t))| \leq N|h_1(t) - h_2(t)|.$$

We now estimate the components of relation (3.2):

$$I_1 = k_2 \int_{g(t)}^{y(t)} \xi |v_1^2(t, \xi) - v_2^2(t, \xi)| d\xi \leq 2M_1 k_2 \int_{g(t)}^{y(t)} \xi |v_1(t, \xi) - v_2(t, \xi)| d\xi, \quad (3.3)$$

$$I_2 = k_2 \int_{y(t)}^{\chi(t)} \xi |v_i^2(t, \xi)| d\xi \leq \frac{M_1 k_2}{2} y(t) |h_1(t) - h_2(t)|^2 + \frac{M_1 k_2}{2} |h_1(t) - h_2(t)|^3, \quad (3.4)$$

$$I_3 = k_2 \int_0^t d\tau \int_{g(\tau)}^{y(\tau)} |v_1^2(\tau, \xi) - v_2^2(\tau, \xi)| d\xi \leq 2M_1 k_2 \int_0^t d\tau \int_{g(\tau)}^{y(\tau)} |v_1(\tau, \xi) - v_2(\tau, \xi)| d\xi, \quad (3.5)$$

$$I_4 = k_2 \int_0^t d\tau \int_{y(\tau)}^{\chi(\tau)} v_i^2(\tau, \xi) d\xi \leq k_2 M_1 t |h_1(t) - h_2(t)|. \quad (3.6)$$

Let $A(t_0) = \max_{0 \leq t \leq t_0} |h_1(t) - h_2(t)| > 0$. If $A(t_0) > 0$, taking into account the obtained inequalities from (3.3)–(3.6), we have:

$$\begin{aligned} (1 - k_1 t_0) A(t_0) &\leq 2M_1 k_2 \int_{g(t_0)}^{y(t_0)} \xi |v_1(t, \xi) - v_2(t, \xi)| d\xi + \frac{M_1 k_2}{2} y(t_0) A^2(t_0) + \frac{M_1 k_2}{2} A^3(t_0) \\ &\quad + 2M_1 k_2 \int_0^{t_0} d\tau \int_{g(\tau)}^{y(\tau)} |v_1(\tau, \xi) - v_2(\tau, \xi)| d\xi + k_2 M_1 t_0 A(t_0), \end{aligned} \quad (3.7)$$

where $t_0 < \frac{1}{k_1}$.

If (3.7) is divided by $A(t_0)$, we get:

$$\begin{aligned} (1 - k_1 t_0) \leq & 2M_1 k_2 \int_{g(t_0)}^{y(t_0)} \frac{\xi |v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi + \frac{M_1 k_2}{2} y(t_0) A(t_0) + \frac{M_1 k_2}{2} A^2(t_0) \\ & + 2M_1 k_2 \int_0^{t_0} d\tau \int_{g(\tau)}^{y(\tau)} \frac{|v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi + k_2 M_1 t_0. \end{aligned} \quad (3.8)$$

Let us rewrite (3.8) in the form

$$\begin{aligned} (1 - k_1 t_0) \leq & 2M_1 k_2 \int_{g(t_0)}^{y(0)} \frac{\xi |v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi + 2M_1 k_2 \int_{y(0)}^{y(t_0)} \frac{\xi |v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi \\ & + \frac{M_1 k_2}{2} y(t_0) A(t_0) + \frac{M_1 k_2}{2} A^2(t_0) + k_2 M_1 t_0 \\ & + 2M_1 k_2 \int_0^{t_0} d\tau \int_{g(\tau)}^{y(0)} \frac{|v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi + 2M_1 k_2 \int_0^{t_0} d\tau \int_{y(0)}^{y(\tau)} \frac{|v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi. \end{aligned} \quad (3.9)$$

and consider the function $V(t, x) = v_1(t, x) - v_2(t, x)$. Then we have the problem

$$\begin{cases} V_t = dV_{zz} + 2v_1 V_z + 2v_2 V, & 0 < t \leq t_0, \quad g_0 < z \leq y_0, \\ V(0, z) = 0, & g_0 \leq z \leq y_0, \\ V(t, g(t)) = mV(t, z_0), & 0 \leq t \leq t_0, \\ V(t, y(t)) \leq NA(t_0), & 0 \leq t \leq t_0. \end{cases}$$

Hence, according to the maximum principle $|V(z, t)| \leq NA(t_0)$.

Let us estimate the integral terms in formula (3.8):

$$\begin{aligned} 2M_1 k_2 \int_{g(t_0)}^{y(t_0)} \frac{\xi |V(t, \xi)|}{A(t_0)} d\xi & \leq 2M_1 k_2 \left(\int_{g(t_0)}^{y(0)} \frac{\xi |V(t, \xi)|}{A(t_0)} d\xi + \int_{y(0)}^{y(t_0)} \frac{\xi |V(t, \xi)|}{A(t_0)} d\xi \right) \\ & \leq 2M_1 k_2 \left(\int_{g(t_0)}^{y(0)} \frac{\xi |V(t, \xi)|}{A(t_0)} d\xi + 2M_1 k_2 \int_{y(0)}^{y(t_0)} \xi d\xi \right) \leq 2M_1 k_2 \int_{g(t_0)}^{y(0)} \frac{\xi |V(t, \xi)|}{A(t_0)} d\xi + M_7 t_0, \\ 2M_1 k_2 \int_0^{t_0} d\tau \int_{g(\tau)}^{y(\tau)} \frac{|V(t, \xi)|}{A(t_0)} d\xi & \leq 2M_1 k_2 \left(\int_0^{t_0} d\tau \int_{g(\tau)}^{y(0)} \frac{|V(t, \xi)|}{A(t_0)} d\xi + \int_0^{t_0} d\tau \int_{y(0)}^{y(\tau)} \frac{|V(t, \xi)|}{A(t_0)} d\xi \right) \\ & \leq 2M_1 k_2 \left(\int_0^{t_0} d\tau \int_{g(\tau)}^{y(0)} \frac{|V(t, \xi)|}{A(t_0)} d\xi + \int_0^{t_0} (y(\tau) - y(0)) d\tau \right) \leq 2M_1 k_2 \int_0^{t_0} d\tau \int_{g(\tau)}^{y(0)} \frac{|V(t, \xi)|}{A(t_0)} d\xi + M_8 t_0^2. \end{aligned}$$

Now let's consider the auxiliary problem:

$$\begin{cases} w_t = dw_{zz} + 2v_1 w_z + 2v_2 w, & 0 < t \leq t_0, \quad g_0 < z \leq y_0, \\ w(0, z) = 0, & g_0 \leq z \leq y_0, \\ w(t, g(t)) = mw(t, z_0), & 0 \leq t \leq t_0, \\ w(t, y_0) = 1, & 0 \leq t \leq t_0, \end{cases}$$

We introduce a function

$$W(t, z) = \frac{V(t, z)}{NA(t_0 \Delta t)} - w(t, z), \quad 0 < t \leq t_0, \quad g_0 < z \leq y_0.$$

This enables us to write

$$\begin{cases} W_t = dW_{zz} + 2v_1 W_z + 2v_2 W, & 0 < t \leq t_0, \quad g_0 < z \leq y_0, \\ W(0, z) = 0, & g_0 \leq z \leq y_0, \\ W(t, g(t)) = mW(t, z_0), & 0 \leq t \leq t_0, \\ W(t, y_0) = \frac{V(t, y_0)}{NA(t_0)} - w(t, y_0), & 0 \leq t \leq t_0. \end{cases}$$

By the maximum principle, we find

$$\frac{V(t, z)}{NA(t_0)} \leq w(t, z), \quad 0 \leq t \leq t_0.$$

Since $\lim_{t \rightarrow 0} W(t, z) = 0$, $g_0 \leq x \leq y_0$, we conclude that

$$\lim_{t \rightarrow 0} \int_{g_0}^{y(0)} W(t, z) dz = 0.$$

Therefore,

$$\lim_{t \rightarrow 0} \int_{g_0}^{y_0} \frac{|V(t, \xi)|}{A(t_0)} d\xi \rightarrow 0.$$

Since the right-hand side of (3.2) tends to zero as $t_0 \rightarrow 0$, then (3.2) does not hold for sufficiently small t_0 and we arrive at a contradiction. Consequently, $h_1(t) \equiv h_2(t)$ and further $v_1(t, z) = v_2(t, z)$, $0 \leq t \leq t_0$.

The uniqueness of the solution to the problem for any $0 < t < \infty$ is established as follows. Let $t_1 = \sup\{t : s_1(\eta) = s_2(\eta), 0 \leq \eta \leq t\}$. If $t_1 = \infty$, then the question will be solved. Otherwise, assuming that the parameter t_1 is bounded and repeating the above calculations in the interval $t_1 \leq t \leq t_1 + \Delta t$, we again arrive at a contradiction. $\square$

Theorem 3.2. *Suppose that the conditions of Theorem 2.2 are satisfied. Then, in Q there exists a solution $v(t, z) \in C^{2+\alpha}(\bar{Q})$, $h(t) \in C^{1+\alpha}(0 \leq t \leq T)$ of problem (1.7)–(1.12).*

Proof: To prove the solvability of a nonlinear problem, you can use various theorems from the theory of nonlinear equations, remembering that the uniqueness theorem of the classical solution is valid for it. Let us use the Leray-Schauder principle [22], established by a priori estimates $|\cdot|_{1+\alpha}$ for all possible solutions of nonlinear problems, and the solvability theorem in Hölder classes for linear problems. A more detailed exposition of the technique can be found, for example, in (Section VI, [5]; Section VII, [22]). $\square$

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