

Regularization of the boundary value problem for an inhomogeneous high-order parabolic equation with one degeneracy line

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Abstract. In this paper, we study an ill-posed initial-boundary value problem for a high-order inhomogeneous parabolic equation with one degeneracy line. We obtain a priori estimate of the solution of the problem, determine the correctness set, and prove the uniqueness and conditional stability of the solution in this set. As a result, we construct a regularized approximate solution in the correctness set according to the theory of A.N. Tikhonov. In this case, an approximate solution is constructed depending on the evenness or oddness of the parameter n , which determines the order of the equation.

Keywords: equation with one line of degeneracy, ill-posed problem, a priori estimate, conditional stability, uniqueness, set of correctness, regularized solution

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1. INTRODUCTION

In this research work, the initial-boundary value problem for the high-order inhomogeneous parabolic equation that changes direction is studied.

Let $Q_T = \{(x, y, t) : (x, y) \in \Omega, 0 < t < T\}$, $\Omega = \{(x, y) : |x| < l, 0 < y < d\}$. We consider the differential equation

$$u_t(x, y, t) + k(x)u_{xx}(x, y, t) + \partial_y^{2n}u(x, y, t) + cu(x, y, t) = f(x, y, t) \quad (1.1)$$

in the region $Q_T \cap \{x \neq 0\}$, where $k(x) = \begin{cases} \alpha, & x > 0, \\ -\beta, & x < 0, \end{cases}$ $\alpha > 0$, $\beta > 0$, c are given real numbers and $f(x, y, t) \in L_2(Q_T)$ is a sufficiently smooth, source function, $n \in N$.

Problem statement. Find a function $u(x, y, t) \in L_2(Q_T)$ in the region $Q_T \cap \{x \neq 0\}$ satisfying equation (1.1) and initial

$$u|_{t=0} = \varphi(x, y), (x, y) \in \bar{\Omega}, \quad (1.2)$$

boundary

$$u|_{x=-l} = u|_{x=l} = 0, 0 \leq y \leq d, 0 \leq t \leq T, \quad (1.3)$$

$$\partial_y^{2j}u|_{y=0} = \partial_y^{2j}u|_{y=d} = 0, j = 0, 1, \dots, (n-1), -l \leq x \leq l, 0 \leq t \leq T, \quad (1.4)$$

gluing

$$u(-0, y, t) = u(+0, y, t), \alpha u_x(-0, x, t) = -\beta u_x(+0, y, t), 0 \leq y \leq d, 0 \leq t \leq T \quad (1.5)$$

conditions, where $\varphi(x, y) \in L_2(\Omega)$ is sufficiently smooth function.

The theory of parabolic equations with changing direction plays an important role in both mathematics and applied sciences. Unlike the classical heat equation, these models describe processes where diffusion and conductivity vary depending on spatial regions or material properties, and in some cases, the direction of diffusion itself may change. Such equations arise naturally in the study of heat transfer in layered or composite materials, in groundwater filtration through soils with variable permeability, and in the modeling of phase transitions in physics. They are also applied in population dynamics, environmental sciences, and even in financial mathematics, where degenerate or direction-changing diffusion appears in certain pricing models. However, such equations lead to serious mathematical difficulties: the classical existence, uniqueness, and stability results for the equations no longer hold, and boundary value problems often become ill-posed in the Hadamard sense.

High-order parabolic equations are of particular interest because they describe more delicate physical processes than second-order models. Examples include higher-order diffusion, vibration of viscoelastic

plates, phase transition phenomena, and other systems where simple diffusion models are insufficient. The inhomogeneous case is especially important, as external forces or source terms inevitably appear in realistic applications. The presence of a degeneracy line in such equations substantially complicates the analysis, since the standard maximum principles, energy estimates, and semigroup methods cannot be applied directly.

The study of equations with changing direction began with the famous work of M. Gevrey[1]. This theory was developed by L. Cattabriga[2], M.S. Baouendi and P. Grisvard[3], C.D. Pagani and G. Talenti[4]. Parabolic equations whose direction changes with time have been extensively studied by C.D. Pagani[5], S.A. Tersenov[6], A.A. Kerefov[7], N.V. Kislov[8], S.G. Pyatkov[9], and other researchers. I.E. Egorov[10], D. Amanov[11], S.V. Popov and V.G. Markov[12] worked with parabolic equations of high-order with changing direction. Most of these works are devoted to the study of the existence and uniqueness of solutions to various boundary value problems.

Analytical calculation of the solution of initial-boundary value problems for high-order equations with changing direction create serious difficulties. Especially in cases where there is no classical solution to such problems, finding solutions based on the regularization approach is of great importance. Important scientific results in this regard were reflected in the scientific research of K.S. Fayazov, I.O. Khajiev, Y.K. Khudaiberganov and E.B. Shobdarov [13, 14, 15, 16, 17], [18]. In their works, they conducted research on higher-order parabolic equations with changing direction, focusing on the derivation of a priori estimates, the proof of uniqueness and stability theorems, and the construction of regularized solutions.

2. EIGENVALUE PROBLEM

Find such values of the parameter λ that the problem

$$k(x) \Phi_{xx}(x, y) + \frac{\partial^{2n} \Phi(x, y)}{\partial y^{2n}} + \lambda \Phi(x, y) = 0, (x, y) \in \Omega, n \in N, \quad (2.1)$$

$$\begin{cases} \Phi(-l, y) = \Phi(l, y) = 0, 0 \leq y \leq d, \\ \frac{\partial^{2i} \Phi(x, 0)}{\partial y^{2i}} = \frac{\partial^{2i} \Phi(x, d)}{\partial y^{2i}} = 0, -l \leq x \leq l, i = 0, 1, \dots, n-1, \\ \Phi(-0, y) = \Phi(+0, y), \quad \beta \Phi_x(-0, y) = -\alpha \Phi_x(+0, y), 0 \leq y \leq d. \end{cases} \quad (2.2)$$

has a nontrivial solution. In the problem (2.1)–(2.2), the differential operator is given by

$$L \equiv k(x) \partial_x^2 + \partial_y^{2n}.$$

This operator is self-adjoint under the boundary and gluing conditions with respect to the variables x and y . Since the coefficients of the operator L , as well as the boundary and gluing conditions (2.2), allow for separation with respect to the variables x and y , the solution can be sought in the form $\Phi(x, y) = X(x)Y(y)$, by applying the method of separation of variables.

Let $\lambda = \mu + \eta$. As a result, we obtain the eigenvalue problems

$$\begin{cases} k(x) X''(x) + \mu X(x) = 0, \\ X(-l) = X(l) = 0, \\ X(-0) = X(+0), \\ \beta X'(-0) = -\alpha X'(+0), \end{cases} \quad (2.3)$$

$$\begin{cases} \frac{\partial^{2n} Y(y)}{\partial y^{2n}} + \eta Y(y) = 0, \\ \frac{\partial^{2k} Y(0)}{\partial y^{2k}} = \frac{\partial^{2k} Y(d)}{\partial y^{2k}} = 0, k = 0, \dots, n-1, \end{cases} \quad (2.4)$$

with respect to functions $X(x)$, $Y(y)$, respectively. The eigenvalue problem 2.3 has positive eigenvalues μ_i^+ and negative eigenvalues μ_i^- , which are solutions of the transcendental equations

$$\sqrt{\beta} \tan \sqrt{\frac{\mu_i^+}{\alpha}} l - \sqrt{\alpha} \tanh \sqrt{\frac{\mu_i^+}{\beta}} l = 0,$$

$$\sqrt{\alpha} \tan \sqrt{\frac{-\mu_i^-}{\beta}} l - \sqrt{\beta} \tanh \sqrt{\frac{-\mu_i^-}{\alpha}} l = 0,$$

respectively, $i \in N$. We emphasize that the numbers μ_i^+ and $-\mu_i^-$ are non-decreasing sequences. The eigenfunctions corresponding to the eigenvalues are expressed in the form

$$X_i^+(x) = \begin{cases} \frac{1}{\sqrt{C(\mu_i^+)}} \frac{\sin \sqrt{\frac{\mu_i^+}{b}}(l+x)}{\cos \sqrt{\frac{\mu_i^+}{b}}l}, & 0 < x < -l, \\ \frac{1}{\sqrt{C(\mu_i^+)}} \sqrt{\frac{b}{a}} \frac{\sinh \left(\sqrt{\frac{\mu_i^+}{a}}(l-x) \right)}{\cosh \sqrt{\frac{\mu_i^+}{a}}l}, & 0 < x < l, \end{cases}$$

$$X_i^-(x) = \begin{cases} -\frac{1}{\sqrt{C(\mu_i^-)}} \sqrt{\frac{a}{b}} \frac{\sinh \sqrt{\frac{-\mu_i^-}{b}}(l+x)}{\cosh \sqrt{\frac{-\mu_i^-}{b}}l}, & -l < x < 0, \\ -\frac{1}{\sqrt{C(\mu_i^-)}} \frac{\sin \sqrt{\frac{-\mu_i^-}{a}}(l-x)}{\cos \sqrt{\frac{-\mu_i^-}{a}}l}, & 0 < x < l, \end{cases}$$

where $C(\mu_i^+) = \frac{l}{2} \left(1 + \tan^2 \sqrt{\frac{\mu_i^+}{\alpha}} l + \frac{\alpha}{\beta} \left(\tanh^2 \sqrt{\frac{\mu_i^+}{\beta}} l - 1 \right) \right)$, $C(\mu_i^-) = \frac{l}{2} \left(1 + \tan^2 \sqrt{\frac{-\mu_i^-}{\beta}} l + \frac{\beta}{\alpha} \left(\tanh^2 \sqrt{\frac{-\mu_i^-}{\alpha}} l - 1 \right) \right)$.

The eigenvalues of the problem (2.4) are determined by formula $\eta_j = (-1)^{n+1} \left(\frac{\pi j}{d} \right)^{2n}$, and the corresponding eigenfunctions

$$Y_j(y) = \sqrt{\frac{2}{d}} \sin \left(\frac{\pi j}{d} y \right), j \in N.$$

Therefore, the eigenvalues of the problem (2.1)-(2.2) are expressed by the formulas

$$\bar{\lambda}_{i,j} = \mu_i^+ + (-1)^{n+1} \left(\frac{\pi j}{d} \right)^{2n}, \quad \tilde{\lambda}_{i,j} = \mu_i^- + (-1)^{n+1} \left(\frac{\pi j}{d} \right)^{2n}$$

and the eigenfunctions are expressed by the formula $\Phi_{i,j}(x, y) = X_i(x) Y_j(y)$, $i, j \in N$, (see [15], p. 123).

Let the dot product be $(u, v) = \int_{-l}^l u v dx$ and the norm be $\|u\|^2 = (u, u)$ in the space $L_2[-l, l]$. Then the eigenfunctions have orthonormal properties

$$\int_{\Omega} \tilde{\Phi}_{i,j} \tilde{\Phi}_{p,q} dx dy = \begin{cases} 1, & i = p \wedge j = q, \\ 0, & i \neq p \vee j \neq q, \end{cases} \quad \int_{\Omega} \bar{\Phi}_{i,j} \bar{\Phi}_{p,q} dx dy = \begin{cases} 1, & i = p \wedge j = q, \\ 0, & i \neq p \vee j \neq q, \end{cases}$$

$$\int_{\Omega} \bar{\Phi}_{i,j} \tilde{\Phi}_{p,q} dx dy = 0.$$

Under the conditions (2.2), the operator L is self-adjoint, and since the domain Ω is bounded, its eigenvalues are discrete. According to the properties of the above eigenfunctions and by the Hilbert–Schmidt theorem, the system of eigenfunctions $\bar{\Phi}_{i,j}(x, y)$ and $\tilde{\Phi}_{i,j}(x, y)$ are complete and orthogonal in the space $L_2(\Omega)$ (see [18], p. 117).

Lemma 2.1. Suppose that a function $u(x, y, t)$ satisfies equation

$$u_t(x, y, t) + k(x) u_{xx}(x, y, t) + \partial_y^{2n} u(x, y, t) + cu(x, y, t) = 0 \quad (2.5)$$

and conditions $u(-l, y, t) = u(l, y, t) = 0$, $\partial_y^{2j} u(x, y, t)|_{y=0} = \partial_y^{2j} u(x, y, t)|_{y=d} = 0$, $j = 0, 1, \dots, (n-1)$, $u(-0, y, t) = u(+0, y, t)$, $\beta u_x(-0, y, t) = -\alpha u_x(+0, y, t)$, then the estimate

$$\|u(x, y, t)\| \leq \sqrt{2}(\|u(x, y, 0)\|)^{1-\frac{t}{T}} (\|u(x, y, T)\|)^{\frac{t}{T}} \quad (2.6)$$

is valid, $t \in (0, T)$.

Proof: According to the properties of the eigenfunctions $\bar{\Phi}_{i,j}(x, y)$, $\tilde{\Phi}_{i,j}(x, y)$, $(i, j = 1, 2, \dots)$ of the problem (2.1)-(2.2), the function $u(x, y, t)$ satisfying the conditions of Lemma 2.1, that is, the solution of equation (2.5), can be written in the form

$$u(x, y, t) = \sum_{i,j=1}^{\infty} \bar{u}_{i,j}(t) \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{u}_{i,j}(t) \tilde{\Phi}_{i,j}(x, y)$$

where the functions $\bar{u}_{i,j}(t)$, $\tilde{u}_{i,j}(t)$, $(i, j = 1, 2, \dots)$ are the solutions of problems

$$\begin{cases} \{\bar{u}_{i,j}(t)\}_t - (\bar{\lambda}_{i,j} - c) \bar{u}_{i,j}(t) = 0, & \{\tilde{u}_{i,j}(t)\}_t - (\tilde{\lambda}_{i,j} - c) \tilde{u}_{i,j}(t) = 0, \\ \bar{u}_{i,j}(0) = \bar{\varphi}_{i,j}(x, y), & \tilde{u}_{i,j}(0) = \tilde{\varphi}_{i,j}(x, y). \end{cases}$$

We know that for each function $\bar{u}_{i,j}(t)$, $\tilde{u}_{i,j}(t)$, the inequalities

$$\{\bar{u}_{i,j}(t)\}^2 \leq \left(\{\bar{u}_{i,j}(0)\}^2\right)^{1-\frac{t}{T}} \left(\{\bar{u}_{i,j}(T)\}^2\right)^{\frac{t}{T}}, \quad \{\tilde{u}_{i,j}(t)\}^2 \leq \left(\{\tilde{u}_{i,j}(0)\}^2\right)^{1-\frac{t}{T}} \left(\{\tilde{u}_{i,j}(T)\}^2\right)^{\frac{t}{T}}$$

hold, respectively (see [19], p. 820). By Hölder's inequality, we have either

$$\begin{aligned} \sum_{i,j=1}^{\infty} \{\bar{u}_{i,j}(t)\}^2 + \sum_{i,j=1}^{\infty} \{\tilde{u}_{i,j}(t)\}^2 &\leq 2 \left(\sum_{i,j=1}^{\infty} \{\bar{u}_{i,j}(0)\}^2 + \sum_{i,j=1}^{\infty} \{\tilde{u}_{i,j}(0)\}^2 \right)^{1-\frac{t}{T}} \times \\ &\times \left(\sum_{i,j=1}^{\infty} \{\bar{u}_{i,j}(T)\}^2 + \sum_{i,j=1}^{\infty} \{\tilde{u}_{i,j}(T)\}^2 \right)^{\frac{t}{T}} \end{aligned}$$

or

$$\|u(x, y, t)\| \leq \sqrt{2}(\|u(x, y, 0)\|)^{1-\frac{t}{T}} (\|u(x, y, T)\|)^{\frac{t}{T}}.$$

□

Lemma 2.2. For the solution $u(x, y, t)$ of the problem (1.1)-(1.5), the inequality

$$\|u(x, y, t)\| \leq \sqrt{2}(\|u(x, y, 0)\| + \gamma)^{1-\frac{t}{T}} (\|u(x, y, T)\| + \gamma)^{\frac{t}{T}} + \gamma \quad (2.7)$$

is valid, where $\gamma = \sqrt{T} e^{|c|(T+t)} \left(\int_0^T \|f(x, y, t)\|^2 dt \right)^{\frac{1}{2}}$, $t \in (0, T)$.

Proof: By introducing substitution $u(x, y, t) = w(x, y, t) e^{-ct}$ into equation (1.1), we can obtain equation

$$w_t(x, y, t) + k(x) w_{xx}(x, y, t) + \frac{\partial^{2n} w(x, y, t)}{\partial y^{2n}} = p(x, y, t),$$

where $p(x, y, t) = f(x, y, t) e^{ct}$, the function $w(x, y, t)$ satisfies conditions $w(-0, y, t) = w(+0, y, t)$, $\beta w_x(-0, y, t) = -\alpha w_x(+0, y, t)$. The solution of equation (2.3) can be expressed in the form

$$w(x, y, t) = \vartheta(x, y, t) + \omega(x, y, t).$$

Where, the functions $\vartheta(x, y, t)$ and $\omega(x, y, t)$ satisfy boundary conditions

$$\vartheta(-l, y, t) = \vartheta(l, y, t) = 0,$$

$$\omega(-l, y, t) = \omega(l, y, t) = 0,$$

and the gluing conditions

$$\vartheta(-0, y, t) = \vartheta(+0, y, t), \quad \beta \vartheta_x(-0, y, t) = -\alpha \vartheta_x(+0, y, t),$$

$$\omega(-0, y, t) = \omega(+0, y, t), \quad \beta \omega_x(-0, y, t) = -\alpha \omega_x(+0, y, t).$$

Let $\vartheta(x, y, t)$ be the solution of the equation

$$\vartheta_t(x, y, t) + k(x) \vartheta_{xx}(x, y, t) + \frac{\partial^{2n} \vartheta(x, y, t)}{\partial y^{2n}} = 0$$

and $\omega(x, y, t)$ be the solution of the equation

$$\omega_t(x, y, t) + k(x) \omega_{xx}(x, y, t) + \frac{\partial^{2n} \omega(x, y, t)}{\partial y^{2n}} = p(x, y, t).$$

Suppose $\vartheta(x, y, t)$ and $\omega(x, y, t)$ exist. Then, according to the results obtained, the solutions $\vartheta(x, y, t)$ and $\omega(x, y, t)$ are expressed as:

$$\begin{aligned} \vartheta(x, y, t) &= \sum_{i,j=1}^{\infty} \bar{\vartheta}_{i,j}(t) \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{\vartheta}_{i,j}(t) \tilde{\Phi}_{i,j}(x, y), \\ \omega(x, y, t) &= \sum_{i,j=1}^{\infty} \bar{\omega}_{i,j}(t) \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{\omega}_{i,j}(t) \tilde{\Phi}_{i,j}(x, y). \end{aligned}$$

Here, each $\bar{\vartheta}_{i,j}(t)$, $\tilde{\vartheta}_{i,j}(t)$, $\bar{\omega}_{i,j}$, $\tilde{\omega}_{i,j}$ are solutions to the following problems, respectively:
 $\bar{\lambda}_{i,j} > 0$ and $\tilde{\lambda}_{i,j} > 0$

$$\begin{cases} \{\bar{\vartheta}_{i,j}(t)\}_t - \bar{\lambda}_{i,j} \bar{\vartheta}_{i,j}(t) = 0, \\ \bar{\vartheta}_{i,j}(0) = \bar{\varphi}_{i,j} + \int_0^T e^{-\bar{\lambda}_{i,j}\tau} \bar{f}_{i,j}(\tau) d\tau, \end{cases} \quad \begin{cases} \{\tilde{\vartheta}_{i,j}(t)\}_t - \tilde{\lambda}_{i,j} \tilde{\vartheta}_{i,j}(t) = 0, \\ \tilde{\vartheta}_{i,j}(0) = \tilde{\varphi}_{i,j} + \int_0^T e^{-\tilde{\lambda}_{i,j}\tau} \tilde{f}_{i,j}(\tau) d\tau, \end{cases}$$

$$\begin{cases} \{\bar{\omega}_{i,j}(t)\}_t - \bar{\lambda}_{i,j} \bar{\omega}_{i,j}(t) = \bar{p}_{i,j}, \\ \bar{\omega}_{i,j}(0) = -\int_0^T e^{-\bar{\lambda}_{i,j}\tau} \bar{f}_{i,j}(\tau) d\tau, \end{cases} \quad \begin{cases} \{\tilde{\omega}_{i,j}(t)\}_t - \tilde{\lambda}_{i,j} \tilde{\omega}_{i,j}(t) = \tilde{p}_{i,j}, \\ \tilde{\omega}_{i,j}(0) = -\int_0^T e^{-\tilde{\lambda}_{i,j}\tau} \tilde{f}_{i,j}(\tau) d\tau, \end{cases}$$

$\bar{\lambda}_{i,j} < 0$ and $\tilde{\lambda}_{i,j} < 0$

$$\begin{cases} \{\bar{\vartheta}_{i,j}(t)\}_t - \bar{\lambda}_{i,j} \bar{\vartheta}_{i,j}(t) = 0, \\ \bar{\vartheta}_{i,j}(0) = \bar{\varphi}_{i,j}, \end{cases} \quad \begin{cases} \{\tilde{\vartheta}_{i,j}(t)\}_t - \tilde{\lambda}_{i,j} \tilde{\vartheta}_{i,j}(t) = 0, \\ \tilde{\vartheta}_{i,j}(0) = \tilde{\varphi}_{i,j}, \end{cases}$$

$$\begin{cases} \{\bar{\omega}_{i,j}(t)\}_t - \bar{\lambda}_{i,j} \bar{\omega}_{i,j}(t) = \bar{p}_{i,j}, \\ \bar{\omega}_{i,j}(0) = 0, \end{cases} \quad \begin{cases} \{\tilde{\omega}_{i,j}(t)\}_t - \tilde{\lambda}_{i,j} \tilde{\omega}_{i,j}(t) = \tilde{p}_{i,j}, \\ \tilde{\omega}_{i,j}(0) = 0, \end{cases}$$

where $\bar{p}_{i,j}(t) = \int_{-l}^l \int_0^d p(x, y, t) \bar{\Phi}(x, y) dx dy$, $\tilde{p}_{i,j}(t) = \int_{-l}^l \int_0^d p(x, y, t) \tilde{\Phi}(x, y) dx dy$, ($i, j = 1, 2, \dots$). The solutions to these problems can be expressed as $\bar{\lambda}_{i,j} > 0$ is

$$\bar{\omega}_{i,j}(t) = -\int_t^T e^{\bar{\lambda}_{i,j}(t-\tau)} \bar{p}_{i,j}(\tau) d\tau,$$

$\bar{\lambda}_{i,j} < 0$ is

$$\bar{\omega}_{i,j}(t) = \int_0^t e^{\bar{\lambda}_{i,j}(t-\tau)} \bar{p}_{i,j}(\tau) d\tau,$$

$\tilde{\lambda}_{i,j} > 0$ is

$$\tilde{\omega}_{i,j}(t) = - \int_t^T e^{\tilde{\lambda}_{i,j}(t-\tau)} \tilde{p}_{i,j}(\tau) d\tau,$$

and $\tilde{\lambda}_{i,j} < 0$ is

$$\tilde{\omega}_{i,j}(t) = \int_0^t e^{\tilde{\lambda}_{i,j}(t-\tau)} \tilde{p}_{i,j}(\tau) d\tau.$$

These results lead to inequality

$$\begin{aligned} \|\omega(x, y, t)\|^2 &= \sum_{i,j=1}^{\infty} \bar{\omega}_{i,j}^2(t) + \sum_{i,j=1}^{\infty} \tilde{\omega}_{i,j}^2(t) = \\ &= \sum_{\bar{\lambda}_{i,j} > 0} \bar{\omega}_{i,j}^2(t) + \sum_{\bar{\lambda}_{i,j} < 0} \bar{\omega}_{i,j}^2(t) + \sum_{\tilde{\lambda}_{i,j} > 0} \tilde{\omega}_{i,j}^2(t) + \sum_{\tilde{\lambda}_{i,j} < 0} \tilde{\omega}_{i,j}^2(t) \leq \\ &\leq \sum_{\bar{\lambda}_{i,j} > 0} \left(\int_0^T e^{\bar{\lambda}_{i,j}(t-\tau)} \bar{p}_{i,j}(\tau) d\tau \right)^2 + \sum_{\bar{\lambda}_{i,j} < 0} \left(\int_0^t e^{\bar{\lambda}_{i,j}(t-\tau)} \bar{p}_{i,j}(\tau) d\tau \right)^2 + \\ &+ \sum_{\tilde{\lambda}_{i,j} > 0} \left(\int_t^T e^{\tilde{\lambda}_{i,j}(t-\tau)} \tilde{p}_{i,j}(\tau) d\tau \right)^2 + \sum_{\tilde{\lambda}_{i,j} < 0} \left(\int_0^t e^{\tilde{\lambda}_{i,j}(t-\tau)} \tilde{p}_{i,j}(\tau) d\tau \right)^2 \leq \\ &\leq T \int_0^T \|p(x, y, t)\|^2 dt \end{aligned} \quad (2.8)$$

We also express the solution of equation (1.1) in the form $u(x, y, t) = h(x, y, t) + g(x, y, t)$, where function $h(x, y, t)$ is the solution of the equation $h_t(x, y, t) + k(x) h_{xx}(x, y, t) + \frac{\partial^{2n} h(x, y, t)}{\partial y^{2n}} + c h(x, y, t) = 0$, and function $g(x, y, t)$ is the solution of the equation $g_t(x, y, t) + k(x) g_{xx}(x, y, t) + \frac{\partial^{2n} g(x, y, t)}{\partial y^{2n}} + c g(x, y, t) = f(x, y, t)$. Substituting $u(x, y, t) = w(x, y, t) e^{-ct}$ and from inequality 2.8 we get

$$\|g(x, y, t)\| \leq \sqrt{T} e^{|c|(T+t)} \left(\int_0^T \|f(x, y, t)\|^2 dt \right)^{\frac{1}{2}}.$$

By the result of Lemma 2.1, inequality (2.6) holds for function $h(x, y, t)$ and taking into account equality $h(x, y, t) = u(x, y, t) - g(x, y, t)$, we obtain inequality

$$\|u(x, y, t)\| \leq \sqrt{2} (\|u(x, y, 0)\| + \gamma)^{1-\frac{t}{T}} (\|u(x, y, T)\| + \gamma)^{\frac{t}{T}} + \gamma,$$

where $\gamma = \sqrt{T} e^{|c|(T+t)} \left(\int_0^T \|f(x, y, t)\|^2 dt \right)^{\frac{1}{2}}$. □

So, we introduce the correctness set as follows:

$$M = \{u(x, y, t) : \|u(x, y, T)\| \leq m, \ m < \infty\}.$$

3. UNIQUENESS AND CONDITIONAL STABILITY THEOREMS

Theorem 3.1. *Let the solution of the problem (1.1)–(1.5) exists in the set M , $u(x, y, t) \in M$. Then it is unique.*

Proof: Let there are two solutions $u_1(x, y, t)$ and $u_2(x, y, t)$ of the problem (1.1) – (1.5). Let $U(x, y, t) = u_1(x, y, t) - u_2(x, y, t)$, then function $U(x, y, t)$ satisfies equation

$$U_t(x, y, t) + k(x)U_{xx}(x, y, t) + \frac{\partial^{2n}U(x, y, t)}{\partial y^{2n}} + cU(x, y, t) = 0$$

and $U(x, y, 0) = 0$ and conditions (1.3)–(1.5). From this, $\gamma = 0$ is deduced, and according to the inequality (2.7), it is established that either $\|U(x, y, t)\| \leq 0$ or $U(x, y, t) \equiv 0$ holds, which leads to $u_1(x, y, t) \equiv u_2(x, y, t)$, thereby proving the uniqueness of the solution of the problem. $\square$

Let $u(x, y, t)$ be the exact solution of the problem (1.1)–(1.5) corresponding to the precisely given initial data, and let $u_\varepsilon^N(x, y, t)$ be the approximate solution of the problem (1.1)–(1.5) corresponding to the approximately given initial data.

Theorem 3.2. *Let the solution of the problem (1.1)–(1.5) exists and $(u(x, y, t), u_\varepsilon(x, y, t)) \in M$. Furthermore, let $\|\varphi(x, y) - \varphi_\varepsilon(x, y)\| \leq \varepsilon$, $\|f(x, y, t) - f_\varepsilon(x, y, t)\| \leq \varepsilon$. Then the inequality*

$$\|u(x, y, t) - u_\varepsilon(x, y, t)\| \leq \sqrt{2}(\varepsilon + \gamma_\varepsilon)^{1 - \frac{t}{T}}(2m + \gamma_\varepsilon)^{\frac{t}{T}} + \gamma_\varepsilon$$

holds, where $\gamma_\varepsilon = Te^{|\varepsilon|(T+t)}\varepsilon$.

Proof: Let $U(x, y, t) = u(x, y, t) - u_\varepsilon(x, y, t)$. The function $U(x, y, t)$ satisfies equation

$$U_t(x, y, t) + k(x)U_{xx}(x, y, t) + \frac{\partial^{2n}U(x, y, t)}{\partial y^{2n}} + cU(x, y, t) = f(x, t) - f_\varepsilon(x, t)$$

in the region $Q_T \cap \{x \neq 0\}$ and initial

$$u(x, y, 0) = \varphi(x, y) - \varphi_\varepsilon(x, y), -l \leq x \leq l, 0 \leq y \leq d,$$

boundary

$$U(-l, y, t) = U(l, y, t) = 0, 0 \leq y \leq d, 0 \leq t \leq T,$$

$$\left. \frac{\partial^{2j}U(x, y, t)}{\partial y^{2j}} \right|_{y=0} = \left. \frac{\partial^{2j}U(x, y, t)}{\partial y^{2j}} \right|_{y=d} = 0, j = 0, 1, \dots, (n-1), -l \leq x \leq l, 0 \leq t \leq T,$$

and gluing

$$U(-0, y, t) = U(+0, y, t), \beta U_x(-0, x, t) = -\alpha U_x(+0, y, t), 0 \leq y \leq d, 0 \leq t \leq T,$$

conditions. Hence, function $U(x, y, t)$ satisfies inequality (2.7),

$$\|U(x, y, t)\| \leq \sqrt{2}(\|U(x, y, 0)\| + \gamma)^{1 - \frac{t}{T}}(\|U(x, y, T)\| + \gamma)^{\frac{t}{T}} + \gamma.$$

In this inequality,

$$\|U(x, y, 0)\| = \|\varphi(x, y) - \varphi_\varepsilon(x, y)\| \leq \varepsilon,$$

$$\|U(x, y, T)\| \leq \|u(x, y, T)\| + \|u_\varepsilon(x, y, T)\| \leq 2m,$$

$$\gamma_\varepsilon = Te^{|\varepsilon|(T+t)}\varepsilon.$$

Then we get

$$\|u(x, y, t) - u_\varepsilon(x, y, t)\| \leq \sqrt{2}(\varepsilon + \gamma_\varepsilon)^{1 - \frac{t}{T}}(2m + \gamma_\varepsilon)^{\frac{t}{T}} + \gamma_\varepsilon.$$

$\square$

4. REGULARIZATION

Let us assume that the solution of the problem (1.1)-(1.5) exists and it can be written in form

$$u(x, y, t) = \sum_{i,j=1}^{\infty} \bar{u}_{i,j}(t) \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{u}_{i,j}(t) \tilde{\Phi}_{i,j}(x, y).$$

We will consider the regularized solution of the problem (1.1)-(1.5) in two cases.

Case 1. Let $f(x, y, t) = 0$, $\|\varphi(x, y) - \varphi_{\varepsilon}(x, y)\| \leq \varepsilon$ and $u(x, y, t) \in M$. Then

$$\begin{cases} \bar{u}_{i,j}(t) = \bar{\varphi}_{i,j} e^{(\bar{\lambda}_{i,j}-c)t}, \\ \tilde{u}_{i,j}(t) = \tilde{\varphi}_{i,j} e^{(\tilde{\lambda}_{i,j}-c)t}. \end{cases}$$

From this equality, the solution of the problem is found in form

$$u(x, y, t) = \sum_{i,j=1}^{\infty} \bar{\varphi}_{i,j} e^{(\bar{\lambda}_{i,j}-c)t} \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{\varphi}_{i,j} e^{(\tilde{\lambda}_{i,j}-c)t} \tilde{\Phi}_{i,j}(x, y).$$

According to the eigenvalues of $\bar{\lambda}_{i,j}$ and $\tilde{\lambda}_{i,j}$, we consider separately when there are odd and even numbers of n :

n odd number. We obtain the approximate solution with respect to the initial data as

$$u^N(x, y, t) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \bar{\varphi}_{i,j} e^{(\bar{\lambda}_{i,j}-c)t} \bar{\Phi}_{i,j}(x, y) + \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} \tilde{\varphi}_{i,j} e^{(\tilde{\lambda}_{i,j}-c)t} \tilde{\Phi}_{i,j}(x, y),$$

and the approximate solution with respect to the initial approximate data as

$$u_{\varepsilon}^N(x, y, t) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \bar{\varphi}_{\varepsilon i,j} e^{(\bar{\lambda}_{i,j}-c)t} \bar{\Phi}_{i,j}(x, y) + \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} \tilde{\varphi}_{\varepsilon i,j} e^{(\tilde{\lambda}_{i,j}-c)t} \tilde{\Phi}_{i,j}(x, y),$$

where N_1, N_2 are regularization parameters.

Using inequality

$$\|u - u_{\varepsilon}^N\| \leq \|u - u^N\| + \|u^N - u_{\varepsilon}^N\|, \quad (4.1)$$

we calculate the norm of the difference between the exact solution and the approximate solution with respect to the initial approximate data.

We evaluate the second term $\|u^N - u_{\varepsilon}^N\|$ of the right-hand side of inequality (4.1) as follows:

$$\begin{aligned} \|u^N - u_{\varepsilon}^N\|^2 &= \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (\bar{\varphi}_{i,j} - \bar{\varphi}_{\varepsilon i,j})^2 e^{2(\bar{\lambda}_{i,j}-c)t} + \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} (\tilde{\varphi}_{i,j} - \tilde{\varphi}_{\varepsilon i,j})^2 e^{2(\tilde{\lambda}_{i,j}-c)t} \leq \\ &\leq \left(e^{2(\bar{\lambda}_{N_1, N_2}-c)t} + e^{2(\tilde{\lambda}_{1, N_2}-c)t} \right) \varepsilon^2. \end{aligned} \quad (4.2)$$

Now, we get

$$\begin{aligned} \|u - u^N\|^2 &= \sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t} + \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{N_2} \tilde{\varphi}_{i,j}^2 e^{2(\tilde{\lambda}_{i,j}-c)t} + \\ &+ \sum_{i=1}^{N_1} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t} + \sum_{i=1}^{\infty} \sum_{j=N_2+1}^{\infty} \tilde{\varphi}_{i,j}^2 e^{2(\tilde{\lambda}_{i,j}-c)t} \end{aligned} \quad (4.3)$$

for the first term. Under condition $\|u(x, y, T)\| \leq m$, the maximum value of each of the series

$$\sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t}, \quad \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{N_2} \tilde{\varphi}_{i,j}^2 e^{2(\tilde{\lambda}_{i,j}-c)t}, \quad \sum_{i=1}^{N_1} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t}, \quad \sum_{i=1}^{\infty} \sum_{j=N_2+1}^{\infty} \tilde{\varphi}_{i,j}^2 e^{2(\tilde{\lambda}_{i,j}-c)t} \text{ in}$$

equality (4.3) is found using the Lagrange multiplier method. First, we evaluate for the series

$\sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t}$. We have

$$\|u(x, y, T)\|^2 = \sum_{i,j=1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)T} + \sum_{i,j=1}^{\infty} \tilde{\varphi}_{i,j}^2 e^{2(\tilde{\lambda}_{i,j}-c)T} \leq m^2.$$

Based on the above information, we construct the Lagrangian function $F(\bar{\varphi}_{i,j}, \xi)$ as follows.

$$F(\bar{\varphi}_{i,j}, \xi) = \sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t} + \xi \left(\sum_{i,j=1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t} - m^2 \right),$$

where ξ is a parameter. Next, $\frac{\partial F(\bar{\varphi}_{i,j}, \xi)}{\partial \bar{\varphi}_{i,j}}$, $\frac{\partial F(\bar{\varphi}_{i,j}, \xi)}{\partial \xi}$ are calculated and equal to zero, and we find $\bar{\varphi}_{i,j}$ in the form

$$\bar{\varphi}_{i,j} = \begin{cases} m e^{-(\bar{\lambda}_{N_1+1, N_2+1}-c)T}, & i = N_1 + 1, j = N_2 + 1, \\ 0, & i \neq N_1 + 1, j \neq N_2 + 1. \end{cases}$$

By calculating the other series in the same way and putting the result in 4.3, we obtain

$$\begin{aligned} \|u - u^N\|^2 &\leq \left(e^{2(\bar{\lambda}_{N_1+1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{N_1+1, 1}-c)(t-T)} + \right. \\ &\quad \left. + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} \right) m^2. \end{aligned} \quad (4.4)$$

By putting the obtained values (4.2) and (4.4) into inequality (4.1), we get

$$\begin{aligned} \|u - u_\varepsilon^N\| &\leq \left(e^{2(\bar{\lambda}_{N_1+1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{N_1+1, 1}-c)(t-T)} + \right. \\ &\quad \left. + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} \right)^{1/2} m + \\ &\quad + \left(e^{2(\bar{\lambda}_{N_1, N_2}-c)t} + e^{2(\bar{\lambda}_{1, N_2}-c)t} \right)^{1/2} \varepsilon. \end{aligned} \quad (4.5)$$

n even number. The approximate solution with respect to the initial data is

$$u^N(x, y, t) = \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} \bar{\varphi}_{i,j} e^{(\bar{\lambda}_{i,j}-c)t} \bar{\Phi}_{i,j}(x, y) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tilde{\varphi}_{i,j} e^{(\tilde{\lambda}_{i,j}-c)t} \tilde{\Phi}_{i,j}(x, y),$$

the approximate solution with respect to the initial approximate data is

$$u_\varepsilon^N(x, y, t) = \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} \bar{\varphi}_{\varepsilon i,j} e^{(\bar{\lambda}_{i,j}-c)t} \bar{\Phi}_{i,j}(x, y) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tilde{\varphi}_{\varepsilon i,j} e^{(\tilde{\lambda}_{i,j}-c)t} \tilde{\Phi}_{i,j}(x, y),$$

where N_1 is regularization parameter. From the second term in inequality (4.1), we get the estimate

$$\begin{aligned} \|u^N - u_\varepsilon^N\|^2 &= \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} (\bar{\varphi}_{i,j} - \bar{\varphi}_{\varepsilon i,j})^2 e^{2(\bar{\lambda}_{i,j}-c)t} + \\ &+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (\tilde{\varphi}_{i,j} - \tilde{\varphi}_{\varepsilon i,j})^2 e^{2(\tilde{\lambda}_{i,j}-c)t} \leq \left(e^{2(\bar{\lambda}_{N_1, 1}-c)t} + e^{2(\bar{\lambda}_{1, 1}-c)t} \right) \varepsilon^2. \end{aligned} \quad (4.6)$$

We find the estimate

$$\|u - u^N\|^2 = \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t} \leq m^2 e^{2(\bar{\lambda}_{N_1+1, 1}-c)(t-T)} \quad (4.7)$$

for the first term as above, using the Lagrange multiplier method. Substituting the estimates (4.6) and (4.7) into inequality (4.1), we obtain the result. The estimate is

$$\|u - u_\varepsilon^N\| \leq m e^{(\bar{\lambda}_{N_1+1,1}-c)(t-T)} + \left(e^{2(\bar{\lambda}_{N_1,1}-c)t} + e^{2(\bar{\lambda}_{1,1}-c)t} \right)^{1/2} \varepsilon. \quad (4.8)$$

Case 2. Let $\varphi(x, y) = 0$, $\|f(x, y, t) - f_\varepsilon(x, y, t)\| \leq \varepsilon$ and $u(x, y, t) \in M$. In this case, we perform the substitution $u(x, y, t) = w(x, y, t) e^{-ct}$ and find the solution of the problem $w(x, y, t)$ in the form

$$w(x, y, t) = \sum_{i,j=1}^{\infty} \bar{w}_{i,j}(t) \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{w}_{i,j}(t) \tilde{\Phi}_{i,j}(x, y)$$

and from the equation

$$\begin{cases} \bar{w}_{i,j}(t) = \int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau, \\ \tilde{w}_{i,j}(t) = \int_0^t \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau, \end{cases}$$

we find the solution of the problem

$$\begin{aligned} w(x, y, t) &= \sum_{i,j=1}^{\infty} \int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \bar{\Phi}_{i,j}(x, y) + \\ &+ \sum_{i,j=1}^{\infty} \int_0^t \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \tilde{\Phi}_{i,j}(x, y), \end{aligned}$$

where $p(x, y, t) = f(x, y, t) e^{ct}$. In this case, we also consider separately the values of $\bar{\lambda}_{i,j}$ and $\tilde{\lambda}_{i,j}$ when n is an odd and even number.

n odd number. The approximate solution with respect to the initial data is

$$\begin{aligned} w^N(x, y, t) &= \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \bar{\Phi}_{i,j}(x, y) + \\ &+ \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} \int_0^t \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \tilde{\Phi}_{i,j}(x, y), \end{aligned}$$

and the approximate solution with respect to the initial approximate data is

$$\begin{aligned} w_\varepsilon^N(x, y, t) &= \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \int_0^t \bar{p}_{\varepsilon i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \bar{\Phi}_{i,j}(x, y) + \\ &+ \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} \int_0^t \tilde{p}_{\varepsilon i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \tilde{\Phi}_{i,j}(x, y). \end{aligned}$$

Using inequality (4.1), we calculate the norm of the difference between the exact solution and the approximate solution with respect to the initial approximations.

We evaluate the expression $\|w^N - w_\varepsilon^N\|$ on the right-hand side of inequality (4.1) as follows:

$$\|w^N - w_\varepsilon^N\|^2 = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \left(\int_0^t (\bar{p}_{i,j}(\tau) - \bar{p}_{\varepsilon i,j}(\tau)) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 +$$

$$+ \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} \left(\int_0^t (\tilde{p}_{i,j}(\tau) - \tilde{p}_{\varepsilon i,j}(\tau)) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \leq \frac{1}{2} \left(\frac{e^{2\tilde{\lambda}_{N_1, N_2} t}}{\tilde{\lambda}_{1,1}} + \frac{e^{2\tilde{\lambda}_{1, N_2} t}}{\sigma_1} \right) \varepsilon^2, \quad (4.9)$$

where $\sigma_1 = \min_{j=1, \dots, N_2} \{|\tilde{\lambda}_{1,j}|\}$.

Now, under condition $\|w(x, y, T)\| \leq me^{cT}$ and using the Lagrange multiplier method, we obtain the estimate

$$\begin{aligned} \|w - w^N\|^2 &= \sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(\int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \\ &+ \sum_{i=1}^{N_1} \sum_{j=N_2+1}^{\infty} \left(\int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{N_2} \left(\int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \\ &+ \sum_{i=1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(\int_0^t \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \leq \\ &\leq \sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(2 \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \left(\int_t^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \right) + \\ &+ \sum_{i=1}^{N_1} \sum_{j=N_2+1}^{\infty} \left(2 \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \left(\int_t^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \right) + \\ &+ \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{N_2} \left(2 \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \left(\int_t^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \right) + \\ &+ \sum_{i=1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(2 \left(\int_0^T \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \left(\int_t^T \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \right) \leq \\ &\leq 2 \sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \sum_{i=1}^{N_1} \sum_{j=N_2+1}^{\infty} \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \\ &+ 2 \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{N_2} \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \sum_{i=1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(\int_0^T \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \\ &+ 2(T-t) \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_t^T \bar{p}_{i,j}^2(\tau) d\tau + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_t^T \tilde{p}_{i,j}^2(\tau) d\tau \right) \leq \\ &\leq 2m^2 \left(e^{2\tilde{\lambda}_{N_1+1, N_2+1}(t-T)+2cT} + e^{2\tilde{\lambda}_{1, N_2+1}(t-T)+2cT} + \right. \\ &\left. + e^{2\tilde{\lambda}_{N_1+1, 1}(t-T)+2cT} + e^{2\tilde{\lambda}_{1, N_2+1}(t-T)+2cT} \right) + 2(T-t) \frac{1}{(N_1 + N_2)^2}. \quad (4.10) \end{aligned}$$

Substituting the obtained estimates (4.9) and (4.10) into inequality (4.1), we get the estimate

$$\|u - u_{\varepsilon}^N\| \leq \left(\frac{e^{2(\tilde{\lambda}_{N_1, N_2} - c)t}}{2\tilde{\lambda}_{1,1}} + \frac{e^{2(\tilde{\lambda}_{1, N_2} - c)t}}{2\sigma_1} \right)^{1/2} \varepsilon +$$

$$\begin{aligned}
& + \sqrt{2}m \left(e^{2(\bar{\lambda}_{N_1+1, N_2+1}-)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-)(t-T)} + \right. \\
& \left. + e^{2(\bar{\lambda}_{N_1+1, 1}-)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-)(t-T)} \right)^{1/2} + \sqrt{2(T-t)} \frac{1}{(N_1 + N_2)}. \quad (4.11)
\end{aligned}$$

n even number. We obtain the approximate solution with respect to the initial data as

$$\begin{aligned}
w^N(x, y, t) &= \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} \int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \bar{\Phi}_{i,j}(x, y) + \\
&+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^t \tilde{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \tilde{\Phi}_{i,j}(x, y),
\end{aligned}$$

and the approximate solution with respect to the initial approximate data as

$$\begin{aligned}
w_{\varepsilon}^N(x, y, t) &= \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} \int_0^t \bar{p}_{\varepsilon i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \bar{\Phi}_{i,j}(x, y) + \\
&+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^t \tilde{p}_{\varepsilon i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \tilde{\Phi}_{i,j}(x, y).
\end{aligned}$$

From the second sum in inequality (4.1), we obtain an estimate of the form

$$\begin{aligned}
\|w^N - w_{\varepsilon}^N\|^2 &= \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} \left(\int_0^t (\bar{p}_{i,j}(\tau) - \bar{p}_{\varepsilon i,j}(\tau)) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \\
&+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\int_0^t (\tilde{p}_{i,j}(\tau) - \tilde{p}_{\varepsilon i,j}(\tau)) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \leq \\
&\leq \left(\frac{e^{2\bar{\lambda}_{N_1,1}t}}{2\sigma_2} + \frac{e^{2\bar{\lambda}_{1,1}t}}{2|\bar{\lambda}_{1,1}|} \right) \varepsilon^2. \quad (4.12)
\end{aligned}$$

where $\sigma_2 = \min_{i=1, \dots, N_1} \{|\bar{\lambda}_{i,1}|\}$.

We also find $\|w - w^N\|$ using the Lagrange multiplier method as above, and obtain the estimate

$$\begin{aligned}
\|w - w^N\|^2 &= \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{\infty} \left(\int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \leq \\
&\leq m^2 e^{2\bar{\lambda}_{N_1+1,1}(t-T)+2cT}. \quad (4.13)
\end{aligned}$$

We substitute the estimates (4.12) and (4.13) with inequality (4.1) and obtain the result

$$\|u - u_{\varepsilon}^N\| \leq \left(\frac{e^{2(\bar{\lambda}_{N_1,1}-c)t}}{2\sigma_2} + \frac{e^{2(\bar{\lambda}_{1,1}-c)t}}{2|\bar{\lambda}_{1,1}|} \right)^{1/2} \varepsilon + m e^{(\bar{\lambda}_{N_1+1,1}-c)(t-T)}. \quad (4.14)$$

By minimizing the above results, which are the two cases we choose as $f(x, y, t) = 0$ and $\varphi(x, y) = 0$, and the right-hand side of the evaluations for the even and odd numbers n in them, the available regularization parameter N is found.

For the first case, we show that when $\|\varphi(x, y) - \varphi_\varepsilon(x, y)\| \leq \varepsilon$, it is possible to choose the regularization parameter N such that $\|u - u_\varepsilon^N\| \leq C\varepsilon$, where C is a constant. From estimate (4.5), as $\varepsilon \rightarrow 0$, there exists such an N that the following expression tends to zero:

$$D(t, \varepsilon) = \left(e^{2(\bar{\lambda}_{N_1+1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{N_1+1, 1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} \right)^{1/2} m + \\ + \left(e^{2(\bar{\lambda}_{N_1, N_2}-c)t} + e^{2(\bar{\lambda}_{1, N_2}-c)t} \right)^{1/2} \varepsilon.$$

Then one can show that

$$\lim_{\varepsilon \rightarrow 0} D(t, \varepsilon) = 0. \quad (4.15)$$

Assume that δ is a sufficiently small positive number. From the relation

$$\lim_{N \rightarrow \infty} J(t, m, N) = 0,$$

it follows that there exists $N(\delta)$ such that for all $N \geq N(\delta)$,

$$J(t, m, N) \leq \frac{\delta}{2},$$

where

$$J(t, m, N) = \left(e^{2(\bar{\lambda}_{N_1+1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{N_1+1, 1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} \right)^{1/2} m.$$

Now define $Z(\delta) = \inf_{N \geq N(\delta)} \left\{ \left(e^{2(\bar{\lambda}_{N_1, N_2}-c)t} + e^{2(\bar{\lambda}_{1, N_2}-c)t} \right)^{1/2} \right\}$. If

$$\varepsilon \leq \frac{1}{2} \cdot \frac{\delta}{Z(\delta)},$$

then the function $D(t, \varepsilon)$ satisfies the inequality $D(t, \varepsilon) \leq \delta$. This proves relation (4.15). Using the same approach, and relying on estimates (4.8), (4.11), and (4.14), one can similarly choose the parameter N depending on ε .

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