

## Approximate Solution for Two-Dimensional Free Surface Flow Using Perturbation Methods

Amroune B., Gasmi A.

**Abstract.** The two-dimensional free-surface flow of an inviscid and incompressible fluid passing over a triangular hollow at the bottom of a channel is studied in this paper. The flow is considered irrotational and steady. The effect of surface tension is taken into account, while the force of gravity is neglected. The region in the potential  $W$  plane is mapped onto the auxiliary  $\lambda$  plane using the Schwarz–Christoffel transformation. The primary objective of this research is to provide an approximate solution for this problem using the perturbation approach together with the Hilbert transform; the results are computed at high Weber numbers and small inclination interior angles. The solution is provided and illustrated up to first-order approximation. The shape of the free surface for different values of: the Weber number  $\alpha$ , the depth of the hollow  $l$  and the inclination interior angles  $\gamma$  and  $\delta$  are plotted. The findings demonstrate the simplicity of the method used and give an approximate solution for similar problems.

**Keywords:** Surface tension, conformal mapping, Weber number, Hilbert’s transforms, perturbation technique

**MSC (2020):** 76B10, 30E25, 30C20, 76M45

### 1. INTRODUCTION

The phenomenon of fluid flow over an irregular terrain when there is a free surface has significant theoretical and practical interest. This class of phenomena, often referred to as Neumann-Kelvin problems, is distinguished by the nonlinear boundary conditions on the free surface, as dictated by the Bernoulli equation and as the free surface’s shape is undetermined, it must be taken into account when solving the problem. For more information about this subject see: Bhimsen & Shivamoggi [1], Batchelor [2]. Different methods have been proposed to solve this type of problem numerically, such as the boundary integral equation method, series truncation method, relaxation method and Hilbert transformation. Various researchers have studied the free-surface topics numerically considering the influence of gravity, surface tension or both, such as Doak and Vanden-Broeck [3], studied the problem of flow past a vertical flat plate in a channel by using series truncation, Laiadi and Merzougui [4], solved the problem of the flows over successive obstacles.

Several authors have successfully used the Hilbert method in tandem with the perturbation approach, for example: M. B. Abd-el-Malek & S. N. Hanna [5], resolved the issue of a flow over a ramp with gravity effect, M. B & Abd-el-Malek [6], found an Approximate solution of gravity-affected flow from planar sluice gate, M. M. Bounif and A. Gasmi [7], determined the numerical solution for free surface flow over a step and others.

In this paper, we have considered the issue of two dimensional, irrotational and stationary flow of a fluid passing over a triangular hollow at the bottom of a channel. It is believed that the liquid is both inviscid and incompressible. In this work, gravity and viscosity are neglected in order to study a free-surface flow dominated by surface tension. Under these conditions, surface tension governs the interface dynamics and controls the free-surface shape through curvature effects. This simplification permits analytical treatment. However, since the flow is assumed inviscid and irrotational, viscous boundary layers and possible vortical motions inside the cavity are not described in the present formulation. Under the stipulated assumptions, we derive simplifications for the governing Bernoulli’s equation, enabling the application of complex variable theory, specifically the properties inherent to analytical functions.

The objective of this study is to find the numerical solution to the specified problem, characterized by high Weber number, by employing the Hilbert method for a mixed boundary value problem within this novel plane, thus finding a system of nonlinear integral equations and the perturbation technique

to solve the obtained system.

This paper is organized according to this plan: First, section 2 explains the mathematical formulation of the problem in dimensionless variables. Second, in section 3 we derive the approximate equations of the problem. Next, section 4 we find the numerical solution by using the perturbation technique. Then, section 5 provides a graphical representation and discussion. Finally, this study is concluded in section 6.

## 2. PROBLEM FORMULATION

Let us examine the movement of two-dimensional, irrotational and steady flow of an incompressible and non-viscous fluid. The influence of surface tension is taken into account. Far upstream and downstream, the flow is proposed as uniform with constant flux:  $U_1 h_1 = U_2 h_2$ , where  $U_i, i = 1, 2$  and  $h_i, i = 1, 2$  are velocities and depths of the flow upstream and downstream, respectively. The bottom is composed of inclined walls  $BC, CD$  and horizontal walls  $AB, DE$ , the lengths of the step-inclined walls  $BC, CD$  and its inclination angles are indicated by the numbers  $r, d, \gamma$  and  $\delta$  respectively. As reference, we designate the point  $C$  as the origin in the  $(x', y')$ -plane, The  $x'$ -axis from left to right and the  $y'$ -axis is vertically at the point  $C$  (see Figure 1) .

We introduce the two harmonic functions  $\psi'$  and  $\phi'$ . Next, we introduce the complex potential

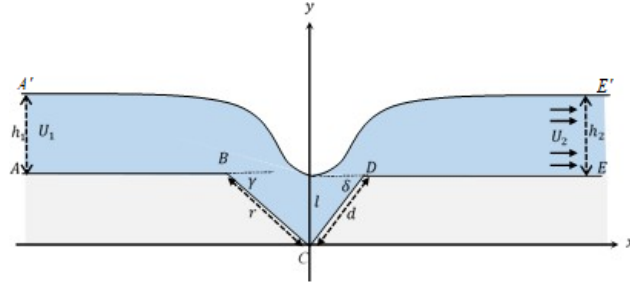


FIGURE 1. Sketch of the flow and of the coordinates.

function:

$$W' = \phi' + i\psi',$$

where  $\psi'$  and  $\phi'$  are the stream function and the potential function respectively.

Within the flow space, the function  $W'(z')$  represents an analytical function of  $z' = x' + iy'$ .

The  $x'$  and  $y'$  directions' velocity components are denoted by  $u'$  and  $v'$ , then:

$$\frac{dW'}{dz'} = u' - iv' = q' e^{-i\theta}. \quad (2.1)$$

Under the assumption of irrotationality and incompressibility, the 2D steady motion of an inviscid fluid is governed by Laplace's equation for the velocity potential in the flow domain [2],

$$\Delta\phi' = 0,$$

When the pressure is uniform, Bernoulli's equation along the free surface  $A'E'$  has the following form:

$$\frac{1}{2}q'^2 + \frac{p'}{\rho} = \frac{1}{2}U_1^2 + \frac{p_a}{\rho}, \quad (2.2)$$

where  $\rho$  is the fluid's density and  $p$  is the fluid pressure.

Using the variables without dimensions, we make the analysis simpler by selecting  $h_1$  as the unit length and  $U_1$  as the unit velocity. The new dimensionless variables are as follows

$$z = \frac{z'}{h_1}, \quad q = \frac{q'}{U_1}, \quad W = \frac{W'}{U_1 h_1}. \quad (2.3)$$

From Sekhri et al [8], equation (2.2) can be written in dimensionless form

$$q^2 + \frac{2}{\alpha} \left| \frac{\partial \theta}{\partial \phi} \right| q = 1, \quad (2.4)$$

where the Weber number, denoted by  $\alpha$ , is defined by:

$$\alpha = \frac{\rho U_1^2 h_1}{T}. \quad (2.5)$$

$T$  represent the surface tension.

By using (2.3), the complex conjugate velocity in (2.1) has the following form:

$$\zeta = \frac{dW}{dz} = \frac{1}{U_1} \frac{dW'}{dz'} = \frac{q'}{U_1} e^{-i\theta} = q e^{-i\theta}. \quad (2.6)$$

Moreover, when utilizing the logarithmic hodograph variable

$$\omega = \ln \zeta = \ln q - i\theta. \quad (2.7)$$

from (2.6) and (2.7), we get

$$z = \int e^{-\omega} dW. \quad (2.8)$$

For the streamline  $ABCDE$ , we choose  $\psi = 0$ .  $\psi = 1$  on the streamline  $A'E'$  can be inferred from the choosing of dimensionless variables.

The flow space in the  $W$ -plane is mapped onto the top half of an auxiliary  $\lambda$ -plane using the transformation of Schwarz-Christoffel, see Figures 2 and 3.

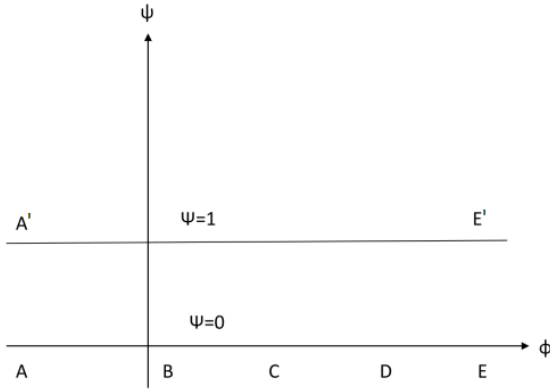


FIGURE 2. The normalized complex a potential-plane.

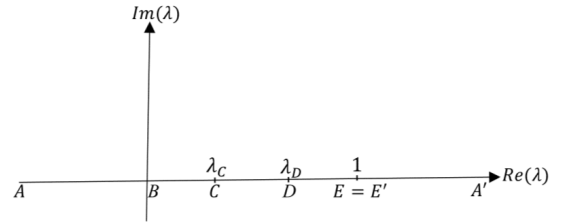


FIGURE 3. The upper a half plane,  $\lambda$ -plane.

The obtained function is

$$W(\lambda) = -\frac{1}{\pi} \ln(1 - \lambda). \quad (2.9)$$

We now introduce the transformation of Hilbert for a mixed boundary-value problem on the  $\lambda$ -plane, which allows us to describe  $W$  as a function of  $\lambda$ , the following provides the Hilbert problem's general solution:

$$\chi(\lambda) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{Im[\chi(\tau)]}{\tau - \lambda} d\tau + \sum_{j=0}^{\infty} R_j \lambda^j. \quad (2.10)$$

where  $R_j$  and  $p.v.$  represent real constants and the integral's principal value respectively.

We can derive a relation between  $\omega(\lambda)$  and  $\chi(\lambda)$  by introducing an auxiliary function  $H(\lambda)$ . In our

issue, we choose:

$$H(\lambda) = \begin{cases} \sqrt{1-\lambda}, & \lambda < 1, \\ -i\sqrt{\lambda-1}, & \lambda > 1. \end{cases} \quad (2.11)$$

With

$$\chi(\lambda) = \frac{\omega(\lambda)}{H(\lambda)} = \frac{\ln q(\lambda) - i\theta(\lambda)}{H(\lambda)} = U(\lambda) + iV(\lambda), \quad (2.12)$$

the value of  $\theta$  is

$$\theta(\lambda) = \begin{cases} 0 & \lambda < 0, \\ -\gamma & 0 < \lambda < \lambda_c, \\ \delta & \lambda_c < \lambda < \lambda_d, \\ 0 & \lambda_d < \lambda < 1, \\ \theta(\lambda) & \lambda > 1. \end{cases} \quad (2.13)$$

Using (2.11), (2.12) and (2.13), we have

$$Im(\chi(\lambda)) = \begin{cases} 0 & \lambda < 0, \\ \frac{\gamma}{\sqrt{1-\lambda}} & 0 < \lambda < \lambda_c, \\ -\frac{\delta}{\sqrt{1-\lambda}} & \lambda_c < \lambda < \lambda_d, \\ 0 & \lambda_d < \lambda < 1, \\ \frac{\ln q(\lambda)}{\sqrt{\lambda-1}} & \lambda > 1. \end{cases} \quad (2.14)$$

We look at the situation upstream. Thus, (2.10) takes the form

$$\chi(\lambda) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{Im[\chi(\tau)]}{\tau - \lambda} d\tau, \quad (2.15)$$

from (2.12) and (2.15), we obtain the equations of  $U(\lambda)$  and  $V(\lambda)$ :

$$U(\lambda) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{V(\tau)}{\tau - \lambda} d\tau. \quad (2.16)$$

And

$$V(\lambda) = \frac{-1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{U(\tau)}{\tau - \lambda} d\tau. \quad (2.17)$$

We create a system of integral equations by using (2.16) and (2.17). The equations below are all that we need for our study:

$$\frac{\theta(\lambda)}{\sqrt{\lambda-1}} = \frac{1}{\pi} \int_1^{\infty} \frac{\ln q(\tau)}{(\tau - \lambda)\sqrt{\tau-1}} d\tau + \frac{\gamma}{\pi} \int_0^{\lambda_c} \frac{1}{(\tau - \lambda)\sqrt{1-\tau}} d\tau - \frac{\delta}{\pi} \int_{\lambda_c}^{\lambda_d} \frac{1}{(\tau - \lambda)\sqrt{1-\tau}} d\tau, \quad \lambda > 1, \quad (2.18)$$

and

$$\frac{\ln(q_{k_i}(\lambda))}{\sqrt{1-\lambda}} = \frac{1}{\pi} \int_1^{\infty} \frac{\ln q(\tau)}{(\tau - \lambda)\sqrt{\tau-1}} d\tau + \frac{\gamma}{\pi} \int_0^{\lambda_c} \frac{1}{(\tau - \lambda)\sqrt{1-\tau}} d\tau - \frac{\delta}{\pi} \int_{\lambda_c}^{\lambda_d} \frac{1}{(\tau - \lambda)\sqrt{1-\tau}} d\tau, \quad \lambda < 1, \quad (2.19)$$

where  $q_{k_i}$ ,  $i = 1, 2, 3, 4$  are defined in  $\lambda < 1$ .

By integration, (2.18) and (2.19) take the forms

$$\begin{aligned} \theta(\lambda) = & \frac{\sqrt{\lambda-1}}{\pi} \int_1^{\infty} \frac{\ln q(\tau)}{(\tau - \lambda)\sqrt{\tau-1}} d\tau + \frac{2\gamma}{\pi} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right) \\ & - \frac{2\delta}{\pi} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \quad \lambda > 1, \end{aligned}$$

and

$$\ln(q_{k_i}(\lambda)) = \frac{\sqrt{1-\lambda}}{\pi} \int_1^\infty \frac{\ln q(\tau)}{(\tau-\lambda)\sqrt{\tau-1}} d\tau + \frac{1}{\pi} \ln \left( \frac{\left[ \frac{(\sqrt{1-\lambda_c}-\sqrt{1-\lambda})(1+\sqrt{1-\lambda})}{(\sqrt{1-\lambda_c}+\sqrt{1-\lambda})(1-\sqrt{1-\lambda})} \right]^\gamma}{\left[ \frac{(\sqrt{1-\lambda_d}-\sqrt{1-\lambda})(\sqrt{1-\lambda_c}+\sqrt{1-\lambda})}{(\sqrt{1-\lambda_d}+\sqrt{1-\lambda})(\sqrt{1-\lambda_c}-\sqrt{1-\lambda})} \right]^\delta} \right), \quad \lambda < 1$$

using (2.8) and (2.9) to find the coordinates of  $z$  on  $A'E'$ , we get:

$$z(\lambda) = z_\infty - \frac{1}{\pi} \int_\lambda^\infty \frac{e^{i\theta(\tau)}}{(1-\tau)q(\tau)} d\tau, \quad \lambda > 1,$$

dividing imaginary and real part, we obtain:

$$x(\lambda) = x_\infty - \frac{1}{\pi} \int_\lambda^\infty \frac{\cos \theta(\tau)}{(1-\tau)q(\tau)} d\tau, \quad \lambda > 1,$$

$$y(\lambda) = 1 + l - \frac{1}{\pi} \int_\lambda^\infty \frac{\sin \theta(\tau)}{(1-\tau)q(\tau)} d\tau, \quad \lambda > 1.$$

Along  $BC'$ , we have (  $0 < \lambda < \lambda_c$ ,  $q = q_{k_2}$  and  $\theta = -\gamma$  ), then we obtain:

$$r = \frac{1}{\pi} \int_0^{\lambda_c} \frac{1}{(1-\tau)q_{k_2}(\tau)} d\tau. \quad (2.20)$$

Along  $CD$ , we have (  $\lambda_c < \lambda < \lambda_d$ ,  $q = q_{k_3}$  and  $\theta = \delta$  ), then we obtain

$$d = \frac{1}{\pi} \int_{\lambda_c}^{\lambda_d} \frac{1}{(1-\tau)q_{k_3}(\tau)} d\tau, \quad (2.21)$$

using (2.21), we get:

$$l = \frac{\sin \delta}{\pi} \int_{\lambda_c}^{\lambda_d} \frac{1}{(1-\tau)q_{k_3}(\tau)} d\tau. \quad (2.22)$$

### 3. THE APPROXIMATE EQUATIONS

The solution to equation (2.4) is provided, when the Weber number is high and first-order Taylor development is applied, the result is:

$$q(\lambda) \approx 1 - \frac{1}{\alpha} \left| \frac{\partial \theta}{\partial \phi} \right|,$$

with

$$\left| \frac{\partial \theta}{\partial \phi} \right| = \pi(\lambda - 1) \left| \frac{\partial \theta}{\partial \lambda} \right|, \quad \lambda > 1.$$

Then

$$q(\lambda) \approx 1 - \frac{\pi}{\alpha} (\lambda - 1) \left| \frac{\partial \theta}{\partial \lambda} \right|, \quad \lambda > 1,$$

$$\ln(q(\lambda)) \approx -\frac{\pi}{\alpha} (\lambda - 1) \left| \frac{\partial \theta}{\partial \lambda} \right|, \quad \lambda > 1,$$

$$\frac{1}{q(\lambda)} \approx 1 + \frac{\pi}{\alpha} (\lambda - 1) \left| \frac{\partial \theta}{\partial \lambda} \right|, \quad \lambda > 1.$$

Slight interior degrees of inclination cause a slight variance in  $\theta$ , As a result,  $\theta$  can approximate  $\sin \theta$ , while unity can approximate  $\cos \theta$ , we obtain the approximate equations

$$q(\lambda) \approx 1 - \frac{\pi}{\alpha} (\lambda - 1) \frac{\partial \theta}{\partial \lambda}, \quad \lambda > 1,$$

$$\begin{aligned}
\theta(\lambda) &\approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^\infty \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta}{\partial \tau}(\tau) d\tau + \frac{2\gamma}{\pi} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right) \\
&\quad - \frac{2\delta}{\pi} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \quad \lambda > 1, \\
x(\lambda) &\approx x_\infty - \frac{1}{\pi} \int_\lambda^\infty \frac{1}{1-\tau} d\tau - \frac{\theta(\lambda)}{\alpha}, \quad \lambda > 1, \\
y(\lambda) &\approx 1+l - \frac{1}{\pi} \int_\lambda^\infty \frac{\theta(\tau)}{1-\tau} d\tau - \frac{\theta^2(\lambda)}{2\alpha}, \quad \lambda > 1, \\
l &\approx \frac{\delta}{\pi} \int_{\lambda_c}^{\lambda_d} \frac{1}{(1-\tau)q_3(\tau)} d\tau.
\end{aligned}$$

Along the solid boundaries  $BC$  and  $CD$ , we can approximate  $q_{k_i}$  by one, (2.20), (2.21) and (2.22) become:

$$r \approx \frac{1}{\pi} \ln \left( \frac{1}{1-\lambda_c} \right), \quad d \approx \frac{1}{\pi} \ln \left( \frac{1-\lambda_c}{1-\lambda_d} \right), \quad \text{and, the height } l \approx \frac{\delta}{\pi} \ln \left( \frac{1-\lambda_c}{1-\lambda_d} \right).$$

The perturbation method is used to solve this resulting nonlinear system.

#### 4. PERTURBATION TECHNIQUE

This series represents the expansion of  $X(\lambda)$  in terms of the tiny parameters  $\gamma$  and  $\delta$ :

$$X(\lambda) = X_0(\lambda) + \sum_{m=1}^{\infty} (\gamma^m X_{m,\gamma}(\lambda) + \delta^m X_{m,\delta}(\lambda)), \quad (4.1)$$

in this case,  $X(\lambda)$  represents  $q(\lambda)$ ,  $\theta(\lambda)$ ,  $x(\lambda)$  and  $y(\lambda)$ .

**4.1. Zeroth-order approximation.** The situation of flow is corresponding to this zeroth-order approximation ( $\gamma = \delta = 0$ ). Then,

$$q_0(\lambda) \approx 1 - \frac{\pi}{\alpha}(\lambda-1) \frac{\partial \theta_0}{\partial \lambda}(\lambda), \quad \lambda > 1, \quad (4.2)$$

$$\theta_0(\lambda) \approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^\infty \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta_0}{\partial \tau}(\tau) d\tau, \quad \lambda > 1, \quad (4.3)$$

$$x_0(\lambda) \approx x_\infty - \frac{1}{\pi} \int_\lambda^\infty \frac{1}{1-\tau} d\tau - \frac{\theta_0(\lambda)}{\alpha}, \quad \lambda > 1, \quad (4.4)$$

$$y_0(\lambda) \approx 1+l - \frac{1}{\pi} \int_\lambda^\infty \frac{\theta_0(\tau)}{1-\tau} d\tau - \frac{\theta_0^2(\lambda)}{2\alpha}, \quad \lambda > 1, \quad (4.5)$$

**4.2. First-order approximation.** The first order's can be found by using the series (4.1), as well as the equations (4.2), (4.3), (4.4) and (4.5). we get

$$\begin{aligned}
q_{1,\gamma}(\lambda) &\approx \frac{-\pi}{\alpha}(\lambda-1) \frac{\partial \theta_{1,\gamma}}{\partial \lambda}(\lambda), \quad \lambda > 1, \\
q_{1,\delta}(\lambda) &\approx \frac{-\pi}{\alpha}(\lambda-1) \frac{\partial \theta_{1,\delta}}{\partial \lambda}(\lambda), \quad \lambda > 1.
\end{aligned}$$

$$\theta_{1,\gamma}(\lambda) \approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^\infty \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta_{1,\gamma}}{\partial \tau}(\tau) d\tau + \frac{2}{\pi} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right), \quad \lambda > 1 \quad (4.6)$$

$$\theta_{1,\delta}(\lambda) \approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^\infty \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta_{1,\delta}}{\partial \tau}(\tau) d\tau - \frac{2}{\pi} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{1-\lambda_c}\sqrt{1-\lambda_d}} \right), \quad \lambda > 1 \quad (4.7)$$

$$\begin{aligned}
x_{1,\gamma}(\lambda) &\approx \frac{-1}{\alpha} \theta_{1,\gamma}(\lambda), \quad \lambda > 1, \\
x_{1,\delta}(\lambda) &\approx \frac{-1}{\alpha} \theta_{1,\delta}(\lambda), \quad \lambda > 1. \\
y_{1,\gamma}(\lambda) &\approx \frac{-1}{\pi} \int_{\lambda}^{\infty} \frac{1}{1-\tau} \theta_{1,\gamma}(\tau) d\tau - \frac{1}{2\alpha} \theta_{1,\gamma}^2(\lambda), \quad \lambda > 1. \\
y_{1,\delta}(\lambda) &\approx \frac{-1}{\pi} \int_{\lambda}^{\infty} \frac{1}{1-\tau} \theta_{1,\delta}(\tau) d\tau - \frac{1}{2\alpha} \theta_{1,\delta}^2(\lambda), \quad \lambda > 1.
\end{aligned}$$

**4.3. Nth-order approximation.** For  $n \geq 2$ , we get

$$\begin{aligned}
q_{n,\gamma}(\lambda) &\approx \frac{-\pi}{\alpha} (\lambda - 1) \frac{\partial \theta_{n,\gamma}(\lambda)}{\partial \lambda}, \quad \lambda > 1, \\
q_{n,\delta}(\lambda) &\approx \frac{-\pi}{\alpha} (\lambda - 1) \frac{\partial \theta_{n,\delta}(\lambda)}{\partial \lambda}, \quad \lambda > 1. \\
\theta_{n,\gamma}(\lambda) &\approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^{\infty} \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta_{n,\gamma}}{\partial \tau}(\tau) d\tau, \quad \lambda > 1, \\
\theta_{n,\delta}(\lambda) &\approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^{\infty} \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta_{n,\delta}}{\partial \tau}(\tau) d\tau, \quad \lambda > 1, \\
x_{n,\gamma}(\lambda) &\approx \frac{-1}{\alpha} \theta_{n,\gamma}(\lambda), \quad \lambda > 1, \quad \text{and} \quad x_{n,\delta}(\lambda) \approx \frac{-1}{\alpha} \theta_{n,\delta}(\lambda), \quad \lambda > 1.
\end{aligned}$$

**4.4. Solution of the problem.** solution of the zeroth-order is given as: the direction of velocity  
Considering the horizontal orientation

$$\theta_0(\lambda) \approx 0, \quad \lambda > 1,$$

the flow's velocity

$$q_0(\lambda) \approx 1, \quad \lambda > 1,$$

on the other hand, we have

$$x_0(\lambda) \approx x_{\infty} - \frac{1}{\pi} \int_{\lambda}^{\infty} \frac{1}{1-\tau} d\tau - \frac{1}{\alpha} \theta_0(\lambda), \quad \lambda > 1,$$

at the point  $C$ , we have ( $\lambda = \lambda_c, x = 0$ ), therefore

$$x_{\infty} = \frac{1}{\pi} \int_{\lambda_c}^{\infty} \frac{1}{1-\tau} d\tau,$$

Then

$$x_0(\lambda) \approx \frac{-1}{\pi} \ln \left( \frac{\lambda - 1}{1 - \lambda_c} \right), \quad \lambda > 1,$$

and

$$y_0(\lambda) \approx 1 + l, \quad \lambda > 1.$$

For high Weber number  $\alpha$ , the first term in (4.6) and (4.7) may be disregarded and we obtain the solution to first-order in  $\gamma$  and  $\delta$

$$\begin{aligned}
\theta_{1,\gamma}(\lambda) &\approx \frac{2}{\pi} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right), \quad \lambda > 1, \\
\theta_{1,\delta}(\lambda) &\approx \frac{-2}{\pi} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \quad \lambda > 1, \\
x_{1,\gamma}(\lambda) &\approx \frac{-2}{\pi\alpha} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right), \quad \lambda > 1,
\end{aligned}$$

$$\begin{aligned}
x_{1,\delta}(\lambda) &\approx \frac{2}{\pi\alpha} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \quad \lambda > 1, \\
y_{1,\gamma}(\lambda) &\approx \frac{4(1-\sqrt{1-\lambda_c})}{\pi^2(1-\lambda_c)^{\frac{1}{4}}} \left( \arctan \left( \frac{\sqrt{\lambda-1}}{(1-\lambda_c)^{\frac{1}{4}}} \right) - \frac{\pi}{2} \right), \quad \lambda > 1, \\
y_{1,\delta}(\lambda) &\approx \frac{4(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\pi^2((1-\lambda_c)(1-\lambda_d))^{\frac{1}{4}}} \left( \arctan \left( \frac{\sqrt{\lambda-1}}{((1-\lambda_c)(1-\lambda_d))^{\frac{1}{4}}} \right) - \frac{\pi}{2} \right), \quad \lambda > 1,
\end{aligned}$$

Using an approximate first-order solution, and for  $\lambda > 1$ , we arrive at:

$$\begin{aligned}
q(\lambda) &\approx 1 + \frac{\gamma\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)(\lambda-1-\sqrt{1-\lambda_c})}{\alpha\lambda(\lambda-\lambda_c)} - \frac{\delta\sqrt{\lambda-1}}{\alpha} \\
&\quad \frac{(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})(\lambda-1-\sqrt{(\lambda_c-1)(\lambda_d-1)})}{((\lambda-1+\sqrt{(\lambda_c-1)(\lambda_d-1)})^2+(\lambda-1)(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})^2)}, \\
\theta(\lambda) &\approx \frac{2\gamma}{\pi} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right) - \frac{2\delta}{\pi} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \\
x(\lambda) &\approx \frac{1}{\pi} \ln \left( \frac{1-\lambda_c}{\lambda-1} \right) - \frac{2\gamma}{\pi\alpha} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right) \\
&\quad + \frac{2\delta}{\pi\alpha} \arctan \left( \frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \\
y(\lambda) &\approx 1 + l + \frac{4\gamma(1-\sqrt{1-\lambda_c})}{\pi^2(1-\lambda_c)^{\frac{1}{4}}} \left( \arctan \left( \frac{\sqrt{\lambda-1}}{(1-\lambda_c)^{\frac{1}{4}}} \right) - \frac{\pi}{2} \right) \\
&\quad + \frac{4\delta(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\pi^2((1-\lambda_c)(1-\lambda_d))^{\frac{1}{4}}} \left( \arctan \left( \frac{\sqrt{\lambda-1}}{((1-\lambda_c)(1-\lambda_d))^{\frac{1}{4}}} \right) - \frac{\pi}{2} \right)
\end{aligned}$$

## 5. RESULTS AND DISCUSSIONS

**5.1. Results for  $\gamma \neq 0$  and  $\delta \neq 0$ .** In this case, the flow configuration as is shown in figure 1, models flow passing over a triangular hollow at the bottom of a channel. Using the previous method illustrated in section 4 and after calculation, the shape of the free surface with the fixed values of the Weber number  $\alpha = 10000$ , the depth of the hollow  $l = 0.05$  and the angles  $\gamma = \frac{\pi}{8}$  and  $\delta = \frac{\pi}{8}$  is shown in figure 4.a.

The free surface profiles of the flow are plotted in figure 4.b, for different values of the depth  $l$ , for a fixed values of  $\alpha = 200$ ,  $\gamma = \frac{\pi}{8}$  and  $\delta = \frac{\pi}{8}$ .

As it appears in this figure, we see that the results show that as the depth  $l$  increases, the elevation of the free surface also increases, indicating a correlation between the depth of the hollow and the variation of the surface profile.

Figures 4.c, 4.d and 4.e illustrate the free surface profiles for various values of the angles  $\gamma$  and  $\delta$ , at a fixed  $\alpha = 10000$  and  $l = 0.1$ .

As shown in figure 4.c, increasing the value of the angle  $\gamma$  causes decreases in the radius of curvature of the free surface above the hollow.

The variation of the angle  $\delta$  causes changes in the height of the free surface and increasing of the flow's velocity at higher angles causes the free surface to rise significantly see figure 4.d. Thus, the flow behavior can be affected by variations in the shape of the free surface as a function of  $\delta$ . The figure 4.e present the variation of the angles in the symmetrical case when  $\gamma = \delta$  on the free surface forms.

The following table presents some values of  $\lambda_c$  and  $\lambda_d$  for different angles  $\gamma$ ,  $\delta$  and depth  $l$ .

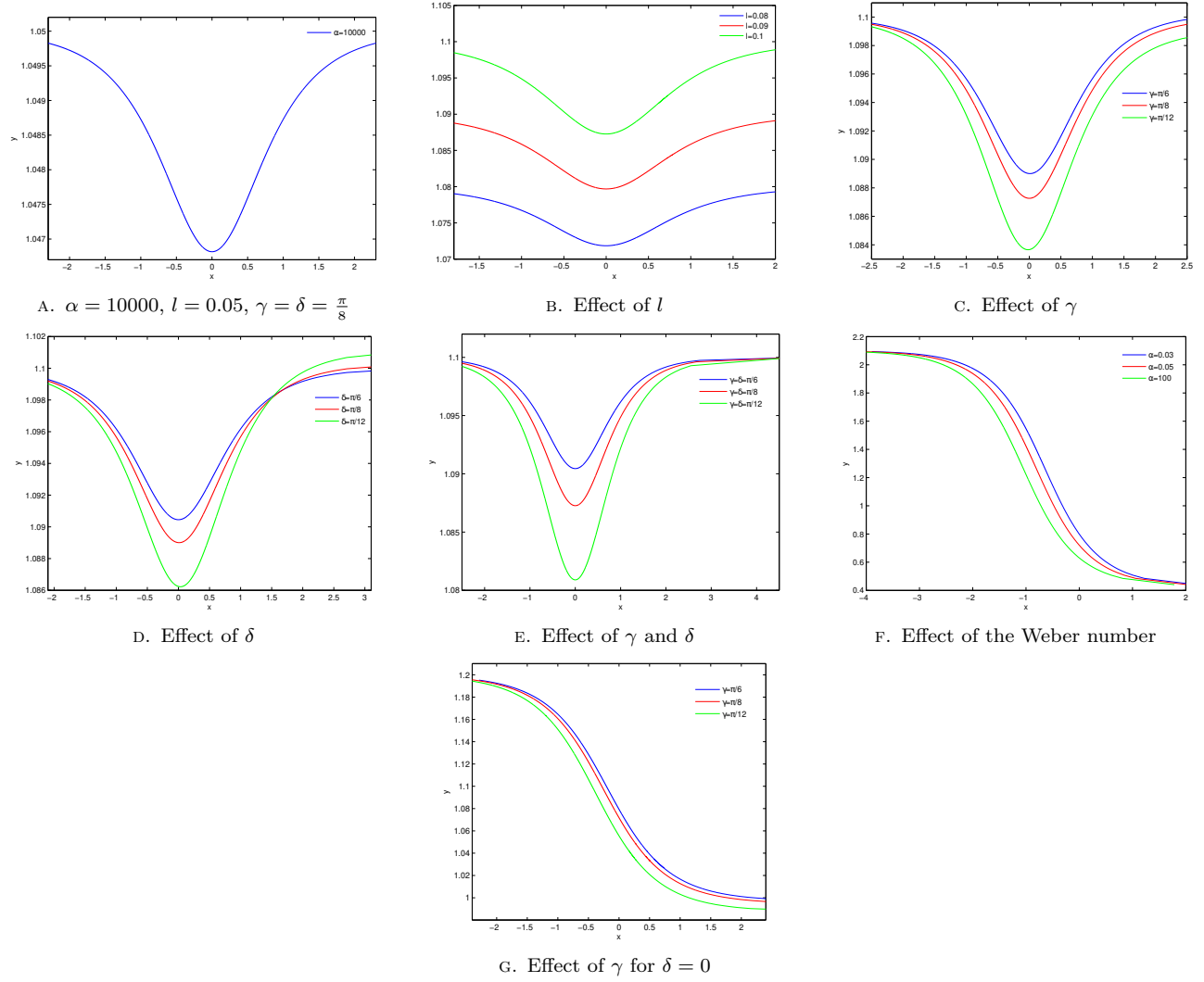


FIGURE 4. Free surface profiles for different values of the Weber number, depth, and inclination angles: (a) the shape of the free surface for  $\alpha = 10000$ ,  $l = 0.05$ , and  $\gamma = \delta = \frac{\pi}{8}$ ; (b) the effect of the depth of the hollow  $l$  for  $\alpha = 200$  and  $\gamma = \delta = \frac{\pi}{8}$ ; (c) the effect of the angle  $\gamma$  for  $\alpha = 10000$ ,  $l = 0.1$ , and  $\delta = \frac{\pi}{8}$ ; (d) the effect of the angle  $\delta$  for  $\alpha = 10000$ ,  $l = 0.1$ , and  $\gamma = \frac{\pi}{6}$ ; (e) the effect of the angles  $\gamma$  and  $\delta$  for  $\alpha = 1000$  and  $l = 0.1$ ; (f) the effect of the Weber number for  $l = 1.1$  and  $\gamma = \frac{\pi}{6}$ ; (g) the effect of the angle  $\gamma$  for  $\alpha = 10000$ ,  $l = 0.2$ , and  $\delta = 0$ .

TABLE 1. The values of  $\lambda_c$  and  $\lambda_d$  for various angles  $\gamma$ ,  $\delta$ , and depth  $l$ .

| $\gamma$         | $\delta$         | $l$  | $\lambda_c$ | $\lambda_d$ |
|------------------|------------------|------|-------------|-------------|
| $\frac{\pi}{6}$  | $\frac{\pi}{6}$  | 0.50 | 0.9502      | 0.9975      |
|                  |                  | 1.30 | 0.9996      | 1.0000      |
| $\frac{\pi}{8}$  | $\frac{\pi}{8}$  | 0.05 | 0.3297      | 0.5507      |
|                  |                  | 1.02 | 0.9997      | 1.0000      |
| $\frac{\pi}{12}$ | $\frac{\pi}{12}$ | 0.30 | 0.9727      | 0.9993      |
|                  |                  | 0.80 | 0.9999      | 1.0000      |

**5.2. Results for  $\delta = 0$  or  $\gamma = 0$ .** For the case when  $\delta = 0$  and several values of  $\gamma$ , the problem is reduced to studying flow over a descending step and the figure 4.f illustrates the effect of different values of the Weber number  $\alpha$  on the free-surface profiles. It can be noted from this figure that as surface tension increases, the curvature of the free surface decreases. This is consistent with the theoretical aspect. Figure 4.g shows the effect of different values of the inclination angle  $\gamma$  on the free-surface profiles for  $\delta = 0$ . It highlights how varying the angle can influence the shape and behavior of the free surface. Finally, when  $\gamma = 0$  the flow configuration reduce to a problem of flow over a step. This problem is treated numerically in previous paper presented by Bounif and Gasmi [7].

## 6. CONCLUSION

The study formulates the flow passing over a trapezoidal hollow at the bottom of a channel as a system of nonlinear integral equations and uses perturbation techniques to derive approximate solutions at high Weber numbers. The results illustrate the free surface profiles and show the impact of parameters like inclination angles, depth, and the Weber number on the flow. The findings highlight the effectiveness of the approximate scheme and provide valuable insights into flow characteristics, demonstrating how these parameters influence the free-surface profiles. We emphasize that the present analysis neglects gravity and viscosity; under these conditions the surface tension dominates the flow dynamics, but the inviscid and irrotational assumptions preclude the observation of viscous boundary layers and possible vortical structures within the hollow.

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Amroune B.,  
Laboratory of Pure and Applied Mathematics,  
University of M'sila, Msila, Algeria  
e-mail: bochra.amroune@univ-msila.dz

Gasmi A.,  
Laboratory of Pure and Applied Mathematics,  
University of M'sila, Msila, Algeria,  
e-mail: abdelkader.gasmi@univ-msila.dz