

The discrete spectrum of the three-particle Schrödinger operator on $\mathbb{Z}^2$

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Abstract. We study a three-particle lattice Hamiltonian describing two identical bosons and one fermion on $\mathbb{Z}^2$, interacting via attractive or repulsive zero-range pair potentials. We analyze the point spectrum of the associated three-particle lattice discrete Schrödinger operator and describe the existence and multiplicity of bound states. Particular attention is devoted to the case in which the bosons have infinite mass.

Keywords: Schrödinger operator, dispersion functions, zero-range pair potentials, discrete spectrum, essential spectrum

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1. INTRODUCTION

The Efimov effect, first discovered by V. Efimov in 1970 [1], is one of the most remarkable and counterintuitive phenomena in quantum physics. It arises in three-body systems in three-dimensional space where the particles interact via short-range pairwise potentials. Notably, it is possible to tune the interaction strengths such that none of the two-particle subsystems supports a negative-energy bound state, yet at least two of the pairs exhibit a zero-energy resonance.

The existence of the Efimov effect for three-particle Schrödinger operators was rigorously established by D. R. Yafaev in the continuous case [2], and later extended to the discrete setting by several authors [3, 4, 5, 6, 7, 8]. The analysis of this phenomenon is closely linked to the asymptotic behavior of the eigenvalue counting function, which in turn is reduced to the study of compact integral operators of Faddeev type.

In recent studies [9, 10, 11, 12, 13, 14], systems of three quantum particles on the one- two- and three-lattices, interacting via pairwise attractive point potentials, were examined. These works focused on configurations where two of the three particles are assumed to have infinite mass. It was shown that, under such conditions, the associated three-particle Schrödinger operators admit an infinite sequence of negative eigenvalues for any nonzero strength of the interaction. Furthermore, the author of [11] derived precise asymptotic formulas describing the behavior of these eigenvalues in the strong coupling limit.

It should be noted that the operator investigated in the present paper has not been considered in these earlier works.

In physics, stable composite objects often result from attractive forces, enabling constituents the constituents to lower their energy by binding together. In contrast, repulsive forces tend to scatter particles when they are in free space. However, within structured environments such as a periodic potential, and in the absence of dissipation, stable composite objects can exist even with repulsive interactions arising from the lattice band structure [15].

The work referenced by [15] exemplifies the crucial link between the Bose-Hubbard model [16, 17] and atoms in optical lattices, and helps pave the way for many more interesting developments and applications, as discussed in [18]. In cold atom lattice physics, stable repulsively bound objects are expected to be common, which leading to potential composites with fermions [19] or Bose-Fermi mixtures [20], and can even be formed with more than two particles in a similar manner.

The main goal of the paper is to investigate the existence of infinite number of bound states of the three-particle discrete Schrödinger operator associated to a system of two identical bosons and a fermion, where the bosons have infinite mass and the fermion has a finite mass. This investigation is conducted on the two-dimensional lattice $\mathbb{Z}^2$ and involves repulsive or attractive zero-range pairwise interactions. The problem is reduced to the study the Fredholm determinant of a diagonal operator.

It should be noted that, unlike the last three articles [9, 10, 11], we study eigenvalues below and above the essential spectrum of the unperturbed operator for all repulsive or attractive zero-range pairwise interactions.

The paper is organized as follows. In Section 2, we introduce the three-particle discrete Schrödinger operator $H(K)$ and the two-particle discrete Schrödinger operators associated with subsystems of the system of two identical bosons and a fermion. In Section 3, we study the essential spectrum of the three-particle discrete Schrödinger operator $H(K)$. The eigenvalues of $H(K)$ below and above the spectrum of the non-perturbed operator $H_0(K)$ are investigated in Section 4. Section 5 is devoted to showing main result, Theorem 5.4.

2. THREE-PARTICLE DISCRETE SCHRÖDINGER OPERATOR ON THE LATTICE $\mathbb{Z}^2$

Let $L_s^2((\mathbb{T}^2)^2)$ be the linear subspace of the symmetric functions of the Hilbert space $L^2((\mathbb{T}^2)^2)$.

Let us consider the discrete Schrödinger operator $H(K)$, where $K \in \mathbb{T}^2$, associated with a system consisting of two identical bosons and a fermion moving on the two-dimensional lattice $\mathbb{Z}^2$ (see [9, 11]).

$$H(K) = H_0(K) - V \quad (2.1)$$

with zero-range attractive potentials

$$V = V_1 + V_2 + V_3,$$

where

$$\begin{aligned} (V_1 f)(p, q) &= \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} f(p, t) dt, & (V_2 f)(p, q) &= \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} f(t, q) dt \\ (V_3 f)(p, q) &= \frac{\mu}{(2\pi)^2} \int_{\mathbb{T}^2} f(t, p + q - t) dt, & f &\in L_s^2((\mathbb{T}^2)^2), \quad p, q \in \mathbb{T}^2, \end{aligned}$$

and numbers λ and μ serve as the parameters of boson-fermion interaction and boson-boson interaction, respectively.

In the physical literature the parameter K , $K \in \mathbb{T}^2$, is called the *total quasi-momentum* of the three-particle system ([21]).

Here the numbers λ and μ indicate repulsive pair-wise interaction when $\lambda < 0$ and $\mu < 0$, and attractive pair-wise interaction when $\lambda > 0$ and $\mu > 0$. The operator $H_0(K)$ is defined on the Hilbert space $L_s^2((\mathbb{T}^2)^2)$ by

$$(H_0(K)f)(p, q) = E(K; p, q)f(p, q), \quad f \in L_s^2((\mathbb{T}^2)^2),$$

and

$$E(K; p, q) = \varepsilon_b(p) + \varepsilon_b(q) + \varepsilon_f(K - p - q), \quad p, q \in \mathbb{T}^2. \quad (2.2)$$

Here, the real-valued continuous function $\varepsilon_b(\cdot)$ and $\varepsilon_f(\cdot)$, referred to as *the dispersion relation* associated with the free boson and fermion is defined as

$$\varepsilon_b(p) = \frac{1}{m} \varepsilon(p), \quad \varepsilon_f(p) = \frac{1}{\mathbf{m}} \varepsilon(p), \quad \varepsilon(p) = 2 - \cos p_1 - \cos p_2, \quad p = (p_1, p_2) \in \mathbb{T}^2, \quad (2.3)$$

respectively, and m and $\mathbf{m}$ represents the mass of the boson and fermion, respectively.

2.1. Statement of the main result in the case $m = \infty$ and $0 < \mathbf{m} < \infty$. With $m = \infty$ and $0 < \mathbf{m} < \infty$ and the equality (2.3), the three-particle dispersion function (2.2) can be written as

$$E(K; p, q) = \varepsilon_f(K - p - q), \quad p, q \in \mathbb{T}^2. \quad (2.4)$$

2.1.1. *Zeros of Birman–Schwinger determinants.* In this subsection we study the zeros of three functions that play the role of Birman–Schwinger determinants, i.e., determinants associated with Birman–Schwinger operators.

Firstly, for $n \in \mathbb{Z}^2$ we define the integral

$$d_n(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{e^{i(n,t)}}{\varepsilon_f(t) - z} dt, \quad z \in \mathbb{C} \setminus [0, 4/\mathfrak{m}]. \quad (2.5)$$

The first determinant is defined on $\mathbb{R} \setminus [0, 4/\mathfrak{m}]$ by

$$\Delta(\lambda; z) = 1 - \lambda d_0(z), \quad z \in \mathbb{R} \setminus [0, 4/\mathfrak{m}] \quad (2.6)$$

and proof of the next lemma is presented on Page 124

Lemma 2.1. (a) If $\lambda < 0$, then the function $\Delta(\lambda; z)$ has a unique simple zero z_1^0 in the interval $(4/\mathfrak{m}, \infty)$.

(b) If $\lambda > 0$, then $\Delta(\lambda; z)$ has a unique simple zero z_1^0 in the interval $(-\infty, 0)$.

Next, consider the second determinant

$$\tilde{\Delta}(\lambda, \mu; z) = \Delta(2\lambda; \mu + z) = 1 - 2\lambda d_0(\mu + z), \quad z \in \mathbb{R} \setminus [-\mu, 4/\mathfrak{m} - \mu] \quad (2.7)$$

and the proof of the next lemma is given on Page 126

Lemma 2.2. Assume $\mu \in \mathbb{R}$.

(a) If $\lambda < 0$, then $\tilde{\Delta}(\lambda, \mu; z)$ has a unique zero η in the interval $(4/\mathfrak{m} - \mu, \infty)$.

(b) If $\lambda = 0$, then $\tilde{\Delta}(\lambda, \mu; z)$ has no zeros.

(c) If $\lambda > 0$, then $\tilde{\Delta}(\lambda, \mu; z)$ has a unique zero η in the interval $(-\infty, -\mu)$.

Finally, for $n \in \mathbb{Z}^2$, define the third determinant

$$\Delta_n^+(\lambda; z) = 1 - \lambda(d_0(z) + d_n(z)), \quad z \in \mathbb{R} \setminus [0, 4/\mathfrak{m}], \quad (2.8)$$

and the proof of the following lemma can be found on Page 132.

Lemma 2.3. (a) Let $\lambda > 0$. Then $\Delta_n^+(\lambda; z)$ has a unique zero z_n in the interval $(-\infty, 0)$ such that $z_n < z_1^0 < 0$. Moreover,

$$\lim_{|n| \rightarrow \infty} z_n = z_1^0. \quad (2.9)$$

(b) Let $\lambda = 0$. Then $\Delta_n^+(\lambda; z)$ has no zeros.

(c) Let $\lambda < 0$. Then $\Delta_n^+(\lambda; z)$ has a unique zero z_n in the interval $(4/\mathfrak{m}, \infty)$ such that

$$4/\mathfrak{m} < z_1^0 < z_n.$$

Moreover, a similar limit to that in (2.9) holds.

2.1.2. *The essential spectrum of $H(K)$ for $m = \infty$ and $\mathfrak{m} < \infty$.*

Theorem 2.1. The essential spectrum of $H(K)$ is given by

$$\sigma_{\text{ess}}(H(K)) = \{z_1^0\} \cup \left([-\mu, 4/\mathfrak{m} - \mu] \cup [0, 4/\mathfrak{m}] \right).$$

Proof: The proof lies in Page 125. □

2.1.3. *Main result: the point spectrum of $H(K)$ for $m = \infty$ and $\mathfrak{m} < \infty$.* The following assertion states that three-particle Schrödinger operators admit an infinite sequence of eigenvalues for any nonzero strength of the interaction.

Theorem 2.2. *Assume that $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.*

(a) *Let $\lambda < 0$. Then*

$$\sigma_{pp}(H(K)) = \bigcup_{n \in \mathbb{Z}^2} \{z_n\} \cup \{\eta\},$$

where $4/\mathfrak{m} < z_1^0 < z_n$ and $4/\mathfrak{m} - \mu < \eta$.

(b) *If $\lambda = 0$, then*

$$\sigma_{pp}(H(K)) = \emptyset.$$

(c) *Let $\lambda > 0$. Then*

$$\sigma_{pp}(H(K)) = \bigcup_{n \in \mathbb{Z}^2} \{z_n\} \cup \{\eta\},$$

where $z_n < z_1^0 < 0$ and $\eta < -\mu$.

Proof: See Page 133. □

Remark 2.1. By Theorem 2.1, one branch of the essential spectrum, $[-\mu, 4/\mathfrak{m} - \mu]$, depends on μ . Theorem 2.2 then implies that some or all of the eigenvalues z_n and η may lie either in the essential spectrum $\sigma_{\text{ess}}(H(K))$, or in the gaps of this essential spectrum.

Remark 2.2. Let $U_K : L_s^2(\mathbb{T}^2 \times \mathbb{T}^2) \rightarrow L_s^2(\mathbb{T}^2 \times \mathbb{T}^2)$ be a unitary operator defined by

$$(U_K f)(p, q) = f(-K/2 + p, -K/2 + q). \quad (2.10)$$

When $m = \infty$ and $\mathfrak{m} < \infty$, the operator U_K establishes a unitary equivalence between $H(K)$ and $H(0)$. Therefore, it suffices to prove all subsequent results for the operator $H(0)$.

3. TWO-PARTICLE DISCRETE SCHRÖDINGER OPERATORS

In addition to the three-particle Schrödinger operator, we also examine the discrete Schrödinger operators associated with the two-particle boson–fermion and boson–boson subsystems, as these are necessary for analyzing the essential spectrum of $H(K)$.

Let $k \in \mathbb{T}^2$ and $L_k^2(\mathbb{T}^2)$ be a linear subspace of the Hilbert space $L^2(\mathbb{T}^2)$ defined by

$$L_k^2(\mathbb{T}^2) = \{f \in L^2(\mathbb{T}^2) \mid f(p) = f(k - p)\}.$$

The two-particle discrete Schrödinger operators corresponding to the boson–fermion and boson–boson subsystems of the three-particle system act on the Hilbert spaces $L^2(\mathbb{T}^2)$ and $L_k^2(\mathbb{T}^2)$ as

$$h_1(k) = h_1^0(k) - \lambda v_1, \quad \text{and} \quad h_2(k) = h_2^0(k) - \mu v_2, \quad k \in \mathbb{T}^2, \quad (3.1)$$

respectively.

In the physics literature, the parameter k , $k \in \mathbb{T}^2$, is known as the *total quasi-momentum* of the two-particle system ([21]).

Here, the operators $h_\alpha^0(k)$ are defined as

$$(h_1^0(k)f)(p) = E_k^{(1)}(p)f(p), \quad f \in L^2(\mathbb{T}^2),$$

and

$$(h_2^0(k)f)(p) = E_k^{(2)}(p)f(p), \quad f \in L_k^2(\mathbb{T}^2),$$

where

$$E_k^{(1)}(p) = \varepsilon_{\mathfrak{f}}(p) + \varepsilon_{\mathfrak{f}}(k - p), \quad E_k^{(2)}(p) = \varepsilon_{\mathfrak{b}}(p) + \varepsilon_{\mathfrak{b}}(k - p), \quad p \in \mathbb{T}^2. \quad (3.2)$$

The operators v_1 and v_2 are defined as

$$(v_1 f)(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(q) dq, \quad f \in L^2(\mathbb{T}^2), \quad p \in \mathbb{T}^2, \quad (3.3)$$

and

$$(v_2 f)(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(q) dq, \quad f \in L_k^2(\mathbb{T}^2), \quad p \in \mathbb{T}^2,$$

respectively. Observe that $v_1 f$ and $v_2 f$ are constant functions on $\mathbb{T}^2$.

3.1. Spectral properties of the two-particle discrete Schrödinger operators when $m = \infty$ and $0 < m < \infty$. With $m = \infty$ and $0 < m < \infty$ and the equality (2.3), the functions (3.2) can be written as

$$E_k^{(1)}(p) = \varepsilon_f(k - p) = \varepsilon(k - p)/m, \quad E_k^{(2)}(p) = 0, \quad p \in \mathbb{T}^2.$$

Consequently, since the potentials v_α , $\alpha = 1, 2$ have a convolution-type property, all three two-particle Schrödinger operators do not depend on the quasi-momentum $k \in \mathbb{T}^2$,

$$h_1 := h_1(k), \quad \text{and} \quad h_2 = h_2(k).$$

Then, the operators $h_1(k)$ and $h_2(k)$ act as

$$(h_1(k)f)(p) = \varepsilon_f(p)f(p) - \lambda(v_1 f)(p), \quad f \in L^2(\mathbb{T}^2) \quad (3.4)$$

and

$$(h_2(k)f)(p) = -\mu(v_2 f)(p), \quad f \in L_k^2(\mathbb{T}^2). \quad (3.5)$$

As v_1 is a finite rank operator, according to the Weyl theorem, the essential spectrum $\sigma_{ess}(h_1(k))$ of the operator $h_1(k)$ in (3.1) coincides with the spectrum $\sigma(h_1^0(k))$ of the non-perturbed operator $h_1^0(k)$. More specifically,

$$\sigma_{ess}(h_1(k)) = [E_{\min}^{(1)}(k), E_{\max}^{(1)}(k)],$$

where

$$E_{\min}^{(1)}(k) \equiv \min_{p \in \mathbb{T}^2} E_k^{(1)}(p), \quad E_{\max}^{(1)}(k) \equiv \max_{p \in \mathbb{T}^2} E_k^{(1)}(p).$$

Remark 3.1. Note that, under the given mass condition, statements concerning the properties of the operators $h_1(k)$ and $h_2(k)$ formally depend on the total quasi-momentum k , as discussed in Remark 2.2.

Therefore, in our case we have

$$\sigma_{ess}(h_1(k)) = [0, 4/m] \quad \text{and} \quad \sigma_{ess}(h_2(k)) = \{0\}.$$

The Fredholm determinant $\Delta(\lambda; z)$ defined in (2.6) is associated with the Birman–Schwinger operator

$$I - \lambda(h_1^0(0) - z)^{-1} v_1, \quad z \in \mathbb{C} \setminus [0, 4/m].$$

Here, $(h_1^0(0) - z)^{-1}$ is the resolvent of the unperturbed operator $h_1^0(0)$, that is, the operator of multiplication by the function $(\varepsilon_f(p) - z)^{-1}$.

3.1.1. Proof of Lemma 2.1. In the first place, we show that the number $z \in \mathbb{C} \setminus [0, 4/m]$ is an eigenvalue of $h_1(k)$ if and only if $\Delta(\lambda; z) = 0$.

The equation

$$h_1(0)f = zf \quad \text{i.e.,} \quad f = \lambda(h_1^0(0) - z)^{-1} v_1 f$$

has a non-trivial solution if and only if

$$\Delta(\lambda; z)C = 0, \quad C \in \mathbb{C},$$

has a non-trivial solution.

Therein, the solutions $C \in \mathbb{C}$ and $f \in L^2(\mathbb{T}^2)$ are connected by the following relations

$$C = v_1 f \quad \text{and} \quad f = \lambda(h_1^0(0) - z)^{-1} C.$$

(a) Let $\lambda > 0$. The function $\Delta(\lambda; z)$ is monotonic decreasing in $(-\infty, 0)$ and $\Delta(\lambda; z) > 1$ in $(4/\mathfrak{m}, \infty)$. Since

$$\lim_{z \rightarrow -\infty} \Delta(\lambda; z) = 1 \quad \text{and} \quad \lim_{z \rightarrow 0-} \Delta(\lambda; z) = -\infty,$$

the intermediate-value theorem implies the existence of a unique simple zero $z = z_1^0, z_1^0 \in (-\infty, 0) \cup (4/\mathfrak{m}, \infty)$ of the function $\Delta(\lambda; \cdot)$, and furthermore $z_1^0 \in (-\infty, 0)$.

(b) The case $\lambda < 0$ can be treated using the same method. $\square$

Since $h_2(k)$ defined in (3.5) is a rank-one integral operator with norm $|\mu|$, it follows that

$$\sigma_{\text{ess}}(h_2(k)) = \{0\} \quad \text{and} \quad \sigma_{\text{disc}}(h_2(k)) = \{-\mu\}.$$

Therefore, the results of this section can be summarized in the following lemma.

Lemma 3.1. *We have*

$$\begin{aligned} \sigma_{\text{disc}}(h_1(k)) &= \{z_1^0\}, \quad \text{if } \lambda \neq 0, \\ \sigma(h_1(k)) &= \{z_1^0\} \cup [0, 4/\mathfrak{m}], \quad \text{if } \lambda \neq 0 \end{aligned}$$

and

$$\sigma_{\text{disc}}(h_2(k)) = \{-\mu\}, \quad \sigma(h_2(k)) = \{-\mu\} \cup \{0\}.$$

4. PROOF OF THEOREM 2.1: ESSENTIAL SPECTRUM OF $H(K)$

One of the notable results in the spectral theory of multi-particle continuous Schrödinger operators is the characterization of the essential spectrum in terms of cluster operators (the HVZ theorem); see, for example, [22, 23, 24, 25, 26] for the discrete case and [27] for the pseudo-relativistic case.

Lemma 4.1. *The essential spectrum of $H(K)$ satisfies the relation*

$$\sigma_{\text{ess}}(H(K)) = \bigcup_{k \in \mathbb{T}^2} \left\{ \sigma(h_1(K - k)) + \varepsilon_{\mathfrak{b}}(k) \right\} \cup \bigcup_{k \in \mathbb{T}^2} \left\{ \sigma(h_2(K - k)) + \varepsilon_{\mathfrak{f}}(k) \right\}.$$

Proof: The proof can be found in [11, 24]. $\square$

4.0.1. *The essential spectrum of $H(K)$ with $m = \infty$, and $\mathfrak{m} < \infty$.* Due to Lemma 3.1 and the relations $\varepsilon_{\mathfrak{b}}(p) = 0$ and $\varepsilon_{\mathfrak{f}}(p) = \varepsilon(p)/\mathfrak{m}$, we obtain

$$\begin{aligned} \bigcup_{k \in \mathbb{T}^2} \left\{ \sigma(h_1(K - k)) + \varepsilon_{\mathfrak{f}}(k) \right\} &= \bigcup_{k \in \mathbb{T}^2} \left\{ \sigma(h_1(k)) + \varepsilon_{\mathfrak{f}}(k) \right\} = \{z_1^0\} \cup [0, 4/\mathfrak{m}], \\ \bigcup_{k \in \mathbb{T}^2} \left\{ \sigma(h_2(K - k)) + \varepsilon_{\mathfrak{f}}(k) \right\} &= \bigcup_{k \in \mathbb{T}^2} (\{-\mu\} + \varepsilon_{\mathfrak{f}}(k)) \cup \bigcup_{k \in \mathbb{T}^2} (\{0\} + \varepsilon_{\mathfrak{f}}(k)) \\ &= [-\mu, 4/\mathfrak{m} - \mu] \cup [0, 4/\mathfrak{m}]. \end{aligned}$$

According the last two relations and Lemma 4.1 we have the proof of Theorem 2.1

5. PROOF OF THEOREM 2.2: ANALYSIS OF THE POINT SPECTRUM OF $H(K)$ FOR $m = \infty$ AND $\mathfrak{m} < \infty$

Since the integral kernels of the partial integral operators V_1 and V_2 are constant functions, it is straightforward to verify that the subspace

$$\mathcal{A}_0 = \{f \in L_s^2(\mathbb{T}^2 \times \mathbb{T}^2) \mid f(p, q) = g(p + q), \ g \in L^2(\mathbb{T}^2)\}$$

is invariant under the action of the operator $H(K)$.

Therefore, we have

$$\sigma_{pp}(H(K)) = \sigma_{pp}(A_0(K)) \cup \sigma_{pp}(A_1(K)),$$

where $A_0(K)$ and $A_1(K)$ are restrictions of $H(K)$ on the linear subspaces $\mathcal{A}_0$ and $\mathcal{A}_0^\perp$, respectively.

Since $\mathcal{A}_0$ and $L^2(\mathbb{T}^2)$ are isomorphic, the operator $A_0(K)$ is unitarily equivalent to the operator B_0 on $L^2(\mathbb{T}^2)$, where

$$B_0 = E_0(K) - \mu I - 2\lambda v_1, \quad (5.1)$$

$E_0(K)$ denotes the multiplication by the function $\varepsilon_f(K - p)$, I is the identity operator, and v_1 is an integral operator defined by (3.3).

The operator $A_1(K)$ takes the form

$$A_1(K) = H_0(K) - V_1 - V_2.$$

5.1. The Spectrum of the Operator $A_0(K)$. Since the equivalence (5.1) and the Weyl theorem imply that

$$\sigma_{ess}(A_0(K)) = [-\mu, 4/\mathfrak{m} - \mu].$$

The second determinant $\tilde{\Delta}(\lambda, \mu; z)$ defined by (2.7) associated with the Birman-Schwinger operator

$$I - 2\lambda(h_1^0(0) - (\mu + z))^{-1}v_1, \quad z \in \mathbb{C} \setminus [-\mu, 4/\mathfrak{m} - \mu]$$

where $(h_1^0(0) - z)^{-1}$ is the resolvent of the unperturbed operator $h_1(k)$ in (3.4).

5.1.1. Proof of Lemma 2.2. Firstly, we show that $z \in \mathbb{C} \setminus [-\mu, 4/\mathfrak{m} - \mu]$ is an eigenvalue of $H(0)$ in $\mathcal{A}_0$ if and only if $\tilde{\Delta}(\lambda, \mu; z) = 0$.

The equation

$$H(0)f = zf, \quad f \in \mathcal{A}_0,$$

has a solution if and only if the equation

$$C(1 - 2\lambda d_0(\mu + z)) = 0, \quad (5.2)$$

has one, and their solutions are related by

$$f(p, q) := g(p + q) = \frac{2\lambda}{\varepsilon_f(-p - q) - \mu - z} C,$$

where

$$C = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} g(t) dt.$$

Equation (5.2) has a nontrivial solution C if and only if the equation

$$\Delta(2\lambda; \mu + z) := 1 - 2\lambda d_0(\mu + z) = 0.$$

has a zero.

The function $\Delta(2\lambda; \mu + z)$ is defined on

$$(-\infty, -\mu) \cup (4/\mathfrak{m} - \mu, \infty),$$

and, according to Lemma 2.1, the following assertions hold:

(a) If $\lambda > 0$, then $\Delta(2\lambda; \mu + z) = 1 - 2\lambda d_0(\mu + z)$ has a unique zero in $(-\infty, -\mu)$, but does not have one in $(4/\mathfrak{m} - \mu, \infty)$;

(b) If $\lambda < 0$, then $\Delta(2\lambda; \mu + z)$ has a unique zero in $(4/\mathfrak{m} - \mu, \infty)$, but does not have one in $(-\infty, -\mu)$;

(c) If $\lambda = 0$, then $B_0 = E_0(K) - \mu I$ is a multiplication operator, and so $A_0(K)$ has no eigenvalues for any $\mu \in \mathbb{R}$. $\square$

5.2. Spectrum of $A_1(K)$. Recall that

$$A_1(K) = H_0(K) - V_1 - V_2,$$

where $H_0(K)$ is the multiplication operator defined by the function $E(K; p, q) := \varepsilon_f(K - p - q)$ in (2.4):

$$(H_0(K)f)(p, q) = \varepsilon_f(K - p - q)f(p, q), \quad f \in \mathcal{A}_0^\perp.$$

The operators V_1 and V_2 are partially integral operators given in (2.1):

$$(V_1 f)(p, q) = \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} f(p, t) dt, \quad (V_2 f)(p, q) = \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} f(t, q) dt,$$

for $f \in \mathcal{A}_0^\perp$ and $p, q \in \mathbb{T}^2$.

Since the operator $A_1(K)$ does not depend on the parameter μ , Lemma 2.1 yields

$$\sigma_{\text{ess}}(A_1(K)) = \{z_1^0\} \cup [0, 4/\mathfrak{m}].$$

The function $d_n(z)$ defined in (2.5) plays a role in the investigation of the eigenvalues of $A_1(k)$. In this connection, we introduce the following lemma.

Lemma 5.1. (a) For any $z < 0$, the following asymptotics holds

$$\begin{aligned} d_n(z) &= \frac{\mathfrak{m}}{2^{|n|_1}} \frac{1}{|2 - \mathfrak{m}z|^{|n|_1+1}} \sum_{\xi=0}^{\infty} \frac{1}{(2 - \mathfrak{m}z)^{2\xi}} \frac{1}{2^{2\xi}} \\ &\quad \times \sum_{s_1+s_2=\xi} \frac{(|n|_1 + \xi)!}{s_1!s_2!(|n_1|! + s_1)!(|n_2|! + s_2)!}, \quad z < 0, \end{aligned} \quad (5.3)$$

where

$$|n|_1 = |n_1| + |n_2|.$$

(b) For any fixed $n \in \mathbb{Z}^2$, $d_n(z)$ is positive and monotonically increasing on $(-\infty, 0)$.

(c) The following asymptotics holds:

$$d_0(z) = -\frac{\sqrt{2}\mathfrak{m}}{(2\pi)^2} \ln(-\mathfrak{m}z) + C + O(1), \quad \text{as } z \rightarrow 0-. \quad (5.4)$$

Proof: We have the following representation:

$$\frac{e^{i(n,t)}}{\varepsilon_f(t) - z} = \mathfrak{m} \sum_{\eta=0}^{\infty} \frac{e^{i(n,t)}}{(2 - \mathfrak{m}z)^{\eta+1}} (e(t))^\eta, \quad (5.5)$$

valid for $z < 0$, since $2 - \mathfrak{m}z > e(t)$, where

$$e(t) = \cos t_1 + \cos t_2, \quad p = (p_1, p_2) \in \mathbb{T}^2.$$

We represent the function $d_n(z)$ as

$$d_n(z) = \sum_{\eta=0}^{\infty} \frac{1}{(2 - \mathfrak{m}z)^{\eta+1}} A_\eta, \quad \text{when } z < 0. \quad (5.6)$$

where

$$\begin{aligned} A_\eta &= \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{i(n,t)} (\cos t_1 + \cos t_2)^\eta dt \\ &= \frac{1}{2^\eta (2\pi)^2} \sum_{\substack{0 \leq k, k' \in \mathbb{N}_0^2, \\ k_1 + k_2 + k'_1 + k'_2 = \eta}} \frac{\eta!}{k_1! k_2! k'_1! k'_2!} \int_{\mathbb{T}^2} e^{i(n,t)} e^{i(k-k',t)} dt \end{aligned} \quad (5.7)$$

and

$$\mathbb{N}_0^2 = \mathbb{N}_0 \times \mathbb{N}_0$$

of the natural set

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}.$$

According to the identity

$$\frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{i(n,t)} e^{i(k-k',t)} dt = \begin{cases} 1, & \text{for } n + k - k' = 0 \\ 0, & \text{for } n + k - k' \neq 0, \end{cases} \quad k, k' \in \mathbb{N}_0^2, \quad (5.8)$$

the number A_η is non-negative for any $\eta \in \mathbb{N}$ and strictly positive if there exist $k, k' \in \mathbb{N}_0^2$ such that $n + k - k' = 0$ with $k_1 + k_2 + k'_1 + k'_2 = \eta$.

Let us take new expression for $k = (k_1, k_2)$, as

$$k_j = \begin{cases} -n_j + s_j, & \text{if } n_j < 0 \\ s_j, & \text{if } n_j \geq 0 \end{cases} \quad (5.9)$$

where $s_j \geq 0$ and for

$$\begin{aligned} k' &= (k'_1, k'_2), \\ k'_j &= \begin{cases} n_j + s_j, & \text{if } n_j > 0, \\ s_j, & \text{if } n_j \leq 0. \end{cases} \end{aligned} \quad (5.10)$$

Due to the conditions $n + k - k' = 0$ with $k_1 + k_2 + k'_1 + k'_2 = \eta$ we get

$$k_1 + k_2 + k'_1 + k'_2 = |n|_1 + 2(s_1 + s_2) = \eta$$

which implies that the number $\eta - |n|_1$ must be even, and moreover

$$\begin{aligned} A_\eta &= 0, \quad \text{for } \eta < |n|_1 \quad \text{or} \quad \eta - |n|_1 \text{ is odd,} \\ A_\eta &> 0, \quad \text{for } \eta \geq |n|_1 \quad \text{and} \quad \eta - |n|_1 \text{ is even} \end{aligned} \quad (5.11)$$

Taking into account the relations $\eta \geq |n|_1$, the evenness of $\eta - |n|_1$ and the equality $k_1 + k_2 + k'_1 + k'_2 = |n|_1 + 2(s_1 + s_2) = \eta$ we get

$$A_\eta = \frac{1}{2^\eta} \sum_{2(s_1 + s_2) = \eta - |n|_1} \frac{\eta!}{s_1! s_2! (|n|_1! + s_1)! (|n|_2! + s_2)!} \quad (5.12)$$

Introducing a new variable $\xi = (\eta - |n|_1)/2$ and with regard to equations (5.6) and (5.11) we receives

$$d_n(z) = \mathfrak{m} \sum_{\xi=0}^{\infty} \frac{1}{(2 - \mathfrak{m}z)^{2\xi + |n|_1 + 1}} \frac{1}{2^{2\xi + |n|_1}} A_{|n|_1 + 2\xi}, \quad \text{when } z < 0. \quad (5.13)$$

The relation (5.12), and the expansion (5.6) and (5.13) prove the first part of the lemma.

(b) This part is an immediate consequence of the part (a), as every term of (5.3) is positive and monotonically increasing in the interval $(-\infty, 0)$.

(c) The proof of the following asymptotics can be found in [28]:

$$d_0(z) = -\frac{\sqrt{2}\mathfrak{m}}{2\pi} \ln(-\mathfrak{m}z) + \mathfrak{m} \left(\frac{1}{2} - \frac{\sqrt{2}}{\pi} \right) + O(1), \quad \text{as } z \rightarrow 0- \quad (5.14)$$

As the inequality

$$\left| \frac{e^{ikx} - 1}{e^{ix} - 1} \right| \leq |k| \quad \text{for } x \neq 0,$$

we get

$$\left| \frac{e^{i(n_1 t_1 + n_2 t_2)} - 1}{\varepsilon(t)/\mathfrak{m} - z} \right| \leq \mathfrak{m}|n|_1 \quad \text{for } n \in \mathbb{Z}^2 \quad \text{and } t \in \mathbb{T}^2.$$

Thus, the integrand of the right hand side of the difference

$$d_n(z) - d_0(z)$$

is bounded, and $|d_n(z) - d_0(z)|$ is bounded and continuous at 0 from the left.

Consequently, the last fact, the expansion (5.14) and the relation

$$d_n(z) = d_0(z) + (d_n(z) - d_0(z))$$

complete the proof of (c). $\square$

The operators V_1 and V_2 can be expressed as $V_1 = \lambda \varphi_1^* \varphi_1$ and $V_2 = \lambda \varphi_2^* \varphi_2$, respectively, where $\varphi_1, \varphi_2 : L^2(\mathbb{T}^2 \times \mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$ are defined by

$$(\varphi_1 f)(p) = \frac{1}{2\pi} \int_{\mathbb{T}^2} f(p, t) dt, \quad (\varphi_2 f)(q) = \frac{1}{2\pi} \int_{\mathbb{T}^2} f(t, q) dt.$$

Their adjoint operators $\varphi_1^*, \varphi_2^* : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2 \times \mathbb{T}^2)$ are given by

$$(\varphi_1^* g)(p, q) = \frac{1}{2\pi} g(p), \quad (\varphi_2^* g)(p, q) = \frac{1}{2\pi} g(q), \quad g \in L^2(\mathbb{T}^2).$$

Lemma 5.2. *The number $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(A_1(K))$ is an eigenvalue of the operator $A_1(K)$ if and only if $D(z) = 0$, where*

$$D(z) = \frac{1}{\Delta^2(\lambda; z)} \prod_{n \in \mathbb{Z}^2} \Delta_n^+(\lambda; z), \quad (5.15)$$

with $\Delta_n^+(\lambda; z)$ defined in (2.8), i.e.

$$\Delta_n^+(\lambda; z) = 1 - \lambda(d_0(z) + d_n(z)).$$

If $z_n \in \mathbb{C} \setminus \sigma_{\text{ess}}(A_1(K))$ is an eigenvalue of $A_1(K)$, then $D(z_n) = 0$, $n \in \mathbb{Z}^2$ and the corresponding eigenfunction is of the form

$$f_n(p, q) = \frac{\lambda}{\varepsilon_f(K - p - q) - z_n} \left(\cos(n, (-K/2 + p)) + \cos(n, (-K/2 + q)) \right) \quad (5.16)$$

Proof: Given the unitary equivalence of $H(K)$ and $A_1(K)$ to $H(0)$ and $A_1(0)$, respectively, we first establish the claim for the latter operators.

Let $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(A_1(0))$ be an eigenvalue of $A_1(0)$, and let f be the corresponding eigenfunction, then

$$f = R_0(z)[V_1 + V_2]f, \quad (5.17)$$

where $R_0(z) = (H_0(0) - zI)^{-1}$ is a resolvent of $H_0(0)$.

This equation has a non-trivial solution if and only if the system of two linear equations

$$\tilde{\phi}_\alpha = \varphi_\alpha(R_0(z)[\lambda \varphi_1^* \tilde{\phi}_1 + \lambda \varphi_2^* \tilde{\phi}_2]), \quad \alpha = 1, 2, \quad \tilde{\phi}_1, \tilde{\phi}_2 \in L^2(\mathbb{T}^2) \quad (5.18)$$

on the space $L^2(\mathbb{T}^2) \oplus L^2(\mathbb{T}^2)$ has a non-zero solution.

Solutions of equations (5.17) and (5.18) are linked by

$$f(p, q) = R_0(z)[\lambda \varphi_1^* \tilde{\phi}_1 + \lambda \varphi_2^* \tilde{\phi}_2], \quad (5.19)$$

and

$$\tilde{\phi}_\alpha = \varphi_\alpha f, \quad \alpha = 1, 2.$$

Since f is symmetric, the functions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ coincide; only their arguments are denoted differently. In particular,

$$\tilde{\phi}_2(p) = \tilde{\phi}_1(p). \quad (5.20)$$

We note that the operators

$$\Delta_\alpha := I - \lambda \varphi_\alpha R_0(z) \varphi_\alpha^*, \quad z \in \mathbb{C} \setminus \sigma_{ess}(H_0(0)), \quad \alpha = 1, 2,$$

are invertible for any $z \in \mathbb{C} \setminus \sigma_{ess}(A_1(0))$. Therefore,

$$\Delta_\alpha^{-1} = (I - \lambda \varphi_\alpha R_0(z) \varphi_\alpha^*)^{-1}.$$

Then the functions $\tilde{\phi}_\alpha$, $\alpha = 1, 2$, satisfying equation (5.18), also satisfy the following system of integral equations:

$$\begin{cases} \tilde{\phi}_1 = \lambda \Delta_1^{-1} Q \tilde{\phi}_2, \\ \tilde{\phi}_2 = \lambda \Delta_2^{-1} Q^* \tilde{\phi}_1, \end{cases} \quad (5.21)$$

where

$$Q = \varphi_1 R_0(z) \varphi_2^*$$

is the integral operator on $L^2(\mathbb{T}^2)$ defined by

$$(Q\phi)(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{\phi(t) dt}{\varepsilon_f(-p-t) - z}, \quad \phi \in L^2(\mathbb{T}^2)$$

and Q^* is the adjoint operator of Q .

Using the substitution method, system (5.21) can be reduced to

$$\tilde{\phi}_1 = Q(z) \tilde{\phi}_1, \quad Q(z) = \lambda^2 \Delta_1^{-1} Q \Delta_2^{-1} Q^*. \quad (5.22)$$

Moreover, if

$$\Phi = (\tilde{\phi}_1, \tilde{\phi}_2)$$

is a solution of (5.21), then $\tilde{\phi}_1$ is an eigenfunction of the operator $Q(z)$ corresponding to the eigenvalue 1.

Then

$$\Phi = (\tilde{\phi}_1, \tilde{\phi}_2), \quad \tilde{\phi}_2 = \frac{\lambda}{\Delta(\lambda; z)} Q \tilde{\phi}_1,$$

is a solution of (5.21) (equivalently, of (5.18)).

Notice that the multiplicities of the linearly independent eigenvectors $\tilde{\phi}_1$ and Φ coincide.

We also note that the function f defined by (5.19) is an eigenfunction of $A_1(0)$ corresponding to the eigenvalue $z \in \mathbb{C} \setminus \sigma_{ess}(A_1(0))$.

Moreover, the multiplicity of the eigenvalue z of $A_1(0)$ coincides with the multiplicity of the eigenvalue 1 of the operator $Q(z)$.

The operator $Q(z)$ is a convolution-type trace-class integral operator. The standard Fourier transform

$$\mathcal{F}_1 : L^2(\mathbb{T}^2) \rightarrow \ell^2(\mathbb{Z}^2)$$

defined by

$$(\mathcal{F}_1 g)(n) = \frac{1}{2\pi} \int_{\mathbb{T}^2} e^{i(n,p)} g(p) dp, \quad g \in L^2(\mathbb{T}^2), \quad n \in \mathbb{Z}^2,$$

transforms $Q(z)$ into the multiplication operator

$$\hat{Q}(z) := \mathcal{F}_1 Q(z) \mathcal{F}_1^*$$

on $\ell^2(\mathbb{Z}^2)$.

Since the kernel of $Q(z)$ is of convolution type, a direct computation of its Fourier coefficients yields

$$\kappa_n(z) = \frac{\lambda^2}{(\Delta(\lambda; z))^2} d_n(z) d_{-n}(z), \quad n \in \mathbb{Z}^2. \quad (5.23)$$

Thus, the spectrum of $\hat{Q}(z)$ consists of the following union

$$\sigma(\hat{Q}(z)) = \{0\} \cup \bigcup_{n \in \mathbb{Z}^2} \{\kappa_n(z)\},$$

with the space of eigenfunctions

$$\hat{\varphi}_n(m) = \delta_{n,m}, \quad n, m \in \mathbb{Z}^2,$$

where $\delta_{\cdot, \cdot}$ is the Kronecker delta function on $\mathbb{Z}^2$.

Note that the compact operator $Q(z)$ has eigenvalues $\kappa_n(z)$, $n \in \mathbb{Z}^2$ with the corresponding eigenfunctions

$$\psi_n(p) = e^{i(n,p)}, \quad n \in \mathbb{Z}^2, p \in \mathbb{T}^2.$$

Therefore, the determinant of the operator $I - Q(z)$ can be written as the following product

$$\det(I - Q(z)) = \prod_{n \in \mathbb{Z}^2} (1 - \kappa_n(z)), \quad (5.24)$$

which takes the form (5.15), since $d_{-n}(z) = d_n(z)$ and (5.23).

Let $\kappa_n(z_n) = 1$ be an eigenvalue of $Q(z_n)$, then ψ_n is the first component of the solution $\varphi_n = (\psi_n, \tilde{\psi}_n)$ of equation (5.18), and the second component is defined as

$$\tilde{\psi}_n(q) := \tilde{\varphi}_2(q) = \lambda \Delta_2^{-1} Q^* \psi_n(q) = \frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} \tilde{\psi}_{-n}(q). \quad (5.25)$$

Using equality (5.20) in the last relation, we get

$$e^{i(n,q)} = \lambda \Delta_2^{-1} Q^* \psi_n(q) = \frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} e^{-i(n,q)},$$

which is contradiction.

However, if $\vartheta_n(p) = (e^{i(n,p)} + e^{-i(n,p)})/2 = \cos((n,p))$, it satisfies (5.20) and (5.25) if and only if

$$\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} = 1, \quad \text{i.e.} \quad \Delta_n^+(\lambda; z) = 1 - \lambda(d_0(z_n) + d_n(z_n)) = 0.$$

However, for the functions $\theta_n(p) = (e^{i(n,p)} - e^{-i(n,p)})/2i = \sin((n,p))$ the relation

$$\theta_n = \lambda \Delta_2^{-1} Q^* \theta_n = -\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} \theta_n,$$

holds, which contradicts with (5.20). It implies

$$-\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} = 1, \quad \text{i.e.} \quad \Delta_n^-(\lambda; z) = 1 - \lambda(d_0(z_n) - d_n(z_n)) = 0.$$

Since the system $\{\vartheta_n, \theta_n\}$, $n \in \mathbb{Z}^2$ is complete in $\ell^2(\mathbb{Z}^2)$, the last three relations and (5.23) allow us to use the Fredholm determinant (5.15) instead of (5.24).

Consequently, the number z_n is an eigenvalue of $A_1(0)$, and the corresponding eigenfunction is given by (5.19):

$$f_n^0 = R_0(z_n) [\lambda \varphi_1^* \psi_n + \lambda \varphi_2^* \tilde{\psi}_n].$$

i.e.,

$$f_n^0(p, q) = \frac{\lambda}{\varepsilon_{\mathbb{F}}(-p - q) - z} (\cos(n, p) + \cos(n, q)).$$

According to the relation $H(0) = U_K^* H(K) U_K$, the number z_n is also an eigenvalue of $H(K)$ with the eigenfunction $f_n = U_K f_n^0$ in (5.16), where the unitary operator U_K is defined in (2.10). $\square$

5.3. Eigenvalues of $H(K)$ below and above the spectrum of $H_0(K)$. Let z_1^0 be a zero of $\Delta_\alpha(\cdot) = 0$ in Lemma 2.1.

Lemma 5.3. *For any fixed $n \in \mathbb{Z}^2$, the function $d_0(z) + d_n(z)$ is positive and monotonically increasing on the interval $(-\infty, 0)$.*

Additionally,

$$\lim_{z \rightarrow -\infty} (d_0(z) + d_n(z)) = 0, \quad (5.26)$$

$$\lim_{z \rightarrow 0-} (d_0(z) + d_n(z)) = \infty, \quad (5.27)$$

and for any $z \in (-\infty, 0)$

$$\lim_{|n| \rightarrow \infty} d_n(z) = 0 \quad (5.28)$$

hold.

Proof: The equalities

$$d_0(z) + d_n(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{1 + \cos(nt)}{\varepsilon_{\mathfrak{f}}(t) - z} dt$$

and part (b) of Lemma 5.1 imply the positivity and monotonicity of $d_0(z) + d_n(z)$ in $(-\infty, 0)$.

Limits (5.26) and (5.28) follows from part (a) in Lemma 5.1.

Part (c) provides (5.27). □

5.3.1. Proof of Lemma 2.3. (a) Let z_1^0 be a zero of $\Delta(\lambda; \cdot) = 0$ (see Lemma 2.1).

According to Lemma 5.3, the function $\Delta_n^+(\lambda; z)$ is monotonically increasing in $(-\infty, 0)$ and

$$\begin{aligned} \lim_{z \rightarrow -\infty} \Delta_n^+(\lambda; z) &= 1, \\ \Delta_n^+(\lambda; z_1^0) &< \Delta(\lambda; z_1^0) = 0, \end{aligned}$$

holds.

Consequently, the function $\Delta_n^+(\lambda; z)$ has a unique zero z_n in the interval $(-\infty, z_1^0)$ if $\lambda > 0$.

The equalities

$$\begin{aligned} \Delta_n^+(\lambda; z_1^0) &= (\Delta_n^+(\lambda; \xi_n))'(z_1^0 - z_n), \\ \Delta_n^+(\lambda; z_1^0) &= -\lambda d_n(z_1^0), \\ (\Delta_n^+(\lambda; z))' &= -\lambda(d_0'(z) + d_n'(z)) \end{aligned}$$

where $z_n < \xi_n < z_1^0$ is a number obtained due to the intermediate value theorem, imply

$$z_n - z_1^0 = -\frac{d_n(z_1^0)}{d_0'(\xi_n) + d_n'(\xi_n)}$$

Applying the limit (5.28) in the last inequality, the last equality, we get the proof of the limit in (2.9), i.e.

$$\lim_{|n| \rightarrow \infty} z_n = z_1^0.$$

(b) If $\lambda = 0$, then $\Delta_n^+(\lambda; z) = 1$.

(c) The unitary operator $U_{\pi/2}$ in (2.10) is used to establish the equalities

$$U_{\pi/2} H_0(K) U_{\pi/2} = \frac{2}{\mathfrak{m}} - H_0(K) \quad \text{and} \quad U_{\pi/2} V U_{\pi/2} = V$$

which implies the relation

$$U_{\pi/2} (H_0(K) - V) U_{\pi/2} = \frac{2}{\mathfrak{m}} - (H_0(K) + V). \quad (5.29)$$

The final relation allows us to reduce the investigation of the eigenvalues of $H(K)$ above the interval $[0, 4/\mathfrak{m}]$ to the study of those below it, and vice versa.

The proof of part (b) is a direct consequence of part (a) and the identity (5.29). □

5.4. **Proof of the main theorem: Theorem 2.2.** Lemmas 2.2 and 2.3 and

$$\sigma_{pp}(H(K)) = \sigma_{pp}(A_0(K)) \bigcup \sigma_{pp}(A_1(K)) \quad (5.30)$$

provide the proof.

For example, if $\lambda < 0$, part (a) of Lemma 2.2 provides that $A_0(K)$ has a simple eigenvalue η , $\eta \in (4/\mathfrak{m} - \mu, \infty)$ and part (c) of Lemma 2.3 implies that $A_1(K)$ has infinitely many eigenvalues $\{z_n\} \in (4/\mathfrak{m} - \mu, \infty)$.

The last facts together with (5.30) give the proof of the part (a) of the theorem. $\square$

CONCLUSION

We have considered the discrete Schrödinger operator associated with the Hamiltonian of a system of three quantum particles (two identical bosons and one fermion) on a two-dimensional lattice.

The bosons are assumed to have infinite mass, while the fermion has finite mass:

$$m = \infty, \quad \mathfrak{m} < \infty.$$

The system was studied for all nonzero interaction parameters. We proved that the corresponding three-particle discrete Schrödinger operator possesses infinitely many eigenvalues.

The appearance of infinitely many eigenvalues is related to the singular behavior of the associated Birman–Schwinger type operator near the threshold of the essential spectrum. In the case $m = \infty$, the kinetic energy contribution of the bosons disappears, which changes the threshold behavior of the corresponding resolvent and leads to an accumulation of eigenvalues near the edge of the essential spectrum.

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