

Coupled coincidence point sets of α -dominated mappings and stability analysis

Kundu S., Dutta S., Mondal S. P., Metiya N., Bandyopadhyay C.

Abstract. In this paper we utilize a rational metric inequality for the purpose of establishing a coupled coincidence point theorem for a coupled mapping and a single-valued mapping on a metric space. We define some regularity condition on the mappings which are extensions of ideas involved in the well known admissibility condition. In the second part of the paper we make a stability analysis of the set of coupled coincidence points obtained here. All the main results are supported with illustrative examples. The work is in the domain of set valued analysis.

Keywords: Metric space; dominated maps; Hausdorff distance; coupled coincidence point, stability
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1. INTRODUCTION AND MATHEMATICAL BACKGROUND

The present paper is about multi-valued coupled coincidence points and stability associated with coupled coincidence point sets of certain multi-valued mappings and single-valued mapping defined on complete metric spaces. The idea of a coupled fixed point originated in the work of Guo et al [1] in 1987. It attracts much attention after the appearance of the work of Bhaskar et al [2] in 2006 where a coupled contraction mapping theorem was established in partially ordered metric spaces. Several other works followed in a large number exploring the coupled fixed point and related properties of different types of coupled mappings in various types of spaces. Some instances of works in the extensive literature created on this research area in the last decade are [3, 4, 5]. The concept of a coupled fixed point was extended to the domain of set-valued analysis by [6] after which several works have appeared in this direction like those noted in [7, 8, 9].

Fixed point theory in metric spaces equipped with a partial order has developed rapidly in recent years. Although an early result appeared in the work of Turinici [10] in 1986, the actual development of this line of research took place following the works of Ran et al. [11] and Nieto et al. [12]. Some recent works in this area are [13, 14, 15, 16]. The logical content of this study is that for the existence of fixed points, the various types of contractive inequality conditions to be satisfied by the operators for pairs of points collected from the metric space can be restricted. Several of these results are still valid in cases where the corresponding contractive conditions are assumed for those pairs of points related through the partial order. In fact, upon close examination of the proofs of several fixed point theorems of various types of contractive operators including the celebrated Banach's contraction mapping principle [17] one finds that the satisfaction of contraction inequality conditions are not used between every pair of points from the metric space. It is possible to restrict it to suitable pairs of points without disturbing the conclusions of the theorems. This is what is accomplished through the introduction of the partial order relation. In particular, in their work Bhaskar et al [2] considered partially ordered metric spaces and used the mixed-monotone property of the coupled maps corresponding to the partial ordering. Some works in this subject area are [3, 4, 5].

It has been shown by Samet et al [18] that the above purpose of introducing the partial order can be also served by a set of conditions called admissibility conditions. Originally α -admissibility was introduced in [18]. Later, several other types of admissibilities have been introduced and used in the domain of fixed point theory. These conditions, rather than introducing a new structure like partial order in metric spaces, are requirements on the operator under consideration. We note a few works from this recently originated branch of study in [19, 20, 21, 22, 23, 24].

Stability is a concept associated with the limiting behaviors of a system. It has been studied in various contexts of discrete and continuous dynamical systems [25, 26]. The study of the relationship between the convergence of a sequence of mappings and their fixed points, known as the stability of fixed points has also been widely studied in various settings [27, 28, 29, 30, 31]. The fixed point sets

of a sequence of mappings are said to be stable if they converge to the set of fixed points of the limit mapping in the Hausdorff metric. Multi-valued mappings often have more fixed points than their single-valued counterparts [29, 30, 31]. Therefore, the set of fixed points of multi-valued mappings becomes larger and hence more interesting for the study of stability.

In the present context, putting together some of the ideas mentioned above, we work out a fixed point result for coupled multi-valued mappings on complete metric spaces. In the last part of the paper we perform the stability analysis of the fixed point sets corresponding to a convergent sequence of multi-valued coupled mappings of the type considered here.

In the following, we discuss some concepts and definitions that we subsequently use in this paper.

Let (X, d) be a metric space. Let $N(X) :=$ the collection of all non-empty subsets of X ; $CL(X) :=$ the collection of all non-empty closed and bounded subsets of X ; and $K(X) :=$ the collection of all compact subset of X . Let $A, B \in CL(X)$, we use following notations and definitions

$$D(x, B) = \inf \{d(x, y) : y \in B\}, \text{ where } x \in X,$$

$$d(A, B) = D(A, B) = \inf \{d(a, b) : a \in A, b \in B\},$$

$$H(A, B) = \max \left\{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A) \right\},$$

H is known as the Hausdorff metric on $CL(X)$ [31]. Furthermore, if (X, d) is complete then $(CL(X), H)$ is also complete.

Lemma 1.1. Let (X, d) be a metric space and $B \in K(X)$. Then for every $x \in X$ there exists $y \in B$ such that $d(x, y) = D(x, B)$.

Proof. Let $B \in K(X)$ and $x \in X$. From the definition, $D(x, B) = \inf \{d(x, b) : b \in B\}$. Then there exists a sequence $\{y_n\}$ in B such that $d(x, y_n) \rightarrow D(x, B)$, as $n \rightarrow \infty$. Since B is compact, $\{y_n\}$ has a convergent subsequence $\{y_{n(k)}\}$. Hence, there exists $y \in X$ such that $y_{n(k)} \rightarrow y$ as $k \rightarrow \infty$. As B is compact, it is closed and hence $y \in B$. Now, $\lim_{n \rightarrow \infty} d(x, y_n) = D(x, B)$ implies that $\lim_{k \rightarrow \infty} d(x, y_{n(k)}) = D(x, B)$, that is, $d(x, y) = D(x, B)$.

Definition 1.1. [5] Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$. Then X is said to have α -regular property if $\{x_n\}$ is a sequence with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x$ then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

We propose the following three definitions for our purpose in the next section. These are conceptual extensions of the admissibility conditions[18].

Definition 1.2. Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping. A mapping $F : X \rightarrow X$ is said to be α -dominated if $\alpha(x, Fx) \geq 1$, for all $x \in X$.

Definition 1.3. Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping. A multi-valued mapping $F : X \rightarrow N(X)$ is said to be α -dominated if $\alpha(x, y) \geq 1$, for $x \in X$ and $y \in Fx$.

Definition 1.4. Let X be a non-empty set and $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping. A multi-valued mapping $F : X \times X \rightarrow N(X)$ is said to be α -dominated with respect to a function $f : X \rightarrow X$ if for $x, y \in X$, $\alpha(fx, u) \geq 1$ and $\alpha(fy, v) \geq 1$ whenever $u \in F(x, y)$ and $v \in F(y, x)$.

Definition 1.5. [7] Let X be a non-empty set. An element $(x, y) \in X \times X$ is said to be a coupled coincidence point of a multi-valued mapping $F : X \times X \rightarrow N(X)$ and a single-valued $f : X \rightarrow X$ if $fx \in F(x, y)$ and $fy \in F(y, x)$.

We denote the set of all coupled coincidence points of F and f by the notation $C(F, f)$.

We propose the following definition of stability of coupled coincidence point sets. If (X, d) is a metric space, then $X \times X$ is also a metric space under the metric ρ defined by $\rho((x, y), (u, v)) = \max \{d(x, u), d(y, v)\}$, for all (x, y) and $(u, v) \in X \times X$. Let H_ρ be the Hausdorff metric induced by ρ .

Definition 1.6. Let (X, d) be a metric space, $F: X \times X \rightarrow CL(X)$ be a multi-valued mapping and $f: X \rightarrow X$. Let $\{F_n: X \times X \rightarrow CL(X)\}$ be a sequence of multi-valued mappings with $F = \lim_{n \rightarrow \infty} F_n$. Suppose that $\{C(F_n, f)\}$ is the sequence of coupled coincidence point sets of the sequence of pairs of mappings $\{(F_n, f) : n \in \mathbb{N}\}$ and $C(F, f)$ is the coupled coincidence point set of the pair of mappings (F, f) . We say that the coupled coincidence point sets of the sequence $\{(F_n, f) : n \in \mathbb{N}\}$ are stable if $\lim_{n \rightarrow \infty} H_\rho(C(F_n, f), C(F, f)) = 0$.

The above definition is proposed by incorporating the same ideas and in the same vein of Nadler on stability in his work [31] which is incidentally the first paper on set-valued fixed point theory.

2. MAIN RESULTS

Theorem 2.1. Let (X, d) be a complete metric space, $F: X \times X \rightarrow K(X)$ be a multi-valued mapping, $f: X \rightarrow X$ and $\alpha: X \times X \rightarrow [0, \infty)$. Suppose that (i) (X, d) has α -regular property, (ii) F is α -dominated with respect to f , (iii) $F(x, y) \subseteq f(X)$ for all $(x, y) \in X \times X$, (iv) $f(X)$ is a closed subset of X , (v) there exist nonnegative real numbers a, b, c, e with $(a + b + 2c + 2e) < 1$ such that for all $x, y, u, v \in X$ with $\alpha(fx, fu) \geq 1$ and $\alpha(fy, fv) \geq 1$,

$$\begin{aligned} H(F(x, y), F(u, v)) &\leq a \frac{D(fx, F(x, y)) D(fu, F(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\ &\quad + c [D(fx, F(x, y)) + D(fu, F(u, v))] \\ &\quad + e [D(fx, F(u, v)) + D(fu, F(x, y))]. \end{aligned}$$

Then the set of coupled coincidence point of F and f is non-empty, that is, $C(F, f) \neq \emptyset$.

Proof. Let $x_0, y_0 \in X$ be arbitrary. The assumption (iii) implies that for every $z \in F(x, y)$ there exists $p \in X$ such that $fp = z$. Suppose $x_1, y_1 \in X$ such that $fx_1 \in F(x_0, y_0)$ and $fy_1 \in F(y_0, x_0)$. By the assumption (ii), we have

$$\alpha(fx_0, fx_1) \geq 1 \quad \text{and} \quad \alpha(fy_0, fy_1) \geq 1.$$

By Lemma 1.1, there exist $x_2, y_2 \in X$ with $fx_2 \in F(x_1, y_1)$ and $fy_2 \in F(y_1, x_1)$ such that

$$d(fx_1, fx_2) = D(fx_1, F(x_1, y_1)) \quad \text{and} \quad d(fy_1, fy_2) = D(fy_1, F(y_1, x_1)).$$

By the assumption (ii), we have

$$\alpha(fx_1, fx_2) \geq 1 \quad \text{and} \quad \alpha(fy_1, fy_2) \geq 1.$$

Again, by Lemma 1.1, there exist $x_3, y_3 \in X$ with $fx_3 \in F(x_2, y_2)$ and $fy_3 \in F(y_2, x_2)$ such that

$$d(fx_2, fx_3) = D(fx_2, F(x_2, y_2)) \quad \text{and} \quad d(fy_2, fy_3) = D(fy_2, F(y_2, x_2)).$$

Using the assumption (ii), we have

$$\alpha(fx_2, fx_3) \geq 1 \quad \text{and} \quad \alpha(fy_2, fy_3) \geq 1.$$

Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $n \geq 0$,

$$fx_{n+1} \in F(x_n, y_n) \quad \text{and} \quad fy_{n+1} \in F(y_n, x_n), \quad (2.1)$$

$$d(fx_{n+1}, fx_{n+2}) = D(fx_{n+1}, F(x_{n+1}, y_{n+1})), \quad (2.2)$$

$$d(fy_{n+1}, fy_{n+2}) = D(fy_{n+1}, F(y_{n+1}, x_{n+1})) \quad (2.3)$$

and

$$\alpha(fx_n, fx_{n+1}) \geq 1 \quad \text{and} \quad \alpha(fy_n, fy_{n+1}) \geq 1. \quad (2.4)$$

We first prove that the sequences $\{fx_n\}$ and $\{fy_n\}$ are Cauchy sequences. By (2.3), (2.4) and the assumption (v), we have

$$\begin{aligned}
 d(fx_{n+1}, fx_{n+2}) &= D(fx_{n+1}, F(x_{n+1}, y_{n+1})) \leq H(F(x_n, y_n), F(x_{n+1}, y_{n+1})) \\
 &\leq a \frac{D(fx_n, F(x_n, y_n)) D(fx_{n+1}, F(x_{n+1}, y_{n+1}))}{1 + d(fx_n, fx_{n+1})} + b d(fx_n, fx_{n+1}) \\
 &\quad + c [D(fx_n, F(x_n, y_n)) + D(fx_{n+1}, F(x_{n+1}, y_{n+1}))] \\
 &\quad + e [D(fx_n, F(x_{n+1}, y_{n+1})) + D(fx_{n+1}, F(x_n, y_n))] \\
 &\leq a \frac{d(fx_n, fx_{n+1}) d(fx_{n+1}, fx_{n+2})}{1 + d(fx_n, fx_{n+1})} + b d(fx_n, fx_{n+1}) \\
 &\quad + c [d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2})] \\
 &\quad + e [d(fx_n, fx_{n+2}) + d(fx_{n+1}, fx_{n+1})] \\
 &\leq a d(fx_{n+1}, fx_{n+2}) + b d(fx_n, fx_{n+1}) \\
 &\quad + c [d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2})] \\
 &\quad + e [d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2})], \tag{2.5}
 \end{aligned}$$

which implies that

$$d(fx_{n+1}, fx_{n+2}) \leq \frac{[b + c + e]}{[1 - a - c - e]} d(fx_n, fx_{n+1}). \tag{2.6}$$

Similarly, we have

$$d(fy_{n+1}, fy_{n+2}) \leq \frac{[b + c + e]}{[1 - a - c - e]} d(fy_n, fy_{n+1}). \tag{2.7}$$

Let $k = \frac{(b+c+e)}{1-(a+c+e)}$, then $0 \leq k < 1$. By repeated application of (2.6) and (2.7), we respectively have

$$d(fx_{n+1}, fx_{n+2}) \leq k d(fx_n, fx_{n+1}) \leq k^2 d(fx_{n-1}, fx_n) \leq \dots \leq k^{n+1} d(fx_0, fx_1) \tag{2.8}$$

and

$$d(fy_{n+1}, fy_{n+2}) \leq k d(fy_n, fy_{n+1}) \leq k^2 d(fy_{n-1}, fy_n) \leq \dots \leq k^{n+1} d(fy_0, fy_1). \tag{2.9}$$

Using (2.8), we have for $m > n > 0$ that

$$\begin{aligned}
 d(fx_n, fx_m) &\leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \dots + d(fx_{m-1}, fx_m) \\
 &\leq [k^n + k^{n+1} \dots + k^{m-1}] d(fx_0, fx_1) \\
 &\leq \frac{k^n [1 - k^{n-m+1}]}{1 - k} d(fx_0, fx_1) \leq \frac{k^n}{1 - k} d(fx_0, fx_1). \tag{2.10}
 \end{aligned}$$

Since $0 < k < 1$, taking limit as $m, n \rightarrow \infty$ in (2.10), we have

$$\lim_{m, n \rightarrow \infty} d(fx_n, fx_m) = 0, \tag{2.11}$$

which implies that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Similarly, we can show that $\{fy_n\}$ is a Cauchy sequence in $f(X)$. Since X is complete, by the assumption (iv), $(f(X), d)$ is also complete. So there exist $x, y \in X$ such that

$$\lim_{n \rightarrow \infty} fx_n = fx \quad \text{and} \quad \lim_{n \rightarrow \infty} fy_n = fy. \tag{2.12}$$

We now prove that this (x, y) is a coupled coincidence point of F and f . Using (2.4), (2.12) and the assumption (i), we have $\alpha(fx_n, fx) \geq 1$ and $\alpha(fy_n, fy) \geq 1$ for all n . Now applying the assumption

(v), we have

$$\begin{aligned}
 D(fx_{n+1}, F(x, y)) &\leq H(F(x_n, y_n), F(x, y)) \\
 &\leq a \frac{D(fx_n, F(x_n, y_n)) D(fx, F(x, y))}{1 + d(fx_n, fx)} + b d(fx_n, fx) \\
 &\quad + c [D(fx_n, F(x_n, y_n)) + D(fx, F(x, y))] \\
 &\quad + e [D(fx_n, F(x, y)) + D(fx, F(x_n, y_n))] \\
 &\leq a \frac{d(fx_n, fx_{n+1}) D(fx, F(x, y))}{1 + d(fx_n, fx)} + b d(fx_n, fx) \\
 &\quad + c [d(fx_n, fx_{n+1}) + D(fx, F(x, y))] \\
 &\quad + e [D(fx_n, F(x, y)) + d(fx, fx_{n+1})].
 \end{aligned}$$

Now taking limit as $n \rightarrow \infty$ in the above inequality and using (2.12), we have

$$D(fx, F(x, y)) \leq c D(fx, F(x, y)) + e D(fx, F(x, y)),$$

that is,

$$(1 - c - e) D(fx, F(x, y)) \leq 0.$$

Since $1 - c - e > 0$, it follows that $D(fx, F(x, y)) = 0$. Since $F(x, y) \in K(X)$. So $fx \in F(x, y)$. Similarly we can prove that $fy \in F(y, x)$. Hence (x, y) is a coupled coincidence point of F and f , that is, $C(F, f) \neq \emptyset$.

Example 2.1. Let $X = [0, 1]$ and d be the usual metric on X . Define $F : X \times X \rightarrow K(X)$ by $F(x, y) = [0, \frac{x}{9}]$ for $(x, y) \in X \times X$ and $f : X \rightarrow X$ by $f(x) = \frac{x}{3}$ for $x \in X$. Let $\alpha(x, y) = 2$ for all $x, y \in X$ and $a = \frac{1}{10}$, $b = \frac{1}{2}$, $c = e = \frac{1}{12}$. All the conditions of Theorem 2.1 are satisfied and $(0, 0) \in C(F, f)$, that is, $C(F, f)$ is non-empty.

3. STABILITY OF COINCIDENCE POINT SETS

In this section, we investigate the stability of coupled coincidence point sets of the set valued contractions.

Theorem 3.1. Let (X, d) be a complete metric space, $F_j : X \times X \rightarrow K(X)$, $j = 1, 2$ be two multi-valued mappings, $f : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. Suppose that (i) (X, d) has α -regular property, (ii) each F_j ($j = 1, 2$) is α -dominated with respect to f , (iii) $F_j(x, y) \subseteq f(X)$ for all $(x, y) \in X \times X$ and $j = 1, 2$, (iv) $f(X)$ is a closed subset of X , (v) there exist nonnegative real numbers a, b, c, e with $(a + b + 2c + 2e) < 1$ such that for all $x, y, u, v \in X$ with $\alpha(fx, fu) \geq 1$ and $\alpha(fy, fv) \geq 1$,

$$\begin{aligned}
 H(F_j(x, y), F_j(u, v)) &\leq a \frac{D(fx, F_j(x, y)) D(fu, F_j(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\
 &\quad + c [D(fx, F_j(x, y)) + D(fu, F_j(u, v))] \\
 &\quad + e [D(fx, F_j(u, v)) + D(fu, F_j(x, y))].
 \end{aligned}$$

Then $C(F_j, f) \neq \emptyset$ for $j = 1, 2$. Suppose $M_j = \sup\{d(x, fx), d(y, fy) : (x, y) \in C(F_j, f)\}$, where $j = 1, 2$, and $\Delta = \sup_{x, y \in X} H(F_1(x, y), F_2(x, y))$ exist and $M = \max\{M_1, M_2\}$. Then

$$H_\rho(C(F_1, f), C(F_2, f)) \leq 2M + \frac{\Delta}{1 - k}, \text{ where } k = \frac{(b + c + e)}{1 - (a + c + e)}.$$

Proof. By Theorem 2.1, $C(F_j, f) \neq \emptyset$ for $j = 1, 2$. Let $(x_0, y_0) \in C(F_1, f)$; that is

$$fx_0 \in F_1(x_0, y_0) \text{ and } fy_0 \in F_1(y_0, x_0). \quad (3.1)$$

By the assumption (iii) and Lemma 1.1 there exist $x_1, y_1 \in X$ with $fx_1 \in F_2(x_0, y_0)$ and $fy_1 \in F_2(y_0, x_0)$ such that

$$d(fx_0, fx_1) = D(fx_0, F_2(x_0, y_0)) \leq H(F_1(x_0, y_0), F_2(x_0, y_0)) \leq \Delta \quad (3.2)$$

and

$$d(fy_0, fy_1) = D(fy_0, F_2(y_0, x_0)) \leq H(F_1(y_0, x_0), F_2(y_0, x_0)) \leq \Delta. \quad (3.3)$$

By the assumption (ii) on F_2 , we have

$$\alpha(fx_0, fx_1) \geq 1 \text{ and } \alpha(fy_0, fy_1) \geq 1.$$

By the assumption (iii) and Lemma 1.1 there exist $x_2, y_2 \in X$ with $fx_2 \in F_2(x_1, y_1)$ and $fy_2 \in F_2(y_1, x_1)$ such that

$$d(fx_1, fx_2) = D(fx_1, F_2(x_1, y_1)) \leq H(F_2(x_0, y_0), F_2(x_1, y_1))$$

and

$$d(fy_1, fy_2) = D(fy_1, F_2(y_1, x_1)) \leq H(F_2(y_0, x_0), F_2(y_1, x_1)).$$

By the assumption (ii) on F_2 , we have

$$\alpha(fx_1, fx_2) \geq 1 \text{ and } \alpha(fy_1, fy_2) \geq 1.$$

Then arguing as in the proof of Theorem 2.1 we construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$fx_{n+1} \in F_2(x_n, y_n) \text{ and } fy_{n+1} \in F_2(y_n, x_n), \quad (3.4)$$

$$d(fx_{n+1}, fx_{n+2}) = D(fx_{n+1}, F_2(x_{n+1}, y_{n+1})) \leq H(F_2(x_n, y_n), F_2(x_{n+1}, y_{n+1})), \quad (3.5)$$

$$d(fy_{n+1}, fy_{n+2}) = D(fy_{n+1}, F_2(y_{n+1}, x_{n+1})) \leq H(F_2(y_n, x_n), F_2(y_{n+1}, x_{n+1})) \quad (3.6)$$

and also (2.4) is satisfied. Arguing similarly as in the proof of Theorem 2.1 we prove that (1) (2.8) and (2.9) are satisfied; (2) $\{fx_n\}$ and $\{fy_n\}$ are Cauchy sequences in $f(X)$; (3) there exist $u, v \in X$ such that $\lim_{n \rightarrow \infty} fx_n = fu$ and $\lim_{n \rightarrow \infty} fy_n = fv$; and (4) (u, v) is a coupled coincidence of F_2 and f , that is, $(u, v) \in C(F_2, f)$, that is,

$$fu \in F_2(u, v) \text{ and } fv \in F_2(v, u). \quad (3.7)$$

Using (2.8), (3.2), we have

$$\begin{aligned} d(x_0, u) &\leq d(x_0, fx_0) + d(fx_0, u) \leq d(x_0, fx_0) + \sum_{i=0}^n d(fx_i, fx_{i+1}) + d(fx_{n+1}, u) \\ &\leq d(x_0, fx_0) + \sum_{i=0}^n k^i d(fx_0, fx_1) + d(fx_{n+1}, u) \\ &\leq M_1 + \frac{1}{1-k} d(fx_0, fx_1) + d(fx_{n+1}, u) \leq M + \frac{\Delta}{1-k} + d(fx_{n+1}, u), \end{aligned}$$

where $k = \frac{(b+c+e)}{1-(a+c+e)}$ is taken in the proof of Theorem 2.1. Taking the limit as $n \rightarrow \infty$ in the above inequality, we have

$$d(x_0, u) \leq M + \frac{\Delta}{1-k} + d(fu, u) \leq M + \frac{\Delta}{1-k} + M_2 \leq 2M + \frac{\Delta}{1-k}. \quad (3.8)$$

Similarly

$$d(y_0, v) \leq 2M + \frac{\Delta}{1-k}. \quad (3.9)$$

Combining (3.8) and (3.9), we have

$$\max \{d(x_0, u), d(y_0, v)\} \leq 2M + \frac{\Delta}{1-k}.$$

that is,

$$\rho(d(x_0, y_0), d(u, v)) \leq 2M + \frac{\Delta}{1-k}.$$

So we have for $(x_0, y_0) \in C(F_1, f)$ there exists $(u, v) \in C(F_2, f)$ such that $\rho(d(x_0, y_0), d(u, v)) \leq 2M + \frac{\Delta}{1-k}$. Similarly we can show that for $(s_0, t_0) \in C(F_2, f)$ there exists $(p, q) \in C(F_1, f)$ such that $\rho(d(s_0, t_0), d(p, q)) \leq 2M + \frac{\Delta}{1-k}$. Then it follows that $H_\rho(C(F_1, f), C(F_2, f)) \leq 2M + \frac{\Delta}{1-k}$.

Lemma 3.1. *Let (X, d) be a complete metric space, $\{F_n : X \times X \rightarrow K(X)\}$ be a sequence of multi-valued mappings, $f : X \rightarrow X$ with $f(X)$ is closed in X and $\alpha : X \times X \rightarrow [0, \infty)$. Suppose that the sequence $\{F_n\}$ converges uniformly to a multi-valued mapping $F : X \times X \rightarrow K(X)$. Suppose that (i) each F_n ($n \in \mathbb{N}$) is α -dominated with respect to f ; (ii) $F_n(x, y) \subseteq f(X)$ for all $(x, y) \in X \times X$ and $n \in \mathbb{N}$; (iii) there exist nonnegative real numbers a, b, c, e with $(a + b + 2c + 2e) < 1$ such that for every $n \in \mathbb{N}$ and for all $x, y, u, v \in X$ with $\alpha(fx, fu) \geq 1$ and $\alpha(fy, fv) \geq 1$,*

$$\begin{aligned} H(F_n(x, y), F_n(u, v)) &\leq a \frac{D(fx, F_n(x, y)) D(fu, F_n(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\ &\quad + c [D(fx, F_n(x, y)) + D(fu, F_n(u, v))] \\ &\quad + e [D(fx, F_n(u, v)) + D(fu, F_n(x, y))]. \end{aligned}$$

Also, suppose that for any $u \in X$ and for any sequence $\{x_n\}$ in X the following condition is satisfied.

$$\alpha(u, x_n) \geq 1 \text{ for every } n \in \mathbb{N} \Rightarrow \alpha(u, x) \geq 1, \text{ whenever } \{x_n\} \rightarrow x \text{ as } n \rightarrow \infty. \quad (3.10)$$

Then F also satisfies the assumptions (ii), (iii) and (v) of Theorem 2.1.

Proof. Let $(x, y) \in X \times X$ and $u \in F(x, y)$. Since $F_n \rightarrow F$ uniformly, there exists a sequence $\{x_n \in F_n(x, y)\}$ such that $x_n \rightarrow u$ and as $n \rightarrow \infty$. By the assumption (i), $\alpha(fx, x_n) \geq 1$ for every $n \in \mathbb{N}$. By (3.10), it follows that $\alpha(fx, u) \geq 1$. Thus we have that $\alpha(fx, u) \geq 1$ for all $u \in F(x, y)$. Similarly, we can show that $\alpha(fy, v) \geq 1$, for all $v \in F(y, x)$. Hence F satisfies the assumption (ii) of Theorem 2.1.

Let $(x, y) \in X \times X$ and $t \in F(x, y)$. Since $F_n \rightarrow F$ uniformly, there exists a sequence $\{x_n\}$ with $x_n \in F_n(x, y)$ such that $x_n \rightarrow t$, as $n \rightarrow \infty$. By the assumption (ii), $\{x_n\}$ is a sequence in $f(X)$. As $f(X)$ is closed, we have $t \in f(X)$. So $t \in F(x, y) \implies t \in f(X)$. Hence $F(x, y) \subseteq f(X)$, that is, F satisfies the assumption (iii) of Theorem 2.1.

Let $(x, y), (u, v) \in X \times X$ with $\alpha(fx, fu) \geq 1$ and $\alpha(fy, fv) \geq 1$. Then applying the assumption (iii), we have

$$\begin{aligned} H(F_n(x, y), F_n(u, v)) &\leq a \frac{D(fx, F_n(x, y)) D(fu, F_n(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\ &\quad + c [D(fx, F_n(x, y)) + D(fu, F_n(u, v))] \\ &\quad + e [D(fx, F_n(u, v)) + D(fu, F_n(x, y))]. \end{aligned}$$

Since F_n converges uniformly to F , taking limit $n \rightarrow \infty$, we get

$$\begin{aligned} H(F(x, y), F(u, v)) &\leq a \frac{D(fx, F(x, y)) D(fu, F(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\ &\quad + c [D(fx, F(x, y)) + D(fu, F(u, v))] \\ &\quad + e [D(fx, F(u, v)) + D(fu, F(x, y))]. \end{aligned}$$

Hence F satisfies the assumption (v) of Theorem 2.1.

Theorem 3.2. Let (X, d) be a complete metric space, $\{F_n : X \times X \rightarrow K(X)\}$ be a sequence of multi-valued mappings, $f : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. Suppose that the sequence $\{F_n\}$ converges uniformly to a multi-valued mapping $F : X \times X \rightarrow K(X)$. Suppose that (i) (X, d) has α -regular property; (ii) each F_n ($n \in \mathbb{N}$) is α -dominated with respect to f ; (iii) $F_n(x, y) \subseteq f(X)$ for all $(x, y) \in X \times X$ and $n \in \mathbb{N}$; (iv) $f(X)$ is closed in X ; (v) there exist nonnegative real numbers a, b, c, e with $(a + b + 2c + 2e) < 1$ such that for every $n \in \mathbb{N}$ and for all $x, y, u, v \in X$ with $\alpha(fx, fu) \geq 1$ and $\alpha(fy, fv) \geq 1$,

$$\begin{aligned} H(F_n(x, y), F_n(u, v)) &\leq a \frac{D(fx, F_n(x, y)) D(fu, F_n(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\ &\quad + c [D(fx, F_n(x, y)) + D(fu, F_n(u, v))] \\ &\quad + e [D(fx, F_n(u, v)) + D(fu, F_n(x, y))]. \end{aligned}$$

Also, suppose that (3.10) is satisfied. Then $C(F_n, f) \neq \emptyset$ for every $n \in \mathbb{N}$ and $C(F, f) \neq \emptyset$. Suppose that $M_n = \sup\{d(x, fx), d(y, fy) : (x, y) \in C(F_n, f) \cup C(F, f)\}$ and $\Delta_n = \sup_{x, y \in X} H(F_n(x, y), F(x, y))$ exist and $M_n \rightarrow 0$ as $n \rightarrow \infty$. Then $\lim_{n \rightarrow \infty} H_\rho(C(F_n, f), C(F, f)) = 0$, that is, the coincidence point sets of the sequence of pairs of mappings $\{(F_n, f)\}$ are stable.

Proof. By Lemma 3.1 and Theorem 2.1, $C(F_n, f) \neq \emptyset$ for every $n \in \mathbb{N}$ and $C(F, f) \neq \emptyset$. By Theorem 3.1, we have

$$H_\rho(C(F_n, f), C(F, f)) \leq 2M_n + \frac{\Delta_n}{1 - k}.$$

Since T_n converges to T uniformly, we get

$$\lim_{n \rightarrow \infty} H_\rho(C(F_n, f), C(F, f)) \leq 0,$$

which implies that $\lim_{n \rightarrow \infty} H_\rho(C(F_n, f), C(F, f)) = 0$, that is, the coincidence point sets of the sequence of pairs of mappings $\{(F_n, f)\}$ are stable.

Example 3.1. Let $X = [0, 1]$ be a set and d be the usual metric. Define $F : X \times X \rightarrow K(X)$ and $F_n : X \times X \rightarrow K(X)$ respectively by $F(x, y) = [0, \frac{x}{9}]$ and $F_n(x, y) = [0, \frac{x}{9} + \frac{2}{9n}]$ for $(x, y) \in X \times X$. Let $f(x) = \frac{x}{3}$ for all $x \in X$, $\alpha(x, y) = 2$ for all $x, y \in X$ and $a = \frac{1}{10}$, $b = \frac{1}{2}$, $c = e = \frac{1}{12}$. Then all the conditions of the Theorem 3.2 are satisfied. Here $C(F_n, f) = \{(x, y) : x, y \in [0, \frac{1}{n}]\}$ for every n , $C(F, f) = \{(0, 0)\}$ and the coincidence point sets of the sequence of pairs of mappings $\{(F_n, f)\}$ are stable.

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S. Kundu,
Government General Degree College, Salboni, West Bengal,
India
e-mail: sunirmalkundu2009@rediffmail.com

S. Dutta,
Maulana Abul Kalam Azad University of Technology, West
Bengal, India
e-mail: sd83online@gmail.com

S. P. Mondal,
Maulana Abul Kalam Azad University of Technology, West
Bengal, India
e-mail: sankar.mondal02@gmail.com

N. Metiya,
Sovarani Memorial College, Howrah, India
e-mail: metiya.nikhilesh@gmail.com

C. Bandyopadhyay,
Indian Institute of Engineering Science and Technology,
Shibpur, Howrah, India
e-mail: chaitali.math@gmail.com