UZBEKISTAN ACADEMY OF SCIENCES
V.I.ROMANOVSKIY INSTITUTE OF MATHEMATICS

UZBEK
MATHEMATICAL JOURNAL

Journal was founded in 1957. Until 1991 it was named by
"Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk". Since 1991 it is known as
"Uzbek Mathematical Journal". It has 4 issues annually.

Volume 70 Issue 1 2026

Uzbek Mathematical Journal is abstracting and indexing by

SCOPUS EBSCO CrossRef
MathSciNet Zentralblatt Math VINITI

TASHKENT - 2026

Address: University str., 9, Tashkent 100174, Uzbekistan

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 5–18
DOI: 10.29229/uzmj.2026-1-1

Lyapunov stability of an upwind difference scheme for a quasilinear hyperbolic system

Aloev R., Nematova D., Abdullah I., Mohd M.

Abstract. The paper considers a mixed problem for a quasilinear system of hyperbolic equations in Riemann invariants with dissipative nonlinear boundary conditions. To numerically solve the mixed problem, an initial-boundary difference problem based on an upwind difference scheme is proposed. A discrete Lyapunov function is constructed for the numerical solution of the initial-boundary difference problem for nonlinear problems. A theorem on the exponential stability of the stationary state of a quasilinear system is proven.

Keywords: Exponential stability, hyperbolic system, mixed problem
MSC (2020): 53C12, 57R25, 57R35

1. Introduction

One of the promising directions in the theory of stability of nonlinear difference schemes seems to us to be a generalization of the direct Lyapunov method (see, for example, [1], which allows us to reduce the study of the stability of systems of nonlinear hyperbolic equations to the construction of a positive definite function that monotonically decreases on the solution of this system (function Lyapunov). The method of [2] Lyapunov functions has found its application in the theory of stability of ordinary difference equations [3, 4]. There are generalizations of the direct Lyapunov method to partial differential equations [5, 6, 7, 8, 9] although this method is not so popular here. Note, however, that the method of energy inequalities, widely used in the theory of linear partial differential equations, is, in essence, a special case of the direct [2] Lyapunov method (in this case, the Lyapunov function is constructed in the form of some quadratic form from the solution of the problem). All this allows us to hope to generalize the method of Lyapunov functions to nonlinear difference schemes.

Note that works [10, 11, 12, 13, 14, 15, 16, 17] are devoted to the construction of the Lyapunov function for linear difference schemes.

2. Method

Extensive literature is devoted to the issues of studying the stability of difference schemes. The most complete results were obtained for linear schemes with constant (in time) operators; linear circuits with variable operators, and especially nonlinear circuits, have been studied much less well. Two methods for identifying the stability of linear difference schemes are widely used: the spectral method and the method of energy inequalities, the latter being, in essence, a difference analogue of the Lyapunov method of functions. This paper sets out stability criteria for nonlinear difference schemes. The proposed criterion is a sufficient condition for the stability of the circuit in terms of Lyapunov vector functions.

Well-known methods of Lyapunov stability analysis are based on the qualitative theory of ordinary differential equations. Research on this basis is necessary for the theory and practice of automatic regulation, control and monitoring, and super-operational control. As a rule, sustainability analysis is carried out either a priori, before creating a control system, or a posteriori, based on the results of operation. However, stability monitoring is important for the current state of the system.

The need to study the stability of motion or some state arises at all stages of the design or study of physical systems. For the first time, a rigorous mathematical definition of stability and exact methods for solving the issue of stability for a fairly wide class of systems were given by A.M. Lyapunov in his famous [2].

This work was the logical conclusion of the entire previous stage in the development of the theory of stability. With its advent, stability theory reached the level of an independent discipline, taking its rightful place among other mathematical disciplines. A.M. Lyapunov proposed two methods for analyzing the stability of solutions to ordinary differential equations. The first method consists in constructing the solutions of the differential equations of perturbed motions themselves in the form of certain series. Based on subsequent qualitative research of these solutions, conclusions are drawn about sustainability or instability. The second method is to find some auxiliary function, the properties of which determine the stability or instability of the solution. Currently, these functions are called Lyapunov functions, and the method is called the Lyapunov function method, the second Lyapunov method, or the direct Lyapunov method.

Lyapunov's work was the starting point for research of this kind. His ideas develop and deepen in many directions. New theorems have been established that expand these methods, many questions of the existence of Lyapunov functions and their effective construction have been solved, questions of stability of unsteady and periodic motions, stability of the first approximation, in critical cases, with constantly acting disturbances, and many others have been studied.

A development of the theory of stability in relation to automatic control and regulation systems is the theory of motion stabilization, which explores such system control modes in which some program motion (unperturbed motion) of the system will be stable in one sense or another. In many cases, along with the requirement of stability of undisturbed motion, additional requirements are imposed both on the nature of transient processes and on control actions. Often these requirements can be expressed in the form of a minimum of some integral functional. Stabilization problems with these additional requirements are called optimal stabilization problems, or analytical design of regulators.

3. Results and Discussions

Statement of a quasilinear mixed problem. According to the work [18] in the region $\bar{\omega} \triangleq \{(t,x) : 0 \le t \le T, 0 \le x \le L\}$ we consider a mixed problem for the following quasilinear hyperbolic system

$$\begin{cases} \xi_t + \varphi(\xi,\eta)\,\xi_x = 0, \quad \gamma_t + \varphi(\xi,\eta)\,\gamma_x + \gamma f = 0, \quad \rho_t + \varphi(\xi,\eta)\,\rho_x + \rho f + \gamma f_x = 0, \\ \eta_t - \psi(\xi,\eta)\,\eta_x = 0, \quad \delta_t - \psi(\xi,\eta)\,\delta_x - \delta p = 0, \quad \theta_t - \psi(\xi,\eta)\,\theta_x - \theta p - \delta p_x = 0, \\ 0 < t \le T,\ 0 < x < L \end{cases} \tag{3.1}$$

with boundary conditions

$$\begin{gathered} \text{at } x = 0: \\ \begin{cases} \xi(t,0) = a(\eta(t,0)), \quad \varphi(t,0)\,\gamma(t,0) = -a'(\eta(t,0))\,\psi(t,0)\,\delta(t,0), \\ \varphi(t,0)\,\rho(t,0) + \gamma(t,0)\,f(t,0) = -e'(t)\,\delta(t,0) - e(t)\left[\psi(t,0)\,\theta(t,0) + \delta(t,0)\,p(t,0)\right], \end{cases} \\ \text{at } x = L: \\ \begin{cases} \eta(t,L) = b(\xi(t,L)), \quad \psi(t,L)\,\delta(t,L) = -b'(\xi(t,L))\,\varphi(t,L)\,\gamma(t,L), \\ \psi(t,L)\,\theta(t,L) + \delta(t,L)\,p(t,L) = -h'(t)\,\gamma(t,L) + h(t)\left[\varphi(t,L)\,\rho(t,L) + \gamma(t,L)\,f(t,L)\right], \end{cases} \end{gathered} \tag{3.2}$$

and with initial data

$$\begin{cases} \xi(0,x) = \xi_0(x), \quad \gamma(0,x) = \xi_0'(x), \quad \rho(0,x) = \xi_0''(x), \\ \eta(0,x) = \eta_0(x), \quad \delta(0,x) = \eta_0'(x), \quad \theta(0,x) = \eta_0''(x), \end{cases} \quad 0 < x < L \tag{3.3}$$

Here $\xi = \xi(t,x)$, $\eta = \eta(t,x)$, $\gamma = \gamma(t,x) = \xi_x$, $\delta = \delta(t,x) = \eta_x$, $\rho = \rho(t,x) = \gamma_x$, $\theta = \theta(t,x) = \delta_x$ are unknown functions to be determined, and $\varphi = \varphi(\xi,\eta)$, $\psi = \psi(\xi,\eta)$ are given functions that have continuous derivatives up to the second order inclusive. We will assume that $a, b \in C^2(\mathrm{R})$.

$$f = \gamma\frac{\partial\varphi}{\partial\xi} + \delta\frac{\partial\varphi}{\partial\eta}, \quad p = \gamma\frac{\partial\psi}{\partial\xi} + \delta\frac{\partial\psi}{\partial\eta}.$$

Here

$$\begin{aligned} \varphi(t,0) &= \varphi(\xi(t,0),\eta(t,0)), \qquad \psi(t,0) = \psi(\xi(t,0),\eta(t,0)), \\ \varphi(t,L) &= \varphi(\xi(t,L),\eta(t,L)), \qquad \psi(t,L) = \psi(\xi(t,L),\eta(t,L)), \end{aligned}$$

$$\begin{aligned} f(t,0) &= f(\xi(t,0),\ \eta(t,0),\ \gamma(t,0),\ \delta(t,0)), \qquad p(t,0) = p(\xi(t,0),\ \eta(t,0),\ \gamma(t,0),\ \delta(t,0)), \\ f(t,L) &= f(\xi(t,L),\ \eta(t,L),\ \gamma(t,L),\ \delta(t,L)), \qquad p(t,L) = p(\xi(t,L),\ \eta(t,L),\ \gamma(t,L),\ \delta(t,L)). \end{aligned}$$

The functions $e(t)$ and $h(t)$ are defined as

$$e(t) := \frac{a'(\eta(t,0))\,\psi(t,0)}{\varphi(t,0)}, \quad h(t) := \frac{b'(\xi(t,L))\,\varphi(t,L)}{\psi(t,L)}.$$

Exponential stability of the numerical solution of a nonlinear initial-boundary difference problem. In this section we establish the exponential stability of the numerical solution of the initial-boundary difference problem for the mixed problem (3.1), (3.2), (3.3).

To obtain the initial-boundary difference problem, we will use an upwind difference scheme for the numerical calculation of system (3.1).

To do this, we cover the spatial region [0, 1] using a uniform grid $\Omega_{\Delta x} = \left\{x_j = j \cdot \Delta x,\ \ j = \overline{0, J}\right\}$, Δx step by x.

To numerically solve the mixed problem (3.1), (3.2), (3.3), we propose the following upwind explicit difference scheme

$$\begin{cases} \xi_j^{k+1} = \left(1 - [C_\varphi]_{j-1}^k\right)\xi_j^k + [C_\varphi]_{j-1}^k\,\xi_{j-1}^k,\ k = \overline{0, K-1},\ \ j = \overline{1, J}, \\ \eta_j^{k+1} = \left(1 - [C_\psi]_{j+1}^k\right)\eta_j^k + [C_\psi]_{j+1}^k\,\eta_{j+1}^k,\ k = \overline{0, K-1},\ \ j = \overline{0, J-1}, \\ \gamma_j^{k+1} = \left(1 - [C_\varphi]_{j-1}^k\right)\gamma_j^k + [C_\varphi]_{j-1}^k\,\gamma_{j-1}^k - \Delta t \cdot \gamma_j^k f_j^k,\ k = \overline{0, K-1},\ \ j = \overline{1, J}, \\ \delta_j^{k+1} = \left(1 - [C_\psi]_{j+1}^k\right)\delta_j^k + [C_\psi]_{j+1}^k\,\delta_{j+1}^k + \Delta t \cdot \delta_j^k p_j^k,\ k = \overline{0, K-1},\ \ j = \overline{0, J-1}, \\ \rho_j^{k+1} = \left(1 - [C_\varphi]_{j-1}^k\right)\rho_j^k + [C_\varphi]_{j-1}^k\,\rho_{j-1}^k - \Delta t \cdot \left[\rho_j^k f_j^k + \gamma_j^k \left(\frac{\partial f}{\partial x}\right)_j^k\right],\ j = \overline{1, J}, \\ \theta_j^{k+1} = \left(1 - [C_\psi]_{j+1}^k\right)\theta_j^k + [C_\psi]_{j+1}^k\,\theta_{j+1}^k + \Delta t \cdot \left[\theta_j^k p_j^k + \delta_j^k \left(\frac{\partial p}{\partial x}\right)_j^k\right],\ j = \overline{0, J-1}, \end{cases} \tag{3.4}$$

with boundary conditions

$$\begin{cases} \xi_0^k = a\left(\eta_0^k\right), \quad \varphi_0^k\gamma_0^k = -a'\left(\eta_0^k\right)\psi_0^k\delta_0^k, \quad \varphi_0^k\rho_0^k + \gamma_0^k f_0^k = -\left(e'(t)\right)^k\delta_0^k - e^k\left[\psi_0^k\theta_0^k + \delta_0^k p_0^k\right], \\ \eta_J^k = b\left(\xi_J^k\right), \quad \psi_J^k\delta_J^k = -b'\left(\xi_J^k\right)\varphi_J^k\gamma_J^k, \quad \psi_J^k\theta_J^k + \delta_J^k p_J^k = -\left(h'(t)\right)^k\gamma_J^k + h^k\left[\varphi_J^k\rho_J^k + \gamma_J^k p_J^k\right], \end{cases} \tag{3.5}$$

and with initial data

$$\begin{gathered} \xi_j^0 = \xi_0(x_j),\ \ \eta_j^0 = \eta_0(x_j),\ \ \gamma_j^0 = \xi_0'(x_j),\ \ \delta_j^0 = \eta_0'(x_j), \\ \rho_j^0 = \xi_0''(x_j),\ \ \theta_j^0 = \eta_0''(x_j),\ \ j \in \{0, 1, 2, \cdots, J\}. \end{gathered} \tag{3.6}$$

Let us introduce the following vectors into consideration

$$\vec{\xi} = (\xi, \gamma, \rho),\ \vec{\eta} = (\eta, \delta, \theta),\ \vec{\xi^*} = (\xi^*, \gamma^*, \rho^*),\ \vec{\eta^*} = (\eta^*, \delta^*, \theta^*),\ \vec{\phi}_1 = (\xi_0, \xi_0', \xi''),\ \vec{\phi}_2 = (\eta_0, \eta_0', \eta'')$$

and the following matrices:

$$\mathsf{U}^k \triangleq diag\left(\vec{\eta}_0^k, \vec{\xi}_1^k, \vec{\eta}_1^k, \cdots\cdots, \vec{\xi}_{J-1}^k, \vec{\eta}_{J-1}^k, \vec{\xi}_J^k\right), \qquad \mathsf{U}^* \triangleq diag\left(\vec{\eta^*}, \overbrace{\vec{\xi^*}, \vec{\eta^*}, \cdots\cdots, \vec{\xi^*}, \vec{\eta^*}}^{6J}, \vec{\xi^*}\right),$$

$$\mathsf{U}^0 \triangleq diag\left(\vec{\phi}_2(x_0), \vec{\phi}_1(x_1), \vec{\phi}_2(x_1), \cdots\cdots, \vec{\phi}_1(x_{J-1}), \vec{\phi}_2(x_{J-1}), \vec{\phi}_1(x_J)\right).$$

Definition 3.1. The equilibrium state $\mathbf{U}^*$ of the initial-boundary difference problem (3.4), (3.5), (3.6) is stable in the l^2 -norm if there exist positive real constants $n_1,\ n_2$ such that for any initial condition $\mathbf{\Phi}$ the solution to $\mathbf{U}^k,\ \ k\in\{1,2,\cdots\}$ the initial-boundary difference problem (3.4), (3.5), (3.6) satisfies the inequality

$$\left\|\mathbf{U}^k-\mathbf{U}^*\right\|_{l^2}\le n_2 e^{-n_1 t^k}\left\|\mathbf{\Phi}-\mathbf{U}^*\right\|_{l^2},\quad k\in\{1,2,\cdots\},\tag{3.7}$$

where

$$\mathbf{U}^k\triangleq\left(\vec{\eta}_0^k,\vec{\xi}_1^k,\vec{\eta}_1^k,\cdots\cdots,\vec{\xi}_{J-1}^k,\vec{\eta}_{J-1}^k,\vec{\xi}_J^k\right)^T,\qquad \mathbf{U}^*\triangleq\overbrace{\left(\vec{\eta}^*,\vec{\xi}^*,\vec{\eta}^*,\cdots\cdots,\vec{\xi}^*,\vec{\eta}^*,\vec{\xi}^*\right)^T}^{6J},$$

$$\mathbf{\Phi}\triangleq\left(\vec{\phi}_2(x_0),\vec{\phi}_1(x_1),\vec{\phi}_2(x_1),\cdots\cdots,\vec{\phi}_1(x_{J-1}),\vec{\phi}_2(x_{J-1}),\vec{\phi}_1(x_J)\right).$$

and

$$\begin{aligned}
\left\|\mathbf{U}^k-\mathbf{U}^*\right\|_{l^2}^2 &\triangleq \Delta x\left(\left[\vec{\eta}_0^k-\vec{\eta}^*\right]^T,\left[\vec{\eta}_0^k-\vec{\eta}^*\right]^T\right)+h\left(\left[\vec{\xi}_J^k-\vec{\xi}^*\right]^T,\left[\vec{\xi}_J^k-\vec{\xi}^*\right]^T\right)+\\
&+\Delta x\sum_{j=1}^{J-1}\left\{\left(\left[\vec{\xi}_j^k-\vec{\xi}^*\right]^T,\left[\vec{\xi}_j^k-\vec{\xi}^*\right]^T\right)+\left(\left[\vec{\eta}_j^k-\vec{\eta}^*\right]^T,\left[\vec{\eta}_j^k-\vec{\eta}^*\right]^T\right)\right\},\\
\left\|\mathbf{\Phi}-\mathbf{U}^*\right\|_{l^2} &\triangleq \Delta x\left(\left[\vec{\phi}_2(x_0)-\vec{\eta}^*\right]^T,\left[\vec{\phi}_2(x_0)-\vec{\eta}^*\right]^T\right)+\Delta x\left(\left[\vec{\phi}_1(x_J)-\vec{\xi}^*\right]^T,\left[\vec{\phi}_1(x_J)-\vec{\xi}^*\right]^T\right)+\\
&+\Delta x\sum_{j=1}^{J-1}\left\{\left(\left[\vec{\phi}_1(x_j)-\vec{\xi}^*\right]^T,\left[\vec{\phi}_1(x_j)-\vec{\xi}^*\right]^T\right)+\left(\left[\vec{\phi}_2(x_j)-\vec{\eta}^*\right]^T,\left[\vec{\phi}_2(x_j)-\vec{\eta}^*\right]^T\right)\right\},\\
&k\in\{0,1,\cdots\}.
\end{aligned}$$

Definition 3.2. (Discrete Lyapunov function). It is said that the function $\mathbf{L}^k:\mathrm{R}^{n\times J}\to\mathrm{R}_0^+$ is a discrete Lyapunov function for the initial-boundary difference problem (3.4), (3.5), (3.6), if

(1) there are positive constants h_1 and h_2 that for all $k\in\{0,1,\cdots\}$:

$$h_1\left\|\mathbf{U}^k-\mathbf{U}^*\right\|_{l^2}^2\le\mathbf{L}^k\left(\mathbf{U}^k\right)\le h_2\left\|\mathbf{U}^k-\mathbf{U}^*\right\|_{l^2}^2,\tag{3.8}$$

(2) there is a positive constant n such that for all $k\in\{0,1,\cdots\}$:

$$\frac{\mathbf{L}^k\left(\mathbf{U}^{k+1}\right)-\mathbf{L}^k\left(\mathbf{U}^k\right)}{\Delta t}\le -n\mathbf{L}^k\left(\mathbf{U}^k\right).\tag{3.9}$$

To simplify the notation, in what follows we define a sequence of discrete values $\mathcal{L}^k$ as

$$\mathcal{L}^k=\mathbf{L}^k\left(\mathbf{U}^k\right),\ \ k\in\{0,1,\cdots\}$$

where $\mathbf{U}^k$ a given solution of the initial-boundary difference problem (3.4), (3.5), (3.6).

It should be noted that the presence of a discrete Lyapunov function ensures the stability of the equilibrium state of $\mathbf{U}^*$ the initial-boundary difference problem (3.4), (3.5), (3.6) in the l^2 -norm.

First, let us separately consider the difference equations for ξ_j^k and η_j^k system (3.4):

$$\left\{\begin{array}{l}
\xi_j^{k+1}=\left(1-[C_\varphi]_{j-1}^k\right)\xi_j^k+[C_\varphi]_{j-1}^k\,\xi_{j-1}^k,\qquad k=\overline{0,K-1},\quad j=\overline{1,J},\\
\eta_j^{k+1}=\left(1-[C_\psi]_{j+1}^k\right)\eta_j^k+[C_\psi]_{j+1}^k\,\eta_{j+1}^k,\quad k=\overline{0,K-1},\ \ j=\overline{0,J-1},
\end{array}\right.\tag{3.10}$$

where

$$[C_\varphi]_j^k=\varphi_j^k\frac{\Delta t}{\Delta x},\ \ [C_\psi]_j^k=\psi_j^k\frac{\Delta t}{\Delta x},\ \ C_j^k=\max\left([C_\varphi]_j^k,[C_\psi]_j^k\right).$$

$$\bar{\varphi}=\varphi(0,0)>0,\ \ \bar{\psi}=\psi(0,0)>0$$

As a discrete Lyapunov function for (3.10), we consider discrete function

$$\mathcal{L}^k=\mathcal{L}_1^k+\mathcal{L}_2^k,\qquad \mathcal{L}_1^k=\frac{\mathrm{A}}{\bar{\varphi}}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right),\ \ \mathcal{L}_2^k=\frac{\mathrm{B}}{\bar{\psi}}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\exp\left(\frac{m}{\bar{\psi}}x_{j+1}\right).$$

The difference time derivative of the discrete Lyapunov function on solutions of difference equations (3.4) is

$$\frac{\mathcal{L}^{k+1}-\mathcal{L}^{k}}{\Delta t}=\frac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}+\frac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^{k}}{\Delta t} \tag{3.11}$$

Let us calculate the difference ratios the right side of (3.11) separately.

Lemma 3.3. *For grid functions* ξ_j^k *satisfying difference equations* (3.10), *the following inequality holds:*

$$\begin{aligned}\tfrac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t} \le -\tfrac{\mathrm{A}}{\bar{\varphi}}\left[\left(\xi_j^k\right)^2\varphi_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_j\right)\right]\Big|_0^J - m\tfrac{\mathrm{A}}{\bar{\varphi}^2}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\varphi_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)+\\ +\tfrac{\mathrm{A}}{\bar{\varphi}}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2 f_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right).\end{aligned} \tag{3.12}$$

with

$$f_j^k=\frac{\varphi_j^k-\varphi_{j-1}^k}{\Delta x}=\frac{\partial\varphi\left(\xi_j^k,\eta_j^k\right)}{\partial\xi}\gamma_j^k+\frac{\partial\varphi\left(\xi_{j-1}^k,\eta_j^k\right)}{\partial\eta}\delta_j^k$$

Proof. Substituting the value of the expression for ξ_j^{k+1} from (3.10) into expression $\frac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}$ we get

$$\begin{aligned}\tfrac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}=\tfrac{\mathrm{A}}{\bar{\varphi}}\Delta x\sum_{j=1}^{J}\left[\tfrac{\left(\xi_j^{k+1}\right)^2-\left(\xi_j^k\right)^2}{\Delta t}\right]\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)=\tfrac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left[\left(\xi_j^{k+1}\right)^2-\left(\xi_j^k\right)^2\right]\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)=\\ =\tfrac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left[\left\{\left(1-[C_\varphi]_{j-1}^k\right)\xi_j^k+[C_\varphi]_{j-1}^k\,\xi_{j-1}^k\right\}^2-\left(\xi_j^k\right)^2\right]\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right).\end{aligned} \tag{3.13}$$

Using Jensen's inequality to evaluate the expression $\left\{\left(1-[C_\varphi]_{j-1}^k\right)\xi_j^k+[C_\varphi]_{j-1}^k\,\xi_{j-1}^k\right\}^2$ from above we have

$$\frac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}= \frac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left[\left(\xi_{j-1}^k\right)^2-\left(\xi_j^k\right)^2\right][C_\varphi]_{j-1}^k\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right). \tag{3.14}$$

Using the difference formula

$$\begin{aligned}\left(u_j-u_{j-1}\right)v_{j-1}w_{j-1}=\left(u_jv_jw_j-u_{j-1}v_{j-1}w_{j-1}\right)-u_j\left(v_jw_j-v_{j-1}w_{j-1}\right)=\\ =\left(u_jv_jw_j-u_{j-1}v_{j-1}w_{j-1}\right)-u_j\left[\left(v_j-v_{j-1}\right)w_j+v_{j-1}\left(w_j-w_{j-1}\right)\right]\end{aligned} \tag{3.15}$$

and assuming $u_j=\left(\xi_j^k\right)^2,\ v_j=[C_\varphi]_j^k,\ w_j=\exp\left(-\frac{m}{\bar{\varphi}}x_j\right)$ from (3.14) we obtain the inequality

$$\begin{aligned}\tfrac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}\le\tfrac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left[\left(\xi_{j-1}^k\right)^2[C_\varphi]_{j-1}^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)-\left(\xi_j^k\right)^2[C_\varphi]_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_j\right)\right]+\\ +\tfrac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left(\xi_j^k\right)^2\left\{\left[[C_\varphi]_j^k-[C_\varphi]_{j-1}^k\right]\exp\left(-\tfrac{m}{\bar{\varphi}}x_j\right)+[C_\varphi]_{j-1}^k\left[\exp\left(-\tfrac{m}{\bar{\varphi}}x_j\right)-\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)\right]\right\}.\end{aligned}$$

Therefore, taking into account equality

$$\begin{aligned}\left[\exp\left(-\tfrac{m}{\bar{\varphi}}x_j\right)-\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)\right]=\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)\left(\exp\left(-\tfrac{m}{\bar{\varphi}}\Delta x\right)-1\right)=\\ =\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)\left[-\tfrac{m}{\bar{\varphi}}\Delta x+O\left((\Delta x)^2\right)\right].\end{aligned}$$

with accuracy $O\left(\Delta x\right)$ from inequality (3.14) we obtain

$$\begin{aligned}\tfrac{\mathcal{L}_1^{k+1}-\mathcal{L}_1^{k}}{\Delta t}\le -\tfrac{\mathrm{A}}{C_\varphi}\left[\left(\xi_j^k\right)^2[C_\varphi]_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_j\right)\right]\Big|_0^J - m\tfrac{\mathrm{A}}{\bar{\varphi}C_\varphi}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2[C_\varphi]_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)+\\ +\tfrac{\mathrm{A}}{C_\varphi}\sum_{j=1}^{J}\left(\xi_j^k\right)^2\left[[C_\varphi]_j^k-[C_\varphi]_{j-1}^k\right]\exp\left(-\tfrac{m}{\bar{\varphi}}x_j\right)=-\tfrac{\mathrm{A}}{\bar{\varphi}}\left[\left(\xi_j^k\right)^2\varphi_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_j\right)\right]\Big|_0^J-\\ -m\tfrac{\mathrm{A}}{\bar{\varphi}^2}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\varphi_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)+\tfrac{\mathrm{A}}{\bar{\varphi}}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2 f_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_j\right)\end{aligned}$$

Lemma 3.3 is proven. □

Lemma 3.4. *For grid functions* η_j^k *satisfying difference equations* (3.10), *the following inequality holds:*

$$\frac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t} \le \frac{\mathrm{B}}{\psi}\left[\left(\eta_j^k\right)^2\psi_j^k\exp\left(\frac{m}{\psi}x_j\right)\right]\Big|_0^J - m\frac{\mathrm{B}}{\psi^2}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\psi_{j+1}^k\exp\left(\frac{m}{\psi}x_{j+1}\right) - \\ -\frac{\mathrm{B}}{\psi}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2 p_j^k\exp\left(\frac{m}{\psi}x_j\right) \tag{3.16}$$

with

$$p_j^k = \frac{\psi_{j+1}^k-\psi_j^k}{\Delta x} = \frac{\partial\psi\left(\xi_j^k,\eta_{j+1}^k\right)}{\partial\xi}\gamma_j^k + \frac{\partial\psi\left(\xi_j^k,\eta_j^k\right)}{\partial\eta}\delta_j^k.$$

Proof. Substituting the value of the expression for η_j^{k+1} from (3.10) into expression $\frac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t}$ we get

$$\frac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t} = \frac{\mathrm{B}}{\psi}\Delta x\sum_{j=0}^{J-1}\left[\frac{\left(\eta_j^{k+1}\right)^2-\left(\eta_j^k\right)^2}{\Delta t}\right]\exp\left(\frac{m}{\psi}x_{j+1}\right) = \frac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left[\left(\eta_j^{k+1}\right)^2-\left(\eta_j^k\right)^2\right]\exp\left(\frac{m}{\psi}x_{j+1}\right) = \\ = \frac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left[\left\{\left(1-\left[C_\psi\right]_{j+1}^k\right)\eta_j^k + \left[C_\psi\right]_{j+1}^k\eta_{j+1}^k\right\}^2 - \left(\eta_j^k\right)^2\right]\exp\left(\frac{m}{\psi}x_{j+1}\right). \tag{3.17}$$

Using Jensen's inequality (for convex mappings $y\to y^2$ the inequality $\left[q_1y_1+q_2y_2\right]^2 \le q_1\left(y_1\right)^2+q_2\left(y_2\right)^2$ holds, $q_1,\ q_2>0$ and $q_1+q_2=1$), to evaluate the expression $\left\{\left(1-\left[C_\psi\right]_{j+1}^k\right)\eta_j^k+\left[C_\psi\right]_{j+1}^k\eta_{j+1}^k\right\}^2$ from above we have

$$\frac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t} \le \frac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left[\left\{\left(1-\left[C_\psi\right]_{j+1}^k\right)\left(\eta_j^k\right)^2+\left[C_\psi\right]_{j+1}^k\left(\eta_{j+1}^k\right)^2\right\}-\left(\eta_j^k\right)^2\right]\exp\left(\frac{m}{\psi}x_{j+1}\right) = \\ = \frac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left[\left(\eta_{j+1}^k\right)^2-\left(\eta_j^k\right)^2\right]\left[C_\psi\right]_{j+1}^k\exp\left(\frac{m}{\psi}x_{j+1}\right) \tag{3.18}$$

Using the difference formula

$$\left(u_{j+1}-u_j\right)v_{j+1}w_{j+1} = \left(u_{j+1}v_{j+1}w_{j+1}-u_jv_jw_j\right)-u_j\left(v_{j+1}w_{j+1}-v_jw_j\right) = \\ = \left(u_{j+1}v_{j+1}w_{j+1}-u_jv_jw_j\right)-u_j\left[\left(v_{j+1}-v_j\right)w_j+v_{j+1}\left(w_{j+1}-w_j\right)\right] \tag{3.19}$$

and taking into $u_j=\left(\eta_j^k\right)^2,\ \ v_j=\left[C_\psi\right]_j^k,\ \ w_j=\exp\left(\frac{m}{\psi}x_{j+1}\right)$ account equality (3.19) from inequality (3.18) we obtain

$$\frac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t} \le \frac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left[\left(\eta_{j+1}^k\right)^2\left[C_\psi\right]_{j+1}^k\exp\left(\frac{m}{\psi}x_{j+1}\right)-\left(\eta_j^k\right)^2\left[C_\psi\right]_j^k\exp\left(\frac{m}{\psi}x_j\right)\right] - \\ -\frac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\left[C_\psi\right]_{j+1}^k-\left[C_\psi\right]_j^k\right]\exp\left(\frac{m}{\psi}x_j\right)-\frac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[C_\psi\right]_{j+1}^k\left[\exp\left(\frac{m}{\psi}x_{j+1}\right)-\exp\left(\frac{m}{\psi}x_j\right)\right].$$

we obtain with accuracy $O\left(\Delta x\right)$

$$\frac{\mathcal{L}_2^{k+1}-\mathcal{L}_2^k}{\Delta t} \le \frac{\mathrm{B}}{C_\psi}\left[\left(\eta_j^k\right)^2\left[C_\psi\right]_j^k\exp\left(\frac{m}{\psi}x_j\right)\right]\Big|_0^J - \frac{\mathrm{B}}{C_\psi}\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\left[C_\psi\right]_{j+1}^k-\left[C_\psi\right]_j^k\right]\exp\left(\frac{m}{\psi}x_j\right) - \\ -\frac{\mathrm{B}}{C_\psi}\frac{m}{\psi}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[C_\psi\right]_{j+1}^k\exp\left(\frac{m}{\psi}x_{j+1}\right) = \frac{\mathrm{B}}{\psi}\left[\left(\eta_j^k\right)^2\psi_j^k\exp\left(\frac{m}{\psi}x_j\right)\right]\Big|_0^J - \\ -m\frac{\mathrm{B}}{\psi^2}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\psi_{j+1}^k\exp\left(\frac{m}{\psi}x_{j+1}\right) - \frac{\mathrm{B}}{\psi}\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\psi_{j+1}^k-\psi_j^k\right]\exp\left(\frac{m}{\psi}x_j\right) = \\ = \frac{\mathrm{B}}{\psi}\left[\left(\eta_j^k\right)^2\psi_j^k\exp\left(\frac{m}{\psi}x_j\right)\right]\Big|_0^J - m\frac{\mathrm{B}}{\psi^2}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\psi_{j+1}^k\exp\left(\frac{m}{\psi}x_{j+1}\right) - \\ -\frac{\mathrm{B}}{\psi}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\frac{\partial\psi\left(\xi_j^k,\eta_{j+1}^k\right)}{\partial\xi}\gamma_{j+1}^k-\frac{\partial\psi\left(\xi_j^k,\eta_j^k\right)}{\partial b}\delta_j^k\right]\exp\left(\frac{m}{\psi}x_j\right).$$

Lemma 3.4 is proven. □

From Lemmas 3.3-3.4 taking into account equality (3.11), we obtain the following inequality

$$\frac{\mathcal{L}^{k+1}-\mathcal{L}^{k}}{\Delta t} \leq \Upsilon_1^k+\Upsilon_2^k+\Upsilon_3^k. \tag{3.20}$$

Here

$$\begin{aligned}
\Upsilon_1^k &= -\tfrac{\mathrm{A}}{\bar{\varphi}}\left[\left(\xi_j^k\right)^2\varphi_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_j\right)\right]\Big|_0^J+\tfrac{\mathrm{B}}{\bar{\psi}}\left[\left(\eta_j^k\right)^2\psi_j^k\exp\left(\tfrac{m}{\bar{\psi}}x_j\right)\right]\Big|_0^J,\\
\Upsilon_2^k &= -m\left[\tfrac{\mathrm{A}}{\bar{\varphi}^2}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\varphi_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)+\tfrac{\mathrm{B}}{\bar{\psi}^2}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\psi_{j+1}^k\exp\left(\tfrac{m}{\bar{\psi}}x_{j+1}\right)\right],\\
\Upsilon_3 &= \tfrac{\mathrm{A}}{\bar{\varphi}}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2 f_j^k\exp\left(-\tfrac{m}{\bar{\varphi}}x_{j-1}\right)-\tfrac{\mathrm{B}}{\bar{\psi}}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2 p_j^k\exp\left(\tfrac{m}{\bar{\psi}}x_j\right).
\end{aligned}$$

Let us assume that the functions of boundary conditions (3.5) $a, b:\ \mathrm{R}\to\mathrm{R}$, satisfy the inequalities:

$$\begin{array}{ll}
\max\limits_{0\leq k\leq K}\left|a\left(\eta_0^k\right)\right|<+\infty, & \max\limits_{0\leq k\leq K}\left|a'\left(\eta_0^k\right)\right|<+\infty,\\
\max\limits_{0\leq k\leq K}\left|b\left(\xi_J^k\right)\right|<+\infty, & \max\limits_{0\leq k\leq K}\left|b'\left(\xi_J^k\right)\right|<+\infty
\end{array} \tag{3.21}$$

and denote

$$\kappa_0=a'(0)\,,\qquad \kappa_L=b'(0)\,.$$

Then we have the following lemma

Lemma 3.5. *If* $|\kappa_0\kappa_L|<1$, *if coefficients* m, A, B *satisfy inequalities* $\mathrm{A}\kappa_0^2-\mathrm{B}<0$ *and* $\mathrm{B}\exp\left(\frac{m}{\bar{\psi}}L\right)\kappa_L^2-\mathrm{A}\exp\left(-\frac{m}{\bar{\varphi}}L\right)<0$, *then there are positive real constants* $\mathrm{K}_1,\ d_1,\ l_1$ *such that if* $\left|\xi_j^k\right|+\left|\eta_j^k\right|\leq d_1\quad\forall j\in\{0,1,\cdots,J\}$, *then along the solution of system* (3.4) *with boundary conditions* (3.5) *the following inequality holds*

$$\frac{\mathcal{L}^{k+1}-\mathcal{L}^{k}}{\Delta t}\leq -l_1\mathcal{L}^k+\mathrm{K}_1\left\{\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\left[\left|\gamma_j^k\right|+\left|\delta_j^k\right|\right]+\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\left|\gamma_j^k\right|+\left|\delta_j^k\right|\right]\right\}.$$

Proof. For expression Υ_1^k the following equalities are true

$$\begin{aligned}
\Upsilon_1^k &= \mathrm{A}\frac{\left(\xi_0^k\right)^2\varphi_0^k}{\bar{\varphi}}-\mathrm{A}\frac{\left(\xi_J^k\right)^2\varphi_J^k}{\bar{\varphi}}\exp\left(-\tfrac{m}{\bar{\varphi}}L\right)+\mathrm{B}\frac{\left(\eta_J^k\right)^2\psi_J^k}{\bar{\psi}}\exp\left(\tfrac{m}{\bar{\psi}}L\right)-\mathrm{B}\frac{\left(\eta_0^k\right)^2\psi_0^k}{\bar{\psi}}=\\
&= A\frac{a^2\left(\eta_0^k\right)\varphi_0^k}{\bar{\varphi}}-\mathrm{A}\frac{\left(\xi_J^k\right)^2\varphi_J^k}{\bar{\varphi}}\exp\left(-\tfrac{m}{\bar{\varphi}}L\right)+\mathrm{B}\frac{b^2\left(\xi_J^k\right)\psi_J^k}{\bar{d}}\exp\left(\tfrac{m}{\bar{\psi}}L\right)-\mathrm{B}\frac{\left(\eta_0^k\right)^2\psi_0^k}{\bar{\psi}}.
\end{aligned}$$

This expression can be represented as

$$\Upsilon_1^k=\Upsilon_{01}^k+\Delta\Upsilon_1^k$$

where, through Υ_{01}^k for small ξ_J^k and η_0^k denoted the terms of the second order of smallness with

$$\varphi_j^k\simeq\bar{\varphi},\ \ \psi_j^k\simeq\bar{\psi},\ \ a\left(\eta_0^k\right)\simeq\kappa_0\eta_0^k,\ \ \ b\left(\xi_J^k\right)\simeq\kappa_L\xi_J^k.$$

Then it is obvious that for Υ_{01}^k we have

$$\Upsilon_{01}^k=\left[\mathrm{B}\exp\left(\frac{m}{\bar{\psi}}L\right)\kappa_L^2-\mathrm{A}\exp\left(-\frac{m}{\bar{\varphi}}L\right)\right]\left(\xi_J^k\right)^2+\left[\mathrm{A}\kappa_0^2-\mathrm{B}\right]\left(\eta_0^k\right)^2$$

which, as was shown earlier in [6], is a non-negative expression $\Upsilon_{01}^k\leq 0$. Moreover, the remainder $\Delta\Upsilon_{01}^k$ is denoted by terms of third order of smallness with respect to ξ_J^k and η_0^k (i.e, $\Delta\Upsilon_1^k\approx O\left(\left(\xi_J^k\right)^3,\ \left(\eta_0^k\right)^3\right)$ for small $\left|\xi_J^k\right|$ and $\left|\eta_0^k\right|$. Now let's look at the expression for Υ_2^k. Let us introduce the following notation:

$$\varphi\left(\xi_j^k,\eta_j^k\right)=\bar{\varphi}+\tilde{\varphi}\left(\xi_j^k,\eta_j^k\right),\ \ \psi\left(\xi_j^k,\eta_j^k\right)=\bar{\psi}+\tilde{\psi}\left(\xi_j^k,\eta_j^k\right).$$

Then for the expression Υ_2^k we have

$$\Upsilon_2^k = -mL^k + \Delta\Upsilon_2^k$$

with

$$\Delta\Upsilon_2^k = -m\left[\frac{\mathrm{A}}{\bar{\varphi}^2}\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\tilde{\varphi}_j^k\exp\left(-\frac{m}{\bar{\varphi}}x_{j-1}\right)+\frac{\mathrm{B}}{\bar{\psi}^2}\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\tilde{\psi}_{j+1}^k\exp\left(\frac{m}{\bar{\psi}}x_{j+1}\right)\right].$$

According to the definitions of $\tilde{\varphi}_j^k$ and $\tilde{\psi}_{j+1}^k$, it is easy to verify that the expression for $\Delta\Upsilon_2$ will take place of the members third order of smallness relative to ξ_j^k and η_j^k:

$$\Delta\Upsilon_2^k \approx O\left(\Delta x\sum_{j=1}^{J}\left|\left(\xi_j^k\right)^3\right|,\ \Delta x\sum_{j=0}^{J-1}\left|\left(\eta_j^k\right)^3\right|\right) \text{ for small } \Delta x\sum_{j=1}^{J}\left|\left(\xi_j^k\right)^3\right| \text{ and } \Delta x\sum_{j=0}^{J-1}\left|\left(\eta_j^k\right)^3\right|.$$

For ξ_j^k and η_j^k satisfying inequalities $\left|\xi_j^k\right|+\left|\eta_j^k\right|\le d_1 \quad \forall j\in\{0,1,\cdots,J\}$
Select a parameter l_1 so $l_1>m$ and $\Upsilon_1^k+\Upsilon_2^k\le -l_1\mathcal{L}^k$ for all k.
One can choose a sufficiently large positive real number K_1 such that

$$\Upsilon_3^k \le \mathrm{K}_1\left\{\Delta x\sum_{j=1}^{J}\left(\xi_j^k\right)^2\left[\left|\gamma_j^k\right|+\left|\delta_j^k\right|\right]+\Delta x\sum_{j=0}^{J-1}\left(\eta_j^k\right)^2\left[\left|\gamma_j^k\right|+\left|\delta_j^k\right|\right]\right\}.$$

This completes the proof of Lemma 3.5. □

Now consider the difference equations for γ_j^k and δ_j^k from the system (3.7)

$$\begin{cases} \gamma_j^{k+1} = \left(1-\left[C_\varphi\right]_{j-1}^k\right)\gamma_j^k+\left[C_\varphi\right]_{j-1}^k\gamma_{j-1}^k-\Delta t\cdot\gamma_j^k f_j^k, \quad k=\overline{0,K-1},\ j=\overline{1,J},\\ \delta_j^{k+1} = \left(1-\left[C_\psi\right]_{j+1}^k\right)\delta_j^k+\left[C_\psi\right]_{j+1}^k\delta_{j+1}^k+\Delta t\cdot\delta_j^k p_j^k,\ k=\overline{0,K-1},\ j=\overline{0,J-1}, \end{cases} \tag{3.22}$$

with appropriate boundary conditions

$$\varphi_0^k\gamma_0^k = -a'\left(\eta_0^k\right)\psi_0^k\delta_0^k,\ \ \psi_J^k\delta_J^k = -b'\left(\xi_J^k\right)\varphi_J^k\gamma_J^k. \tag{3.23}$$

Here

$$\varphi_0^k=\varphi\left(\xi_0^k,\eta_0^k\right),\quad \psi_0^k=\psi\left(\xi_0^k,\eta_0^k\right),\ \ \varphi_J^k=\varphi\left(\xi_J^k,\eta_J^k\right),\quad \psi_J^k=\psi\left(\xi_J^k,\eta_J^k\right).$$

Next, consider the system of difference equations for ρ_j^k,θ_j^k

$$\left\{\begin{array}{l} \rho_j^{k+1} = \left(1-\left[C_\varphi\right]_{j-1}^k\right)\rho_j^k+\left[C_\varphi\right]_{j-1}^k\rho_{j-1}^k-\Delta t\cdot\left[\rho_j^k f_j^k+\gamma_j^k\left(\frac{\partial f}{\partial x}\right)_j^k\right],\quad j=\overline{1,J},\\ \theta_j^{k+1} = \left(1-\left[C_\psi\right]_{j+1}^k\right)\theta_j^k+\left[C_\psi\right]_{j+1}^k\theta_{j+1}^k+\Delta t\cdot\left[\theta_j^k p_j^k+\delta_j^k\left(\frac{\partial p}{\partial x}\right)_j^k\right],\ j=\overline{0,J-1}, \end{array}\right. \quad k=\overline{0,K-1}, \tag{3.24}$$

with appropriate boundary conditions

$$\begin{aligned} &\varphi_0^k\rho_0^k+\gamma_0^k f_0^k = -\left(e'(t)\right)^k\delta_0^k-e^k\left[\psi_0^k\theta_0^k+\delta_0^k p_0^k\right],\\ &\psi_J^k\theta_J^k+\delta_J^k p_J^k = -\left(h'(t)\right)^k\gamma_J^k+h^k\left[\varphi_J^k\rho_J^k+\gamma_J^k p_J^k\right]. \end{aligned} \tag{3.25}$$

Here

$$f_0^k=f\left(\xi_0^k,\ \eta_0^k,\ \gamma_0^k,\ \delta_0^k\right),\ \ p_0^k=p\left(\xi_0^k,\ \eta_0^k,\ \gamma_0^k,\ \delta_0^k\right),$$

$$f_J^k=f\left(\xi_J^k,\ \eta_J^k,\ \gamma_J^k,\ \delta_J^k\right),\ \ p_J^k=p\left(\xi_J^k,\ \eta_J^k,\ \gamma_J^k,\ \delta_J^k\right).$$

The functions e^k and h^k are defined as $e^k:=a'\left(\eta_0^k\right)\psi_0^k/\varphi_0^k,\quad h^k:=b'\left(\xi_J^k\right)\varphi_J^k/\psi_J^k$.

Linearization of systems of difference equations (3.24) and (3.25) (around the origin) have the following form:

$$\gamma_j^{k+1} = (1 - C_\varphi)\,\gamma_j^k + C_\varphi \gamma_{j-1}^k,\ k = \overline{0, K-1},\ \ j = \overline{1, J},$$

$$\delta_j^{k+1} = (1 - C_\psi)\,\delta_j^k + C_\psi \delta_{j+1}^k,\ k = \overline{0, K-1},\ \ j = \overline{0, J-1}$$

respectively

$$\rho_j^{k+1} = (1 - C_\varphi)\,\rho_j^k + C_\varphi \rho_{j-1}^k, \qquad k = \overline{0, K-1},\ \ j = \overline{1, J},$$

$$\theta_j^{k+1} = (1 - C_\psi)\,\theta_j^k + C_\psi \theta_{j+1}^k, \qquad k = \overline{0, K-1},\ \ j = \overline{0, J-1}.$$

It is easy to see that the resulting systems coincide with the linear system with other notations for the dependent variables. That is why this circumstance suggests that we should take as the Lyapunov function

$$\mathbf{L}^k = \mathcal{L}^k + \mathsf{L}^k + \mathfrak{L}^k,$$

where L^k and $\mathfrak{L}^k$ have the format $\mathcal{L}^k$:

$$\mathsf{L}^k = \mathsf{L}_1^k + \mathsf{L}_2^k, \quad \mathsf{L}_1^k = \bar\varphi \mathrm{A} \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 \exp\left(-\frac{m}{\bar\varphi} x_{j-1}\right),\ \mathsf{L}_2^k = \bar\psi\, \mathrm{B}\, \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \exp\left(\frac{m}{\bar\psi} x_{j+1}\right)$$

and

$$\mathfrak{L}^k = \mathfrak{L}_1^k + \mathfrak{L}_2^k,\ \ \mathfrak{L}_1^k = \bar\varphi^3 \mathrm{A} \Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 \exp\left(-\frac{m}{\bar\varphi} x_{j-1}\right),\ \ \mathfrak{L}_2^k = \bar\psi^3\, \mathrm{B}\, \Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \exp\left(\frac{m}{\bar\psi} x_{j+1}\right).$$

Now let's study the difference derivatives with respect to time of functions L^k and $\mathfrak{L}^k$ along solutions of the closed-loop system (3.22)-(3.23)-(3.24)-(3.25).

Lemma 3.6. *If* $|\kappa_0 \kappa_L| < 1$*, if positive real constants* $m, \mathrm{A}, \mathrm{B}$ *satisfy inequalities* $\mathrm{A}\kappa_0^2 - \mathrm{B} < 0$ *and* $\mathrm{B}\exp\left(\frac{m}{\bar\psi}L\right)\kappa_L^2 - \mathrm{A}\exp\left(-\frac{m}{\bar\varphi}L\right) < 0$*, then there exist positive real constants* K_2, d_2, l_2 *such that if* $\left|\xi_j^k\right| + \left|\eta_j^k\right| \le d_2 \quad \forall j \in \{0, 1, \cdots, J\}$ *then*

$$\frac{\mathsf{L}^{k+1} - \mathsf{L}^k}{\Delta t} \le -l_2 \mathsf{L}^k + \mathrm{K}_2 \left\{ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 \left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right] \right\}$$

along solutions of systems (3.22),(3.24) *with boundary conditions* (3.23),(3.25).

Proof. The proof of Lemma 3.6 is similar to the proof of Lemma 3.5. Therefore, we omit it. □

Lemma 3.7. *If* $|\kappa_0 \kappa_L| < 1$*, if positive real constants* $m, \mathrm{A}, \mathrm{B}$ *satisfy inequalities* $\mathrm{A}\kappa_0^2 - \mathrm{B} < 0$ *and* $\mathrm{B}\exp\left(\frac{m}{\bar\psi}L\right)\kappa_L^2 - \mathrm{A}\exp\left(-\frac{m}{\bar\varphi}L\right) < 0$*, then there are positive real constants* K_3, d_3, l_3 *such that if* $\left|\xi_j^k\right| + \left|\eta_j^k\right| \le d_3 \quad \forall j \in \{0, 1, \cdots, J\}$ *then*

$$\frac{\mathfrak{L}^{k+1} - \mathfrak{L}^k}{\Delta t} \le -l_3 \mathfrak{L}^k + \mathrm{K}_3 \left\{ \Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 \left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right] + + \Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \left[\left|\gamma_j^k\right| + \left|\delta_j^k\right|\right] \right\} +$$
$$+ \mathrm{K}_3 \left\{ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 \left[\left|\rho_j^k\right| + \left|\theta_j^k\right|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \left[\left|\rho_j^k\right| + \left|\theta_j^k\right|\right] \right\}.$$

along solutions of systems (3.22), (3.24) *with boundary conditions* (3.23), (3.25).

Proof. The proof of Lemma 3.7 is similar to the proof of Lemma 3.5. Therefore, we omit it. □

Lemma 3.8. *If* $|\kappa_0\kappa_L| < 1$, *if positive real constants* $m, \mathrm{A}, \mathrm{B}$ *satisfy inequalities* $\mathrm{A}\kappa_0^2 - \mathrm{B} < 0$ *and* $\mathrm{B}\exp\left(\frac{m}{\bar{\psi}}L\right)\kappa_L^2 - \mathrm{A}\exp\left(-\frac{m}{\bar{\varphi}}L\right) < 0$, *then there are positive real constants* l_0 *such* d_0 *that if* $\mathbf{L}^k < d_0$, *then*

$$\left(\mathbf{L}^{k+1} - \mathbf{L}^k\right) \Big/ \Delta t \le -l_0 \mathbf{L}^k$$

along closed loop solutions system (3.22), (3.23), (3.24),(3.25).

Proof. In the process of proof, we use discrete versions of some inequalities (a continuous analogue of which can be found in [9]), valid for l^2-functions $\sigma, \varsigma: [0, L] \to \mathrm{R}$ and some positive real constant Ξ:

$$\Delta x \sum_{j=0}^{J} \sigma_j^2 |\varsigma_j| \le \max_{0\le j\le J} |\varsigma_j| \, \Delta x \sum_{j=0}^{J} \sigma_j^2, \tag{3.26}$$

$$\Delta x \sum_{j=0}^{J} \sigma_j^2 |\varsigma_j| \le \max_{0\le j\le J} |\sigma_j|^2 \, \Delta x \sum_{j=0}^{J} |\varsigma_j|. \tag{3.27}$$

According to ([3], p.109) we have

$$\|\sigma\|_C^2 \le 2\left(L\| \, \sigma_{\bar{x}}]|^2 + \sigma_0^2\right), \quad \|\sigma\|_C^2 \le 2\left(L\| \, \sigma_{\bar{x}}]|^2 + \sigma_J^2\right), \quad \sigma_{\bar{x}} = \frac{\sigma_j - \sigma_{j-1}}{\Delta x},$$

$$\|\sigma\|_C = \max_{0\le j\le J} |\sigma_j|, \quad \| \, \sigma_{\bar{x}}]| = \left(\sigma_{\bar{x}}, \sigma_{\bar{x}}\right]^{1/2} = \left(\Delta x \sum_{j=1}^{J} (\sigma_{\bar{x}})^2\right)^{1/2}$$

hence

$$\max_{0\le j\le J} |\sigma_j| \le C\left[\left(\Delta x \sum_{j=0}^{J} \sigma_j^2\right)^{1/2} + \left(\Delta x \sum_{j=0}^{J} \left\{\frac{d\sigma(x_j)}{dx}\right\}^2\right)^{1/2}\right]. \tag{3.28}$$

Discrete Cauchy-Schwarz inequality:

$$\Delta x \sum_{j=0}^{J} |\sigma_j| \le \sqrt{L}\left(\Delta x \sum_{j=0}^{J} \sigma_j^2\right)^{1/2}. \tag{3.29}$$

From Lemmas 3.5, 3.6 and 3.7 we conclude that $\mathbf{L}^k = \mathcal{L}^k + \mathsf{L}^k + \mathfrak{L}^k$, it satisfies the following inequality:

$$\begin{aligned}
\tfrac{\mathbf{L}^{k+1}-\mathbf{L}^k}{\Delta t} \le -l_4\mathbf{L}^k + \mathrm{K}_1 &\left\{ \Delta x \sum_{j=1}^{J} \left(\xi_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] \right\} + \\
+\mathrm{K}_2 &\left\{ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] \right\} + \\
+\mathrm{K}_3 &\left\{ \Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] \right\} + \\
+\mathrm{K}_3 &\left\{ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 \left[|\rho_j^k| + |\theta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \left[|\rho_j^k| + |\theta_j^k|\right] \right\}.
\end{aligned}$$

with $l_4 = \min\{l_1, l_2, l_3\}$. Next we use the following inequalities. Regardless, $\xi, \eta, \gamma, \delta, \rho, \theta$ there are real positive constants Ξ_1, Ξ_2 such that

$$\begin{aligned}
&\left\{ \Delta x \sum_{j=1}^{J} \left(\xi_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] \right\} \leq \\
&\leq \left(\|\gamma^k\|_C + \|\delta^k\|_C\right) \left\{ \Delta x \sum_{j=1}^{J} \left(\xi_j^k\right)^2 + h \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2 \right\} \leq \\
&\leq \Xi_1 \left\{ \begin{array}{l} \left[\Delta x \sum\limits_{j=1}^{J} \left(\gamma_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum\limits_{j=1}^{J} \left(\rho_j^k\right)^2\right]^{\frac{1}{2}} + \\ + \left[\Delta x \sum\limits_{j=0}^{J-1} \left(\delta_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum\limits_{j=0}^{J-1} \left(\theta_j^k\right)^2\right]^{\frac{1}{2}} \end{array} \right\} \left\{ \Delta x \sum_{j=1}^{J} \left(\xi_j^k\right)^2 + \Delta x \sum_{j=0}^{J-1} \left(\eta_j^k\right)^2 \right\} \leq \\
&\leq \ \Xi_2 \left[\left(\mathsf{L}_1^k\right)^{1/2} + \left(\mathfrak{L}_1^k\right)^{1/2} + \left(\mathsf{L}_2^k\right)^{1/2} + \left(\mathfrak{L}_2^k\right)^{1/2}\right] \left[\mathcal{L}_1^k + \mathcal{L}_2^k\right].
\end{aligned} \tag{3.30}$$

We obtain this chain of inequalities as a result of applying inequalities (3.26) and (3.28). Likewise, regardless of $\xi, \eta, \gamma, \delta, \rho, \theta$ $\exists\ \Xi_1' > 0$ and $\exists\ \Xi_2' > 0$ such that

$$\begin{aligned}
&\left\{ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] \right\} \leq \\
&\leq \left(\|\gamma^k\|_C + \|\delta^k\|_C\right) \left[\Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \right] \leq \\
&\leq \Xi'_1 \left\{ \begin{array}{l} \left[\Delta x \sum\limits_{j=1}^{J} \left(\gamma_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum\limits_{j=1}^{J} \left(\rho_j^k\right)^2\right]^{\frac{1}{2}} + \\ + \left[\Delta x \sum\limits_{j=0}^{J-1} \left(\delta_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum\limits_{j=0}^{J-1} \left(\theta_j^k\right)^2\right]^{\frac{1}{2}} \end{array} \right\} \left\{ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \right\} \leq \\
&\leq \Xi'_2 \left[\left(\mathsf{L}_1^k\right)^{1/2} + \left(\mathfrak{L}_1^k\right)^{1/2} + \left(\mathsf{L}_2^k\right)^{1/2} + \left(\mathfrak{L}_2^k\right)^{1/2}\right] \left[\mathsf{L}_1^k + \mathsf{L}_2^k\right].
\end{aligned} \tag{3.31}$$

Inequalities (3.26) and (3.28) were also applied here. Likewise, regardless of $\xi, \eta, \gamma, \delta, \rho, \theta$ $\exists\ \Xi''_1 > 0$ and $\exists\ \Xi_2'' > 0$ such that

$$\begin{aligned}
&\left\{ \Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \left[|\gamma_j^k| + |\delta_j^k|\right] \right\} \ \leq \\
&\leq \ \left(\|\gamma^k\|_C + \|\delta^k\|_C\right) \left[\Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 + \Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \right] \leq \\
&\leq \ \Xi''_1 \left\{ \begin{array}{l} \left[\Delta x \sum\limits_{j=1}^{J} \left(\gamma_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum\limits_{j=1}^{J} \left(\rho_j^k\right)^2\right]^{\frac{1}{2}} + \\ + \left[\Delta x \sum\limits_{j=0}^{J-1} \left(\delta_j^k\right)^2\right]^{\frac{1}{2}} + \left[\Delta x \sum\limits_{j=0}^{J-1} \left(\theta_j^k\right)^2\right]^{\frac{1}{2}} \end{array} \right\} \left\{ \Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 + \Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \right\} \leq \\
&\leq \Xi''_2 \left[\left(\mathsf{L}_1^k\right)^{1/2} + \left(\mathfrak{L}_1^k\right)^{1/2} + \left(\mathsf{L}_2^k\right)^{1/2} + \left(\mathfrak{L}_2^k\right)^{1/2}\right] \left[\mathfrak{L}_1^k + \mathfrak{L}_2^k\right].
\end{aligned} \tag{3.32}$$

To obtain this inequality, we applied inequalities (3.26) and (3.28). Regardless $\xi, \eta, \gamma, \delta, \rho, \theta \; \exists \; \Xi_1''' > 0$ and $\exists \; \Xi_2''' > 0$ such that

$$\begin{gathered}
\left\{ \Delta x \sum_{j=1}^{J} \left(\gamma_j^k\right)^2 \left[|\rho_j^k| + |\theta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left(\delta_j^k\right)^2 \left[|\rho_j^k| + |\theta_j^k|\right] \right\} \le \\
\le \left(\left\| \left(\gamma^k\right)^2 \right\|_C + \left\| \left(\delta^k\right)^2 \right\|_C \right) \left\{ \Delta x \sum_{j=1}^{J} \left[|\rho_j^k| + |\theta_j^k|\right] + \Delta x \sum_{j=0}^{J-1} \left[|\rho_j^k| + |\theta_j^k|\right] \right\} \le \\
\le \Xi'''_1 \left\{ \begin{array}{c} \Delta x \sum\limits_{j=1}^{J} \left(\gamma_j^k\right)^2 + \Delta x \sum\limits_{j=1}^{J} \left(\rho_j^k\right)^2 + \\ + \Delta x \sum\limits_{j=0}^{J-1} \left(\delta_j^k\right)^2 + \Delta x \sum\limits_{j=0}^{J-1} \left(\theta_j^k\right)^2 \end{array} \right\} \left\{ \left[\Delta x \sum_{j=1}^{J} \left(\rho_j^k\right)^2 \right]^{\frac{1}{2}} + \left[\Delta x \sum_{j=0}^{J-1} \left(\theta_j^k\right)^2 \right]^{\frac{1}{2}} \right\} \le \qquad (3.33) \\
\le \Xi'''_2 \left[\left(\mathsf{L}_1^k\right)^{1/2} + \left(\mathfrak{L}_1^k\right)^{1/2} + \left(\mathsf{L}_2^k\right)^{1/2} + \left(\mathfrak{L}_2^k\right)^{1/2} \right] \left[\mathfrak{L}_1^k + \mathfrak{L}_2^k \right].
\end{gathered}$$

Here we also applied inequalities (3.26), (3.27) and (3.28). So from inequality (3.30)-(3.31)-(3.32)-(3.33), we obtain the inequality

$$\frac{\mathbf{L}^{k+1} - \mathbf{L}^k}{\Delta t} \le -l_4 \mathbf{L}^k + \mathrm{K}_4 \left(\mathbf{L}^k\right)^{3/2}.$$

$\forall l_0, \quad 0 < l_0 < l_4 \quad \exists d_0$ that

$$K_4 \left(\mathbf{L}^k\right)^{3/2} < (l_4 - l_0) \, \mathbf{L}^k \qquad \forall \mathbf{L}^k < d_0.$$

Taking into account this inequality we have

$$\frac{\mathbf{L}^{k+1} - \mathbf{L}^k}{\Delta t} \le -l_0 \mathbf{L}^k \qquad \forall \mathbf{L}^k < d_0. \qquad (3.34)$$

Note that if d_0 we take small enough, then from (3.34) we conclude that $\left|\xi_j^k\right| + \left|\eta_j^k\right| < \min(d_1, d_2, d_3) \quad \forall j \in \{0, 1, \cdots, J\}$. Finally, this fact gives us the right to use Lemmas 3.1, 3.2, 3.3 in the process of proving Lemma 3.4. □

Theorem 3.9. *(Discrete stability for the case* $\mathsf{U}^* \ge 0$*). Let us assume that the Courant Friedrichs Lewy (CFL) type condition*

$$C = \max(C_\varphi, C_\psi) < 1, \quad \text{where} \quad C_\varphi = \bar{\varphi} \frac{\Delta t}{\Delta x}, \quad C_\psi = \bar{\psi} \frac{\Delta t}{\Delta x},$$

is satisfied for (3.4)*. For each* $\mathbf{U}^*$ *satisfying the matrix inequality* $\mathsf{U}^* \ge 0$*, each* κ_0, κ_L *satisfying the inequality* $0 < |\kappa_0 \kappa_L| < 1$*, each* $\mathcal{U} > 0$ *and for any initial vector function* $\mathbf{\Phi}$ *satisfying the matrix inequality* $\mathsf{U}^{\mathbf{0}} \ge \mathbf{0}$*, and*

$$\left\| \mathbf{\Phi} - \mathbf{U}^* \right\|_{l^2} < \mathcal{U} \qquad (3.35)$$

the solution $\mathbf{U}^k$ *of the initial-boundary value problem* (3.4), (3.5), (3.6) *satisfies the matrix inequalities* $\mathsf{U}^k \ge 0, \quad k \in \{0, 1, \cdots\}$ *, and the stationary state* $\mathbf{U}^*$*of the initial-boundary difference problem* (3.4), (3.5), (3.6) *is stable in the* l^2 *-norm.*

Let's go to $\mathsf{U}^* = 0$. Then inequality (3.35) in Theorem 3.9 is now expressed as

$$\left\| \mathbf{\Phi} \right\|_{l^2} < \mathcal{U}. \qquad (3.36)$$

Note that inequality (3.7) can be rewritten as

$$\left\| \mathbf{U}^k \right\|_{l^2} \le \nu_2 e^{-\nu_1 t^k} \left\| \mathbf{\Phi} \right\|_{l^2}, \quad k \in \{1, 2, \cdots\},$$

Proof. Further, in the process of proving Theorem 3.9, we consider only the case of the matrix inequality

$$\mathsf{U}^* > 0. \tag{3.37}$$

Since the initial data are $\mathsf{U}^0 \geq 0$, then according to the discrete system (3.4), (3.5), (3.6) and the CFL condition in equation (3.4), (3.5), (3.6) , we have $\mathsf{U}^k \geq 0,\ \ k \in \{0, 1, \cdots\}$.

Consider the following candidate for the discrete Lyapunov function for any $\mathbf{U}^k \in \mathrm{R}^{6\times J}$

$$\mathbf{L}\left(\mathbf{U}^k\right) = \Delta x \sum_{j=1}^{J} \left[\tfrac{\mathrm{A}}{\bar\varphi}\left(\xi_j^k\right)^2 + \bar\varphi \mathrm{A}\left(\gamma_j^k\right)^2 + \bar\varphi^3 \mathrm{A}\left(\rho_j^k\right)^2\right] \exp\left(-\tfrac{m}{\bar\varphi} x_{j-1}\right) +$$

$$+ \Delta x \sum_{j=0}^{J-1} \left[\tfrac{\mathrm{B}}{\bar\psi}\left(\eta_j^k\right)^2 + \bar\psi \mathrm{B}\left(\delta_j^k\right)^2 + \bar\psi^3 \mathrm{B}\left(\theta_j^k\right)^2\right] \exp\left(\tfrac{m}{\bar\psi} x_{j+1}\right).$$

Discrete weight norm $\mathbf{L}\left(\mathbf{U}^k\right)$ is equivalent to the discrete W_2^2-norm , for all $k \geq 0$.

$$\min\left\{\exp\left(-\tfrac{m}{\bar\varphi} L\right) \min\left(\tfrac{\mathrm{A}}{\bar\varphi}, \bar\varphi \mathrm{A}, \bar\varphi^3 \mathrm{A}\right) + \min\left(\tfrac{\mathrm{B}}{\bar\psi}, \bar\psi \mathrm{B}, \bar\psi^3 \mathrm{B}\right)\right\} \left\|\mathbf{U}^k\right\|^2_{w_2^2} \leq \mathbf{L}\left(\mathbf{U}^k\right) \leq$$

$$\max\left\{\exp\left(\tfrac{m}{\bar\psi} L\right) \max\left(\tfrac{\mathrm{A}}{\bar\varphi}, \bar\varphi \mathrm{A}, \bar\varphi^3 \mathrm{A}\right) + \max\left(\tfrac{\mathrm{B}}{\bar\psi}, \bar\psi \mathrm{B}, \bar\psi^3 \mathrm{B}\right)\right\} \left\|\mathbf{U}^k\right\|^2_{w_2^2}$$

where

$$\left\|\mathbf{U}^k\right\|^2_{w_2^2} = \Delta x \sum_{j=1}^{J} \left[\left(\xi_j^k\right)^2 + \left(\gamma_j^k\right)^2 + \left(\rho_j^k\right)^2\right] + \Delta x \sum_{j=0}^{J-1} \left[\left(\eta_j^k\right)^2 + \left(\delta_j^k\right)^2 + \left(\theta_j^k\right)^2\right].$$

As a second step, we evaluate a finite difference approximation of the time derivative of $\mathbf{L}\left(\mathbf{U}^k\right)$ time. For this purpose, we use inequality (3.34)

$$\frac{\mathbf{L}\left(\mathbf{U}^{k+1}\right) - \mathbf{L}\left(\mathbf{U}^k\right)}{\Delta t} \leq -e\, \mathbf{L}\left(\mathbf{U}^k\right),$$

Inequality (3.34) means the existence of a discrete Lyapunov function $\mathbf{L}\left(\mathbf{U}^k\right)$, which provides an exponential decrease $\mathbf{L}\left(\mathbf{U}^k\right)$.

This completes the proof of Theorem 3.9. □

4. Conclusion

So, in this work we studied the problem of exponential stability of the numerical solution of an upwind difference scheme for a quasilinear hyperbolic system with dissipative boundary conditions. An upwind difference scheme is constructed for the numerical solution of the initial boundary value problem. The definition of exponential stability of a numerical solution with respect to the equilibrium state of an initial-boundary difference problem is given. For the first time, a discrete Lyapunov function for a numerical solution was constructed and a theorem on the exponential stability of the equilibrium state of an initial-boundary difference problem for a quasilinear hyperbolic system was proved.

References

[1] Demidovich B., Lectures on the mathematical theory of stability. Nauka, Moscow, (1967).

[2] Lyapunov A., General problem of motion stability. Gostekh-izdat, Moscow, (1950).

[3] Martynyuk D., Lectures on the qualitative theory of difference equations. Nuokova Dimka, Kyiv, (1972).

[4] Evtushenko Y., Zhadan V., Application of the method of Lyapunov functions to study the convergence of numerical methods. Zhurnal vichislitelnoy matematiki i matematicheckoy fiziki, (1975) Vol. 15, Iss. 1, P. 101–112.

[5] Zubov V., Lyapunov's methods and their application. Leningrad State University Publishing House, Leningrad, (1957).

[6] Banda M., Herty M., Numerical discretization of stabilization problems with boundary controls for systems of hyperbolic conservation laws. Math. Control Relat. Fields, (2013) Vol. 3, Iss. 2, P. 121–142.

[7] Gottlich S., Schillen P., Numerical discretization of boundary control problems for systems of balance laws: Feedback stabilization. European Journal of Control, (2017) Vol. 35, P. 11–18.

[8] Weldegiyorgis G. Numerical stabilization with boundary controls for hyperbolic systems of balance laws. PhD thesis. 2017.

[9] Mapundi K., Banda M., Gediyon Y., Numerical boundary feedback stabilization of non-uniform hyperbolic systems of balance laws. International Journal of Control, (2020) Vol. 93, Iss. 6, P. 1428–1441.

[10] Aloev R., Nematova D., Lyapunov Numerical Stability of a Hyperbolic System of Linear Balance Laws with Inhomogeneous Coefficients. AIP Conference Proceedings, (2021), P. 020001.

[11] Aloev R., Nematova D., Three-Dimensional Linear Hyperbolic System. AIP Conference Proceedings, (2021), P. 020002.

[12] Aloev R., Nematova D., An Algebraic Condition For The Exponential Stability Of An Upwind Difference Scheme For Hyperbolic Systems. Journal of Mathematical Sciences, (2024) Vol. 278, Iss. 4,

[13] Aloev R., Berdyshev A., Bliyeva D., Dadabayev S., Baishemirov Z., Stability Analysis of an Upwind Difference Splitting Scheme for Two-Dimensional Saint-Venant Equations. Symmetry, (2022) Vol. 14, Iss. 10, P. 1–21.

[14] Aloev R., Dadabaev S., Stability of the upwind difference splitting scheme for symmetric t-hyperbolic systems with constant coefficients. Results in Applied Mathematics, (2022) Vol. 15, P. 1–20.

[15] Aloev R., Hudayberganov M., A Discrete Analogue of the Lyapunov Function for Hyperbolic Systems. Journal of Mathematical Sciences, (2022) Vol. 264, P. 661–671.

[16] Aloev R., Eshkuvatov Z., Khudoyberganov M., Nematova D., The Difference Splitting Scheme for n-Dimensional Hyperbolic Systems. Malaysian Journal of Mathematical Sciences, (2022) Vol. 16, Iss. 1, P. 1–10.

[17] Aloev R., Berdyshev A., Akbarova A., Baishemirov Z., Development of an algorithm for calculating stable solutions of the Saint-Venant equation using an upwind implicit difference scheme. Eastern-European Journal of Enterprise Technologies, (2021) Vol. 4, Iss. 112, P. 47–56.

[18] Coron J., d'Andrea-Novel B., Bastin G., A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. IEEE Transactions on Automatic Control, (2007) Vol. 52, Iss. 1, P. 2–11.

Aloev Rakhmatillo,
National University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan
e-mail: aloevr@mail.ru

Nematova Dilfuza,
National University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan
Kimyo international university in Tashkent
Tashkent, Uzbekistan
e-mail: nematova_dilfuza@mail.ru

Ilyani Abdullah,
Universiti Malaysia Terengganu. Malaysia
e-mail: ilyani@umt.edu.my

Shalela Mohd Mahali,
Universiti Malaysia Terengganu. Malaysia
e-mail: shalela@umt.edu.my

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 19–27
DOI: 10.29229/uzmj.2026-1-2

High-accuracy difference schemes for solving non-stationary fourth-order equations and their application to non-classical partial differential equations

Aripov M., Utebaev D., Kdirbaev S.

Abstract. In the article, high-accuracy difference schemes for the Cauchy problem for a system of fourth-order equations are obtained. Based on the method of energy inequalities, the stability of the scheme is proved, a priori estimates of the solution to difference schemes are obtained, and their convergence and accuracy are proved. The results obtained for the system are applied to solve the first initial-boundary value problem for the equation of ion-acoustic waves in a "magnetized" plasma for the generalized potential of the electric field. The schemes constructed for this problem have second-order accuracy in spatial variables and fourth-order accuracy in time variables. In energy norms, convergence and accuracy estimates are obtained in classes of smooth solutions.

Keywords: Cauchy problem, fourth-order system of equations, ion-acoustic wave equation, difference schemes, approximation error, stability, convergence, accuracy
MSC (2020): 65M06, 65M12

1. Introduction

In the spatial approximation of partial differential equations using finite difference or finite element methods, we derive systems of ordinary differential equations with large dimensions. Currently, these semi–discrete methods are frequently employed to numerically solve initial–boundary value problems for differential equations, particularly for non–classical Sobolev–type equations. In reference [1], this approach was applied to a system of nonstationary second–order equations, using piecewise cubic interpolation–the finite element method–for approximation. Similar research was conducted in references [2]–[3] for non-stationary first-and second–order equations, resulting in the development of two– or three-parameter vector difference schemes with fourth-order accuracy. References [4]–[5] examined the application of these schemes for numerically solving various high–order Sobolev–type equations. For instance, high–accuracy difference schemes were constructed and analyzed for equations modeling internal waves in a weakly stratified fluid [4] and for equations describing gravitational–gyroscopic waves in a stratified fluid [6]. Additionally, the three–parameter schemes developed in reference [2] were utilized in references [7]–[8] to address various non–classical Sobolev–type equations, where high–accuracy schemes were constructed and investigated within the context of smooth and non–smooth solutions.

The proposed research focuses on the development and examination of two-parameter difference schemes for non-stationary fourth–order equations. These studies were initially conducted in [9]. This work aims to generalize those findings for the numerical solution of the first initial–boundary value problem related to the equation of ion–sound waves in a "magnetized" plasma, incorporating the generalized potential of the electric field. The research presents theorems concerning the convergence and accuracy of the schemes.

2. Statement of the problem

We consider the Cauchy problem for a system of fourth–order operator differential equations:

$$\mathrm{D}\frac{d^4u}{dt^4} + \mathrm{B}\frac{d^2u}{dt^2} + \mathrm{A}u = f, \quad 0 < t \le T, \tag{2.1}$$

$$u(0) = u_{0,0}, \quad \dot{u}(0) = u_{0,1}, \quad \ddot{u}(0) = u_{0,2}, \quad \dddot{u}(0) = u_{0,3}, \tag{2.2}$$

where $\mathrm{D} \neq \mathrm{D}(t)$, $\mathrm{B} \neq \mathrm{B}(t)$, $\mathrm{A} \neq \mathrm{A}(t)$ are operators from $H \to H$, $u = u(t) \in H$, $f = f(t) \in H$ is the Hilbert space with an inner product (u, ϑ) and norm $\|u\| = \sqrt{(u,u)}$. In what follows, we assume that all necessary derivatives of the sought–for solution $u(t)$ exist.

3. Second–order accuracy scheme

Consider problem (2.1), (2.2). On the interval $0 \le t < \infty$, we introduce uniform grid $\overline{\omega}_\tau = \{t_n = n\tau, \quad n = 0, 1, \ldots\}, \quad \omega_\tau = \overline{\omega}_\tau \cap \{0\}$ with step τ. We will consider abstract functions $y = y(t_n)$ and $\varphi = \varphi(t_n)$ of discrete argument $t_n \in \omega_\tau$ with values from H is grid space. We denote the space consisting of elements of space H with the inner product $(u, \vartheta)_{\mathrm{A}} = (\mathrm{A}u, \vartheta)$ and energy norm $\|u\|_{\mathrm{A}} = \sqrt{(u,u)}_{\mathrm{A}}$ by H_{A}. Now we approximate problem (2.1), (2.2) with the following difference scheme

$$\mathrm{D}y_{\bar{t}t\bar{t}t} + \mathrm{B}y_{\bar{t}t} + \mathrm{A}y = \varphi, \quad t_n \in \omega_\tau, n = 2, 3, ..., \tag{3.1}$$

$$y^0 = u_{0,0}, \quad y^1 = \overline{u}_{0,1}, \quad y^2 = \overline{u}_{0,2}, \quad y^3 = \overline{u}_{0,3}, \tag{3.2}$$

where $y_{\bar{t}t\bar{t}t} = (y^{n+2} - 4y^{n+1} + 6y^n - 4y^{n-1} + y^{n-2})/\tau^4, \quad y_{\bar{t}t} = (y^{n+1} - 2y^n + y^{n-1})/\tau^2, \quad y^n = y(t_n), \quad y^{n\pm1} = y(t_n \pm \tau), \quad y^{n\pm2} = y(t_n \pm 2\tau),$

$$\begin{gathered}\overline{u}_{0,1} = u_{0,1} + 0.5\tau \left[\mathrm{E} - (\tau^2/12)\mathrm{D}^{-1}\mathrm{B}\right] u_{0,2}, \\ \overline{u}_{0,2} = u_{0,2} + \tau u_{0,3}, \\ \overline{u}_{0,3} = u_{0,3} + (3\tau/2)\mathrm{D}^{-1}\left[f(0) - \mathrm{B}u_{0,2} - \mathrm{A}u_{0,0}\right],\end{gathered} \tag{3.3}$$

E is the unit operator.

Let us denote the errors by $z = y - u$, where u is the solution to problem (2.1), and y is the solution to scheme (3.1). Then, substituting $y = z + u$ into scheme (3.1), we obtain the problem for the error

$$\mathrm{D}z_{\bar{t}t\bar{t}t} + \mathrm{B}z_{\bar{t}t} + \mathrm{A}z = \psi, \tag{3.4}$$

where $\psi = O(\tau^2)$ is the approximation error of scheme (3.1). The initial conditions (3.2), considering (3.3), also have second-order approximation error, i.e., $O(\tau^2)$.

To study scheme (3.1), we perform the following transformation:

$$\mathrm{D}y^{n+2} - (4\mathrm{D} - \tau^2\mathrm{B})y^{n+1} + (6\mathrm{D} - 2\tau^2\mathrm{B} + \tau^4\mathrm{A})y^n - (4\mathrm{D} - \tau^2\mathrm{B})y^{n-1} + \mathrm{D}y^{n-2} = \tau^4\varphi.$$

Let $y = y^{n+2}$, then from this equality we obtain:

$$B_4 y^{n+4} + B_3 y^{n+3} + B_2 y^{n+2} + B_1 y^{n+1} + B_0 y^n = \tau^4\varphi. \tag{3.5}$$

Here, $B_0 = B_4 = \mathrm{D}$, $B_1 = B_3 = -4\mathrm{D} + \tau^2\mathrm{B}$, $B_2 = 6\mathrm{D} - 2\tau^2\mathrm{B} + \tau^4\mathrm{A}$. Further, following [10], we write scheme (3.5) in the following canonical form:

$$\mathbb{N}y_{\bar{t}} + \tau^2\Re y_{\overset{\circ\circ}{tt}} + \tau^3\Im y_{\bar{t}t\overset{\circ}{t}} + \tau^4\aleph y_{\bar{t}t\bar{t}t} + \mathbb{R}y = \tau^4\varphi, \tag{3.6}$$

where $\mathbb{N} = \tau(2B_4 + B_3 - B_1 - 2B_0)$, $\Re = 2B_0 + 0.5(B_1 + B_3) + 2B_4$, $\Im = 0.5(B_1 - B_3)$,

$$\aleph = -(1/8)(B_1 + B_3), \quad \mathbb{R} = B_4 + B_3 + B_2 + B_1 + B_0. \tag{3.7}$$

Here we use the following notations $y_{\bar{t}} = (y^n - y^{n-1})/\tau, \quad y_{\overset{\circ\circ}{tt}} = (y^{n+4} - 2y^{n+2} + y^n)/(4\tau^2)$,
$y_{\bar{t}t\overset{\circ}{t}} = (y^{n+4} - 2y^{n+3} + 2y^{n+1} - y^n)/(2\tau^3)$.

Taking into account (3.7), after elementary calculations, from (3.6), we obtain a difference scheme in the canonical form:

$$[\mathrm{D} - (\tau^2/4)\mathrm{B}]y_{\bar{t}t\bar{t}t} + \mathrm{B}y_{\overset{\circ\circ}{tt}} + \mathrm{A}y = \varphi.$$

According to Theorem 2 from [[10], p. 276], an a priori estimate based on the initial data ($\varphi = 0$), holds

$$\|y_{n+1}\|_{\bar{\mathrm{A}}} \le \|y_n\|_{\bar{\mathrm{A}}}, \tag{3.8}$$

if the following conditions are met:

$$\operatorname{Re}\mathbb{N} \ge 0, \tag{3.9}$$

$$\mathbb{R} \ge 0, \tag{3.10}$$

$$\Re - 4\aleph - \mathbb{R} \geq 0, \tag{3.11}$$

$$\mathbb{R} + 16\aleph \geq 0. \tag{3.12}$$

Here

$$\|y_n\|^2_{\bar{\mathrm{A}}} = (1/16)\Big[\|y^n + y^{n+1} + y^{n+2} + y^{n+3}\|^2_{\mathbb{R}} + \|y^{n+3} + y^{n+2} - y^{n+1} - y^n\|^2_{\Re-4\aleph-\mathbb{R}} +$$

$$+ \|y^{n+3} - y^{n+2} - y^{n+1} + y^n\|^2_{\Re-4\aleph-\mathbb{R}} + \|y^{n+3} - y^{n+2} + y^{n+1} - y^n\|^2_{\mathbb{R}+16\aleph}\Big].$$

Let us check the fulfillment of conditions (3.9)-(3.12). Conditions (3.9) and (3.10) are fulfilled, since $\mathbb{N} = 0$ and $\mathbb{R} = \tau^4\mathrm{A}$. ($\mathrm{A} = \mathrm{A}^* > 0$). Condition (3.11) will be fulfilled if $4\mathrm{D} + \tau^4\mathrm{A} \leq 2\tau^2\mathrm{B}$, and, finally, the last condition (3.12) will be fulfilled if $16\mathrm{D} + \tau^4\mathrm{A} \geq 4\tau^2\mathrm{B}$. These two conditions will be fulfilled if

$$\mathrm{D} \geq (\tau^4/4)\mathrm{A}, \tag{3.13}$$

which is the stability condition of scheme (3.1), (3.2).

Thus, the following theorem holds.

Theorem 3.1. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (3.13) are fulfilled, an apriori estimate for the initial data (3.8) is valid for the solution of scheme (3.1), (3.2).

To prove the stability of the right-hand side of scheme (3.1)-(3.3), we represent it as an equivalent two-layer scheme in space H^4 [10]:

$$\mathbb{C}y_t + \mathbb{Q}y = \phi, \tag{3.14}$$

where $y_t = \Big\{y_{\bar{t}},\ \tau y_{\overset{\circ}{t}\overset{\circ}{t}},\ (\tau^2/2)y_{\bar{t}t\overset{\circ}{t}},\ (\tau/2)y_{\overset{\circ}{t}\overset{\circ}{t}} + (\tau^3/8)y_{\bar{t}t\bar{t}t}\Big\}$, $\quad \phi = \{\varphi, 0, 0, 0\}$,

$$\mathbb{Q} = \begin{pmatrix} \mathbb{R} & 0 & 0 & 0 \\ 0 & \Re - 2\aleph - \mathbb{R} & 0 & 0 \\ 0 & 0 & \Re - 2\aleph - \mathbb{R} & 0 \\ 0 & 0 & 0 & \mathbb{R} + 16\aleph \end{pmatrix},$$

$$\mathbb{C} = \begin{pmatrix} \mathbb{N} + 0.5\tau\mathbb{R} & \tau(\Re - 4\aleph - \mathbb{R}) & 0 & 0.5(\mathbb{R} + 16\aleph) \\ -\tau(\Re - 4\aleph - \mathbb{R}) & 0.5\tau(\Re - 4\aleph - \mathbb{R}) & 0.5\tau(\Re - 4\aleph - \mathbb{R}) & 0 \\ 0 & -0.5\tau(\Re - 4\aleph - \mathbb{R}) & 0.5\tau(\Re - 4\aleph - \mathbb{R}) & 0 \\ -0.5(\mathbb{R} + 16\aleph) & 0 & 0 & 0.5(\mathbb{R} + 16\aleph) \end{pmatrix}.$$

By Theorem 4 from ([10], p. 284), for the solution to difference scheme (3.14), the following a priori estimate holds:

$$\|y_{n+1}\|_{\bar{\mathrm{A}}} \leq \|y_0\|_{\bar{\mathrm{A}}} + \|\varphi_0\|_{\bar{\mathrm{A}}^{-1}} + \|\varphi_n\|_{\bar{\mathrm{A}}^{-1}} + \sum_{k=1}^{n} \tau\|\varphi_{k,\bar{t}}\|_{\bar{\mathrm{A}}^{-1}}, \tag{3.15}$$

if the following conditions are met:

$$\operatorname{Re}\mathbb{N} \geq 0, \tag{3.16}$$

$$\mathbb{R} > 0, \tag{3.17}$$

$$\Re - 4\aleph - \mathbb{R} > 0, \tag{3.18}$$

$$\mathbb{R} + 16\aleph > 0. \tag{3.19}$$

In inequality (3.15), $\|\varphi_n\|_{\bar{\mathrm{A}}^{-1}} = \|\varphi^n\|_{\bar{\mathrm{A}}^{-1}}$, $\|\varphi_{k,\bar{t}}\|_{\bar{\mathrm{A}}^{-1}} = \|\varphi^k_{\bar{t}}\|_{\bar{\mathrm{A}}^{-1}} = (\bar{\mathrm{A}}^{-1}\varphi^k_{\bar{t}}, \varphi^k_{\bar{t}})$. Consequently, the following assertion holds.

Theorem 3.2. Let $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$. Then, if (3.13) is satisfied, then for the solution of scheme (3.1)-(3.3), estimate (3.15) holds.

Let us check the fulfillment of conditions (3.16)-(3.19). Conditions (3.16) and (3.17) are fulfilled, since $\mathbb{N} = 0$ and $\mathrm{A} = \mathrm{A}^* > 0$. Conditions (3.18) and (3.19) will also be fulfilled if condition (3.13) is fulfilled.

To prove the convergence of difference scheme (3.1)-(3.3), we obtain a problem for the error $z = y - u$, i.e., substituting $y = z + u$ into (3.1), we obtain:

$$[\mathrm{D} - (\tau^2/4)\mathrm{B}]z_{\bar{t}t\bar{t}t} + \mathrm{B}z_{\overset{\circ}{t}\overset{\circ}{t}} + \mathrm{A}z = \psi$$

with the corresponding initial conditions. Therefore, based on Theorems 3.1 and 3.2, considering (3.4), we have the following result.

Theorem 3.3. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (3.13) are met, the following accuracy estimate for the solution of scheme (3.1)-(3.3) is valid:

$$\|y(t_n) - u(t_n)\|_{\bar{\mathrm{A}}} \leq O(\tau^2), \quad t_n \in \bar{\omega}_\tau. \tag{3.20}$$

4. Fourth-order accuracy scheme

From (3.4), we obtain $\psi = \varphi - \mathrm{D}u_{\bar{t}t\bar{t}t} - \mathrm{B}u_{\bar{t}t} - \mathrm{A}u$ for the error. Then, using the Taylor expansion formula and equation (2.1), we obtain:

$$\mathrm{D}u_{\bar{t}t\bar{t}t} = \mathrm{D}(t_n) + (1/6)\tau^2\mathrm{D}u^{(6)}(t_n) + O(\tau^4), \quad \mathrm{B}u_{\bar{t}t} = \mathrm{B}\ddot{u}(t_n) + (1/12)\tau^2\mathrm{B}\,\ddot{\ddot{u}}\,(t_n) + O(\tau^4).$$

Consequently, $\psi = \varphi - f^n - (\tau^2/6)\ddot{f}^n + (\tau^2/12)\mathrm{B}\,\ddot{\ddot{u}} + (\tau^2/6)\mathrm{A}\ddot{u} + O(\tau^4)$. Then, if we choose

$$\bar{\mathrm{D}} = \mathrm{D} + (\tau^2/12)\mathrm{B}, \quad \bar{\mathrm{B}} = \mathrm{B} + (\tau^2/6)\mathrm{A}, \quad \bar{\varphi} = \varphi + (\tau^2/6)\ddot{f}, \tag{4.1}$$

then we obtain the following difference scheme:

$$\bar{\mathrm{D}}y_{\bar{t}t\bar{t}t} + \bar{\mathrm{B}}y_{\bar{t}t} + \mathrm{A}y = \bar{\varphi}, \ t_n \in \omega_\tau, \ n = 2, 3, ..., \tag{4.2}$$

which has the order of approximation $\psi = O(\tau^4)$. We choose the initial conditions for (4.2) in the form (3.2):

$$y^0 = u_{0,0}, \ y^1 = \bar{u}_{0,1}, \ y^2 = \bar{u}_{0,2}, \ y^3 = \bar{u}_{0,3}, \tag{4.3}$$

where

$$\begin{gathered}
\bar{u}_{0,1} = u_{0,1} + 0.5\tau[\mathrm{E} - (\tau^2/12)\mathrm{D}^{-1}\mathrm{B}]u_{0,2} + (\tau^2/6)u_{0,3} + (\tau^3/24)\mathrm{D}^{-1}[f(0) - \mathrm{A}u_{0,0}], \\
\bar{u}_{0,2} = u_{0,2} + \tau u_{0,3} + (\tau^2/2)\mathrm{D}^{-1}[f(0) - \mathrm{B}u_{0,2} - \mathrm{A}u_{0,0}] + \\
+(\tau^3/4)\mathrm{D}^{-1}[\dot{f}(0) - \mathrm{B}\dot{u}_{0,2} - \mathrm{A}\dot{u}_{0,0}], \\
\bar{u}_{0,3} = u_{0,3} + (3\tau/2)\mathrm{D}^{-1}[f(0) - \mathrm{B}u_{0,2} - \mathrm{A}u_{0,0}] + (5\tau^2/4)\mathrm{D}^{-1}[\dot{f}(0) - \mathrm{B}\dot{u}_{0,2} - \mathrm{A}\dot{u}_{0,0}] + \\
+(3\tau^2/4)\mathrm{D}^{-1}[\ddot{f}(0) - \mathrm{B}\ddot{u}_{0,2} - \mathrm{A}\ddot{u}_{0,0}].
\end{gathered} \tag{4.4}$$

The approximation error of the initial conditions coincides with the approximation error of scheme (4.2), i.e. $\psi = O(\tau^4)$.

To study the stability of the initial data of scheme (4.2), we write it in the canonical form:

$$[\bar{\mathrm{D}} - (\tau^2/4)\bar{\mathrm{B}}]y_{\bar{t}t\bar{t}t} + \bar{\mathrm{B}}y_{\overset{\circ}{t}\overset{\circ}{t}} + \mathrm{A}y = \varphi. \tag{4.5}$$

Then, for scheme (4.5), the a priori estimate (3.8) holds if conditions (3.9)-(3.12) with operators (3.7) are satisfied, where:

$$B_0 = B_4 = \bar{\mathrm{D}}, \quad B_1 = B_3 = -4\bar{\mathrm{D}} + \tau^2\bar{\mathrm{B}}, \quad B_2 = 6\bar{\mathrm{D}} - 2\tau^2\bar{\mathrm{B}} + \tau^4\mathrm{A}. \tag{4.6}$$

Checking the fulfillment of conditions (3.9)-(3.12) and considering (4.6), we arrive at the stability condition of difference scheme (4.2):

$$\bar{\mathrm{D}} \geq (\tau^4/4)\mathrm{A}. \tag{4.7}$$

Consequently, the following theorem holds.

Theorem 4.1. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (4.7) are satisfied, for the solution of scheme (4.2)-(4.4), an a priori estimate based on the initial data (3.8) with operators (4.1), (4.6) holds.

Similarly to the second-order accuracy scheme, the following assertion is proved.

Theorem 4.2. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (4.7) are satisfied, for the solution of scheme (4.2)-(4.4) with operators (4.1), (4.6), estimate (3.15) holds.

To prove the convergence of scheme (4.2)-(4.4), we obtain a problem for the error:

$$\bar{\mathrm{D}} z_{\bar{t}t\bar{t}t} + \bar{\mathrm{B}} z_{\bar{t}t} + \mathrm{A} z = \bar{\psi}, \ t_n \in \bar{\omega}_\tau, \ n = 2, 3, ..., \tag{4.8}$$

with the corresponding initial conditions. Here, $\bar{\psi} = O(\tau^4)$. Therefore, based on Theorems 4.1 and 4.2, taking into account (4.8), we obtain the following result.

Theorem 4.3. Under conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (4.7), the accuracy estimate (3.20) is valid for solving scheme (4.2)-(4.4).

5. Scheme with weights

Based on difference schemes (4.2)-(4.4) with operators (4.6), we consider the following family of difference schemes with weights

$$\bar{\mathrm{D}} y_{\bar{t}t\bar{t}t} + \bar{\mathrm{B}} y_{\bar{t}t}^{(\sigma_1,\sigma_2)} + \mathrm{A} y^{(\sigma_3,\sigma_4)} = \bar{\varphi}, \quad t_n \in \bar{\omega}_\tau. \tag{5.1}$$

Here, $y^{(\sigma_1,\sigma_2)} = \sigma_1 \hat{y} + (1-\sigma_1-\sigma_2) y + \sigma_2 \overset{\vee}{y}$, $y^{(\sigma_3,\sigma_4)} = \sigma_3 \hat{y} + (1-\sigma_3-\sigma_4) y + \sigma_4 \overset{\vee}{y}$, where $\sigma_1, \ \sigma_2, \ \sigma_3, \ \sigma_4$ are some constants, the weights of the scheme, the presence of which allows us to select various explicit and implicit schemes and regulate their accuracy.

Let us study the stability and convergence of scheme (5.1) with initial conditions (4.3), (4.4). To do this, we will reduce (5.1) to canonical form. We will perform the following transformation with scheme (5.1):

$$\begin{gathered}\bar{\mathrm{D}} + \tau^2\sigma_1\bar{\mathrm{B}}) y^{n+2} - [4\bar{\mathrm{D}} - \tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} + 2\tau^2\sigma_1\bar{\mathrm{B}} - \tau^4\sigma_3\mathrm{A}] y^{n+1} + \\ +[6\bar{\mathrm{D}} + \tau^2\sigma_2\bar{\mathrm{B}} - 2\tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} + \tau^2\sigma_1\bar{\mathrm{B}} + \tau^4(1-\sigma_3-\sigma_4)\mathrm{A}] y^n - \\ - [4\bar{\mathrm{D}} + 2\tau^2\sigma_2\bar{\mathrm{B}} - \tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} - \tau^4\sigma_4\mathrm{A}] y^{n-1} + (\bar{\mathrm{D}} + \tau^2\sigma_2\bar{\mathrm{B}}) y^{n-2} = \tau^4\bar{\varphi}.\end{gathered} \tag{5.2}$$

Let in (5.2) $y = y^{n+2}$. Then (5.2) has the following form:

$$B_4 y^{n+4} + B_3 y^{n+3} + B_2 y^{n+2} + B_1 y^{n+1} + B_0 y^n = \tau^4 \bar{\varphi}^n,$$

where

$$\begin{gathered}B_4 = \bar{\mathrm{D}} + \tau^2\sigma_1\bar{\mathrm{B}}, \quad B_3 = -[4\bar{\mathrm{D}} - \tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} + 2\tau^2\sigma_1\bar{\mathrm{B}} - \tau^4\sigma_3\mathrm{A}], \\ B_2 = 6\bar{\mathrm{D}} + \tau^2\sigma_2\bar{\mathrm{B}} - 2\tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} + \tau^2\sigma_1\bar{\mathrm{B}} + \tau^4(1-\sigma_3-\sigma_4)\mathrm{A}, \\ B_1 = -[4\bar{\mathrm{D}} + 2\tau^2\sigma_2\bar{\mathrm{B}} - \tau^2(1-\sigma_1-\sigma_2)\bar{\mathrm{B}} - \tau^4\sigma_4\mathrm{A}], B_0 = \bar{\mathrm{D}} + \tau^2\sigma_2\bar{\mathrm{B}}.\end{gathered} \tag{5.3}$$

Now we write scheme (5.3) in the following canonical form:

$$M y_{\bar{t}} + \tau^2 R y_{\overset{\circ}{t}\overset{\circ}{t}} + \tau^3 P y_{\bar{t}t\overset{\circ}{t}} + \tau^4 Q y_{\bar{t}t\bar{t}t} + N y = \tau^4 \bar{\varphi}, \tag{5.4}$$

where

$$\begin{gathered}M = \tau(2B_4 + B_3 - B_1 - 2B_0), \quad R = 2B_0 + 0.5(B_1 + B_3) + 2B_4, \\ P = 0.5(B_1 - B_3), \quad Q = -(1/8)(B_1 + B_3), \quad N = B_4 + B_3 + B_2 + B_1 + B_0.\end{gathered}$$

From here, taking into account (5.3), we obtain

$$\begin{gathered}M = \tau^5(\sigma_3 - \sigma_4)\mathrm{A}, \quad R = \tau^2\left[1 - 2(\sigma_1+\sigma_2)\right]\bar{\mathrm{B}} + 0.5\tau^4(\sigma_4+\sigma_3)\mathrm{A}, \\ P = -(\sigma_2-\sigma_1)\tau^2\bar{\mathrm{B}} + 0.5\tau^4(\sigma_4-\sigma_3)\mathrm{A}, \\ Q = \bar{\mathrm{D}} - (\tau^2/4)\left[1 - 2(\sigma_2+\sigma_1)\right]\bar{\mathrm{B}} - (\tau^4/8)(\sigma_4+\sigma_3)\mathrm{A}, \quad N = \tau^4\mathrm{A}.\end{gathered}$$

Let $\sigma_1 = \sigma_2 = \sigma$, $\sigma_3 = \sigma_4 = \theta$, then $M = P = 0$. Consequently, after elementary calculations from (5.4), we obtain the difference scheme in the canonical form:

$$\tilde{Q} y_{\bar{t}t\bar{t}t} + \tilde{R} y_{\overset{\circ}{t}\overset{\circ}{t}} + \mathrm{A} y = \bar{\varphi}, \tag{5.5}$$

where

$$\tilde{Q} \equiv \mathrm{D} - (\tau^2/4)(1-4\sigma)\bar{\mathrm{B}} - (\tau^4/8)\theta\mathrm{A}, \quad \tilde{R} = \tau^2(1-4\sigma)\bar{\mathrm{B}} + \tau^4\theta\mathrm{A}.$$

Then, an a priori estimate based on the initial data (3.8) ($\varphi = 0$) holds, if the following conditions are satisfied:

$$\operatorname{Re} M \geq 0, \quad N \geq 0, \quad R - 4Q - N \geq 0, \quad N + 16Q \geq 0, \tag{5.6}$$

where $\|Y^n\|^2_{\bar{A}}$ is defined in section 3.

Let us check the fulfillment of conditions (5.6). The first condition $\operatorname{Re} M \geq 0$ is fulfilled, since $M = 0$. The second condition is $N \geq 0$, since the operator is $\mathrm{A} > 0$. Condition $R - 4Q - N \geq 0$ will be fulfilled if

$$4\bar{\mathrm{D}} + \tau^4(1-2\sigma)\mathrm{A} \leq 2\tau^2(1-4\sigma)\bar{\mathrm{B}} \tag{5.7}$$

and finally the last condition $N + 16Q \geq 0$ will be fulfilled if

$$16\bar{\mathrm{D}} + \tau^4(1-2\theta)\mathrm{A} \geq 4\tau^2(1-4\sigma)\bar{\mathrm{B}}. \tag{5.8}$$

Conditions (5.7) and (5.8) will be fulfilled if

$$\theta \leq 1/2, \quad \sigma \leq 1/4, \quad \bar{\mathrm{D}} \geq (\tau^4/8)\mathrm{A}, \tag{5.9}$$

which are the stability conditions for scheme (5.1), (4.3), (4.4).

Thus, the following theorem is proved.

Theorem 5.1. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (5.9) are fulfilled, an a priori estimate based on the initial data (3.8) holds for the solution of scheme (5.1), (4.3), (4.4).

To prove the stability of the right-hand side of scheme (5.1), (4.3), (4.4)., we will represent it as an equivalent two-layer scheme in space H^4:

$$\mathfrak{A} y_t + \mathfrak{M} y = \Psi,$$

where $y_t = \left\{ y_{\bar{t}},\ \tau y_{\overset{\circ}{t}\overset{\circ}{t}},\ (\tau^2/2) y_{\bar{t}t\overset{\circ}{t}},\ (\tau/2) y_{\overset{\circ}{t}\overset{\circ}{t}} + (\tau^3/8) y_{\bar{t}t\bar{t}t} \right\}$, $\Psi = \{\varphi, 0, 0, 0\}$.

Therefore, the following assertion holds.

Theorem 5.2. Let $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and the following operator inequalities hold:

$$\operatorname{Re} M \geq 0, \quad N > 0, \quad R - 4Q - N > 0, \quad N + 16Q > 0. \tag{5.10}$$

Then, for the solution of difference scheme (5.1), (4.3), (4.4), the following a priori estimate is true:

$$\|y_{n+1}\|_{\bar{\mathrm{A}}} \leq \|y_0\|_{\bar{\mathrm{A}}} + \|\bar{\varphi}_0\|_{\bar{\mathrm{A}}^{-1}} + \|\bar{\varphi}_n\|_{\bar{\mathrm{A}}^{-1}} + \sum_{k=1}^{n} \tau \|\bar{\varphi}_{k,\bar{t}}\|_{\bar{\mathrm{A}}^{-1}}.$$

From (5.10), the first two conditions are satisfied, since $M = 0$ and $\mathrm{A}^* = \mathrm{A} > 0$, and the rest will be satisfied if inequalities (5.9) hold.

To prove the convergence of difference scheme (5.5), (4.3), (4.4), we obtain a problem for the error

$$\tilde{Q} z_{\bar{t}t\bar{t}t} + \tilde{R} z_{\overset{\circ}{t}\overset{\circ}{t}} + \mathrm{A} z = \bar{\psi}, \quad t_n \in \bar{\omega}_\tau, \quad n = 2, 3, ...$$

with the corresponding initial conditions. Here, $\bar{\psi} = O(\tau^4)$. Therefore, based on Theorems 5.1 and 5.2, we obtain the following assertion.

Theorem 5.3. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and (5.9) are satisfied, for the solution of scheme (5.1), (4.3), (4.4), the following accuracy estimate is true:

$$\|y(t_n) - u(t_n)\|_{\bar{\mathrm{A}}} \leq O(\tau^4), \quad t_n \in \bar{\omega}_\tau.$$

6. Difference schemes for partial differential equations

In domain

$$\bar{Q}_T = \{(x,t):\ x=(x_1,x_2,x_3)\in\bar{\Omega}=[0\le x_\alpha\le l_\alpha,\ \alpha=1,2,3],\ t\in[0,T]\}$$

consider equation [11]

$$\frac{\partial^4}{\partial t^4}\left(\Delta_3 u-\frac{1}{r_{\mathrm{D}}^2}u\right)+\frac{\partial^2}{\partial t^2}\left[(\omega_{Bi}^2+\omega_{pi}^2)\Delta_3 u-\frac{\omega_{Bi}^2}{r_{\mathrm{D}}^2}u\right]+\omega_{pi}^2\omega_{Bi}^2\frac{\partial^2 u}{\partial x_3^2}=f(x,t) \tag{6.1}$$

with initial

$$\left.\frac{\partial^k}{\partial t^k}u(x,t)\right|_{t=0}=u_{0,k},\quad k=\overline{0,3},\quad x\in\Omega \tag{6.2}$$

and boundary conditions of the first kind

$$u(x,t)|_\Gamma=\mu(t),\quad t\in[0,T], \tag{6.3}$$

where $u=u(x,t)$ is the generalized potential of the electric field, Δ_3 is the three-dimensional Laplace operator, $\omega_{\mathrm{B}i}^2=e\mathrm{B}_0/(Mc)$ is the ion gyro-frequency, $\omega_{pi}^2=4\pi e^2 n_0/M$ is the Langmuir frequency for ions, $r_{\mathrm{D}}=\left[T_e/(4\pi n_0 e^2)\right]^{1/2}$ is the Debye radius, M is the ion mass, T_e is the electron temperature, n_0 is the unperturbed particle density, e is the absolute value of the electron charge.

To discretize problem (6.1)-(6.3) in space, we rewrite it in the following form

$$L_0\frac{\partial^4 u}{\partial t^4}+L_1\frac{\partial^2 u}{\partial t^2}+L_2 u=f(x,t),\ \ (x,t)\in Q_T,$$
$$\frac{\partial^k}{\partial t^k}u(x,0)=u_{0,k},\ \ k=\overline{0,3},\ \ x\in\Omega,$$
$$u(x,t)|_\Gamma=\mu(t),\ \ t\in[0,T],$$

where

$$L_0=\Delta_3-\frac{1}{r_{\mathrm{D}}^2}E,\ \ L_1=\omega_0^2\Delta_3-\frac{\omega_{Bi}^2}{r_{\mathrm{D}}^2}E,\ \ L_2=\omega_1^2\frac{\partial^2}{\partial x_3^2}. \tag{6.4}$$

Here, $\omega_0^2=\omega_{pi}^2+\omega_{\mathrm{B}i}^2$, $\omega_1^2=\omega_{pi}^2\omega_{\mathrm{B}i}^2$, Γ is the boundary of domain $\bar{\Omega}$, E is the unit operator.

Let us construct subspace $H_h\subset H$, that approximates the Hilbert space H with the corresponding scalar product and norm. Let us introduce into $\bar{\Omega}$ a grid uniform in each direction $\bar{\omega}_h=\bar{\omega}_{h_1}\times\bar{\omega}_{h_2}\times\bar{\omega}_{h_3}$, where $\bar{\omega}_{h_\alpha}=\{x_\alpha=i_\alpha h_\alpha,\ \ i_\alpha=\overline{0,N}_\alpha,\ \ h_\alpha=l_\alpha/N_\alpha\}$, $\alpha=1,2,3$. Here, $\bar{\omega}_h=\omega_h+\gamma_h$, γ_h - are the boundary nodes of the grid. Let us define subspace $H_h=\overset{\circ}{W}{}_2^1(\omega_h)$ with norm

$$\|v\|_1^2=\sqrt{\sum_{i_1}^{N_1}\sum_{i_2}^{N_2}\sum_{i_3}^{N_3}h_1h_2h_3\left[(\vartheta_{\bar{x}_1})^2+(\vartheta_{\bar{x}_2})^2+(\vartheta_{\bar{x}_3})^2\right]}\le M.$$

Here M does not depend on $h_1,\ h_2,\ h_3$, $\vartheta=\vartheta(i_1h_1,i_2h_2,i_3h_3)$,

$$\vartheta_{\bar{x}_1}=\left[\vartheta(i_1h_1,i_2h_2,i_3h_3)-\vartheta((i_1-1)h_1,i_2h_2,i_3h_3)\right]/h_1,$$
$$\vartheta_{\bar{x}_2}=\left[\vartheta(i_1h_1,i_2h_2,i_3h_3)-\vartheta(i_1h_1,(i_2-1)h_2,i_3h_3)\right]/h_2\,,$$
$$\vartheta_{\bar{x}_3}=\left[\vartheta(i_1h_1,i_2h_2,i_3h_3)-\vartheta(i_1h_1,i_2h_2,(i_3-1)h_3)\right]/h_3\,,$$

where $\overset{\circ}{W}{}_2^1(\omega_h)$ is the Sobolev space [12].

Now, approximating operators L_0, L_1, and L_2 by difference relations, we obtain the following problem:

$$\mathrm{D}\frac{d^4u_h}{dt^4}+\mathrm{B}\frac{d^2u_h}{dt^2}+\mathrm{A}u_h(t)=f_h,\quad \frac{d^ku_h}{dt^k}(0)=u_{0,k,h},\qquad k=\overline{0,3}, \tag{6.5}$$

where u_h approximates $u(x,t)$, D, B, and A are linear constant operators from $H_h \to H_h$, $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0 \ \forall \ t \geq 0$, $u_h = u_h(t) \in H_h$, $f_h = f_h(t) \in H_h$. Here operators are:

$$\mathrm{D} = \Lambda - \frac{1}{r_{\mathrm{D}}^2}E, \ \ \mathrm{B} = \omega_0^2\Lambda - \frac{\omega_{\mathrm{B}i}^2}{r_{\mathrm{D}}^2}E, \ \ \mathrm{A} = \omega_1^2\Lambda_3,$$

where $\Lambda = \sum\limits_{\alpha=1}^{3} \Lambda_\alpha$, $\Lambda_m u_h = u_{h,\bar{x}_m x_m}$, $m = 1,2,3$, u_h are the values of function $u(x,t)$ at the fixed node $x = (i_1h_1, i_2h_2, i_3h_3)$,

$$u_{h,\bar{x}_1x_1} = (u_h((i_1+1)h_1, i_2h_2, i_3h_3) - 2u_h(i_1h_1, i_2h_2, i_3h_3) + u_h((i_1-1)h_1, i_2h_2, i_3h_3))/\, h_1^2,$$
$$u_{h,\bar{x}_2x_2} = (u_h(i_1h_1, (i_2+1)h_2, i_3h_3) - 2u_h(i_1h_1, i_2h_2, i_3h_3) + u_h(i_1h_1, (i_2-1)h_2, i_3h_3))/\, h_2^2,$$
$$u_{h,\bar{x}_3x_3} = (u_h(i_1h_1, i_2h_2, (i_3+1)h_3) - 2u_h((i_1h_1, i_2h_2, i_3h_3)) + u_h(i_1h_1, i_2h_2, (i_3-1)h_3))/\, h_3^2.$$

Operators D, B, and A approximate operators L_0, L_1, and L_2 from (6.4) with the second order, respectively, i.e., $O(|h|^2)$, $|h| = \sqrt{h_1^2 + h_2^2 + h_3^2}$.

For approximation of (6.5), we apply difference scheme (5.1) with parameters $\sigma_1 = \sigma_2 = \sigma$, $\sigma_3 = \sigma_4 = \theta$, i.e., we have a two-parameter difference scheme

$$\bar{\mathrm{D}}y_{\bar{t}t\bar{t}t} + \bar{\mathrm{B}}y_{\bar{t}t}^{(\sigma)} + \mathrm{A}y^{(\theta)} = \bar{\varphi}, \quad t_n \in \bar{\omega}_\tau. \tag{6.6}$$

The initial conditions remain the same, i.e., (4.3). Based on Theorem 5.3 and the results of discretization in space, we obtain the following result.

Theorem 6.1. When conditions $\mathrm{D}^* = \mathrm{D} > 0$, $\mathrm{B}^* = \mathrm{B} \geq 0$, $\mathrm{A}^* = \mathrm{A} > 0$ and $\theta \leq 1/2$, $\sigma \leq 1/4$, $\bar{\mathrm{D}} \geq (\tau^4/8)\mathrm{A}$ are satisfied, for the solution of scheme (6.6), (4.3), (4.4), the following accuracy estimate holds:

$$\|y(x_i,t_n) - u(x_i,t_n)\|_{\bar{\mathrm{A}}} \leq O(|h|^2 + \tau^4), \ \ y, u \in H_h, \ \ x_i \in \bar{\omega}_h, \ \ t_n \in \bar{\omega}_\tau.$$

If we choose $h_\alpha = h$, $\alpha = \overline{1,3}$, then condition $\bar{\mathrm{D}} \geq (\tau^4/8)\mathrm{A}$ will be satisfied for $\tau \leq 2h/\sqrt{\omega_1}$.

7. Numerical implementation

If $\sigma = \theta = 0$, then from (6.6) we can derive explicit scheme (4.2), (4.3), which is realized directly, and for other σ and θ, we obtain implicit schemes, which can be implemented using the sweep method.

8. Conclusions

In this article, we have constructed and analyzed fourth-order accurate schemes for a fourth-order non-stationary equation. We established stability conditions and derived a priori estimates. Based on these estimates, we proved theorems regarding the convergence and accuracy of the solutions for the difference schemes. The difference schemes developed for the abstract Cauchy problem were then applied to solve the initial-boundary value problem for the Sobolev sixth-order partial differential equation. We also proved theorems concerning the convergence and accuracy of the constructed difference schemes. These findings pave the way for the development and analysis of difference schemes for other non-classical high-order equations with various boundary conditions.

References

[1] Moskalkov M., Schemes of the Finite Element Method of High Accuracy for Solving Non-Stationary Second-Order Equations. Differential Equations, (1980) Vol. 16, Iss. 7, P. 1283–1292.

[2] Utebaev D., On one method of numerical solution of operator differential equation of the second order. Doklady of the Academy of Science of the Republic of Uzbekistan. Series Mathematics, Technical Sciences, Natural Sciences, (1980), Iss. 1, P. 31–34.

[3] Utebaev D., Moskalkov M., Numerical Modeling of Non-Stationary Processes in Continuum Mechanics. Fan va Technology, Tashkent, (2012).

[4] Moskalkov M., Utebaev D., Convergence of the Finite Element Scheme for the Equation of Internal Waves. Cybernetics and Systems Analysis, (2011) Vol. 47, Iss. 3, P. 459–465.

[5] Aripov M., Utebaev D., Nurullaev Z., Convergence of high-accuracy finite element method schemes for the two-temperature plasma equation. ICAAM 2020, AIP Conference Proceedings, (2021), P. 020059–1–6.

[6] Moskalkov M., Utebaev D., Finite element method for the gravity-gyroscopic wave equation. Journal of Numerical and Applied Mathematics, (2010), Iss. 2(101), P. 97–104.

[7] Moskalkov M., Utebaev D., Comparison of some methods for solving the internal wave propagation problem in a weakly stratified fluid. Mathematical Models and Computer Simulations, (2011) Vol. 3, Iss. 2, P. 264–271.

[8] Utebaev D., Utepbergenova G., Tileuov K., On convergence of schemes of finite element method of high accuracy for the equation of heat and moisture transfer. Bulletin of Karaganda University, (2021), Iss. 2(101), P. 29–43.

[9] Aripov M., Utebaev B., Utebaev D., Kazymbetova M., High-accuracy difference schemes for a fourth-order operator differential equation. Doklady of the Academy of Science of the Republic of Uzbekistan, (2023), Iss. 5, P. 3–9.

[10] Samarskii A., Gulin A., Stability of Difference Schemes. Librokom/URSS, Moscow, (2009).

[11] Alshin A., Korpusov M., Sveshnikov A., Blow Up in Nonlinear Sobolev Type Equations. De Gruyter, Berlin/New York, (2011).

[12] Samarskii A., The Theory of Difference Schemes. Marcel Dekker, New York, (2001).

Aripov M.M.,
National University of Uzbekistan,
Tashkent, Uzbekistan
e-mail: mirsaidaripov@mail.ru

Utebaev D.,
Karakalpak State University, Nukus, Uzbekistan,
V.I. Romanovsky Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
e-mail: dutebaev_56@mail.ru

Kdirbaev S.A.,
Karakalpak State University, Nukus, Uzbekistan
e-mail: kdirbaevsapar@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 28–33
DOI: 10.29229/uzmj.2026-1-3

On a trinomial analogue of Muirhead's inequaty

Azamov A.

Abstract. It is studied inequalities between three elementary symmetric polynomials being trinomial generalization of the well-known Muirhead's inequality. Some sufficient conditions have been derived. A sufficient and necessary condition is given in the case of two variables.

Keywords: Muirhead's inequality, symmetric polynomial, circle division polynomial, arithmetic function
MSC (2020): 26D05; 26E60

1. Introduction

The notable Muirhead's inequality [1]

$$\forall \mathbf{x} \in \mathbb{R}_+^n : \mu_{\boldsymbol{\alpha}}(\mathbf{x}) \geq \mu_{\boldsymbol{\beta}}(\mathbf{x}) \Leftrightarrow \boldsymbol{\alpha} \succeq \boldsymbol{\beta} \tag{1.1}$$

occupies an explicit position in both inequalities and symmetric polynomials theories.

In (1.1), the following denotations are used:

$\mathbb{R}_+^n = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \,|\, x_k > 0,\ k = 1, 2, \ldots, n\}$ is the positive ortant;

$\mu_{\boldsymbol{\alpha}}(\mathbf{x}) = \frac{1}{n!} \sum\limits_{\boldsymbol{\sigma} \in S_n} x_{\sigma(1)}^{\alpha_1} x_{\sigma(2)}^{\alpha_2} \ldots x_{\sigma(n)}^{\alpha_n}$ is an elementary (i.e., one-term) symmetric polynomial;

S_n is the symmetric group;

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an ordered set of integers satisfying the conditions

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0,\ \alpha_1 + \alpha_2 + \cdots + \alpha_n = m, \tag{1.2}$$

where m is a fixed positive integer called a degree of the elementary symmetric polynomial $\mu_{\boldsymbol{\alpha}}(\mathbf{x})$. (Bold letters are used for vector quantities.)

The special order $\succeq$ in the set of collections satisfying (1.2) is determined by the following way: $\boldsymbol{\alpha} \succeq \boldsymbol{\beta}$ means

$$\alpha_1 \geq \beta_1,\ \alpha_1 + \alpha_2 \geq \beta_1 + \beta_2,\ \ldots,,$$
$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \geq \beta_1 + \beta_2 + \cdots + \beta_{n-1}.$$

Several proofs of Muirhead's inequality are known ([2], Sec. 2.18 and 2.74; [3], Sec. 11.3; three proofs were given in [4]).

The subject was generalized to arbitrary positive quotients by R. Rado [3, 5] (for other generalizations, see [6, 7, 8, 9, 10]). Below, only the classical version of Muirhead's inequality with positive integer quotients will be considered. We are interested in the following question: when does the inequality

$$\forall \mathbf{x} \in \mathbb{R}_+^n :\ \mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \geq 2\mu_{\boldsymbol{\beta}}(\mathbf{x}) \tag{1.3}$$

hold?

If (1.3) is true, then it will be called the trinomial Muirhead inequality. Note if $\boldsymbol{\alpha} = \boldsymbol{\gamma}$, then (1.3) turns into the classical (binomial) Muirhead's inequality. Therefore, we will assume $\boldsymbol{\alpha} < \boldsymbol{\gamma}$ further. Besides, cases $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and $\boldsymbol{\gamma} = \boldsymbol{\beta}$ are trivial, therefore, it will be supposed $\boldsymbol{\alpha} > \boldsymbol{\beta} > \boldsymbol{\gamma}$.

Some trinomial generalization of Muirhead's inequality was given by I. Schur ([2], Sec. 2.81) for $n = 3$:

$$\mu_{(\alpha+2\beta,\, 0,\, 0)}(x,\, y,\, z) + \mu_{(\alpha,\, \beta,\, \beta)}(x,\, y,\, z) \geq 2\mu_{(\alpha+\beta,\, \beta,\, 0)}(x,\, y,\, z)\,. \tag{1.4}$$

The simplest case of this looks

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + y^2x + x^2z + z^2x + y^2z + z^2y.$$

The present paper is devoted to the discussion of (1.3). It turned out that, unlike the classical binomial Muirhead's inequality (1.1), the trinomial case meets some surprises. In Section 2, a sufficient condition for (1.3) is given. In Section 3, a sufficient and necessary condition is derived for the case $n = 2$. Properties of two arithmetic functions connected with this condition are considered in Section 4. In Section 5, some inequalities of Muirhead-Schur type are given. Section 5 consists of final notes. Besides, two open problems are formulated.

2. A sufficient condition for the general case

Theorem 2.1. *If $\boldsymbol{\alpha} + \boldsymbol{\gamma} \succeq 2\boldsymbol{\beta}$, then $\mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \geq 2\mu_{\boldsymbol{\beta}}(\mathbf{x})$.*

Proof. Let $\boldsymbol{\alpha} + \boldsymbol{\gamma} \succeq 2\boldsymbol{\beta}$. We set $\bar{\boldsymbol{\alpha}} = 2\boldsymbol{\beta} - \boldsymbol{\gamma}$, i.e. $\bar{\alpha}_k = 2\beta_k - \gamma_k$, $k = 1, 2, \ldots, n$. Then, on one hand

$$\mu_{\bar{\boldsymbol{\alpha}}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) = \tfrac{1}{n!} \sum_{\boldsymbol{\sigma} \in S_n} \left(x_{\sigma(1)}^{\bar{\alpha}_1} x_{\sigma(2)}^{\bar{\alpha}_2} \ldots x_{\sigma(n)}^{\bar{\alpha}_n} + x_{\sigma(1)}^{\gamma_1} x_{\sigma(2)}^{\gamma_2} \ldots x_{\sigma(n)}^{\gamma_n} \right) \geq$$
$$\geq 2 \tfrac{1}{n!} \sum_{\boldsymbol{\sigma} \in S_n} \sqrt{x_{\sigma(1)}^{\bar{\alpha}_1+\gamma_1} x_{\sigma(2)}^{\bar{\alpha}_2+\gamma_2} \ldots x_{\sigma(n)}^{\bar{\alpha}_n+\gamma_n}} = 2\mu_{\boldsymbol{\beta}}(\mathbf{x}).$$

On the other hand, $\boldsymbol{\alpha} + \boldsymbol{\gamma} \succeq 2\boldsymbol{\beta}$ means

$$\begin{aligned}
&\alpha_1 + \gamma_1 \geq 2\beta_1, \\
&\alpha_1 + \gamma_1 + \alpha_2 + \gamma_2 \geq 2\beta_1 + 2\beta_2, \\
&\ldots, \\
&\alpha_1 + \gamma_1 + \alpha_2 + \gamma_2 + \ldots + \alpha_{n-1} + \gamma_{n-1} \geq 2\beta_1 + 2\beta_2 + \ldots + 2\beta_{n-1}, \\
&\alpha_1 + \gamma_1 + \alpha_2 + \gamma_2 + \ldots + \alpha_n + \gamma_n = 2\beta_1 + 2\beta_2 + \ldots + 2\beta_n.
\end{aligned}$$

These relations imply

$$\begin{aligned}
&\alpha_1 \geq 2\beta_1 - \gamma_1 = \bar{\alpha}_1, \\
&\alpha_1 + \alpha_2 \geq 2\beta_1 - \gamma_1 + 2\beta_2 - \gamma_2 = \bar{\alpha}_1 + \bar{\alpha}_2, \ \ldots,
\end{aligned}$$

i.e. $\boldsymbol{\alpha} \succeq \bar{\boldsymbol{\alpha}}$. Therefore, due to (1.1)

$$\mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \succeq \mu_{\bar{\boldsymbol{\alpha}}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \succeq 2\mu_{\boldsymbol{\beta}}(\mathbf{x}).$$

□

At first, we thought that, analogously to Muirhead's theorem, the condition $\mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \succeq 2\mu_{\boldsymbol{\beta}}(\mathbf{x})$ might be not only sufficient but also necessary for (1.2). But (1.4) refutes this supposition. Here is another sample with two variables:

$$x^6 + y^6 + 2x^3y^3 \geq 2\left(x^5y + y^5x\right) \tag{2.1}$$

where $n = 2$, $\boldsymbol{\alpha} = (6, 0)$, $\boldsymbol{\gamma} = (3, 3)$, $\boldsymbol{\beta} = (5, 1)$, and the condition $\boldsymbol{\alpha} + \boldsymbol{\gamma} \succeq 2\boldsymbol{\beta}$ do not hold. Nevertheless

$$x^6 + y^6 + 2x^3y^3 - 2\left(x^5y + y^5x\right) = (x - y)^2\left(x^4 - x^2y^2 + y^4\right) \geq 0.$$

In this regard, it may be of interest
Problem 1. *Find a necessary and sufficient condition in terms of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ for (1.1) to hold.*

3. A necessary and sufficient condition in the case of two variables.

It is natural to begin investigating trinomial inequalities with the simplest case $n = 2$. Further, we will use the notations: $x_1 = x$, $x_2 = y$. A symmetric polynomial in this case contains only a couple of summands:

$$\frac{x^{k_1}y^{k_2} + y^{k_1}x^{k_2}}{2},\ k_1 + k_2 = m.$$

If all quotients in the trinomial Muirhead's inequality are positive, then it can be divided term by term to the greatest quotient of the product xy. Then, after at least one of the vector-quotients $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ will have a lower component equal to 0. If $\beta_2 = 0$, but $\alpha_2 > 0$ and $\gamma_2 > 0$, the trinomial Muirhead's inequality cannot hold, since β_1 would be greater than both α_1 and γ_1. Therefore, we can assume $\boldsymbol{\alpha} = (m, 0)$. Thus, we will deal with the pairs $(\beta, m - \beta)$ and $(\gamma, m - \gamma)$ instead of the vector-quotients $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ respectively.

Now, the trinomial Muirhead's inequality takes the form

$$x^m + y^m + x^\gamma y^{m-\gamma} + y^\gamma x^{m-\gamma} \geq 2\left(x^\beta y^{m-\beta} + y^\beta x^{m-\beta}\right) \tag{3.1}$$

in the considering case. Among all possible relations between the quotients m, β, γ, only the case $m > \beta > \gamma$ is nontrivial. Moreover, one may suppose $\gamma \geq m - \gamma$. In this regard, we will assume

$$m > \beta > \gamma \geq \frac{m}{2}. \tag{3.2}$$

This condition immediately excludes the case $\gamma = 1$, so that $\gamma \geq 2$, $\beta \geq 3$, and $m \geq 4$. Below, unless otherwise stated, condition (3.2) is assumed to be satisfied. In addition, $x \geq y$ can also be supposed due to the symmetry. Then (3.1) will be equivalent to the inequality

$$\left(x^\beta - 1\right)\left(x^{m-\beta} - 1\right) \geq x^{m-\beta}\left(x^{\beta-\gamma} - 1\right)\left(x^{\beta+\gamma-m} - 1\right). \tag{3.3}$$

Reduction by $(x-1)^2$ brings the last to the form

$$G_\beta(x)\, G_{m-\beta}(x) \geq x^{m-\beta} G_{\beta-\gamma}(x)\, G_{\beta+\gamma-m}(x) \tag{3.4}$$

for circle division polynomials $G_l(x) = 1 + x + x^2 + \cdots + x^{l-1}$ ([11].)

Setting $x = 1$, we come to the conclusion that the condition

$$\beta(m - \beta) \geq (\beta - \gamma)(\beta + \gamma - m) \tag{3.5}$$

is necessary in order (3.1) to be held.

It turns out that the quantity $\Delta = \beta(m - \beta) - (\beta - \gamma)(\beta + \gamma - m)$ plays a key role for trinomial Muirhead's inequality in the discussing case.

Theorem 3.1. $\Delta \geq 0$ *is necessary and sufficient for* (3.1).

Proof. Let $\Delta \geq 0$. It is required to establish the inequality

$$\varphi(x) \overset{def}{=} x^m + 1 + x^\gamma + x^{m-\gamma} - 2x^\beta - 2x^{m-\beta} \geq 0$$

for $x \geq 1$. We are going to show, by means of derivatives, that $\varphi(x)$ is increasing. It is more convenient to use the operator $x\frac{d}{dx}$ instead of usual derivation.

We have

$$x\varphi'(x) = mx^m + \gamma x^\gamma + (m-\gamma)\, x^{m-\gamma} - 2\beta x^\beta - 2(m-\beta)\, x^{m-\beta} = x^{m-\beta}\psi(x)$$

where $\psi(x) = mx^\beta + \gamma x^{\beta+\gamma-m} + (m-\gamma)\, x^{\beta-\gamma} - 2\beta x^{2\beta-m} - 2(m-\beta)$.

Further,

$$x\psi'(x) = m\beta x^\beta + \gamma(\beta+\gamma-m)\, x^{\beta+\gamma-m} + (m-\gamma)(\beta-\gamma)\, x^{\beta-\gamma} - 2\beta(2\beta-m)\, x^{2\beta-m} = x^{\beta-\gamma}\chi(x)$$

where $\chi(x) = m\beta x^{\gamma} + \gamma(\beta+\gamma-m)x^{2\gamma-m} + (m-\gamma)(\beta-\gamma) - 2\beta(2\beta-m)x^{\beta+\gamma-m}$.

Similarly

$$x\chi'(x) = m\beta\gamma x^{\gamma} + \gamma(\beta+\gamma-m)(2\gamma-m)x^{2\gamma-m} - 2\beta(2\beta-m)(\beta+\gamma-m)x^{\beta+\gamma-m} = x^{2\gamma-m}\xi(x),$$

where $\xi(x) = m\beta\gamma x^{m-\gamma} + \gamma(\beta+\gamma-m)(2\gamma-m) - 2\beta(2\beta-m)(\beta+\gamma-m)x^{\beta-\gamma}$, and finally,

$$x\xi'(x) = \beta\left[m\gamma(m-\gamma)x^{m-\gamma} - 2(2\beta-m)(2\beta-m)(\beta-\gamma)x^{\beta-\gamma}\right].$$

Since $x \geq 1$ and $m-\gamma > \beta-\gamma > 0$, the inequality $\xi'(x) \geq 0$ is equivalent to

$$m\gamma(m-\gamma) - 2(2\beta-m)(\beta+\gamma-m)(\beta-\gamma) \geq 0.$$

One may verify by direct calculation that the expression on the left side of the last inequality equals $(2\beta-m)\Delta + \gamma(m-\beta)(m-\gamma)$, that is non-negative. Thus $\xi'(x) \geq 0$. Further, it turns out

$$\xi(1) = m\beta\gamma + \gamma(\beta+\gamma-m)(2\gamma-m) - 2\beta(2\beta-m)(\beta+\gamma-m) = (2\beta+2\gamma-m)\Delta \geq 0.$$

(In order to verify, it is enough to open all brackets). Thus, $\xi(x) \geq 0$ implies $\chi'(x) \geq 0$. Moreover, the formula $\chi(1) = 2\Delta$ can be checked as well. Therefore $\chi(x) \geq 0$, that in turn implies $\psi'(x) \geq 0$. However, $\psi(1) = 0$, so $\psi(x) \geq 0$. Now, taking into account the fact that the signs of $\varphi'(x)$ and $\psi(x)$ coincide, as well the value $\varphi(1) = 0$, we obtain $\varphi(x) \geq 0$, which concludes the proof. □

Now we determine the values of m, β and γ for which the condition $\Delta \geq 0$ holds. It is easy to see that the last is equivalent to the relation

$$m \geq m_*(\beta, \gamma) \stackrel{def}{=} \left\lceil \frac{2\beta^2-\gamma^2}{2\beta-\gamma} \right\rceil = \beta + \left\lceil \frac{\beta\gamma-\gamma^2}{2\beta-\gamma} \right\rceil \tag{3.6}$$

Besides, obviously, $2\beta-\gamma \geq \left\lceil \frac{2\beta^2-\gamma^2}{2\beta-\gamma} \right\rceil$. (Here and below $\lceil s \rceil$ denotes the selling function and $\lfloor s \rfloor$ does the floor function).

Similarly, from $\Delta \geq 0$, we can derive the formula for the biggest value of β for given m and γ. But now we get a function containing quadratic irrationality:

$$\beta^*(m, \gamma) \stackrel{def}{=} \left\lfloor \frac{m + \sqrt{m^2 - 2m\gamma + \gamma^2}}{2} \right\rfloor$$

Thus, the condition $\Delta \geq 0$ is equivalent to $\beta \leq \beta^*(m, \gamma)$ as well.

Table 1 illustrates values of the function $m_*(\beta, \gamma)$ for $\beta = 3 \div 33$. If $\gamma \geq \beta$, then inequality (1.3) becomes trivial, and the condition $\Delta \geq 0$ is also evident. Cells corresponding to such pairs (γ, β) are left white in the table. Moreover, $\Delta < 0$ for pairs (γ, β) associated with gray cells.

Nontrivial pairs (γ, β), for those (3.1) is valid, are represented by colored cells. In particular, blue cells reflect Theorem 2.1. Other values for which inequality (1.3) holds are highlighted in red. Inequality (1.3) for values γ, β, m_* corresponding to the light-red cells follows from the inequality associated with the dark-red cell located to the left in the same row. Thus, each dark-red cell expresses a significant three-term Muirhead's inequality.

4. One more inequality of I. Schur's type.

Consider the case $n = 3$. We have mentioned that the inequality (1.4) is usually associated with the name of I. Schur [2]. (It should be noted that (1.4) differs from more essential Schur's inequality on the estimation of the polynomial norm called Schur's Lemma as well [12].) Here is a four-term generalization of (1.4).

Theorem 4.1.

$$\mu_{(\alpha+\beta+\gamma,0,0)}(x, y, z) + \mu_{(\alpha,\beta,\gamma)}(x, y, z) \geq \mu_{(\alpha+\beta,\gamma,0)}(x, y, z) + \mu_{(\alpha+\gamma,\beta,0)}(x, y, z) \tag{4.1}$$

Proof. One can assume $x \geq y \geq z$ without loss of generality. This condition implies

$$x^\alpha \left(x^\beta - y^\beta\right)\left(x^\gamma - z^\gamma\right) \geq y^\alpha \left(x^\beta - y^\beta\right)\left(y^\gamma - z^\gamma\right)$$

$$x^\alpha \left(x^\beta - z^\beta\right)\left(x^\gamma - y^\gamma\right) \geq y^\alpha \left(y^\beta - z^\beta\right)\left(x^\gamma - y^\gamma\right)$$

$$z^\alpha \left(z^\beta - x^\beta\right)\left(z^\gamma - y^\gamma\right) + z^\alpha \left(z^\beta - y^\beta\right)\left(z^\gamma - x^\gamma\right) \geq 0$$

(4.1) implies another trinomial Muirhead's inequality. □

Corollary.

$$\mu_{(\alpha+2\beta+2\gamma,\,0,\,0)}\left(x,\,y,\,z\right) + \mu_{(\alpha,\,2\beta,\,2\gamma)}\left(x,\,y,\,z\right) \geq \mu_{(\alpha+\beta+\gamma,\,\beta+\gamma,\,0)}\left(x,\,y,\,z\right). \tag{4.2}$$

One may note that inequalities (4.1) and (4.2) are not mutually comparable.

5. Final notes

$1^{\#}$. Theorems 2.1 and 4.2 remain valid for any non-negative quotients.

$2^{\#}$. It is easy to show that if inequality (1.1) holds then both sides are equal if and only if $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\gamma}$ or $x_1 = x_2 = \ldots = x_n$.

$3^{\#}$. Each trinomial inequality (1.1) of n variables generates an inequality of $n+1$ variables of the same type by means of the transformation of elementary symmetric polynomials

$$\mu_{\boldsymbol{\alpha}}\left(x_1,\,x_2,\,\ldots,\,x_n\right) \to \mu_{(\boldsymbol{\alpha},\,0)}\left(x_1,\,x_2,\,\ldots,\,x_n,\,x_{n+1}\right) =$$

$$= \tfrac{1}{n+1}\left[\mu_{\boldsymbol{\alpha}}\left(x_1,\,x_2,\,\ldots,\,x_n\right) + \sum_{k=1}^{n} \mu_{\boldsymbol{\alpha}}\left(x_1,\,x_2,\,\ldots,\,x_n\right)|_{x_k=x_{n+1}}\right]$$

For example, the inequality $x^6+y^6+2x^3y^3 \geq 2\left(x^5y+xy^5\right)$ with characteristics $\Delta = 1$ generates a sequence of trinomial Muirhead inequalities

$$x^6 + y^6 + z^6 + x^3y^3 + x^3z^3 + y^3z^3 \geq x^5y + y^5x + x^5z + z^5x + y^5z + z^5y,$$

$$\tfrac{1}{4}\sum_{k=1}^{4} x_k^6 + \tfrac{1}{6}\sum_{1\leq i\leq j\leq 4} x_i^3x_j^3 \geq 2\tfrac{1}{12}\sum_{i,j=1,\,i\neq j}^{4} x_i^5x_j,\ \ldots$$

$4^{\#}$. If $\boldsymbol{\alpha}+\boldsymbol{\gamma} \succeq 2\boldsymbol{\beta}$ and $\boldsymbol{\beta}+\boldsymbol{\delta} \succeq 2\boldsymbol{\gamma}$ then $\mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x}) \geq 2\mu_{\boldsymbol{\beta}}(\mathbf{x})$, $\mu_{\boldsymbol{\beta}}(\mathbf{x}) + \mu_{\boldsymbol{\delta}}$. Adding these inequalities one gets four-term Muirhead's inequality $\mu_{\boldsymbol{\alpha}}(\mathbf{x}) + \mu_{\boldsymbol{\delta}}(\mathbf{x}) \geq \mu_{\boldsymbol{\beta}}(\mathbf{x}) + \mu_{\boldsymbol{\gamma}}(\mathbf{x})$ differ from (4.1).

Thus, we can conclude that the theory of symmetric homogeneous inequalities, initiated by Muirhead more than a century ago, is fraught with many more mysteries. We hope that the present work will serve as a stimulus for further research on the fundamental problem of three-term Muirhead's inequalities: to determine the necessary and sufficient conditions that α, β, γ must satisfy in order for inequality (1.3) to hold.

References

[1] Muirhead R., Some methods applicable to identities and inequalities of symmetric functions of n letters. Proc. Edinburgh Math. Soc. (1903) Vol. 21, P. 144–157.

[2] Hardy G., Littlewood J., Pólya G., Inequalities. Cambridge University Press, Cambridge, (1988).

[3] Prasolov V., Polynomials. Springer-Verlag, Berlin, (2010).

[4] Marshall A., Olkin I., Arnold B., Inequalities: theory of majorization and its applications. Springer, New York, (2011).

[5] Rado R., An inequality. J. London Math. Soc. (1952) Vol. 27, P. 1–6.

[6] Daykin D., Generalization of the Muirhead-Rado inequality. Proc. Amer. Math. Soc. (1971) Vol. 30, P. 84–86.

[7] Aboudi R., Thon D., Refinements of Muirhead's lemma and income inequality. Math. Social Sci. (2006) Vol. 51, Iss. 2, P. 201–216.

[8] Trif T., Monotonicity, comparison and Minkowski's inequality for generalized Muirhead means in two variables. Mathematica, (2006) Vol. 48, Iss. 1 (71), P. 99–110.

[9] Paris J., Vencovska A., A generalization of Muirhead's inequality. J. Math. Inequal. (2009) Vol. 3, Iss. 2, P. 181–187.

[10] Schulman L., Muirhead-Rado inequality for compact groups. Positivity, (2009) Vol. 13, Iss. 3, P. 559–574.

[11] Arnold V., Ordinary differential equation. MIT, Cambridge, (1998).

[12] Komisarski A., Rajba T., Muirhead inequality for convex orders and a problem of I. Raşa on Bernstein polynomials. J. Math. Anal. Appl. (2018) Vol. 458, Iss. 1, P. 821–830.

Table 1

β \ γ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
2																																
3		4																														
4			5																													
5			6	6																												
6				7	7																											
7					9	8																										
8					10	10	9																									
9						11	11	10																								
10						12	12	12	11																							
11							13	13	13	12																						
12								14	14	14	13																					
13								16	16	15	15	14																				
14									17	17	16	16	15																			
15									18	18	18	17	17	16																		
16										19	19	19	19	18	17																	
17										20	20	20	20	20	19	18																
18											22	21	21	21	21	20	19															
19												23	23	22	22	22	21	20														
20												24	24	24	23	23	23	22	21													
21													25	25	25	25	24	24	23	22												
22													26	26	26	26	26	25	25	24	23											
23														27	27	27	27	27	26	26	25	24										
24															29	28	28	28	28	27	27	26	25									
25															30	30	30	29	29	29	28	28	27	26								
26																31	31	31	31	30	30	29	29	28	27							
27																32	32	32	32	32	31	31	30	30	29	28						
28																	33	33	33	33	33	32	32	31	31	30	29					
29																	34	34	34	34	34	34	33	33	33	32	31	30				
30																		36	36	35	35	35	35	34	34	34	33	32	31			
31																			37	37	37	36	36	36	36	35	35	34	33	32		
32																			38	38	38	38	38	37	37	37	36	36	35	34	33	
33																				39	39	39	39	39	38	38	38	37	37	36	35	34

Azamov A.A.,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
e-mail: abdulla.azamov@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 34–48
DOI: 10.29229/uzmj.2026-1-4

On a coefficient inverse problem with nonlocal boundary conditions of periodic type for the three-dimensional Tricomi equation in a parallelepiped

Dzhamalov S., Shokirov A.

Abstract. This article addresses the well-posedness of a coefficient inverse problem for the three-dimensional Tricomi equation in a parallelepiped. For this problem with nonlocal boundary conditions in anisotropic Sobolev spaces, we prove existence and uniqueness theorems for a regular solution using Fourier methods, $\varepsilon-$ regularization, a priori estimates, successive approximations, and contraction mappings

Keywords: three-dimensional Tricomi equation, coefficient inverse problem with nonlocal boundary conditions of periodic type, well-posedness, Fourier methods, $\varepsilon-$ regularization, a priori estimates, successive approximations
MSC (2020): 35M10

1. Introduction

In the study of nonlocal problems, a close relationship has been observed between problems with nonlocal boundary conditions and inverse problems [1]. By now, inverse problems for classical equations of mathematical physics have been studied extensively [2, 3, 4, 5]. Linear inverse problems (involving the determination of solutions and right-hand sides of equations) for equations of mixed type, both of the first and the second kind, have been considered in [6, 7, 8, 9, 10].

In contrast, coefficient inverse problems (involving the determination of solutions, coefficients, and right-hand sides of equations) for equations of mixed type have not been studied in detail. In this work, we aim to partially fill this gap.

In particular, we investigate the unique solvability of a coefficient inverse problem for the three-dimensional Tricomi equation in a parallelepiped. Our approach is based on reducing the coefficient inverse problem to a family of direct problems for loaded Tricomi differential equations with nonlocal boundary conditions of periodic type in a bounded rectangular domain [5, 11].

Recall that a loaded equation is a partial differential equation whose coefficients or right-hand side involve certain functionals of the solution itself [8, 9].

2. Coefficient inverse problem

In the domain

$$G = Q \times (0, \ell) = \{(x, t, y)| \ -1 < x < 1, \ 0 < t < T, \ 0 < y < \ell\}$$

we consider the three-dimensional Tricomi equation:

$$Lu = xu_{tt} - au_{xx} - u_{yy} + \alpha(x, t)u_t = c(x, t)u + \psi(x, t, y), \tag{2.1}$$

here $\psi(x, t, y) = g(x, t, y) + h(x, t) \cdot f(x, t, y)$, where $g(x, t, y)$ and $f(x, t, y)$ are given functions, while $h(x, t)$ and $c(x, t)$ are unknown. For simplicity, we omit function arguments from now on.

Problem statement. Find functions $\{u(x, t, y), \ h(x, t), \ c(x, t)\}$, satisfying equation (2.1) almost everywhere in domain G, such that $u(x, t, y)$ satisfies the following boundary conditions:

1) non-local boundary conditions of periodic type:

$$\gamma D_t^p u|_{t=0} = D_t^p u|_{t=T}, \tag{2.2}$$

$$D_x^p u|_{x=-1} = D_x^p u|_{x=1}, \quad p = 0, 1 \tag{2.3}$$

$$u|_{y=0} = u|_{y=l} = 0, \tag{2.4}$$

where γ is some constant number, such that $|\gamma| > 1$;

2) additional conditions

$$u(x,t,\ell_1) = \varphi_1(x,t), \tag{2.5}$$

$$u(x,t,\ell_2) = \varphi_2(x,t), \tag{2.6}$$

$$0 < \ell_1 < \ell_2 < \ell < +\infty$$

and together with the functions $h(x,t)$, $c(x,t)$ belong to the class

$$U = \left\{(u,\ h,\ c)|\ u \in W_2^{2,3}(G),\ h \in W_2^2(Q),\ c \in W_2^2(Q)\right\}.$$

Here $W_2^{2,3}(G)$ is an anisotropic Sobolev space with norm

$$[u]_{l,3}^2 \equiv |\|u\||^2_{W_2^{l,3}(G)} = \sum_{k=1}^{\infty} \left(1+\lambda_k^2\right)^3 \|u_k\|^2_{W_2^l(Q)}, \quad l = 0,1,2,...$$

where $u_k(x,t)$ denote the coefficients of the Fourier expansion of the functions $u(x,t,y)$ in the system $\{Y_k(y)\} = \left\{\sqrt{\frac{2}{\ell}}\sin\lambda_k y\right\}$, $\lambda_k = \frac{\pi k}{\ell}$, $k = 1,2,3,...$; $W_2^l(Q)$ is the Sobolev space with the norm

$$\|u\|^2_{W_2^l(Q)} = \|u\|_l^2 = \sum_{|\alpha|=0}^{l} \int_Q (D^\alpha u)^2 dxdt,$$

where $l = 0,1,2,...$, $D^\alpha -$ generalized derivative with respect to the variables x and t. Further through $C^l(Q)$ let us denote the spaces of continuously differentiable functions up to order l inclusive with norm

$$\|u\|_{C^l(Q)} = \sum_{|\alpha|=0}^{l} \max_{(x,t)\in Q} |D^\alpha u|.$$

For further study of the inverse problem, we need the following auxiliary theorems and notations in a simplified form.

Theorem A.1. (S.L. Sobolev). There is a continuous embedding $W_2^2(Q) \subset C(Q)$, i.e.

$$\|u\|^2_{C(Q)} \le c_2 \|u\|^2_{W_2^2(Q)},$$

where c_2 is a positive constant [12, 13].

Theorem A.2. For any function $u(x,t) \in W_2^1(Q)$ holds the following inequality holds

$$\|u\|_{L_4(Q)} \le c_3 \|u\|_{W_2^1(Q)},$$

where c_3 is a positive constant [12, 13].

Let us introduce the following notations:

$$g_j(x,t) = g(x,t,\ell_j), \quad f_j(x,t) = f(x,t,\ell_j), \quad \forall j = 1,2; \quad H = \det\begin{pmatrix} \varphi_1 & f_1 \\ \varphi_2 & f_2 \end{pmatrix},$$

$$\gamma_0 = m_1 [g]^2_{1,3}, \quad \gamma_1 = m_2 [f]^2_{2,3}, \quad \gamma_2 = m_2;$$

$$m_1 = 60\delta_0^{-1}\delta^{-1}e^{\mu T}\left(4\left(1+\mu^2\right) + \delta^{-1}e^{\mu T}\left(\|\alpha_x\|_C^2 + \|\alpha_t\|_C^2\right)\right),$$

$$m_2 = 192\delta_0^{-1}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)$$

where $\mu = \frac{2}{T}\ln|\gamma| > 0$, $|\gamma| > 1$, $\delta = min\left\{\frac{\delta_0}{2}, \delta_1, a\mu, \mu\left(\frac{\pi}{\ell}\right)^2\right\}$, δ_1 will be defined in the theorem,

$$\mathfrak{F} = \max\left(\|f_1\|^2_{C^{0,1}_{x,t}(\bar{Q})}, \|f_2\|^2_{C^{0,1}_{x,t}(\bar{Q})}, \|\varphi_1\|^2_{C^{0,1}_{x,t}(\bar{Q})}, \|\varphi_2\|^2_{C^{0,1}_{x,t}(\bar{Q})}\right),$$

$$c_1 = \sum_{m=1}^{\infty} \frac{\lambda_m^4}{\left(1+\lambda_m^2\right)^3}$$

and c_2, c_3 are as defined above.

Definition 2.1. A generalized solution of problem (2.1)-(2.6) will be a functions $\{u, h, c\}$ from class U satisfying equation (2.1) almost everywhere in domain G, with conditions (2.2)-(2.6).

Assume all coefficients of equation (2.1) are sufficiently smooth in $\overline{Q}$, and the following conditions are satisfied with respect to the coefficients, right-hand side, and given functions $\varphi_i(x,t)$, $i = 1, 2$:

Conditions 1:

nonlocal conditions: $\alpha(x,0) = \alpha(x,T), \quad \alpha(-1,t) = \alpha(1,t);$

$$\gamma g(x,0,y) = g(x,T,y), \quad \gamma f(x,0,y) = f(x,T,y);$$

smoothness: $f_i \in C^{0,1}_{x,t}(\bar{Q}),\ g_i \in C^{0,1}_{x,t}(\bar{Q}),\ f \in W_2^{2,3}(G),\ g \in W_2^{2,3}(G).$

coefficient conditions: $2\alpha(x,t) + \mu x > \delta_0 > 1$.

Conditions 2:

Functions $\varphi_i \in W_2^3(Q)$, $i = 1, 2$ is the solution to the following problem:

$$L_1\varphi_i = g_i,$$

$$\gamma D_t^p \varphi_i|_{t=0} = D_t^p \varphi_i|_{t=T},$$

$$D_x^p \varphi_i|_{x=-1} = D_x^p \varphi_i|_{x=1}, p = 0, 1$$

where $L_1\varphi_i = x\varphi_{itt} - a\varphi_{ixx} + \alpha\varphi_{it}$. Besides this let

$$|H| = |\varphi_1 f_2 - \varphi_2 f_1| \geq \eta > 0.$$

Without loss of generality, we may take $\eta = 1$.

To prove the solvability of problem (2.1)−(2.6), we first use the Fourier method. Namely, we search the solution to problem (2.1)-(2.6) in the form

$$u(x,t,y) = \sum_{k=1}^{\infty} u_k(x,t) Y_k(y), \tag{A}$$

where functions $\{Y_k(y)\} = \left\{\sqrt{\frac{2}{\ell}} \sin \lambda_k y\right\}$, $\lambda_k = \frac{\pi k}{\ell}$, $k = 1, 2, 3, ...$ are solutions of the Sturm-Liouville spectral problem with Dirichlet conditions, $u_k(x,t)$ are Fourier coefficients of the function $u(x,t,y)$. It is known that the system of eigenfunctions $\{Y_k(y)\}$ is complete in the space $L_2(0,\ell)$ and forms an orthonormal basis [2, 9, 10, 14, 15]. In order to determine unknown functions $u_k(x,t)$, it is necessary to perform some construction formalities. Let us consider the traces of equation (2.1) at $y = \ell_i$, $i = 1, 2$,

$$Lu|_{y=l_i} = x\varphi_{itt} - a\varphi_{ixx} - u_{yy}(x,t,\ell_i) + \alpha\varphi_{it} = c\varphi_i + hf_i + g_i,\ i = 0, 1. \tag{2.7}$$

Now, taking into account conditions (2.5), (2.6) and condition $H \neq 0$, we define formally unknown functions $h(x,t)$ and $c(x,t)$ from the system of equations (2.7):

$$\begin{cases} c\varphi_1 + hf_1 = \Phi, \\ c\varphi_2 + hf_2 = \Psi, \end{cases}$$

where $\Phi = \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_1$, $\Psi = \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_2$, in the form

$$h(x,t) = \frac{1}{H}\begin{vmatrix} \varphi_1 & \Phi \\ \varphi_2 & \Psi \end{vmatrix} = \frac{1}{H}\left[\varphi_1 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_2 - \varphi_2 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_1\right]$$

and

$$c(x,t) = \frac{1}{H}\begin{vmatrix} \Phi & f_1 \\ \Psi & f_2 \end{vmatrix} = \frac{1}{H}\left[f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_1 - f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_2\right],$$

and to determine the functions $u_k(x,t)$ in the domain Q we obtain infinite systems of loaded nonlinear differential Tricomi equations:

$$\begin{gathered} L\,u_k = xu_{k\,tt} - au_{k\,xx} + \alpha u_{k\,t} + \lambda_k^2 u_k \\ = \frac{u_k}{H}\left[f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t)\sin\lambda_m\ell_1 - f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t)\sin\lambda_m\ell_2\right] \\ +\frac{f_k}{H}\left[\varphi_1 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t)\sin\lambda_m\ell_2 - \varphi_2 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t)\sin\lambda_m\ell_1\right] + g_k \equiv F(u_k) \end{gathered} \tag{2.8}$$

with nonlocal boundary conditions of periodic type

$$\gamma D_t^p u_k|_{t=0} = D_t^p u_k|_{t=T}, \tag{2.9}$$

$$D_x^p u_k|_{x=-1} = D_x^p u_k|_{x=1}, \quad p = 0,1, \tag{2.10}$$

where f_k and g_k are the coefficients of the Fourier expansion of the functions f and g, $k = 1,2,...$.

3. Main result

Let us introduce the following notation

$$q \equiv 4m_1m_2\left(2m_1\,[g]_{1,3}^2 + [f]_{2,3}^2\right).$$

Theorem 3.1. *Let Conditions 1 and 2 above be satisfied for the coefficients of equation* (2.1)*. Suppose there exists a*δ_1 *such that*

$$a - \frac{5\delta_0}{48} > \delta_1 > 1,$$

and assume

$$q < \frac{1}{2}.$$

Then problem (2.1)−(2.6) *admits a unique solution in the class* U*.*

Remark 3.2. The inequality $q < \frac{1}{2}$ can be achieved by choosing the domain sufficiently small and by requiring that g_i and their derivatives are sufficiently small.

Proof. Let us prove the theorem step by step. Let there exist a solution to problem (2.1) − (2.4) from the class U. First, let us show that the function $u(x,t,y)$ satisfies the boundary conditions (2.5) − (2.6) for any $i = 1,2$, i.e.

$$u(x,t,\ell_i) = \varphi_i(x,t), \quad i = 1,2.$$

We prove that these conditions are satisfied using the converse assumptions. Let there exist functions $\vartheta_i(x,t)$ satisfying conditions (2.5), (2.6):

$$u(x,t,\ell_i) = \sum_{k=1}^{\infty} u_k(x,t)\sin\lambda_k\ell_i = \vartheta_i(x,t) \neq \varphi_i(x,t).$$

To make it easier to understand, let's consider each case separately. First, let's consider the case when $i = 1$. Then for the function $z_1(x,t) = \vartheta_1(x,t) - \varphi_1(x,t)$ in the domain Q, taking into account conditions (2.9), (2.10), multiplying the systems of equations (2.8) by $\sin\mu_k\ell_1$ and summing over k from 1 to ∞, we obtain the following loaded equations

$$\begin{gathered} x\vartheta_{1\ tt} - a\vartheta_{1\ xx} + \alpha\vartheta_{1\ t} + \sum_{k=1}^{\infty} \lambda_k^2 u_k \sin\lambda_k\ell_1 \\ = \frac{1}{H}\vartheta_1\left[f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t)\sin\lambda_m\ell_1 - f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t)\sin\lambda_m\ell_2\right] \\ +\frac{1}{H}f_1\left[\varphi_1 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t)\sin\lambda_m\ell_2 - \varphi_2 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t)\sin\lambda_m\ell_1\right] + g_1. \end{gathered} \tag{3.1}$$

Let us substitute expression $z_1 + \varphi_1 = \vartheta_1 = \sum\limits_{m=1}^{\infty} u_m(x,t) \sin \lambda_m \ell_1$ instead of ϑ_1. Then from (3.1) it follows

$$\begin{gathered} x(z_1+\varphi_1)_{tt} - a(z_1+\varphi_1)_{xx} + \alpha(z_1+\varphi_1)_t + \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_1 \\ = g_1 + cz_1 + \frac{1}{H} \sum_{k=1}^{\infty} \lambda_k^2 u_k \sin \lambda_k \ell_1 [\varphi_1 f_2 - f_1 \varphi_2]. \end{gathered} \tag{3.2}$$

Based on formulas (2.8)−(3.1) for the function $z_1(x,t) = \vartheta_1(x,t) - \varphi_1(x,t)$ in the domain Q we obtain the following problem

$$L_0 z_1 \equiv x z_{1\ tt} - a z_{1\ xx} + \alpha z_{1\ t} = c z_1, \tag{3.3}$$

$$\gamma D_t^p z_1|_{t=0} = D_t^p z_1|_{t=T}, \quad p = 0, 1, \tag{3.4}$$

$$z_{1x}|_{x=-1} = z_{1x}|_{x=1} = 0. \tag{3.5}$$

Now we prove the uniqueness of the solution to problem (3.3)−(3.5) by the energy integral method [10]. To do this, consider the identity

$$2(L_0 z_1, e^{-\mu t}(z_{1\,t} + z_1)) = 2(c z_1, e^{-\mu t}(z_{1\,t} + z_1)); \tag{3.6}$$

integrating by parts identity (3.6) taking into account the conditions of Theorem 3.1 and boundary conditions (3.4), (3.5) for $|\gamma| > 1$, and applying Sobolev embedding theorems, we obtain the following inequality

$$\delta e^{\mu T} \|z_1\|_1^2 \le 2 \|c\|_{C(Q)}^2 \|z_1\|_1^2,$$

$$\left(\delta e^{\mu T} - 2 \|c\|_{C(Q)}^2\right) \|z_1\|_1^2 \le 0. \tag{3.7}$$

At the end we will show the correctness of the inequality $\|c\|_{C(Q)}^2 < r$, where $r = \frac{1}{2}\delta e^{\mu T}$. For now we will just use it and obtain $\|z_1\|_1^2 \le 0$, i.e. $z_1 = 0$, from which it follows $\vartheta_1 = \varphi_1$. Thus, for $y = \ell_1$ the solution of equation (2.1) satisfies the condition $u(x,t,\ell_1) = \varphi_1(x,t)$. Similarly, $u(x,t,\ell_2) = \varphi_2(x,t)$ is proved in the case when $i = 2$.

Now we will prove the solvability of problem (2.8)−(2.10) by the methods of "$\varepsilon-$ regularization", a priori estimates and successive approximations [10, 16], namely, in domain Q we will consider the following family of nonlinear equations of the third order with a small parameter:

$$\begin{gathered} L_\varepsilon u_{k,\varepsilon} = -\varepsilon u_{k,\varepsilon\,ttt} + L_0 u_{k,\varepsilon} + \lambda_k^2 u_{k,\varepsilon} \\ = \frac{u_{k,\varepsilon}}{H} \left[f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_1 - f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_2 \right] \\ + \frac{f_k}{H} \left[\varphi_1 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_2 - \varphi_2 \sum_{m=1}^{\infty} \lambda_m^2 u_m(x,t) \sin \lambda_m \ell_1 \right] + g_k \equiv F u_{k,\varepsilon}, \end{gathered} \tag{3.8}$$

with nonlocal boundary conditions of periodic type

$$\gamma D_t^p u_{k,\varepsilon}|_{t=0} = D_t^p u_{k,\varepsilon}|_{t=T}, \quad p = 0, 1, 2, \tag{3.9}$$

$$D_x^q u_{k,\varepsilon}|_{x=-1} = D_x^q u_{k,\varepsilon}|_{x=1}, \quad q = 0, 1, \tag{3.10}$$

where ε is a small positive number.

To prove the unique solvability of problem (3.8)−(3.10), we need the following notations and lemmas.

Let us define the spaces of vector functions

$$W_s(Q) = \{ v = (v_1, v_2, ..., v_k, ...) |\ v_k \in W_2^s(Q),\ k = 1, 2, 3, ... \}, \quad s = 0, 1, 2$$

with the norm

$$\langle\upsilon\rangle_s^2 \equiv \sum_{k=1}^{\infty} \left(1+\lambda_k^2\right)^3 \|\upsilon_k\|^2_{W_2^s(Q)}, \tag{B}$$

where $W_2^s(Q)-$ Sobolev spaces. From the definition of the spaces $W_s(Q)$, $s=0,1,2$ with a certain norm (B) it follows that $W_2(Q)\subset W_1(Q)\subset W_0(Q)$ and they are Banach spaces.

Now by $W(Q)=\{\{\upsilon_k\}_{k=1}^{\infty}| \quad \{\upsilon_{k\,ttt}\}_{k=1}^{\infty}\in L_2(Q),\ \{\upsilon_k\}_{k=1}^{\infty}\in W_2(Q)\}$ we denote the class of vector functions satisfying the corresponding boundary conditions (3.9)–(3.10).

Definition 3.3. A generalized solution of problem (3.8)–(3.10) will be a vector of functions $\{\upsilon_k^{(\theta)}\}_{k=\overline{1,\infty}}^{\theta=\overline{0,\infty}}\subset W(Q)$ satisfying equation (3.8) almost everywhere in Q.

We prove the solvability of problem (3.8)–(3.10) by the method of successive approximations, namely, we consider the following system of nonlinear loaded equations of the third order with a small parameter

$$\begin{gathered} L_\varepsilon u_{k,\varepsilon}^{(\theta)} = -\varepsilon u_{k,\varepsilon\ ttt}^{(\theta)} + L_0 u_{k,\varepsilon}^{(\theta)} + k^2 u_{k,\varepsilon}^{(\theta)} \\ = \frac{u_{k,\varepsilon}^{(\theta-1)}}{H}\left[f_2\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_1 - f_1\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-1)}\sin m\ell_2\right] \\ +\frac{f_k}{H}\left[\varphi_1\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_2 - \varphi_2\sum_{m=1}^{\infty}\lambda_m^2 u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_1\right] + g_k \equiv F(u_{k,\varepsilon}^{(\theta-1)}) \end{gathered} \tag{3.11}$$

with semi-nonlocal boundary conditions

$$\gamma D_t^p u_{k,\varepsilon}^{(\theta)}\Big|_{t=0} = D_t^p u_{k,\varepsilon}^{(\theta)}\Big|_{t=T} \quad p=0,1,2, \tag{3.12}$$

$$u_{k,\varepsilon x}^{(\theta)}\Big|_{x=-1} = u_{k,\varepsilon x}^{(\theta)}\Big|_{x=1} = 0, \tag{3.13}$$

where $\varepsilon>0$, $\theta=0,1,2,...,\quad k=1,2,3...,\ u_{k,\varepsilon}^{(-1)}\equiv 0.$

Lemma 3.4. *Let all the conditions of Theorem 3.1 be satisfied, then the following estimates are valid for the solutions of the problem* (3.11) – (3.13)*:*

I) $\frac{\varepsilon}{\delta}\left\langle u_{\varepsilon\ tt}^{(\theta)}\right\rangle_0^2 + \left\langle u_\varepsilon^{(\theta)}\right\rangle_1^2 < 2\gamma_0,$

II) $\frac{\varepsilon}{\delta}\left\langle u_{\varepsilon\ ttt}^{(\theta)}\right\rangle_0^2 + \left\langle u_\varepsilon^{(\theta)}\right\rangle_2^2 < 2\gamma_0.$

Proof. Consider the identity.

$$2\left(L_\varepsilon u_{k,\varepsilon}^{(\theta)}, e^{-\mu t}u_{k,\varepsilon\ t}^{(\theta)}\right)_0 = 2\left(F_\varepsilon(u_{k,\varepsilon}^{(\theta-1)}), e^{-\mu t}u_{k,\varepsilon\ t}^{(\theta)}\right)_0. \tag{3.14}$$

Let us estimate it from below. Integrating by parts we get:

$$\begin{gathered} \varepsilon e^{-\mu T}\left\|u_{k,\varepsilon\ tt}^{(\theta)}\right\|_0^2 + \int_Q (2\alpha(x,t)+\mu x)u_{k,\varepsilon\ t}^{2(\theta)}e^{-\mu t}dQ + \int_Q a\mu u_{k,\varepsilon\ x}^{2(\theta)}e^{-\mu t}dQ + \mu\left(\frac{\pi}{\ell}\right)^2\int_Q u_{k,\varepsilon}^{2(\theta)}e^{-\mu t}dQ \\ -\int_{\partial Q}(2\varepsilon u_{k,\varepsilon\ t}^{(\theta)}u_{k,\varepsilon\ tt}^{(\theta)} + x u_{k,\varepsilon\ t}^{2(\theta)} - \mu u_{k,\varepsilon\ t}^{2(\theta)} + \lambda_k^2 u_{k,\varepsilon\ t}^{2(\theta)} + u_{k,\varepsilon\ x}^{2(\theta)})e^{-\mu t}e_t dS \\ -\int_{\partial Q} 2u_{k,\varepsilon\ t}^{(\theta)}u_{k,\varepsilon\ x}^{(\theta)}e_x e^{-\mu t}dS \le 2\left(L_\varepsilon u_{k,\varepsilon}^{(\theta)}, e^{-\mu t}u_{k,\varepsilon\ t}^{(\theta)}\right)_0, \end{gathered} \tag{3.15}$$

where $\vec{e}=((e_x,e_t)|\ e_x=(\vec{e},x),\ e_t=(\vec{e},t))$ is the unit internal normal vector to the boundary ∂Q. Let us estimate it from above. Using Cauchy's inequalities with σ to (3.15) we have

$$\begin{gathered} 2\left(F_\varepsilon(u_{k,\varepsilon}^{(\theta-1)}), e^{-\mu t}u_{k,\varepsilon\ t}^{(\theta)}\right)_0 \le \sigma\|g_k\|_0^2 + \sigma c_1 c_3^2\left(\|f_0\|_C^2 + \|f_1\|_C^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_1^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right) \\ +\sigma c_1 c_3^2\left(\|\varphi_0\|_C^2 + \|\varphi_1\|_C^2\right)\|f_k\|_1^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right) + 5\sigma^{-1}\int_Q u_{k,\varepsilon\ t}^{2(\theta)}e^{-\mu t}dQ, \end{gathered} \tag{3.16}$$

where c_3 is the coefficient from the Sobolev embedding theorem, i.e. $\|u\|^2_{L_4(Q)} \leq c_3 \|u\|^2_{W_2^1(Q)}$ [12, 17]. Taking into account the boundary conditions (3.12), (3.13) and $\gamma^2 = e^{\mu T}$, we obtain that the boundary integrals will vanish. After performing some simple operations we get

$$\begin{gathered}\varepsilon e^{-\mu T}\left\|u^{(\theta)}_{k,\varepsilon\ tt}\right\|^2_0 + \delta e^{-\mu T}\left\|u^{(\theta)}_{k,\varepsilon}\right\|^2_1 \leq \sigma\|g_k\|^2_0 + 2\sigma c_1 c_3^2 \mathfrak{F}\|f_k\|^2_1\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u^{(\theta-1)}_{m,\varepsilon}\right\|^2_1\right) \\ +2\sigma c_1 c_3^2\mathfrak{F}\left\|u^{(\theta-1)}_{k,\varepsilon}\right\|^2_1\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u^{(\theta-1)}_{m,\varepsilon}\right\|^2_1\right),\end{gathered} \tag{3.17}$$

where $\delta = \min\left(\delta_0/2, \mu a, \mu(\frac{\pi}{\ell})^2\right)$. Here we took σ equal to $\sigma = \frac{10}{\delta_0}$, which follows $\delta - 5\sigma^{-1} = \frac{\delta_0}{2}$. Now multiplying (3.17) by $(1+\lambda_k^2)^3$ and summing the equations by k from 1 up to ∞ we obtain the first recurrent formula

$$\frac{\varepsilon}{\delta}\left\langle u^{(\theta)}_{\varepsilon\ tt}\right\rangle^2_0 + \left\langle u^{(\theta)}_{\varepsilon}\right\rangle^2_1 < \gamma_{00} + \gamma_{11}\left\langle u^{(\theta-1)}_{\varepsilon}\right\rangle^2_1 + \gamma_{22}\left\langle u^{(\theta-1)}_{\varepsilon}\right\rangle^4_1, \tag{3.18}$$

here

$$\gamma_{00} = 30\delta_0{}^{-1}\delta^{-1}e^{\mu T}[g]^2_{0,3},\ \gamma_{11} = 80\delta_0{}^{-1}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\,[f]^2_{1,3},\ \gamma_{22} = 80\delta_0{}^{-1}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}.$$

For convenience, in inequality (3.18) we introduce the notation through $J_1^{(\theta)} = \langle u^{(\theta)}_{\varepsilon}\rangle^2_1$ and $I_2^{(\theta)} = \left\langle u^{(\theta)}_{\varepsilon\ tt}\right\rangle^2_0$. Then from (3.18) we obtain the following recurrent formula

$$\frac{\varepsilon}{\delta}I_2^{(\theta)} + J_1^{(\theta)} \leq \gamma_{00} + \gamma_{11}J_1^{(\theta-1)} + \gamma_{22}J_1^{2(\theta-1)}. \tag{3.19}$$

Remark 3.5. Note that for linear inverse problems in the recurrent formula (3.19) the coefficient γ_2 will always be zero. However, for nonlinear inverse problems γ_{22} is not zero. This complicates the limited nature of the iterative process.

Now we will prove the boundedness of the iterative process. Let us introduce notations For the initial approximation we take $u^{(-1)}_{k,\varepsilon} = 0$. Then from problem (3.11)–(3.13) for the zero approximation we have

$$\frac{\varepsilon}{\delta}I_2^{(0)} + J_1^{(0)} \leq \gamma_{00} < 2\gamma_{00}.$$

From here, from the recurrent formula (3.19), taking into account the condition of Theorem 3.1, we obtain the following estimate

$$\frac{\varepsilon}{\delta}I_2^{(1)} + J_1^{(1)} < \gamma_{00} + 2\gamma_{00}\gamma_{11} + 4\gamma_0^2\gamma_{22} < 2\gamma_{00}.$$

Continuing this process, by induction from (3.19), taking into account the inequality $\gamma_{11} + 2\gamma_{00}\gamma_{22} < q < \frac{1}{2}$ following from the condition of Theorem 3.1, we obtain the first a priori estimate for $\forall\,\theta$, $\theta = 2, 3, ...,$

$$\frac{\varepsilon}{\delta}I_2^{(\theta)} + J_1^{(\theta)} < \gamma_{00} + \gamma_{11}J_1^{(\theta-1)} + \gamma_{22}J_1^{2(\theta-1)} < \gamma_{00} + 2\gamma_{11}\gamma_{00} + 4\gamma_{22}\gamma_{00}^2 < 2\gamma_{00}. \tag{3.20}$$

Thus, the validity of the first a priori estimate of Lemma 3.4 is proven.
Now we will prove the second a priori estimate.
Let us consider the identities

$$\left|-2\left(L_\varepsilon u^{(\theta)}_{k,\varepsilon}, Pu^{(\theta)}_{k,\varepsilon}e^{-\mu t}\right)_0\right| = \left|-2\left(F_\varepsilon(u^{(\theta-1)}_{k,\varepsilon}), Pu^{(\theta)}_{k,\varepsilon}e^{-\mu t}\right)_0\right|, \tag{3.21}$$

where $Pu^{(\theta)}_{k,\varepsilon} = u^{(\theta)}_{k,\varepsilon\ ttt} - u^{(\theta)}_{k,\varepsilon\ tt} + u^{(\theta)}_{k,\varepsilon\ xx} - u^{(\theta)}_{k,\varepsilon\ t}$. Let us estimate it in the same way as in the case of the first estimate from below and from above. Let us estimate identity (3.21) from below

$$\begin{gathered}\varepsilon e^{-\mu T}\left\|u^{(\theta)}_{k,\varepsilon ttt}\right\|^2_0 + \int\limits_Q\left[\delta_0 u^{2(\theta)}_{k,\varepsilon tt} + a\mu u^{2(\theta)}_{k,\varepsilon xt} + a u^{2(\theta)}_{k,\varepsilon xx} + \delta_0 u^{2(\theta)}_{k,\varepsilon\ t} + a\mu u^{2(\theta)}_{k,\varepsilon\ x} + \mu\left(\frac{\pi}{\ell}\right)^2 u^{2(\theta)}_{k,\varepsilon}\right]e^{-\mu t}dQ \\ -\int\limits_Q\left(\|\alpha_x\|^2_C + \|\alpha_t\|^2_C\right)u^{2(\theta)}_{k,\varepsilon\, t}e^{-\mu t}dQ \leq \left|-2\left(L_\varepsilon u^{(\theta)}_{k,\varepsilon}, Pu^{(\theta)}_{k,\varepsilon}e^{-\mu t}\right)_0\right|.\end{gathered} \tag{3.22}$$

Now let us estimate identity (3.21) from above

$$\left|-2\left(F_\varepsilon(u_{k,\varepsilon}^{(\theta-1)}), Pu_{k,\varepsilon}^{(\theta)}e^{-\mu t}\right)_0\right| \leq 3\sigma\left(1+\mu^2\right)\|g_k\|_1^2 + \left(\|\alpha_x\|_C^2 + \|\alpha_t\|_C^2\right)\int\limits_Q u_{k,\varepsilon\, t}^{2(\theta)}e^{-\mu t}dQ$$
$$+4\sigma c_1 c_3^2 \mathfrak{F}\left(3+\mu^2\right)\|f_k\|_2^2\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2 + 2\sigma c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2$$
$$+24\sigma^{-1}\int\limits_Q u_{k\ tt}^{2(\theta)}e^{-\mu t}dQ + 5\sigma^{-1}\int\limits_Q u_{k\ t}^{2(\theta)}e^{-\mu t}dQ + 5\sigma^{-1}\int\limits_Q u_{k\ xx}^{2(\theta)}e^{-\mu t}dQ. \tag{3.23}$$

Combining (3.22) and (3.23), we get

$$\begin{aligned}\varepsilon e^{-\mu T}\left\|u_{k,\varepsilon\ ttt}^{(\theta)}\right\|_0^2 + \int\limits_Q\Big[&(\delta_0-24\sigma^{-1})u_{k,\varepsilon\ tt}^{2(\theta)} + a\mu u_{k,\varepsilon\ xt}^{2(\theta)} + (a-5\sigma^{-1})u_{k,\varepsilon\ xx}^{2(\theta)}\\
&+(\delta_0-5\sigma^{-1})u_{k,\varepsilon\ t}^{2(\theta)} + a\mu u_{k,\varepsilon\ x}^{2(\theta)} + \mu\left(\frac{\pi}{\ell}\right)^2 u_{k,\varepsilon}^{2(\theta)}\Big]e^{-\mu t}dQ\\
&\leq 3\sigma\left(1+\mu^2\right)\|g_k\|_1^2 + \left(\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\int\limits_Q u_{k,\varepsilon\, t}^{2(\theta)}dQ\\
&+4\sigma c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\|f_k\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right)\\
&+2\sigma c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right).\end{aligned} \tag{3.24}$$

Choosing σ equal to $\sigma = 48\delta_0^{-1}$, it follows that $\delta_0 - 24\sigma^{-1} = \frac{\delta_0}{2}$, $\delta_0 - 5\sigma^{-1} > \frac{\delta_0}{2}$, $a - 5\sigma^{-1} > \delta_1 > 1$. Then from inequality (3.24) we obtain

$$\begin{aligned}\varepsilon e^{-\mu T}\left\|u_{k,\varepsilon\ ttt}^{(\theta)}\right\|_0^2 + \int\limits_Q\Big[&\frac{\delta_0}{2}u_{k,\varepsilon\ tt}^{2(\theta)} + a\mu u_{k,\varepsilon\ xt}^{2(\theta)} + \delta_1 u_{k,\varepsilon\ xx}^{2(\theta)} + \delta_0 u_{k,\varepsilon\ t}^{2(\theta)} + a\mu u_{k,\varepsilon\ x}^{2(\theta)} + \mu\left(\frac{\pi}{\ell}\right)^2 u_{k,\varepsilon}^{2(\theta)}\Big]e^{-\mu t}dxdt\\
&\leq 184\delta_0^{-1}\left(1+\mu^2\right)\|g_k\|_1^2 + \left(\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\int\limits_Q u_{k,\varepsilon\, t}^{2(\theta)}\\
&+192\delta_0^{-1}c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\|f_k\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right)\\
&+192\delta_0^{-1}c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right).\end{aligned} \tag{3.25}$$

Using the notation $\delta = min\left\{\frac{\delta_0}{2}, \delta_1, a\mu, \mu\left(\frac{\pi}{\ell}\right)^2\right\}$, we obtain

$$\begin{aligned}\varepsilon e^{-\mu T}\left\|u_{k,\varepsilon\ ttt}^{(\theta)}\right\|_0^2 + \delta e^{-\mu T}\left\|u_{k,\varepsilon}^{(\theta)}\right\|_2^2 &\leq 184\delta_0^{-1}\left(1+\mu^2\right)\|g_k\|_1^2 + \left(\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\int\limits_Q u_{k,\varepsilon\, t}^{2(\theta)}dQ\\
&+192\delta_0^{-1}c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\|f_k\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right)\\
&+192\delta_0^{-1}c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right).\end{aligned} \tag{3.26}$$

From (3.20) we have $\left\langle u^{(\theta)}_{\varepsilon\, t}\right\rangle_0^2 < 2\gamma_{00}$. Now multiplying (3.26) by $\left(1+\lambda_k^2\right)^3$ and summing the equations by k from 1 to ∞, we get

$$\frac{\varepsilon}{\delta}\left\langle u^{(\theta)}_{\varepsilon\, ttt}\right\rangle_0^2 + \left\langle u^{(\theta)}_{\varepsilon}\right\rangle_2^2 \le \gamma_0 + \gamma_1\left\langle u^{(\theta-1)}_{\varepsilon}\right\rangle_2^2 + \gamma_2\left\langle u^{(\theta-1)}_{\varepsilon}\right\rangle_2^4, \tag{3.27}$$

where

$$\gamma_0 = 60\delta_0^{-1}\delta^{-1}e^{\mu T}\left(4\left(1+\mu^2\right)+\delta^{-1}e^{\mu T}\left(\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\right)[g]_{1,3}^2,$$
$$\gamma_1 = 192\delta_0^{-1}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)[f]_{2,3}^2,$$
$$\gamma_2 = 192\delta_0^{-1}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right).$$

As in the first estimate, we introduce the notation $I_3^{(\theta)} = \left\langle u^{(\theta)}_{\varepsilon\, ttt}\right\rangle_0^2$, $J_2^{(\theta)} = \left\langle u^{(\theta)}_{\varepsilon}\right\rangle_2^2$, then from (3.27) we obtain the second recurrent formula

$$\frac{\varepsilon}{\delta}I_3^{(\theta)} + J_2^{(\theta)} \le \gamma_0 + \gamma_1 J_2^{(\theta-1)} + \gamma_2 J_2^{2(\theta-1)}. \tag{3.28}$$

Let us take $\left\{u^{(-1)}_{k,\varepsilon}\right\} = 0$ as the initial approximation. Then for the zero approximation we have:

$$\frac{\varepsilon}{\delta}I_3^{(0)} + J_2^{(0)} \le \gamma_0 < 2\gamma_0. \tag{3.29}$$

Hence, taking into account the inequality $\gamma_1 + 2\gamma_0\gamma_2 < q < \frac{1}{2}$, from (3.29) for the first approximation we have

$$\frac{\varepsilon}{\delta}I_3^{(1)} + J_2^{(1)} \le \gamma_0 + 2\gamma_1\gamma_0 + 4\gamma_2\gamma_0^2 \le 2\gamma_0. \tag{3.30}$$

Continuing this process, taking into account the conditions of Theorem 3.1, by iteration, we obtain the following second a priori estimate for $\forall\,\theta \ge 2$:

$$\frac{\varepsilon}{\delta}I_3^{(\theta)} + J_2^{(\theta)} < \gamma_0 + \gamma_1 J_2^{(\theta-1)} + \gamma_2 J_2^{2(\theta-1)} < \gamma_0 + 2\gamma_1\gamma_0 + 4\gamma_2\gamma_0^2 < 2\gamma_0.$$

Thus, the validity of the second estimate of Lemma 3.4 is proven.

□

Now let us introduce a new function from U according to the formula $\vartheta^{(\theta)}_{k,\varepsilon} = u^{(\theta)}_{k,\varepsilon} - u^{(\theta-1)}_{k,\varepsilon}$, $k = 1, 2, ...,$ $\theta = 0, 1, ...$ Then the following lemma holds for it.

Lemma 3.6. *Let the conditions of Theorem 3.1 be satisfied. Then the following estimates are valid for the functions*

III) $\frac{\varepsilon}{\delta}\left\langle \vartheta^{(\theta)}_{\varepsilon\ tt}\right\rangle_0^2 + \left\langle \vartheta^{(\theta)}_{\varepsilon}\right\rangle_1^2 < 2\gamma_0 q^\theta$,
VI) $\frac{\varepsilon}{\delta}\left\langle \vartheta^{(\theta)}_{\varepsilon\ ttt}\right\rangle_0^2 + \left\langle \vartheta^{(\theta)}_{\varepsilon}\right\rangle_2^2 < 2\gamma_0 q^\theta$
where $q < \frac{1}{2}$.

Proof. First, we prove the third III) estimate. From (3.11)–(3.13) for functions $\left\{\vartheta^{(\theta)}_{\varepsilon,k}\right\} \in U(Q)$ we obtain the following problem:

$$\begin{aligned} L_\varepsilon \vartheta^{(\theta)}_{k,\varepsilon} &= -\varepsilon\vartheta^{(\theta)}_{k,\varepsilon\, ttt} + L_0\vartheta^{(\theta)}_{k,\varepsilon} + \lambda_k^2\vartheta^{(\theta)}_{k,\varepsilon} \\ &= \frac{1}{H}u^{(\theta-1)}_{k,\varepsilon} f_2\sum_{m=1}^{\infty}\lambda_m^2 u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_1 - \frac{1}{H}u^{(\theta-1)}_{k,\varepsilon} f_1\sum_{m=1}^{\infty}\lambda_m^2 u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_2 \\ &-\frac{1}{H}u^{(\theta-2)}_{k,\varepsilon} f_2\sum_{m=1}^{\infty}\lambda_m^2 u^{(\theta-2)}_{m,\varepsilon}\sin\lambda_m\ell_1 + \frac{1}{H}u^{(\theta-2)}_{k,\varepsilon} f_1\sum_{m=1}^{\infty}\lambda_m^2 u^{(\theta-2)}_{m,\varepsilon}\sin\lambda_m\ell_2 \\ &+\frac{1}{H}f_k\left[\varphi_1\sum_{m=1}^{\infty}\lambda_m^2\vartheta^{(\theta-1)}_m\sin\lambda_m\ell_2 - \varphi_2\sum_{m=1}^{\infty}\lambda_m^2\vartheta^{(\theta-1)}_m\sin\lambda_m\ell_1\right] \end{aligned} \tag{3.31}$$

with nonlocal boundary conditions of periodic type

$$\gamma D_t^q \vartheta_{k,\varepsilon}^{(\theta)}\Big|_{t=0} = D_t^q \vartheta_{k,\varepsilon}^{(\theta)}\Big|_{t=T}, \quad q = 0, 1, 2, \tag{3.32}$$

$$D_x^p \vartheta_{k,\varepsilon}^{(\theta)}\Big|_{x=-1} = D_x^p \vartheta_{k,\varepsilon}^{(\theta)}\Big|_{x=1}, \quad p = 0, 1, \tag{3.33}$$

where $k = 1, 2, ..., \theta = 0, 1, ...$ We change the right-hand side of equation (3.31) adding and subtracting terms $\frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_2 \sum\limits_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_1$ and $\frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_1 \sum\limits_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_1$ as follows:

$$\begin{aligned}
&L_\varepsilon u_{k,\varepsilon}^{(\theta)} - L_\varepsilon u_{k,\varepsilon}^{(\theta-1)} = L_\varepsilon \vartheta_k^{(\theta)} = F_\varepsilon u_{k,\varepsilon}^{(\theta-1)} - F_\varepsilon u_{k,\varepsilon}^{(\theta-2)} \\
&= \frac{1}{H} u_{k,\varepsilon}^{(\theta-1)} f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_1 - \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_1 \\
&- \frac{1}{H} u_{k,\varepsilon}^{(\theta-1)} f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_2 + \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_2 \\
&- \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-2)} \sin \lambda_m \ell_1 + \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_2 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_1 \\
&+ \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-2)} \sin \lambda_m \ell_2 - \frac{1}{H} u_{k,\varepsilon}^{(\theta-2)} f_1 \sum_{m=1}^{\infty} \lambda_m^2 u_{m,\varepsilon}^{(\theta-1)} \sin \lambda_m \ell_2 \\
&+ \frac{1}{H} f_k \left[\varphi_1 \sum_{m=1}^{\infty} \lambda_m^2 \vartheta_m^{(\theta-1)} \sin \lambda_m \ell_2 - \varphi_2 \sum_{m=1}^{\infty} \lambda_m^2 \vartheta_m^{(\theta-1)} \sin \lambda_m \ell_1 \right].
\end{aligned} \tag{3.34}$$

Let us consider identity

$$\left| -2 \left(L_\varepsilon \vartheta_k^{(\theta)}, e^{-\mu t} \vartheta_{t\,k,\varepsilon}^{(\theta)} \right)_0 \right| = \left| -2 \left(F_k(u_{k,\varepsilon}^{(\theta-1)}) - F_k(u_{k,\varepsilon}^{(\theta-2)}), e^{-\mu t} \vartheta_{t\,k,\varepsilon}^{(\theta)} \right)_0 \right|. \tag{3.35}$$

Let us show that the identity (3.35) is bounded below. Integrating by parts we obtain:

$$\begin{aligned}
&\varepsilon e^{-\mu T} \left\| \vartheta_{k,\varepsilon\ tt}^{(\theta)} \right\|_0^2 + \int\limits_Q (2\alpha(x,t) + \mu x) \vartheta_{k,\varepsilon\ t}^{2(\theta)} e^{-\mu t} dQ \\
&+ \int\limits_Q a\mu \vartheta_{k,\varepsilon\ x}^{2(\theta)} e^{-\mu t} dQ + \mu \left(\frac{\pi}{\ell} \right)^2 \int\limits_Q \vartheta_{k,\varepsilon}^{2(\theta)} e^{-\mu t} dQ \\
&+ \int\limits_{\partial Q} (2\varepsilon \vartheta_{k,\varepsilon\ t}^{(\theta)} \vartheta_{k,\varepsilon\ tt}^{(\theta)} + x \vartheta_{k,\varepsilon\ t}^{2(\theta)} - \mu \vartheta_{k,\varepsilon\ t}^{2(\theta)} + \lambda_k^2 \vartheta_{k,\varepsilon\ t}^{2(\theta)} + \vartheta_{k,\varepsilon\ x}^{2(\theta)}) e^{-\mu t} e_t dS \\
&+ \int\limits_{\partial Q} (-2) u_{k,\varepsilon\ t}^{(\theta)} \vartheta_{k,\varepsilon\ x}^{(\theta)} e_x e^{-\mu t} dS \leq \left| -2 \left(L_\varepsilon \vartheta_k^{(\theta)}, e^{-\mu t} \vartheta_{t\,k,\varepsilon}^{(\theta)} \right)_0 \right|.
\end{aligned}$$

Considering the boundary conditions (3.32), (3.33) and $\gamma^2 = e^{\mu T}$, we get that the boundary integrals are equal to zero:

$$\begin{aligned}
&\varepsilon e^{-\mu T} \left\| \vartheta_{k,\varepsilon\ tt}^{(\theta)} \right\|_0^2 + \int\limits_Q ((2\alpha(x,t) + \mu x) \vartheta_{k,\varepsilon\ t}^{2(\theta)} + a\mu \vartheta_{k,\varepsilon\ x}^{2(\theta)} + \mu \left(\frac{\pi}{\ell} \right)^2 \vartheta_{k,\varepsilon}^{2(\theta)}) e^{-\mu t} dQ \\
&\leq \left| -2 \left(L_\varepsilon \vartheta_k^{(\theta)}, e^{-\mu t} \vartheta_{t\,k,\varepsilon}^{(\theta)} \right)_0 \right|.
\end{aligned} \tag{3.36}$$

Let us show the upper bound of expression (3.36). Let us write the right part in expanded form:

$$\left|-2\Big(F_k(u_{k,\varepsilon}^{(\theta-1)})-F_k(u_{k,\varepsilon}^{(\theta-2)}),e^{-\mu t}\vartheta_{t\,k,\varepsilon}^{(\theta)}\Big)_0\right|$$

$$\begin{aligned}=\Bigg|-2\int\limits_Q\Bigg(&\frac{1}{H}u_{k,\varepsilon}^{(\theta-1)}f_2\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_1-\frac{1}{H}u_{k,\varepsilon}^{(\theta-1)}f_1\sum_{m=1}^{\infty}\lambda_m^2\vartheta_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_2\\&-\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_2\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-2)}\sin\lambda_m\ell_1+\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_1\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-2)}\sin\lambda_m\ell_2\\&+\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_2\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_1-\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_2\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_1\\&+\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_1\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m l_2-\frac{1}{H}u_{k,\varepsilon}^{(\theta-2)}f_1\sum_{m=1}^{\infty}\lambda_m^2u_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m\ell_2\\+\frac{1}{H}f_k&\left[\varphi_1\sum_{m=1}^{\infty}\lambda_m^2\vartheta_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m l_2-\varphi_2\sum_{m=1}^{\infty}\lambda_m^2\vartheta_{m,\varepsilon}^{(\theta-1)}\sin\lambda_m l_1\right]\Bigg)e^{-\mu t}\vartheta_{t\,k,\varepsilon}^{(\theta)}dQ\Bigg|.\end{aligned}\tag{3.37}$$

Applying the Cauchy inequality with σ to (3.37) and using the results of Lemma 3.4 we obtain

$$\begin{aligned}\left|-2\left(F_k(u_{k,\varepsilon}^{(\theta-1)})-F_k(u_{k,\varepsilon}^{(\theta-2)}),e^{-\mu t}\vartheta_{t\,k,\varepsilon}^{(\theta)}\right)_0\right|&\le 4\sigma c_1c_3^2\mathfrak{F}\left\|\vartheta_{k,\varepsilon}^{(\theta-1)}\right\|_1^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right)\\&+4\sigma c_1c_3^2\mathfrak{F}\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_1^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right)\\&+4\sigma c_1c_3^2\mathfrak{F}\left\|f_k\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right)+5\sigma^{-1}\int\limits_Q\vartheta_{k,\varepsilon\;t}^{2(\theta)}e^{-\mu t}dQ.\end{aligned}\tag{3.38}$$

Taking into account the conditions of Theorem 3.1, the boundary conditions and we note that the boundary integrals that appear during the calculations vanish. From below, identity (3.35) is estimated similarly to the first estimate of Lemma 3.4. Now, combining the upper and lower bounds for identity (3.36), we obtain

$$\begin{aligned}\frac{\varepsilon}{\delta}\left\|\vartheta_{k,\varepsilon\;tt}^{(\theta)}\right\|_0^2+\left\|\vartheta_{k,\varepsilon}^{(\theta)}\right\|_1^2&\le\frac{40}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left\|\vartheta_{k,\varepsilon}^{(\theta-1)}\right\|_1^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_1^2\right)\\+\frac{40}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_1^2&\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_1^2+\frac{40}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left\|f_k\right\|_2^2\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_1^2.\end{aligned}\tag{3.39}$$

Here we took σ equal to $\sigma=\frac{10}{\delta_0}$, from which follows $\delta_0-5\sigma^{-1}=\frac{\delta_0}{2}$. Multiplying inequalities (3.39) by $(1+\lambda_k^2)^3$ and summing by k from 1 to ∞, we obtain the first recurrent formula

$$\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\;tt}^{(\theta)}\right\rangle_0^2+\left\langle\vartheta_{\varepsilon}^{(\theta)}\right\rangle_1^2\le\frac{40}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(2\gamma_0+[f]_{2,3}^2\right)\left\langle\vartheta_{\varepsilon}^{(\theta-1)}\right\rangle_1^2.\tag{3.40}$$

Let us denote the coefficient before $\langle\vartheta_{\varepsilon}^{(\theta-1)}\rangle_1^2$ by $q_1\equiv\frac{40}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(2\gamma_0+[f]_{2,3}^2\right)$. It is clear that $q_1<q$. Then (3.40) has the form

$$\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\;tt}^{(\theta)}\right\rangle_0^2+\left\langle\vartheta_{\varepsilon}^{(\theta)}\right\rangle_1^2\le q_1\left\langle\vartheta_{\varepsilon}^{(\theta-1)}\right\rangle_1^2<q\left\langle\vartheta_{\varepsilon}^{(\theta-1)}\right\rangle_1^2.$$

Let us consider the zero approximation $\vartheta_{\varepsilon,k}^{(0)}=u_{\varepsilon,k}^{(0)}-u_{\varepsilon,k}^{(-1)}=u_{\varepsilon,k}^{(0)}$. Since $\langle u_{\varepsilon}^{(0)}\rangle_1^2\le 2\gamma_0$ by Lemma 3.4, then $\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\;tt}^{(1)}\right\rangle_0^2+\langle\vartheta_{\varepsilon}^{(1)}\rangle_1^2<2\gamma_0q$ holds for it. Let us now consider the second approximation. The following inequality is valid for it $\frac{\varepsilon}{\delta}\left\langle \vartheta^{(1)}_{\varepsilon\ tt}\right\rangle^2_0 + \langle\vartheta^{(1)}_{\varepsilon}\rangle^2_1 \le 2\gamma_0 q_1 < 2\gamma_0 q$. Continuing this process, we obtain for an arbitrary following relation

$$\frac{\varepsilon}{\delta}\left\langle \vartheta^{(\theta)}_{\varepsilon\ \ tt}\right\rangle^2_0 + \left\langle\vartheta^{(\theta)}_{\varepsilon}\right\rangle^2_1 \le 2\gamma_0 q_1^{\theta} < 2\gamma_0 q^{\theta}.$$

The inequality $q < \frac{1}{2}$ comes from the condition (3.21). Thus, the third *III)* estimate is proven. Now we will prove the fourth *IV)* estimate. Let us consider the identity

$$\left|-2\left(L_\varepsilon\vartheta^{(\theta)}_{k,\varepsilon}, e^{-\mu t}P\vartheta^{(\theta)}_{k,\varepsilon}\right)_0\right| = \left|-2\left(F_k(u^{(\theta-1)}_{k,\varepsilon}) - F_k(u^{(\theta-2)}_{k,\varepsilon}), e^{-\mu t}P\vartheta^{(\theta)}_{k,\varepsilon}\right)_0\right|, \tag{3.41}$$

where $P\vartheta^{(\theta)}_{k,\varepsilon} = \vartheta^{(\theta)}_{k,\varepsilon\,ttt} - \vartheta^{(\theta)}_{k,\varepsilon\,tt} + \vartheta^{(\theta)}_{k,\varepsilon\,xx} - \vartheta^{(\theta)}_{k,\varepsilon\,t}$. Let us show the lower boundedness of expression (3.40). Let us estimate it in a similar way as in the case of the third estimate from below. Then we have

$$\begin{aligned}\varepsilon e^{-\mu T}\left\|\vartheta^{(\theta)}_{k,\varepsilon ttt}\right\|^2_0 + \int\limits_Q \left[\delta_0\vartheta^{2(\theta)}_{k,\varepsilon tt} + a\mu\vartheta^{2(\theta)}_{k,\varepsilon xt} + a\vartheta^{2(\theta)}_{k,\varepsilon xx} + \delta_0\vartheta^{2(\theta)}_{k,\varepsilon\ t} + a\mu\vartheta^{2(\theta)}_{k,\varepsilon\ x} + \mu\vartheta^{2(\theta)}_{k,\varepsilon}\right] e^{-\mu t}dQ \\ - \left(\|\alpha_x\|^2_C + \|\alpha_t\|^2_C\right)\int\limits_Q \vartheta^{2(\theta)}_{k,\varepsilon\,t}e^{-\mu t}dQ \le \left|-2\left(L_\varepsilon\vartheta^{(\theta)}_k, e^{-\mu t}P\vartheta^{(\theta)}_{k,\varepsilon}\right)_0\right|.\end{aligned} \tag{3.42}$$

Now we will show the upper bound (3.41). Let us write the right part in expanded form

$$\begin{aligned}&\left|-2\Big(F_k(u^{(\theta-1)}_{k,\varepsilon}) - F_k(u^{(\theta-2)}_{k,\varepsilon}), e^{-\mu t}P\vartheta^{(\theta)}_{k,\varepsilon}\Big)_0\right| \\ =&\left|-2\int\limits_Q\left(\frac{1}{H}u^{(\theta-1)}_{k,\varepsilon}f_2\sum_{m=1}^{\infty}\lambda_m^2 u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_1 - \frac{1}{H}u^{(\theta-1)}_{k,\varepsilon}f_1\sum_{m=1}^{\infty}\lambda_m^2\vartheta^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_2\right.\right. \\ &-\frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_2\sum_{m=1}^{\infty}\lambda_m^2 u^{(\theta-2)}_{m,\varepsilon}\sin\lambda_m\ell_1 + \frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_1\sum_{m=1}^{\infty}\lambda_m^2 u^{(\theta-2)}_{m,\varepsilon}\sin\lambda_m\ell_2 \\ &+\frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_2\sum_{m=1}^{\infty}\lambda_m^2 u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_1 - \frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_2\sum_{m=1}^{\infty}\lambda_m^2 u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_1 \\ &+\frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_1\sum_{m=1}^{\infty}\lambda_m^2 u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_2 - \frac{1}{H}u^{(\theta-2)}_{k,\varepsilon}f_1\sum_{m=1}^{\infty}\lambda_m^2 u^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_2 \\ &\left.+\frac{1}{H}f_k\left[\varphi_1\sum_{m=1}^{\infty}\lambda_m^2\vartheta^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_2 - \varphi_2\sum_{m=1}^{\infty}\lambda_m^2\vartheta^{(\theta-1)}_{m,\varepsilon}\sin\lambda_m\ell_1\right]\right) \\ &\left.\times\left(\vartheta^{(\theta)}_{k,\varepsilon\ ttt} - \mu\vartheta^{(\theta)}_{k,\varepsilon\ tt} + \vartheta^{(\theta)}_{k,\varepsilon\ xx} - \vartheta^{(\theta)}_{k,\varepsilon\ t}\right)dQ\right|.\end{aligned} \tag{3.43}$$

Let us show the upper bound of expression (3.43). Indeed, applying the Cauchy inequalities with σ and using the previous results we obtain

$$\begin{aligned}&\left|-2\left(F_k(u^{(\theta-1)}_{k,\varepsilon}) - F_k(u^{(\theta-2)}_{k,\varepsilon}), e^{-\mu t}P\vartheta^{(\theta)}_{k,\varepsilon}\right)_0\right| \le 4\sigma c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|\vartheta^{(\theta-1)}_{k,\varepsilon}\right\|^2_2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u^{(\theta-1)}_{m,\varepsilon}\right\|^2_2\right) \\ &+4\sigma c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u^{(\theta-1)}_{k,\varepsilon}\right\|^2_2\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta^{(\theta-1)}_{m,\varepsilon}\right\|^2_2 + 4\sigma c_1 c_3^2\mathfrak{F}\left(3+\mu^2\right)\|f_k\|^2_2\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta^{(\theta-1)}_{m,\varepsilon}\right\|^2_2 \\ &+30\sigma^{-1}\int\limits_Q\vartheta^{2(\theta)}_{k\ tt}e^{-\mu t}dQ + 6\sigma^{-1}\int\limits_Q\vartheta^{2(\theta)}_{k\ xx}e^{-\mu t}dQ + 6\sigma^{-1}\int\limits_Q\vartheta^{2(\theta)}_{k\ t}e^{-\mu t}dQ.\end{aligned} \tag{3.44}$$

Let us choose σ equal to $\sigma = \frac{60}{\delta_0}$, then $\delta_0 - 30\sigma^{-1} = \frac{\delta_0}{2}$, $a - 6\sigma^{-1} > \delta_1$. Taking into account the conditions of Theorem 3.1, the boundary conditions and $\gamma^2 = e^{\mu T}$, we note that the boundary integrals that appear during the calculations vanish. From below, identity (3.41) is estimated similarly to the second estimate of Lemma 3.4. Now combining the upper (3.44) and lower (3.42) estimates for identity (3.41) we obtain

$$\begin{gathered}\frac{\varepsilon}{\delta}\left\|\vartheta_{k,\varepsilon\ ttt}^{(\theta)}\right\|_0^2+\left\|\vartheta_{k,\varepsilon}^{(\theta)}\right\|_2^2\leq\left(\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\int\limits_Q\vartheta_{k,\varepsilon\,t}^{2(\theta)}e^{-\mu t}dQ\\+\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|\vartheta_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|u_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right)\\+\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)\left\|u_{k,\varepsilon}^{(\theta-1)}\right\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right)\\+\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(3+\mu^2\right)\|f_k\|_2^2\left(\sum_{m=1}^{\infty}\left(1+\lambda_m^2\right)^3\left\|\vartheta_{m,\varepsilon}^{(\theta-1)}\right\|_2^2\right).\end{gathered}\tag{3.45}$$

Multiplying the inequalities (3.48) by $(1+\lambda_k^2)^3$ and summing by k from 1 to ∞ we obtain the second recurrent formula

$$\begin{gathered}\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\ ttt}^{(\theta)}\right\rangle_0^2+\left\langle\vartheta_{\varepsilon}^{(\theta)}\right\rangle_2^2\leq\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(3+\mu^2+\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\left(2\gamma_0+[f]_{2,3}^2\right)\left\langle\vartheta_{\varepsilon}^{(\theta-1)}\right\rangle_2^2\\<\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(4(1+\mu^2)+\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\left(2\gamma_0+[f]_{2,3}^2\right)\left\langle\vartheta_{\varepsilon}^{(\theta-1)}\right\rangle_2^2.\end{gathered}\tag{3.46}$$

Here we used inequality (3.40). By the condition of Theorem 3.1, the factor in front of $\langle\vartheta_{\varepsilon}^{(\theta-1)}\rangle_2^2$ is equal to q, i.e.

$$q=\frac{240}{\delta_0}\delta^{-1}e^{\mu T}c_1c_3^2\mathfrak{F}\left(3+\mu^2+\|\alpha_x\|_C^2+\|\alpha_t\|_C^2\right)\left(2\gamma_0+[f]_{2,3}^2\right).$$

Then (3.46) has the form $\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\ ttt}^{(\theta)}\right\rangle_0^2+\langle\vartheta_{\varepsilon}^{(\theta)}\rangle_2^2\leq q\,\langle\vartheta_{\varepsilon}^{(\theta-1)}\rangle_2^2$. Again as in the third estimate, consider the zero approximation $\vartheta_{\varepsilon,k}^{(0)}=u_{\varepsilon,k}^{(0)}-u_{\varepsilon,k}^{(-1)}=u_{\varepsilon,k}^{(0)}$. For him $\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\ ttt}^{(1)}\right\rangle_0^2+\langle\vartheta_{\varepsilon}^{(1)}\rangle_2^2\leq 2\gamma_0 q$ holds, since $\left\langle u_{\varepsilon,k}^{(0)}\right\rangle_2^2\leq 2\gamma_0$ by Lemma 3.4. Let us now consider the second approximation. For him the following inequality is true

$$\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\ ttt}^{(2)}\right\rangle_0^2+\left\langle\vartheta_{\varepsilon}^{(2)}\right\rangle_2^2\leq 2\gamma_0 q^2.$$

Continuing this process, we obtain for an arbitrary $\theta\geq 2$ the following relation

$$\frac{\varepsilon}{\delta}\left\langle\vartheta_{\varepsilon\ \ ttt}^{(\theta)}\right\rangle_0^2+\left\langle\vartheta_{\varepsilon}^{(\theta)}\right\rangle_2^2\leq 2\gamma_0 q^{\theta}<Rq^{\theta}.$$

Thus, estimate *IV)* is proven. *Lemma 3.6 is proven.* □

Theorem 3.7. *Let all the conditions of Theorem 3.1 be satisfied. Then problem* (2.8)−(2.10) *has a unique solution in* $W(Q)$.

Proof. To prove the theorem, we show that the sequence $\{u_k^{(\theta)}\}\in W(Q)$ is fundamental. We assume that $\left\{u_{\varepsilon}^{(\theta)}\right\},\ \left\{u_{\varepsilon}^{(\theta-1)}\right\}\in W(Q)$. Now we consider the new functions $\left\{\vartheta_k^{(\theta)}\right\}=\left\{u_k^{(\theta)}-u_k^{(\theta-1)}\right\}$. For these functions the a priori estimate IV) in Lemma 3.6 is appropriate, i.e.

$$\left[\vartheta_{\varepsilon}^{(\theta)}\right]_{W(Q)}^2<2\gamma_0 q^{\theta}.\tag{3.47}$$

Now we prove that the sequence of functions $\{u_{\varepsilon}^{(\theta)}\}\in W(Q)$ is fundamental. To do this, using the estimate (3.47) and the triangle inequality, we obtain the following inequality

$$\left[u_{\varepsilon}^{(\theta+p)}-u_{\varepsilon}^{(\theta)}\right]_{W(Q)}^2\leq\left[u_{\varepsilon}^{(\theta+p)}-u_{\varepsilon}^{(\theta+p-1)}\right]_{W(Q)}^2+\left[u_{\varepsilon}^{(\theta+p-1)}-u_{\varepsilon}^{(\theta+p-2)}\right]_{W(Q)}^2+\cdots$$

$$+\left[u_\varepsilon^{(\theta+1)}-u_\varepsilon^{(\theta)}\right]^2_{W(Q)} \le 2\gamma_0\left(q^{\theta+p}+q^{\theta+p-1}+...+q^{\theta+1}\right)=2\gamma_0 q^{\theta+1}(1+q+q^2+...+q^p)\le \frac{2\gamma_0 q^{\theta+1}}{1-q}.$$

It is clear that for sufficiently large θ since $q<\frac{1}{2}$, this quantity is arbitrarily small. From this it follows that the sequence $\left\{u_\varepsilon^{(\theta)}\right\}$ is fundamental. Hence, the problem (3.8)-(3.10) has a solution in the space $W(Q)$, i.e. $\{u_\varepsilon^{(\theta)}\}\to\{u_\varepsilon\}$. It follows that the functions $\{u_\varepsilon\}\in W(Q)$ are solutions to the problem (3.8)-(3.10).

Now we show that $\{u_{k,\varepsilon}\}$ has a limit in an appropriate space and the limit functional sequence $\{u_k\}$ is a solution to the problem (2.8)$-$(2.10).

We notice that $F\left(u_{k,\varepsilon}\right)$ in (3.8) consist of nonlinear terms. First we estimate this term. Using Theorem A.2 and Cauchy inequality, we have

$$\begin{gathered}\int\limits_Q\left(\sum_{m=1}^\infty {\lambda_m}^2 u_{m,\varepsilon_j}u_{k,\varepsilon_j}\sin\lambda_m\ell_1\right)^2 dxdt=\int\limits_Q\left(\sum_{m=1}^\infty \frac{1}{\lambda_m}{\lambda_m}^3 u_{m,\varepsilon_j}u_{k,\varepsilon_j}\sin\lambda_m\ell_1\right)^2 dxdt\\
\le\int\limits_Q\sum_{m=1}^\infty\frac{1}{{\lambda_m}^2}\sum_{m=1}^\infty {\lambda_m}^6 u^2_{m,\varepsilon_j}u^2_{k,\varepsilon_j}dxdt=\sum_{m=1}^\infty\frac{1}{\lambda_m^2}\sum_{m=1}^\infty{\lambda_m}^6\int\limits_Q u^2_{m,\varepsilon_j}u^2_{k,\varepsilon_j}dxdt\\
\le c_1\sum_{m=1}^\infty{\lambda_m}^6\left(\int\limits_Q u^4_{m,\varepsilon_j}dxdt\right)^{\frac{1}{2}}\left(\int\limits_Q u^4_{k,\varepsilon_j}dxdt\right)^{\frac{1}{2}}\le c_1c_2^2\sum_{m=1}^\infty{\lambda_m}^6\left\|u_{m,\varepsilon_j}\right\|^2_{W_2^1(Q)}\left\|u_{k,\varepsilon_j}\right\|^2_{W_2^1(Q)}\\
\le c_1c_2^2\sum_{m=1}^\infty\left(1+{\lambda_m}^2\right)^3\left\|u_{m,\varepsilon_j}\right\|^2_{W_2^1(Q)}\sum_{k=1}^\infty\left(1+{\lambda_k}^2\right)^3\left\|u_{k,\varepsilon_j}\right\|^2_{W_2^1(Q)}\le c_1c_2^2(2\gamma_0)^2.\end{gathered}\tag{3.48}$$

Here we have the boundedness of the integrand in $L_2(Q)$, which allows us extract weak convergent subsequence in $L_2(Q)$. So, from estimation of Lemma 3.4 we can pass to the limit in subsequence $\left\{u_{k,\varepsilon_j}\right\}$, such that

$$u_{k,\varepsilon_j}\to u_k \quad weakly \quad in \quad W_2^2;$$
$$u_{k,\varepsilon_j}\to u_k \quad weakly \quad in \quad W;$$

considering above results and from (3.48)

$$F\left(u_{k,\varepsilon_j}\right)\to F\left(u_k\right)\quad weakly\quad in\quad L_2.$$

From Lemma 3.4 we have that $\sqrt{\varepsilon_j}u_{k,\varepsilon_j ttt}$ is bounded, so we have
$\sqrt{\varepsilon_j}\sqrt{\varepsilon_j}u_{k,\varepsilon_j ttt}\to 0$ in $L_2(Q)$. Since, convergence in $L_2(Q)$ is stronger than weak convergence, it follows $\sqrt{\varepsilon_j}\left(\sqrt{\varepsilon_j}u_{k,\varepsilon_j ttt},\vartheta\right)\to 0$. Passing to the weak limit in (3.8) as $\varepsilon_j\to 0$, we obtain $Lu_k=F(u_k)$. This means that the function u_k with fixed k will be the unique solution of the problem (2.8)$-$(2.10) from $W_2^2(Q)$ [12, 18]. This proves Theorem 3.7. □

Now we prove Theorem 3.1.

Since all the conditions of Theorem 3.1 are satisfied, then, using the Parseval-Steklov equalities [20] to solve the problem (2.8)$-$(2.10), we obtain a solution to problem (2.1)$-$(2.6) from the specified class U.

Estimation for $c(x,t)$:

$$\|c\|_{C(Q)}=\max_Q|c(x,t)|=\max_Q\left|\frac{1}{H}\left[f_2\sum_{m=1}^\infty{\lambda_m}^2u_m\sin\lambda_m\ell_1-f_1\sum_{m=1}^\infty{\lambda_m}^2u_m\sin\lambda_m\ell_2\right]\right|$$

$$\le 2\mathfrak{F}\sum_{m=1}^\infty{\lambda_m}^2\max|u_m|<2\mathfrak{F}c_1c_2\sum_{m=1}^\infty(1+{\lambda_m}^2)^3\|u_m\|_2^2<4c_1c_2\mathfrak{F}\gamma_0$$

We can choose data such that number $4c_1c_2\mathfrak{F}\gamma_0$ will be lower than r.

Theorem 3.1 is proved.

□

Remark 3.8. For definiteness and ease of exposition, the analysis is carried out in the three-dimensional framework. Nonetheless, it should be noted that the extension to an arbitrary number of spatial dimensions does not entail essential difficulties, since the underlying arguments and techniques remain valid in the general multidimensional setting.

References

[1] Bitsadze A. V., On nonlocal boundary value problems. Rep. Acad. of Sc. USSR, (1989) Vol. 277, Iss. 1, P. 17–19.

[2] Anikanov Y. E., Some methods for studying multidimensional inverse problems for differential equations. Nauka, Novosibirsk, (1978).

[3] Bubnov B. A., On the question of the solvability of multidimensional inverse problems for parabolic and hyperbolic equations. Novosibirsk, (1987).

[4] Lavrentev M. M., Romanov V. G., Vasilev V. G., Multidimensional inverse problems for differential equations. Nauka, Novosibirsk, (1969).

[5] Kabanixin S. I., Inverse and ill-posed problems. Siberian Scientific Publishing House, Novosibirsk, (2009).

[6] Sabitov K. B., Martem'yanova N. V., Nonlocal inverse problem for a mixed type equation. Izv. Vyssh. Uchebn. Zaved. Mat. (2011), Iss. 2, P. 71–85.

[7] Dzhamalov S. Z., Linear inverse problem for Tricomi equation in three-dimensional space. Bulletin KRASES. Phys. Math.Sci. (2016) Vol. 13, Iss. 2, P. 10–15.

[8] Dzhamalov S. Z., Linear inverse problem for a mixed-type equation of the second kind, second order with nonlocal boundary conditions in three-dimensional space. Bulletin KRASEC, (2017) Vol. 1, Iss. 13, P. 7–13.

[9] Dzhamalov S. Z., Ashurov R. R., On a linear inverse problem for a multidimensional equation of mixed type of the first kind, second order. Izv. Vyssh. Uchebn. Zaved. Mat. (2019) Vol. 6, P. 1–12.

[10] Dzhamalov S. Z., Nonlocal boundary value and inverse problems for equations of mixed type. Monography, Tashkent, (2021).

[11] Nakhushev A. M., Loaded Equations and their Application. Nauka, Moscow, (2012).

[12] Sobolev S. L., Some applications of functional analysis in mathematical physics. Nauka, Moscow, (1988).

[13] Ladizhenskaya O. A., Boundary value problems of mathematical physics. Nauka, Moscow, (1973).

[14] Sabitov K. B., Martem'yanova N. V., Inverse problem for Lavrentiev-Bitsadze equation connected with the search of elements in right-hand side. Izv. Vyssh. Uchebn. Zaved. Mat. (2017), Iss. 2, P. 44–57.

[15] Dzhamalov S. Z., On a nonlocal boundary value problem with constant coefficients for a multidimensional equation of mixed type of the first kind. Bulletin of Samara State Technical University, Series phys. and math. sciences, (2017) Vol. 21, Iss. 4, P. 1–14.

[16] Lions J. L., Magenes E., Inhomogeneous boundary value problems and their applications. Mir, Moscow, (1971).

[17] Trenogin V. A., Functional analysis. FizMatLit, Moscow, (2007).

Dzhamalov S. Z.,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
e-mail: siroj63@mail.ru

Shokirov A. A.,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
e-mail: shokirov.abduvosiq96@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 49–59
DOI: 10.29229/uzmj.2026-1-5

The direct and inverse problems for the fractional equation with the Hilfer derivative

Fayziev Yu., Sadullaeva Sh.

Abstract. In this paper, the Cauchy problem for a differential equation with a fractional Hilfer derivative $D_t^{\alpha,\beta}u(t) + Au(t) = f(t),\ 0 < t \le T$ is studied, where the order of the fractional derivative is $1 < \alpha < 2$. The existence and uniqueness of the solution of the Cauchy problem is proved.

Keywords: Cauchy problem; Hilfer derivatives; Subdiffusion equation; Direct and inverse problems
MSC (2020): 35R11, 34A12

1. Introduction

Fractional calculus plays an important role for the mathematical modeling in many natural and engineering sciences. Fractional calculus is the generalization of ordinary calculus concerned with operations of integration (and differentiation) of non-integer order. Fractional differential equations including Caputo, Riemann-Liouville, Hilfer, and other fractional derivatives have been used in different areas of technological disciplines and concentrated on by numerous mathematicians, see the books [1, 2, 3, 4].

During the last few decades, theoretical foundations and applications of the Hilfer derivative have been explored in a variety of works [5, 6, 7], showing its relevance in anomalous transport, continuum mechanics, and statistical physics (see, for example, [8, 9]). Operational methods for solving such equations have been developed in [10, 11] and the behavior of solutions under different boundary and initial conditions has been analyzed in the works [12, 13]. The Cauchy problem for an ordinary differential equation with Hilfer derivative is studied for parameters $n-1 < \alpha < n$, and in the case $0 < \alpha < 1$, in the works [11] and [6]. The non-local problems for equations with these derivatives were studied in the papers [14, 15, 16, 13].

It should be noted that Hilfer gave a generalization of derivatives of both Riemann-Liouville and Caputo in [2] when he studied fractional-time evolution in physical phenomena. He named it a generalized fractional derivative. This derivative interpolates between the Riemann-Liouville and Caputo derivative in some sense.

The integral of the Riemann-Liouville order α of the function $y(t)$ in the interval $[0, +\infty)$ is defined by the following formula (see, e.g. [17], p. 181):

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} y(\xi) d\xi.$$

The generalized fractional derivative(GFD) of order α and type β is defined as following (see, for example, [11]):

$$D^{\alpha,\beta} y(t) = I^{\beta(n-\alpha)} \frac{d^n}{dt^n} I^{(1-\beta)(n-\alpha)} y(t), \quad t > 0.$$

Note that Hilfer's two-parametric fractional derivative is relatively new, and it interpolates between the Caputo and Riemann-Liouville derivatives: at the value of the parameter $\beta = 1$ we obtain the Caputo derivative, for $\beta = 0$ - the Riemann-Liouville derivative.

Therefore, in this article we study both the direct and the inverse problem of finding the right side of the equation, that is, the source function.

Inverse problems have also been studied by many mathematicians. Here we cite some articles. In the papers [18, 19] the case $Au = u_{xx}$ the unique solvability of the direct and inverse problems for the subdiffusion equation with a fractional Hilfer order derivative is studied. In the paper [20] in the case of $Au = u_{xxxx}$, an equation of mixed type with the participation of the fractional Hilfer derivative is considered. We also note the papers [21, 22], where u_{xx} and $u_{xx}+u_{yy}$ are taken as A on an interval and a rectangle with non-self-adjoint boundary conditions.

Let H be a separable Hilbert space and $A : H \to H$ be a self-adjointed, positive, unbounded arbitrary operator defined in H with the domain of definition $D(A)$. Suppose that A has a complete in system of orthonormal eigenfunctions $\{v_k\}$ and a countable set of positive eigenvalues $\{\lambda_k\}$. It is convenient to assume that the eigenvalues do not decrease as their number increases, i.e. $0<\lambda_1\leq \lambda_2 \leq \cdots \to +\infty$.

Let ε be an arbitrary real number. We introduce the power of operator A, that acting in H according to the rule

$$A^{\varepsilon}h=\sum_{k=1}^{\infty}\lambda_k^{\varepsilon}h_k v_k,$$

where h_k is the the Fourier coefficients of a function $h\in H : h_k=(h,v_k)$. Obviously, the domain of this operator has the form

$$D(A^{\varepsilon})=\{h\in H : \sum_{k=1}^{\infty}\lambda_k^{2\varepsilon} \mid h_k \mid^2 < \infty\}.$$

For elements of $D(A^{\varepsilon})$ we introduce the norm

$$\| h \|_{\varepsilon}^2=\sum_{k=1}^{\infty}\lambda_k^{2\varepsilon} \mid h_k \mid^2 = \| A^{\varepsilon}h \|^2,$$

and together with this norm $D(A^{\varepsilon})$ turns into a Hilbert space.

Let $1<\alpha<2$ and $0\leq\beta\leq 1$ are a fixed number and let $C((a,b);H)$ stands for a set of continuous functions $u(t)$ of $t\in(a,b)$ with values in H. Consider the following problem:

$$\begin{cases} D_t^{\alpha,\beta}u(t)+Au(t)=f(t), \quad 0<t\leq T,\\ \lim\limits_{t\to+0} I^{(1-\beta)(2-\alpha)}u(t)=\varphi,\\ \lim\limits_{t\to+0} \frac{d}{dt}I^{(1-\beta)(2-\alpha)}u(t)=\phi, \end{cases} \tag{1.1}$$

where $\varphi, \phi \in H$ and $f(t)\in C([0,T];H)$ are given functions.

Problem (1.1) also called the *direct problem.*

Definition 1.1. A function $t^{(1-\beta)(2-\alpha)}u(t)\in C([0,T];H)$ with the properties $D_t^{\alpha,\beta}u(t)$, $Au(t)\in C((0,T];H)$ and satisfying conditions (1.1) is called the solution of problem (1.1).

In this work, we study both the direct problem (1.1) and the inverse problem determining the right side of the equation. These tasks are discussed separately in the following two sections respectively.

Let us consider the inverse problem, for this we need an additional condition. We use the following condition as an additional condition for problem (1.1):

$$u(\tau)=\psi, \quad 0<\tau<T. \tag{1.2}$$

In this case φ, $\phi\in H$, $\psi\in D(A)$ are given elements and it should be noted that, when studying the inverse problem, we assume that the unknown element $f\in H$ does not depend on t.

Definition 1.2. A pair of $\{u(t),f\}$ functions $t^{(1-\beta)(2-\alpha)}u(t)\in C([0,T];H)$ and $f\in H$ with the properties $D_t^{\alpha,\beta}u(t)$, $Au(t)\in C((0,T];H)$ and satisfying the conditions (1.1) and (1.2) is called the solution of *inverse problem* (1.1).

2. Preliminaries and Direct problem

In order to find a solution to the direct problem, we introduce some concepts. For α and an arbitrary complex number β, we denote the Mittag-Leffler function with two parameters by $E_{\alpha,\beta}(t)$:

$$E_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)}.$$

Lemma 2.1. *If* $0 < \alpha < 2$, *then for any* $t \geq 0$ *one has:*

$$| E_{\alpha,\mu}(-t) | \leq \frac{C}{1+t},$$

where C *is constant does not depend on* μ *and* t *(see, e.g. [1],p.136).*

Lemma 2.2. *Let* $0 < \alpha < 2$, $\beta > 0$, *for all positive* t, *one has (see, e.g. [1] p.120):*

$$\int_0^t \eta^{\beta-1} E_{\alpha,\beta}(-\lambda\eta^{\alpha}) d\eta = t^{\beta} E_{\alpha,\beta+1}(-\lambda t^{\alpha}). \tag{2.1}$$

Lemma 2.3. *For sufficiently large* t *one has the asymptotic estimation (see, e.g.[1] p.134):*

$$E_{\rho,\rho+1}(-t) = \frac{1}{t}\left(1 + O\left(\frac{1}{t}\right)\right), \ t > 1. \tag{2.2}$$

Lemma 2.4. *The following relation holds:*

$$|t^{\alpha-1} E_{\alpha,\mu}(-\lambda t^{\alpha})| \leq C\lambda^{\varepsilon-1} t^{\varepsilon\alpha-1}, \ t > 0,$$

where λ *is a positive number and* $0 < \varepsilon < 1$.

This lemma is proven in [23].

Lemma 2.5. *Let* $0 < \varepsilon < 1$ *be any fixed number, and* $f(t) \in C([0,T]; D(A^{\varepsilon}))$. *Then the following estimate holds:*

$$\sum_{k=1}^{\infty} \left| \lambda_k \int_0^t \tau^{\alpha-1} E_{\alpha,\mu}(-\lambda_k \tau^{\alpha}) f_k(t-\tau) d\tau \right|^2 \leq C_{\varepsilon} \max_{t\in[0,T]} ||f||_{\varepsilon}^2. \tag{2.3}$$

Proof. By using Lemma 2.4 for any fixed number $0 < \varepsilon < 1$, we take

$$\sum_{k=1}^{n} \lambda_k^2 \left| \int_0^t \tau^{\alpha-1} E_{\alpha,\mu}(-\lambda_k \tau^{\alpha}) f_k(t-\tau) d\tau \right|^2 \leq C \sum_{k=1}^{n} \left[\int_0^t \tau^{\varepsilon\alpha-1} \lambda_k^{\varepsilon} |f_k(t-\tau)| d\tau \right]^2.$$

Using the generalized Minkowski inequality, we have

$$C \sum_{k=1}^{n} \left[\int_0^t \tau^{\varepsilon\alpha-1} \lambda_k^{\varepsilon} |f_k(t-\tau)| d\tau \right]^2 \leq C \left(\int_0^t \tau^{\varepsilon\alpha-1} \left(\sum_{k=1}^{n} \lambda_k^{2\varepsilon} |f_k(t-\tau)|^2 \right)^{\frac{1}{2}} d\tau \right)^2$$

$$\leq CT^{\varepsilon\alpha} \max_{t\in[0,T]} ||f||_{\varepsilon}^2 = C_{\varepsilon} \max_{t\in[0;T]} ||f||_{\varepsilon}^2.$$

Taking the limit as $n \to \infty$, we obtain the estimate (2.3).

Lemma 2.5 has been proved. □

Theorem 2.6. *Let* φ, $\phi \in H$ *and* $f(t) \in C([0,T]; D(A^{\varepsilon}))$ *for some* $\varepsilon \in (0,1)$. *Then the problem* (1.1) *has a unique solution and this solution has the following form:*

$$u(t) = \sum_{k=1}^{\infty} \Bigg[\varphi_k t^{(1-\beta)(\alpha-2)} E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^{\alpha}) + \phi_k t^{\alpha-1+\beta(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^{\alpha}) \tag{2.4}$$

$$+ \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k \tau^{\alpha}) f_k(t-\tau) d\tau \Bigg] v_k,$$

where φ_k, ϕ_k *and* $f_k(t)$ *are the Fourier coefficients of the function* φ, ϕ *and* $f(t)$ *respectively.*

Proof. **Existence**. Assume that a solution to problem (1.1) exists. Then, due to the completeness of the system $\{v_k\}$, the solution can be written in the form:

$$u(t)=\sum_{k=1}^{\infty}T_k(t)v_k, \tag{2.5}$$

where $T_k(t)$ are the Fourier coefficients of the function $u(t)$. Then we put equation (2.5) to problem (1.1), and obtain the following problem:

$$\begin{cases} D_t^{\alpha,\beta}T_k(t)+\lambda_k T_k(t)=f_k(t), \\ \lim\limits_{t\to+0} I^{(1-\beta)(2-\alpha)}T_k(t)=\varphi_k, \\ \lim\limits_{t\to+0} \frac{d}{dt}I^{(1-\beta)(2-\alpha)}T_k(t)=\phi_k, \quad k\geq 1. \end{cases} \tag{2.6}$$

The solution of problem (2.6) has the form (see, for example [11]):

$$T_k(t)=\sum_{k=1}^{\infty}\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^{\alpha})+\phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^{\alpha})$$
$$+\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^{\alpha})f_k(t-\tau)d\tau. \tag{2.7}$$

Thus, according to equalities (2.5) and (2.7), we find the formal solution of problem (1.1) as the form (2.4).

To prove the uniqueness of the solution, we use the standart technique, that is, the solution of problem (1.1) with the homogeneous condition is identically zero. Taking into account $\varphi=0$, $\phi=0$ and $f(t)=0$, then the Fourier coefficients of that functions φ_k, ϕ_k, and $f_k(t)$ would be zero (i.e. $\varphi_k=0$, $\phi_k=0$, and $f_k(t)=0$), respectively. Then it follows $T_k(t)\equiv 0$, for all $k\geq 1$. In that case, $u(t)\equiv 0$ derives from equality (2.5) and the completeness of the system $\{v_k\}$.

We now verify that the formal solution satisfies the conditions of Definition 1.1. We denote the partial sum of series (2.4) by $S_n(t)$.

First of all, we need to show that $t^{(1-\beta)(2-\alpha)}S_n(t)\in C([0,T];H)$. After simplifying the expression, we got following:

$$t^{(1-\beta)(2-\alpha)}S_n(t)=\sum_{k=1}^{n}\Big[\varphi_k E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^{\alpha})+\phi_k t E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^{\alpha})$$
$$+t^{(1-\beta)(2-\alpha)}\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^{\alpha})f_k(t-\tau)d\tau\Big]v_k.$$

Due to Parseval equality, we can write following:

$$\| t^{(1-\beta)(2-\alpha)}S_n(t)\|^2=\sum_{k=1}^{n}\Big|\varphi_k E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^{\alpha})$$
$$+\phi_k t E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^{\alpha})+t^{(1-\beta)(2-\alpha)}\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^{\alpha})f_k(t-\tau)d\tau\Big|^2.$$

Then we obtain this:

$$\| t^{(1-\beta)(2-\alpha)}S_n(t)\|^2\leq C\sum_{k=1}^{n}\Big|\varphi_k E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^{\alpha})\Big|^2+C\sum_{k=1}^{n}\Big|\phi_k t E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^{\alpha})\Big|^2$$
$$+C\sum_{k=1}^{n}t^{2(1-\beta)(2-\alpha)}\Big|\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^{\alpha})f_k(t-\tau)d\tau\Big|^2$$

$$= P_n^1 + P_n^2 + P_n^3.$$

Using inequality in Lemma 2.1, estimate each term:

$$P_n^1 = C\sum_{k=1}^{n}\left|\varphi_k E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha)\right|^2 \leq C\sum_{k=1}^{n} \mid \varphi_k \mid^2,$$

$$P_n^2 = C\sum_{k=1}^{n}\left|\phi_k t E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)\right|^2 \leq CT^2\sum_{k=1}^{n} \mid \phi_k \mid^2 .$$

Using the generalized Minkowski inequality estimate the last term:

$$P_n^3 = Ct^{2(1-\beta)(2-\alpha)}\sum_{k=1}^{n}\left|\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^\alpha)f_k(t-\tau)d\tau\right|^2$$

$$\leq Ct^{2(1-\beta)(2-\alpha)}\sum_{k=1}^{n}\left|\int_0^t \tau^{\alpha-1}f_k(t-\tau)d\tau\right|^2 \leq Ct^{2(1-\beta)(2-\alpha)}\left(\int_0^t \tau^{\alpha-1}\left[\sum_{k=1}^{n} \mid f_k(t-\tau)\mid^2\right]^{\frac{1}{2}}d\tau\right)^2$$

$$\leq Ct^{2(1-\beta)(2-\alpha)}T^\alpha \max_{0\leq t\leq T} \parallel f \parallel^2 \leq CT^{\alpha+4}\max_{0\leq t\leq T}\parallel f \parallel^2 .$$

Hence, we get the following estimation:

$$\parallel t^{(1-\beta)(2-\alpha)}S_n(t)\parallel^2 \leq C\sum_{k=1}^{n}\mid\varphi_k\mid^2 + CT^2\sum_{k=1}^{n}\mid\phi_k\mid^2 + CT^{\alpha+4}\max_{0\leq t\leq T}\parallel f\parallel^2,$$

if $\varphi, \phi \in H$ and $f(t) \in C([0,T];H)$, then $t^{(1-\beta)(2-\alpha)}u(t) \in C([0,T];H)$.

Now we apply the operator A on the partial sum $S_n(t)$, then we have:

$$AS_n(t) = \sum_{k=1}^{n}\Bigg(\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha) + \phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)$$

$$+\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^\alpha)f_k(t-\tau)d\tau\Bigg)\lambda_k v_k.$$

Due to the Parseval equality we may write

$$\parallel AS_n(t)\parallel^2 = \sum_{k=1}^{n}\lambda_k^2\Bigg|\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha) + \phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)$$

$$+\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^\alpha)f_k(t-\tau)d\tau\Bigg|^2.$$

Then, we have

$$\parallel AS_n(t)\parallel^2 \leq C\sum_{k=1}^{n}\lambda_k^2\left|\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha)\right|^2$$

$$+C\sum_{k=1}^{n}\lambda_k^2\left|\phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)\right|^2$$

$$+C\sum_{k=1}^{n}\lambda_k^2\left|\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^\alpha)f_k(t-\tau)d\tau\right|^2 = AS_n^1 + AS_n^2 + AS_n^3,$$

where

$$AS_n^1 = C\sum_{k=1}^{n}\lambda_k^2\left|\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha)\right|^2,$$

$$AS_n^2 = C\sum_{k=1}^{n}\lambda_k^2\Big|\phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)\Big|^2,$$

$$AS_n^3 = C\sum_{k=1}^{n}\lambda_k^2\Big|\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^\alpha)f_k(t-\tau)d\tau\Big|^2.$$

Using inequality in Lemma 2.1, we estimate the first two sums AS_n^1 and AS_n^2:

$$AS_n^1 = C\sum_{k=1}^{n}\lambda_k^2\Big|\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha)\Big|^2 \le C\sum_{k=1}^{n}\lambda_k^2 t^{2(1-\beta)(\alpha-2)} \mid \varphi_k\mid^2 \left|\frac{1}{1+\lambda_k t^\alpha}\right|^2$$

$$\le C\sum_{k=1}^{n}\lambda_k^2\frac{t^{2(1-\beta)(\alpha-2)}}{\lambda_k^2 t^{2\alpha}} \mid \varphi_k\mid^2 \le Ct^{2\beta(2-\alpha)-4}\sum_{k=1}^{n}\mid \varphi_k\mid^2,$$

$$AS_n^2 = C\sum_{k=1}^{n}\lambda_k^2\Big|\phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)\Big|^2 \le C\sum_{k=1}^{n}\lambda_k^2 \mid \phi_k\mid^2 t^{2(\alpha-1+\beta(2-\alpha))}\left|\frac{1}{1+\lambda_k t^\alpha}\right|^2$$

$$\le C\sum_{k=1}^{n}\lambda_k^2\frac{1}{\lambda_k^2 t^{2\alpha}}t^{2(\alpha-1+\beta(2-\alpha))} \mid \phi_k\mid^2 \le Ct^{2\beta(2-\alpha)-2}\sum_{k=1}^{n}\mid \phi_k\mid^2.$$

Let us estimate the sum AS_n^3. According to Lemma 2.5, we have:

$$AS_n^3 = \sum_{n=1}^{j}\lambda_k^2\left|\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^\alpha)f_k(t-\tau)d\tau\right|^2 \le C_\varepsilon \max_{t\in[0,T]}||f||_\varepsilon^2.$$

Therefore,

$$\| AS_n(t)\|^2 \le Ct^{2\beta(2-\alpha)-4}\sum_{k=1}^{n}\mid \varphi_k\mid^2 + Ct^{2\beta(2-\alpha)-2}\sum_{k=1}^{n}\mid \phi_k\mid^2 + C_\varepsilon \max_{t\in[0,T]}||f||_\varepsilon^2.$$

Hence, if $\varphi,\ \phi\in H$ and $f\in C([0,T];D(A^\varepsilon))$ we obtain $Au(t)\in C((0,T];H)$.

Further, from the equation (1.1) one has $D_t^{\alpha,\beta}u(t) = f(t) - Au(t)$. Since $f\in C([0,T];D(A^\varepsilon))$, $Au(t)\in C((0,T];H)$, it follows that $D_t^{\alpha,\beta}u(t)\in C((0,T];H)$.

□

Remark. If the function f does not depend on t, using the equality in Lemma 2.2, $u(t)$ can be written as following:

$$u(t) = \sum_{k=1}^{\infty}\Big[\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha) + \phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha) \tag{2.8}$$
$$+f_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha)\Big]v_k.$$

In that case, according to Definition 1.1, it is sufficient to be $f\in H$, to show that $Au(t)\in C((0,T];H)$. Now we reveal it.

After applying the operator A on the partial sum of equality in (2.8) and in consequence of the Parseval equality we may write:

$$\| AS_n(t)\|^2 = \sum_{k=1}^{n}\lambda_k^2\Big|\varphi_k t^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha) + \phi_k t^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha)$$
$$+f_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha)\Big|^2.$$

Therefore,

$$\| AS_n(t) \|^2 \leq C \sum_{k=1}^{n} \lambda_k^2 \left| \varphi_k t^{(1-\beta)(\alpha-2)} E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k t^\alpha) \right|^2$$

$$+C \sum_{k=1}^{n} \lambda_k^2 \left| \phi_k t^{\alpha-1+\beta(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k t^\alpha) \right|^2 + C \sum_{k=1}^{n} \lambda_k^2 \left| f_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha) \right|^2 = AS_n^1 + AS_n^2 + AS_n^3.$$

The first 2 terms of above were evaluated, so we estimate the last term:

$$AS_n^3 = C \sum_{k=1}^{n} \lambda_k^2 \left| f_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha) \right|^2 \leq C \sum_{k=1}^{n} \lambda_k^2 |\ f_k\ |^2 t^{2\alpha} \left| \frac{1}{1+\lambda_k t^\alpha} \right|^2$$

$$\leq C \sum_{k=1}^{n} \lambda_k^2 |\ f_k\ |^2 t^{2\alpha} \frac{1}{\lambda_k^2 t^{2\alpha}} \leq C \sum_{k=1}^{n} |\ f_k\ |^2 .$$

Hence,

$$\| AS_n(t) \|^2 \leq Ct^{2\beta(2-\alpha)-4} \sum_{k=1}^{n} |\ \varphi_k\ |^2 + Ct^{2(1-\beta(2-\alpha))} \sum_{k=1}^{n} |\ \phi_k\ |^2 + C \sum_{k=1}^{n} |\ f_k\ |^2 .$$

It is clear that, if $\varphi,\ \phi,\ f \in H$ then we obtain $Au(t) \in C((0,T];H)$.

3. Inverse problem

In this section, we study the inverse problem of finding the right-hand side of the equation. Let the functions $u(t)$ and f are unknown in the next problem.

$$\begin{cases} D_t^{\alpha,\beta} u(t) + Au(t) = f, \quad 0 < t \leq T, \\ \lim\limits_{t\to 0} I^{(1-\beta)(2-\alpha)} u(t) = \varphi, \\ \lim\limits_{t\to 0} \frac{d}{dt} I^{(1-\beta)(2-\alpha)} u(t) = \phi; \quad \varphi, \phi \in H. \end{cases} \tag{3.1}$$

Note that f function does not depend on the variable t.

Theorem 3.1. *Let $1 < \alpha < 2$, $0 \leq \beta \leq 1$ and $\varphi, \phi \in H$, $\psi \in D(A)$. Then the inverse problem* (1.1), (1.2) *has a unique solution $\{u(t), f\}$ and this solution has the form as* (2.8), *where*

$$f_k = \frac{\psi_k}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k \tau^\alpha)} - \frac{\varphi_k \tau^{(1-\beta)(\alpha-2)} E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k \tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k \tau^\alpha)}$$

$$-\frac{\phi_k \tau^{\alpha-1+\beta(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k \tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k \tau^\alpha)}$$

and

$$f = \sum_{k=1}^{\infty} f_k v_k. \tag{3.2}$$

Proof. We indicated above, that f is unknown and it does not depend on t. We solve the direct problem by assuming that the unknown function f is a known element. Then the solution to the direct problem has the form (2.4).

Now, using additional condition (1.2) we have:

$$u(\tau) = \sum_{k=1}^{\infty} \Big[\varphi_k \tau^{(1-\beta)(\alpha-2)} E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k \tau^\alpha) + \phi_k \tau^{\alpha-1+\beta(2-\alpha)} E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k \tau^\alpha)$$

$$+ f_k \tau^\alpha E_{\alpha,\alpha+1)}(-\lambda_k \tau^\alpha) \Big] v_k = \psi.$$

from the completeness of the system $\{v_k\}$, we get

$$\varphi_k\tau^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k\tau^\alpha)+\phi_k\tau^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k\tau^\alpha)$$

$$+f_k\tau^\alpha E_{\alpha,\alpha+1)}(-\lambda_k\tau^\alpha)=\psi_k.$$

Hence,

$$f_k=\frac{\psi_k}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}-\frac{\varphi_k\tau^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}$$
$$-\frac{\phi_k\tau^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}.$$

Let us introduce the following notation:

$$f_k^1=\frac{\psi_k}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)},$$

$$f_k^2=\frac{\varphi_k\tau^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)},$$

$$f_k^3=\frac{\phi_k\tau^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}.$$

Then, the equality holds:

$$f=\sum_{k=1}^{\infty}(f_k^1+f_k^2+f_k^3)v_k.$$

Let us reveal the convergence of series (3.2). If F_n the partial sums of series (3.2), then by virtue of the Parseval equality we may write

$$\| F_n \|^2=\sum_{k=1}^{n}\left|f_k^1+f_k^2+f_k^3\right|^2\leq 2\sum_{k=1}^{n}|\ f_k^1\ |^2+2\sum_{k=1}^{n}|\ f_k^2\ |^2+2\sum_{k=1}^{n}|\ f_k^3\ |^2=2M_n^1+2M_n^2+2M_n^3.$$

After using Lemma 2.3 in order to estimate $M_n^i,\ i=(1,2,3)$, then we have:

$$M_n^1\leq\sum_{k=1}^{n}\frac{|\ \psi_k\ |^2}{\left|\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)\right|^2}\leq C\sum_{k=1}^{n}\frac{|\ \psi_k\ |^2}{\left|\tau^\alpha(\lambda_k\tau^\alpha)^{-1}[1+O((\lambda_k\tau^\alpha)^{-1})]\right|^2}$$

$$\leq C\sum_{k=1}^{n}\frac{\lambda_k^2\ |\ \psi_k\ |^2}{\left(1+O((\lambda_k\tau^\alpha)^{-1})\right)^2}\leq C\sum_{k=1}^{n}\lambda_k^2\ |\ \psi_k\ |^2.$$

$$M_n^2\leq\sum_{k=1}^{n}\left|-\frac{\varphi_k\tau^{(1-\beta)(\alpha-2)}E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}\right|^2$$
$$\leq\sum_{k=1}^{n}\frac{|\ \varphi_k\ |^2\ \tau^{2(1-\beta)(\alpha-2)}\ |\ E_{\alpha,\beta+(1-\beta)(\alpha-1)}(-\lambda_k\tau^\alpha)\ |^2}{|\ \tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)\ |^2}$$

$$\leq C\sum_{k=1}^{n}\frac{|\ \varphi_k\ |^2\left|\frac{1}{1+\lambda_k\tau^\alpha}\right|^2\tau^{2(1-\beta)(\alpha-2)}}{\tau^{2\alpha}(\lambda_k\tau^\alpha)^{-2}(1+O((\lambda_k\tau^\alpha)^{-1}))^2}\leq C\sum_{k=1}^{n}\frac{|\ \varphi_k\ |^2\ \tau^{2(\beta(2-\alpha)-2)}}{(1+O((\lambda_k\tau^\alpha)^{-1}))^2}$$

$$\leq C\tau^{2(\beta(2-\alpha)-2)}\sum_{k=1}^{n} \mid \varphi_k \mid^2 .$$

$$M_n^3 \leq \sum_{k=1}^{n}\left| -\frac{\phi_k \tau^{\alpha-1+\beta(2-\alpha)}E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k\tau^\alpha)}{\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)}\right|^2$$

$$\leq C\sum_{k=1}^{n}\frac{\mid \phi_k \mid^2 \tau^{2\beta(2-\alpha)-2} \mid E_{\alpha,\alpha+\beta(2-\alpha)}(-\lambda_k\tau^\alpha)\mid^2}{\mid E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha)\mid^2}$$

$$\leq C\sum_{k=1}^{n}\frac{\mid \phi_k \mid^2 \left|\frac{1}{1+\lambda_k\tau^\alpha}\right|^2 \tau^{2\beta(2-\alpha)-2}}{(\lambda_k\tau^\alpha)^{-2}(1+O((\lambda_k\tau^\alpha)^{-1}))^2} \leq C\sum_{k=1}^{n}\frac{\mid \phi_k \mid^2 \tau^{2\beta(2-\alpha)-2}}{(1+O((\lambda_k\tau^\alpha)^{-1}))^2}$$

$$\leq C\sum_{k=1}^{n}\frac{\mid \phi_k \mid^2 \tau^{2\beta(2-\alpha)-2}}{(1+O((\lambda_k\tau^\alpha)^{-1}))^2} \leq C\tau^{2\beta(2-\alpha)-2}\sum_{k=1}^{n}\mid \phi_k \mid^2 .$$

These estimates derive from the convergence of series (3.2) under the condition $\varphi, \phi \in H$ and $\psi \in D(A)$. From here the existence of the element f determined by series (3.2) follows.

Uniqueness. Suppose that this problem has two solutions $\{u_1(t), f_1\}$ and $\{u_2(t), f_2\}$. It is enough to prove that $u(t) = u_1(t) - u_2(t)$ and $f = f_1 - f_2 = 0$. Using the linearity of the problem conditions, to determine the function $u(t)$ and f we get the following problem:

$$\begin{cases} D_t^{\alpha,\beta}u(t) + Au(t) = f, \ 0 < t \leq T, \\ \lim\limits_{t\to 0} I^{(1-\beta)(2-\alpha)}u(t) = 0, \\ \lim\limits_{t\to +0} \frac{d}{dt}I^{(1-\beta)(2-\alpha)}u(t) = 0, \end{cases} \tag{3.3}$$

and

$$u(\tau) = 0. \tag{3.4}$$

Let $u(t)$ be the solution to this problem. Let us introduce the notation. Then from equation (3.3) and the self-adjointness of operator A, we will have

$$D_t^{\alpha,\beta}u_k(t) = (D_t^{\alpha,\beta}u(t), v_k) = -(Au(t), v_k) + (f, v_k) = -(u(t), Av_k) + (f, v_k) =$$

$$= -(u(t), \lambda_k v_k) + f_k = -\lambda_k(u(t), v_k) + f_k = -\lambda_k u_k(t) + f_k.$$

Thus, taking into account equality (3.4), we have the following problem:

$$\begin{cases} D_t^{\alpha,\beta}u_k(t) + \lambda_k u_k(t) = f_k, \ 0 < t \leq T, \\ \lim\limits_{t\to 0} I^{(1-\beta)(2-\alpha)}u_k(t) = 0, \\ \lim\limits_{t\to +0} \frac{d}{dt}I^{(1-\beta)(2-\alpha)}u_k(t) = 0. \end{cases} \tag{3.5}$$

Then the solution to this problem has the form (see [24]; [12] p.174; [25] p.17):

$$u_k(t) = f_k t^\alpha E_{\alpha,\alpha+1}(-\lambda_k t^\alpha).$$

Using equality in (3.4), we have

$$u_k(\tau) = f_k \tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k \tau^\alpha) = 0.$$

Hence, due to the properties of the Mittag-Leffler function $\tau^\alpha E_{\alpha,\alpha+1}(-\lambda_k\tau^\alpha) \neq 0$, it follows from here $f_k = 0$ for all $k \geq 1$. In consequence, from the completeness of the system of eigenfunctions $\{v_k\}$, we finally obtain $f \equiv 0$ and u(t)$\equiv 0$, as required. Theorem 3.1 is completely proven. □

4. Acknowledgement.

The authors deeply thank Professor R.R. Ashurov for a useful discussion on the article.

References

[1] Dzherbashian M., Integral Transforms and Representation of Functions in the Complex Domain. Nauka, Moscow, (1966).

[2] Hilfer R., Applications of Fractional Calculs in Physics. World Scientific, Singapore, (2000).

[3] Podlubny I., Fractional Differential Equations. Academic, New York, (1999).

[4] Kilbas A., Srivastava H., Trujillo J., Theory and Applications of Fractional Differential Equations. Elsevier, (2006).

[5] Furati K., Kassim N., Tatar N., Existence and uniqueness for a problem involving Hilfer fractional derivative. Comput. Math. Appl. (64), P. 1616–1626.

[6] Hilfer R., Tomovski Z., Srivastava H., Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions. Integr. Transforms Spec. Funct. (2010) Vol. 21, P. 797–814.

[7] Gu H., Trujilo J., Exictense of mild solution for evolution equation with Hilfer fractional derivative. Appl. Math. Comput. (2015) Vol. 257, P. 344–354.

[8] Mainardi F., Carpinteri A., Fractional calculus: some basic problems in continuum and statistical mechanics. Fractals and Fractional Calculus in Continuum Mechanics. Springer, (1997).

[9] Samko S., Kilbas A., Marichev O., Fractional Integrals and derivatives: Theory and Applications. Gordon&Breach, Yverdon, (1993).

[10] Hilfer R., Threefold introduction to fractional derivatives. Anomalous Transport: Foundations and Applications. Wiley-VCH, (2007).

[11] Hilfer R., Luchko Y., Tomovski Z., Operational method for solution of the fractional differential equations with the generalized Riemann-Liouville fractional derivatives. Fractional Calculus and Applied Analysis, (12), Iss. 3, P. 299–318.

[12] Gorenflo R., Kilbas A. A., Mainardi F., Rogozin S. V., Mittag-Leffler functions, related topics and applications. Springer, (2014).

[13] Wang J., Zhang Y., Nonlocal initial value problems for differential equations with Hilfer fractional derivative. Appl. Math. Comput. (2015) Vol. 266, P. 850–859.

[14] Ashurov R., Fayziev Y., Tukhtaeva N., On the nonlocal problem for the equation with the Hilfer fractional derivative. Lobachevskii journal of mathematics, (45), Iss. 3, P. 949–960.

[15] Karimov E., Sobirov Z., Khujakulov J., A non-local problem for a time-fractional differential equation with the Hilfer operator on metric graph. Math. Meth. Appl. Sci. (2024),

[16] Tariboon J., Samadi A., Ntouyas S., Nonlocal boundary value problems for Hilfer generalized proportional fractional differential equations. Fractal Fract. (2022) Vol. 6, Iss. 3, P. 154.

[17] Lizama C., Abstract linear fractional evolution equations. Handbook of Fractional Calculus with Applications. (2019).

[18] Karimov E., Turdiev K., Direct and inverse source problems for sub-diffusion equation involving generalized Hilfer derivative with a non-classical boundary condition. Bull. Inst. Math. (2022) Vol. 5, Iss. 5, P. 53–59.

[19] Karimov E., Toshtemirov B., Nonlocal boundary value problem for a mixed-type equation involving the bi-ordinal Hilfer fractional differential operators. Uzbek Math. J. (2021) Vol. 65, Iss. 2, P. 61–77.

[20] Yuldashev T. K., Kadirkulov B. J., Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator. Ural Math. J. (2020), Iss. 6, P. 153–167.

[21] Furati K., Iyiola O., Kirane M., An inverse problem for a generalized fractional diffusion. Appl. Math. Comput. (2014), Iss. 249, P. 24–31.

[22] Malik S., Aziz S., An inverse source problem for a two parameter anomalous diffusion equation with non-local boundary conditions. Comput. Math. Appl. (2017) Vol. 73, P. 2548–2560.

[23] Ashurov R., Fayziev Y., On the Nonlocal Problems in Time for Time-Fractional Subdiffusion Equations. Fractal Fract. (2022) Vol. 6, P. 41.

[24] Ashurov R., Cabada A., Turmetov B., Operator method for construction of solutions of linear fractional differential equations with constant coefficients. Fractional Calculus and Applied Analysis, (2016), Iss. 1, P. 229–252.

[25] Pskhu A., Fractional partial differential equations. NAUKA, Russian, (2005).

Fayziev Yusuf,
National University of Uzbekistan named after, Mirzo Ulugbek, Tashkent, Uzbekistan.,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan,
Karshi State University, Karshi, Uzbekistan.
e-mail: fayziev.yusuf@mail.ru

Sadullaeva Shahrizoda,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
e-mail: ssh45984@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 60–71
DOI: 10.29229/uzmj.2026-1-6

Cauchy problem for fractional high order equation with singular coefficient

Hasanov A., Yuldashova H.

Abstract. In this paper, we aim to study the Cauchy problem posed for a fractional order equation. General solution of the time-fractional equation is found by the Fourier method. The solution to the given problem is shown to be unique using the generalized Hankel transform. The solution consists of the Bessel function and the Mittag-Leffler function. Unknown coefficients are found by Hankel transformation. It is shown that the constructed solution satisfies the initial condition and equation.

Keywords: Cauchy problem, Riemann–Liouville fractional operator, Hankel transform, Mittag-Leffler function, Bessel function
MSC (2020): 33C10, 33E12, 35A22, 35B30, 35R11

1. Introduction

Third-order partial differential equations are considered when solving problems in the theory of nonlinear acoustics and in the hydrodynamic theory of space plasma and fluid filtration in porous media [1]. In total, all third-order equations occupy a special place due to their specific nature, equations with multiple characteristics. In [2], [3], taking into account the properties of viscosity and thermal conductivity of the gas, a third-order equation with multiple characteristics was obtained from the Navier-Stokes system, containing the second derivative with respect to time

$$u_{xxx} + u_{yy} - \frac{\nu}{y} u_y = u_x u_{xx}, \ \ \nu = const.$$

This equation at $\nu = 1$ describes an axisymmetric flow, and at $\nu = 0$ describes a plane-parallel flow [4].

The first results on a third-order equation with multiple characteristics were obtained in the works of H. Block [5] and E. DelVecehio [6]. In [7] and [8], fundamental solutions of third-order equations with multiple characteristics were constructed, containing second derivatives with respect to time, expressed through degenerate hypergeometric functions, their properties were studied, and estimates for $|t| \to \infty$ were found. In works [9] and [10], boundary value problems for third-order equations with multiple characteristics are considered using the constructed Green function. In recent years, interest in degenerate and singular equations has grown significantly, including equations containing the Bessel differential operator. These equations are often encountered in applications, for example, in problems with axial symmetry in continuum mechanics. Interest in problems related to the Bessel operator is also known from fundamental physics. This is due to its numerous applications in gas dynamics, shell theory, magnetohydrodynamics, and other fields of science and technology [11]. A special place in the theory of degenerate and singular equations is occupied by equations containing the Bessel differential operator

$$B_\nu = x^{-\nu} \frac{d}{dx}\left(x^\nu \frac{d}{dx}\right) = \frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}$$

According to the terminology by the Voronezh mathematician Ivan Aleksandrovich Kipriyanov, equations of three main classes containing the Bessel operator are called B-elliptic, B-hyperbolic, and B-parabolic, respectively. The monograph [12] studies boundary value problems for B-elliptic equations, in addition to this, the account of multi-dimension integral Fourier-Bessel-Hankel transformation theory is given in the monograph. The final chapters contain new results on general weight boundary value problems for singular B-elliptic and B-parabolic equations where parameter may be complex. There it is shown how spectral characteristics of B-elliptic operators including kernels of fractional powers are produced. The theory of boundary value problems for the equations with peculiarity has been reflected there, while the study of B-hyperbolic equations is presented in the monograph by R. Carroll and R. Showalter [13] and of B-parabolic ones, in the monograph by M.I. Matiichuk [14]. A wide range of questions for equations with Bessel operators was studied by I.A. Kipriyanov [15], [12] and his students L.A. Ivanov [16], V.V. Katrakhov [17], [18], [19], [20], V.I. Kononenko [21], L.N. Lyakhov [22], A.B. Muravnik [23], I.P. Polovinkin [24], S.M. Sitnik [25], [26], E.L. Shishkina [27], [28] and others. Mixed problems with integral conditions for hyperbolic equations with the Bessel operator were studied by N. V. Zaitseva [11].

Until now, many mathematicians have conducted many scientific researches on third-order partial differential equations. In contrast to them, in this article, we solve the Cauchy problem for the high order fractional differential equation of the composed type using the Hankel transformation method.

2. Definition of the Hankel Transform

Hermann Hankel is remembered for his numerous contributions to mathematical analysis including the Hankel transformation, which occurs in the study of functions, which depend only on the distance from the origin. The Hankel transform involving Bessel functions as the kernel arises naturally in axisymmetric problems formulated in cylindrical polar coordinates.

By authors Lokenath Debnath and Dambaru Bhatta in [29] engaged with the definition and basic operational properties of the Hankel transform. A large number of axisymmetric problems in cylindrical polar coordinates are solved with the aid of the Hankel transform. The use of the joint Laplace and Hankel transforms is illustrated by several examples of applications to partial differential equations [29]-[30].

Hankel transformation and other transformations are used to solve problems in mechanics, elasticity theory, thermal conductivity, electrodynamics and other branches of theoretical physics. More detailed information about the Hankel transform can be found in [31]-[32].

The Hankel transform is an integral transform and was first developed by the German mathematician Hermann Hankel (1839–1873). It is also known as the Fourier–Bessel transform. Just as the Fourier transform for an infinite interval is related to the Fourier series over a finite interval, so the Hankel transform over an infinite interval is related to the Fourier–Bessel series over a finite interval. The Hankel transform expresses any given function $f(\xi)$ as the weighted sum of an infinite number of Bessel functions of the first kind $J_\nu(\eta\xi)$. The Bessel functions in the sum are all of the same order ν, but differ in a scaling factor η along the ξ axis. The necessary coefficient F_ν of each Bessel function in the sum, as a function of the scaling factor η constitutes the transformed function.

Definition 2.1. Let $f(r)$ be a function defined for $r \geq 0$. The ν^{th} order Hankel transform of $f(r)$ is defined as

$$F_\nu(\eta) = \int_0^\infty r f(r) J_\nu(\eta r)\, dr, \left(\nu > -\frac{1}{2}\right), \tag{2.1}$$

where $J_\nu(x)$ is well known Bessel function of the first kind and defined as

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\nu+1)\, n!} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{2.2}$$

Theorem 2.2. *If the function $f(x)$ piecewise continuous in any finite interval (has bounded variation) belonging to the interval $(0, \infty)$ and integral converges*

$$\int_0^\infty |f(\xi)| \sqrt{\xi} d\xi,$$

then the Hankel transform exists and the inversion of the Hankel transform is given by the following formula

$$f(r) = \int_0^\infty \eta F_\nu(\eta) J_\nu(\eta r)\, d\eta. \tag{2.3}$$

Formulas (2.1) and (2.3) can be written in the following form

$$f(r) = \int_0^\infty J_\nu(r\xi)\, \xi d\xi \int_0^\infty f(\zeta) J_\nu(\xi\zeta)\, \zeta d\zeta = \int_0^\infty f(\zeta)\, \zeta d\zeta \int_0^\infty J_\nu(\xi\zeta) J_\nu(r\xi)\, \xi d\xi. \tag{2.4}$$

From relation (2.4), it follows

$$f(\eta) = \int_0^\infty a(\xi)\,\xi J_\nu(\eta\xi)\,d\xi, \tag{2.5}$$

where

$$a(\xi) = \int_0^\infty f(\rho)\,J_\nu(\rho\xi)\,\rho d\rho, \tag{2.6}$$

formulas (2.1), (2.3) and (2.4) are given in monographs [29]-[33].

From the properties of the Dirac delta function [34]-[35], it follows

$$\int_0^\infty \delta(x-a)\,\Phi(x)\,dx = \Phi(a), \tag{2.7}$$

$$\delta(x-a) = x\int_0^\infty tJ_\mu(xt)\,J_\mu(at)\,dt,\ \ |\mu| < \frac{1}{2}, \tag{2.8}$$

As can be seen from the above properties, we can conclude that the function $f(r)$ is compatible with Dirac delta function.

3. Cauchy problem for high order fractional differential equation

A real valued C^∞-smooth function on an open subset of $\mathbb{R}^n$ is called a Schwartz function, if it and all of its partial derivatives rapidly decay when approaching any boundary point of the subset, including ∞ if the subset is unbounded. The space of all Schwartz functions on a given subset $U \subset \mathbb{R}^n$ is a Fréchet space denoted by $\mathcal{S}(U)$, and is called the Schwartz space of U. Schwartz spaces were first introduced on $\mathbb{R}^n$ by Laurent Schwartz [36] and throughout the years were defined and studied in various contexts on various objects. First introduced in the first half of the 20^{th} century, Schwartz spaces still play an important role in many fields of mathematics, such as harmonic analysis, representation theory and number theory.

Let $\mathbb{N}$ be the set of non-negative integers, and for any $n \in \mathbb{N}$, let $\mathbb{N}^n$ be the n-fold Cartesian product.

Definition 3.1. The Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^n$ is the function space

$$\mathcal{S}(\mathbb{R}^n, \mathbf{C}) = \left\{ f \in C^\infty(\mathbb{R}^n, \mathbf{C}) | \forall p, q \in \mathbb{N}^n, ||f||_{p,q} < \infty \right\},$$

where $C^\infty(\mathbb{R}^n, \mathbf{C})$ is the function space of smooth functions from $\mathbb{R}^n$ into $\mathbf{C}$, and

$$||f||_{p,q} = \sup_{x\in\mathbb{R}^n} |x^p(D^q f)(x)|.$$

Here, sup denotes the supremum, and we used multi-index notation $x^p = x_1^{p_1} x_2^{p_2} ... x_n^{p_n}$ and $D^q = \partial_1^{q_1} \partial_2^{q_2} ... \partial_n^{q_n}$. To put common language to this definition, one could consider a rapidly decreasing function as essentially a function $f(x)$ such that $f(x), f'(x), f''(x), ...$ all exist everywhere on $\mathbb{R}$ and go to zero as $x \to \pm\infty$ faster than any reciprocal power of x. In particular, $\mathcal{S}(\mathbb{R}^n, \mathbf{C})$ is a subspace of the function space $C^\infty(\mathbb{R}^n, \mathbf{C})$ of smooth functions from $\mathbb{R}^n$ into $\mathbf{C}$.

3.1. **Statement of problem.** Let us look for a solution in the space $\mathcal{S}(\Omega)$ of the time-fractional order equation given in the domain $\Omega = \{(x,t) :\ x > 0,\ 0 < t < t_0\}$

$$^{RL}D_{0t}^{\alpha} u(x,t) = u_{xx}(x,t) + \frac{\nu}{x} u_x(x,t),\quad 0 < \nu < 1,\quad 2 < \alpha < 3, \tag{3.1}$$

satisfying initial conditions

$$\left. ^{RL}D_{0t}^{\alpha-1} u(x,t)\right|_{t=0} = \psi(x),\quad 0 \le x < \infty, \tag{3.2}$$

$$\left. ^{RL}D_{0t}^{\alpha-2} u(x,t)\right|_{t=0} = \varphi(x),\quad 0 \le x < \infty, \tag{3.3}$$

$$\left. ^{RL}D_{0t}^{\alpha-3} u(x,t)\right|_{t=0} = \tau(x),\quad 0 \le x < \infty, \tag{3.4}$$

where $\psi(x), \varphi(x)\ \tau(x) \in \mathbf{C}^2(0,\infty)$ and

$$\int_0^\infty |\psi(x)|\, x^{\frac{\nu}{2}} dx < c = const, \quad \int_0^\infty |\varphi(x)|\, x^{\frac{\nu}{2}} dx < c = const, \quad \int_0^\infty |\tau(x)|\, x^{\frac{\nu}{2}} dx < c = const$$

in addition to this

$$\psi(0) = \varphi(0) = \tau(0) = 0, \quad \lim_{x\to\infty} \psi(x) = 0, \ \lim_{x\to\infty} \varphi(x) = 0, \ \lim_{x\to\infty} \tau(x) = 0, \tag{3.5}$$

and for any fixed t we have

$$\lim_{x\to\infty} u(x,t) = 0, \tag{3.6}$$

$$\lim_{x\to 0} u(x,t) = 0, \tag{3.7}$$

where ${}^{RL}D_{0t}^{\alpha}$ is the Riemann–Liouville fractional derivative operator of order α defined by

$${}^{RL}D_{0t}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n \left\{{}^{RL}I_{0t}^{n-\alpha} f(t)\right\}, \quad \mathrm{Re}(\alpha) \geq 0, \ \ n = [\mathrm{Re}(\alpha)] + 1, \tag{3.8}$$

and

$${}^{RL}I_{0t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \quad t > 0, \ \ \mathrm{Re}(\alpha) \geq 0, \tag{3.9}$$

represents Riemann–Liouville fractional integral [32].

Using the generalized Hankel transform, we can show that the solution to the given problem is unique.

Definition 3.2. The generalized Hankel transform [37] of the function $f(r)$ is introduced as follows

$$\mathbf{J}_{a,b;\nu}[f(r)](\eta) = \eta^a \int_0^\infty f(r)\, r^a J_\nu\left(\eta^b r^b\right) dr = F_{a,b;\nu}(\eta), \left(\nu > -\frac{1}{2}\right), \tag{3.10}$$

where $a, b \in \mathbf{R}$, $b \neq 0$, $\eta \in \mathbf{R}_+$.

The generalized Hankel transformation (3.10) is related to the Hankel transformation (2.1) by the equality

$$F_{a,b;\nu}(\eta) = \frac{1}{|b|} \eta^a H_\nu[f(r^{1/b}) r^{-2+(a+1)/b}](\eta^b). \tag{3.11}$$

Relation (3.11) also allows us to obtain the inversion formula

$$f(r) = b^2 r^{2b-a-1} \int_0^\infty F_{a,b;\nu}(\eta)\, \eta^{2b-a-1} J_\nu\left(\eta^b r^b\right) d\eta, \tag{3.12}$$

for the transformation (3.10) of the function $f(\tau)$ from the class $\mathcal{S}(\Omega)$ with weight $\tau^{a-b/2}$, which has bounded variation in the near of the point r.

3.2. **Uniqueness of solution.** Suppose that there are two solution $u_1(x,t)$ and $u_2(x,t)$ to the Cauchy problem (3.1)-(3.4), denote

$$u(x,t) = u_1(x,t) - u_2(x,t).$$

Then, the function $u(x,t)$ clearly satisfy equation (3.1), the conditions in (3.6)-(3.7) and the homogeneous conditions

$${}^{RL}D_{0t}^{\alpha-1} u(x,t)\big|_{t=0} = 0, \quad {}^{RL}D_{0t}^{\alpha-2} u(x,t)\big|_{t=0} = 0, \quad {}^{RL}D_{0t}^{\alpha-3} u(x,t)\big|_{t=0} = 0, \ \ 0 \leq x < \infty. \tag{3.13}$$

Let us the generalized Hankel transform of the function $u(x,t)$ introduce the following

$$U(t,\mu) = \mu^{\frac{1+\nu}{2}} \int_0^\infty u(x,t) x^{\frac{1+\nu}{2}} J_{\frac{1-\nu}{2}}(\mu x)\, dx, \ \ (\mu \in \mathbf{R}\setminus\{0\}). \tag{3.14}$$

Note that the homogeneous conditions in (3.13) lead to

$$ {}^{RL}D_{0t}^{\alpha-1}U\left(t,\mu\right)\big|_{t=0} = {}^{RL}D_{0t}^{\alpha-2}U\left(t,\mu\right)\big|_{t=0} = {}^{RL}D_{0t}^{\alpha-3}u\left(t,\mu\right)\big|_{t=0} = 0 \tag{3.15} $$

and Riemann–Liouville fractional derivative of the expression (3.14) gives

$$ {}^{RL}D_{0t}^{\alpha}U(t,\mu) = \mu^{\frac{1+\nu}{2}} \int_0^\infty x^{-\nu}\frac{\partial}{\partial x}(x^\nu u_x)x^{\frac{1+\nu}{2}}J_{\frac{1-\nu}{2}}\left(\mu x\right)dx, $$

which on integrating by parts and using the conditions in (3.6)-(3.7) and Bessel equation reduces to

$$ {}^{RL}D_{0t}^{\alpha}U\left(t,\mu\right) + \mu^2 U\left(t,\mu\right) = 0, \tag{3.16} $$

In [38], the solution of equation (3.16) is represented

$$ U\left(t,\mu\right) = a_1\left(\mu\right)t^{\alpha-1}E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right) + a_2\left(\mu\right)t^{\alpha-2}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right) + a_3\left(\mu\right)t^{\alpha-3}E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right) \tag{3.17} $$

where $a_i\left(\lambda\right),\ i=1,2,3$ are unknown coefficients and $E_{\alpha,\beta}\left(z\right)$ Mittag-Leffler function with two parameters

$$ E_{\alpha,\beta}\left(z\right) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma\left(\alpha n+\beta\right)},\ \left(\alpha,\beta\in\mathbb{C},\ \ \mathrm{Re}\left(\alpha\right)>0,\ \ \mathrm{Re}\left(\beta\right)>0\right). $$

By (3.17) we determine the unknown function $a_i\left(\mu\right),\ i=1,2,3.$ We obtain Riemann–Liouville fractional derivative of order $\alpha-1$ of the function (3.17)

$$ {}^{RL}D_{0t}^{\alpha-1}U\left(t,\mu\right) = a_1\left(\mu\right)\ E_{\alpha,1}\left(-\mu^2t^\alpha\right) - \mu^2a_2\left(\mu\right)t^{\alpha-1}E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right) - \mu^2a_3\left(\mu\right)t^{\alpha-2}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right) $$

Using the condition (3.15), we get ${}^{RL}D_{0t}^{\alpha-1}U\left(t,\mu\right)\big|_{t=0} = a_1\left(\mu\right) = 0$ The Riemann-Liouville fractional derivative of order $\alpha-2$ $(2<\alpha<3)$ with respect to the variable from the function (3.17) has the following form

$$ {}^{RL}D_{0t}^{\alpha-2}U\left(t,\mu\right) = a_1\left(\mu\right)t\,E_{\alpha,2}\left(-\mu^2t^\alpha\right) + a_2\left(\mu\right)E_{\alpha,1}\left(-\mu^2t^\alpha\right) - \mu^2a_3\left(\mu\right)t^{\alpha-1}E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right). $$

Using the condition (3.15), we get ${}^{RL}D_{0t}^{\alpha-2}U\left(t,\mu\right)\big|_{t=0} = a_2\left(\mu\right) = 0.$ The Riemann-Liouville fractional derivative of order $\alpha-3$ $(2<\alpha<3)$ with respect to the variable from the function (3.17) has the following form

$$ {}^{RL}D_{0t}^{\alpha-3}U\left(t,\mu\right) = a_1\left(\mu\right)t^2E_{\alpha,3}\left(-\mu^2t^\alpha\right) + a_2\left(\mu\right)tE_{\alpha,2}\left(-\mu^2t^\alpha\right) + a_3\left(\mu\right)E_{\alpha,1}\left(-\mu^2t^\alpha\right). $$

Using the condition (3.15), we get ${}^{RL}D_{0t}^{\alpha-3}U\left(t,\mu\right)\big|_{t=0} = a_3\left(\mu\right) = 0$ imply that $U(t,\mu)\equiv 0.$ Therefore, due to the inversion of the generalized Hankel transform (3.12), we get $u(x,t)\equiv 0,\ u(x,t)\in\mathcal{S}(\Omega).$ This ends the proof of uniqueness of solution to the Cauchy problem (3.1)-(3.4).

3.3. **Existence of Solution.**

Theorem 3.3. *If* $2<\alpha<3, \quad 0<\nu<1,$ *then the Cauchy problem* (3.1)-(3.4) *has the following solution defined in Schwartz space*

$$ u\left(x,t\right) = x^{\frac{1-\nu}{2}}t^{\alpha-1}\int_0^\infty\int_0^\infty \rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)\rho\mu $$

$$ \times\left[\psi\left(\rho\right)\ E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right) + \varphi\left(\rho\right)t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right) + \tau\left(\rho\right)t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right)\right]d\rho d\mu \tag{3.18} $$

and this solution satisfied conditions (3.5)-(3.7).

Proof. We look for the solution of equation (3.1) using the Fourier method

$$u(x,t) = X(x)\,T(t)\,. \tag{3.19}$$

Substituting (3.19) in equation (3.1), we have two equations

$$X_{xx} + \frac{\nu}{x}X_x + \mu^2 X = 0, \tag{3.20}$$

$${}^{RL}D_{0t}^{\alpha} T(t) + \mu^2 T(t) = 0, \ \ \mu \in \mathbb{R}\setminus\{0\} \tag{3.21}$$

By substituting the product $X = x^{\frac{1-\nu}{2}}\theta(x\mu)$ into (3.20), we get the following Bessel equation

$$(x\mu)^2\theta_{xx}(x\mu) + x\mu\theta_x(x\mu) + \left((x\mu)^2 - \frac{(\nu-1)^2}{4}\right)\theta(x\mu) = 0.$$

We know that, when $\frac{\nu-1}{2}$ is not integer the functions $J_{\frac{\nu-1}{2}}(\mu x)$ and $J_{\frac{1-\nu}{2}}(\mu x)$ are linear independent solutions of above Bessel equation [39]. From conditions (3.5), we use only the function $J_{\frac{1-\nu}{2}}(\mu x)$. Then the solution of equation (3.20) has the following form

$$X = c(\mu)\,x^{\frac{1-\nu}{2}} J_{\frac{1-\nu}{2}}(\mu x)\,, \tag{3.22}$$

where $J_\mu(z)$-Bessel function of the first kind [34] defined as (2.2).

Using the solutions (3.22) and (3.17), we can write the general solution of the equation (3.1) as follows

$$u(x,t) = x^{\frac{1-\nu}{2}} t^{\alpha-1}\int\limits_0^\infty C_1(\mu)\,E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right) J_{\frac{1-\nu}{2}}(\mu x)\,d\mu$$

$$+x^{\frac{1-\nu}{2}} t^{\alpha-2}\int\limits_0^\infty C_2(\mu)\,E_{\alpha,\alpha-1}\left(-\mu^2 t^\alpha\right) J_{\frac{1-\nu}{2}}(\mu x)\,d\mu$$

$$+x^{\frac{1-\nu}{2}} t^{\alpha-3}\int\limits_0^\infty C_3(\mu)\,E_{\alpha,\alpha-2}\left(-\mu^2 t^\alpha\right) J_{\frac{1-\nu}{2}}(\mu x)\,d\mu,$$

or

$$u(x,t) = x^{\frac{1-\nu}{2}} t^{\alpha-1}\int\limits_0^\infty J_{\frac{1-\nu}{2}}(\mu x)\,\times$$

$$\times\left[C_1(\mu)\,E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right) + C_2(\mu)\,t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2 t^\alpha\right) + C_3(\mu)\,t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2 t^\alpha\right)\right]d\mu. \tag{3.23}$$

By (3.23) we determine the unknown function $C_i(\mu)$, $i = 1,2,3$. We obtain Riemann–Liouville fractional derivative of order $\alpha - 1$ of the function (3.23)

$${}^{RL}D_{0t}^{\alpha-1}u(x,t) = x^{\frac{1-\nu}{2}}\int\limits_0^\infty C_1(\mu)\,E_{\alpha,1}\left(-\mu^2 t^\alpha\right) J_{\frac{1-\nu}{2}}(\mu x)\,d\mu$$

$$-\mu^2 x^{\frac{1-\nu}{2}} t^{\alpha-1}\int\limits_0^\infty C_2(\mu)\,E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right) J_{\frac{1-\nu}{2}}(\mu x)\,d\mu$$

$$-\mu^2 x^{\frac{1-\nu}{2}} t^{\alpha-2}\int\limits_0^\infty C_3(\mu)\,E_{\alpha,\alpha-1}\left(-\mu^2 t^\alpha\right) J_{\frac{1-\nu}{2}}(\mu x)\,d\mu$$

Considering the initial condition (3.2), we determine the unknown function $C_1(\mu)$.

$$ {}^{RL}D_{0t}^{\alpha-1}u\left(x,t\right)\Big|_{t=0} = x^{\frac{1-\nu}{2}}\int\limits_0^\infty C_1\left(\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu = \psi\left(x\right), $$

$$ \psi\left(x\right)x^{\frac{\nu-1}{2}} = \int\limits_0^\infty \frac{C_1\left(\mu\right)}{\mu}\mu J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu\ , $$

$$ C_1\left(\mu\right) = \mu\int\limits_0^\infty \psi\left(\rho\right)\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)\rho d\rho. \tag{3.24} $$

We obtain Riemann–Liouville fractional derivative of order $\alpha-2$ of the function (3.23)

$$ {}^{RL}D_{0t}^{\alpha-2}u\left(x,t\right) = x^{\frac{1-\nu}{2}}t\int\limits_0^\infty C_1\left(\mu\right)E_{\alpha,2}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu $$

$$ +x^{\frac{1-\nu}{2}}\int\limits_0^\infty C_2\left(\mu\right)E_{\alpha,1}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu $$

$$ -\mu^2x^{\frac{1-\nu}{2}}t^{\alpha-1}\int\limits_0^\infty C_3\left(\mu\right)E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu $$

Considering the initial condition (3.3), we determine the unknown function $C_2\left(\mu\right)$.

$$ {}^{RL}D_{0t}^{\alpha-2}u\left(x,t\right)\Big|_{t=0} = x^{\frac{1-\nu}{2}}\int\limits_0^\infty C_2\left(\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu = \varphi\left(x\right), $$

$$ \varphi\left(x\right)x^{\frac{\nu-1}{2}} = \int\limits_0^\infty \frac{C_2\left(\mu\right)}{\mu}\mu J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu, $$

$$ C_2\left(\mu\right) = \mu\int\limits_0^\infty \varphi\left(\rho\right)\rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}\left(\rho\mu\right)\rho d\rho, \tag{3.25} $$

We obtain Riemann–Liouville fractional derivative of order $\alpha-3$ of the function (3.23)

$$ {}^{RL}D_{0t}^{\alpha-3}u\left(x,t\right) = x^{\frac{1-\nu}{2}}t^2\int\limits_0^\infty C_1\left(\mu\right)E_{\alpha,3}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu $$

$$ +x^{\frac{1-\nu}{2}}t\int\limits_0^\infty C_2\left(\mu\right)E_{\alpha,2}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu $$

$$ +x^{\frac{1-\nu}{2}}\int\limits_0^\infty C_3\left(\mu\right)E_{\alpha,1}\left(-\mu^2t^\alpha\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu. $$

Considering the initial condition (3.4), we determine the unknown function $C_3\left(\mu\right)$.

$$ {}^{RL}D_{0t}^{\alpha-3}u\left(x,t\right)\Big|_{t=0} = x^{\frac{1-\nu}{2}}\int\limits_0^\infty C_3\left(\mu\right)J_{\frac{1-\nu}{2}}\left(\mu x\right)d\mu = \tau\left(x\right), $$

$$\tau(x)\, x^{\frac{\nu-1}{2}} = \int_0^\infty \frac{C_3(\mu)}{\mu} \mu J_{\frac{1-\nu}{2}}(\mu x)\, d\mu,$$

$$C_3(\mu) = \mu \int_0^\infty \tau(\rho)\, \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\, \rho d\rho. \tag{3.26}$$

Substituting (3.24), (3.25) and (3.26) in (3.23), we get the solution (3.18) of the Cauchy problem. First, let us show that the constructed function (3.18) satisfies equation (3.1). From the definition of Riemann–Liouville fractional derivative operator of order α (3.8), it is clear that

$$^{RL}D_{0t}^{\alpha} u(x,t) = -x^{\frac{1-\nu}{2}} t^{\alpha-1} \int_0^\infty \int_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\, J_{\frac{1-\nu}{2}}(\mu x)\, \rho\mu^3$$

$$\times \left[\psi(\rho)\ E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right) + \varphi(\rho)\, t^{-1} E_{\alpha,\alpha-1}\left(-\mu^2 t^\alpha\right) + \tau(\rho)\, t^{-2} E_{\alpha,\alpha-2}\left(-\mu^2 t^\alpha\right)\right] d\rho d\mu \tag{3.27}$$

We calculate the first derivative of the constructed function (3.18) with respect to the variable x.

$$u_x(x,t) = x^{\frac{1-\nu}{2}} t^{\alpha-1} \int_0^\infty \int_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu) \left[\frac{1-\nu}{2x} J_{\frac{1-\nu}{2}}(\mu x) + \mu J'_{\frac{1-\nu}{2}}(\mu x)\right] \rho\mu$$

$$\times \left[\psi(\rho)\ E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right) + \varphi(\rho)\, t^{-1} E_{\alpha,\alpha-1}\left(-\mu^2 t^\alpha\right) + \tau(\rho)\, t^{-2} E_{\alpha,\alpha-2}\left(-\mu^2 t^\alpha\right)\right] d\rho d\mu \tag{3.28}$$

Then, we calculate second derivative of the function (3.18) with respect to the variable x . We can make a slight simplification by using the fact that the function $J_{\frac{1-\nu}{2}}(\lambda x)$ are linear independent solution of Bessel equation

$$\lambda^2 J''_{\frac{1-\nu}{2}}(\lambda x) + x^{-1} \lambda J'_{\frac{1-\nu}{2}}(\lambda x) = \left(\frac{(1-\nu)^2}{4x^2} - \lambda^2\right) J_{\frac{1-\nu}{2}}(\lambda x)$$

Then, the second derivative of the function (3.18) with respect to the variable x is as follows

$$u_{xx}(x,t) = x^{\frac{1-\nu}{2}} t^{\alpha-1} \int_0^\infty \int_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\, \rho\mu \times$$

$$\times \left[\left(\frac{\nu^2-\nu}{2x^2} - \mu^2\right) J_{\frac{1-\nu}{2}}(\rho\mu)\, J_{\frac{1-\nu}{2}}(\mu x) - \frac{\nu\mu}{x} J_{\frac{1-\nu}{2}}(\rho\mu)\, J'_{\frac{1-\nu}{2}}(\mu x)\right]$$

$$\times \left[\psi(\rho)\ E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right) + \varphi(\rho)\, t^{-1} E_{\alpha,\alpha-1}\left(-\mu^2 t^\alpha\right) + \tau(\rho)\, t^{-2} E_{\alpha,\alpha-2}\left(-\mu^2 t^\alpha\right)\right] d\rho d\mu. \tag{3.29}$$

Further we can substitute the obtained derivatives (3.27), (3.28) and (3.29) in equation (3.1).

$$-x^{\frac{1-\nu}{2}} t^{\alpha-1} \int_0^\infty \int_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\, J_{\frac{1-\nu}{2}}(\mu x)\, \rho\mu^3$$

$$\times \left[\psi(\rho)\ E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right) + \varphi(\rho)\, t^{-1} E_{\alpha,\alpha-1}\left(-\mu^2 t^\alpha\right) + \tau(\rho)\, t^{-2} E_{\alpha,\alpha-2}\left(-\mu^2 t^\alpha\right)\right] d\rho d\mu$$

$$= x^{\frac{1-\nu}{2}} t^{\alpha-1} \int_0^\infty \int_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu) \left[\left(\frac{\nu^2-\nu}{2x^2} - \mu^2\right) J_{\frac{1-\nu}{2}}(\rho\mu)\, J_{\frac{1-\nu}{2}}(\mu x) - \frac{\nu\mu}{x} J_{\frac{1-\nu}{2}}(\rho\mu)\, J'_{\frac{1-\nu}{2}}(\mu x)\right] \rho\mu$$

$$\times \left[\psi(\rho)\ E_{\alpha,\alpha}\left(-\mu^2 t^\alpha\right) + \varphi(\rho)\, t^{-1} E_{\alpha,\alpha-1}\left(-\mu^2 t^\alpha\right) + \tau(\rho)\, t^{-2} E_{\alpha,\alpha-2}\left(-\mu^2 t^\alpha\right)\right] d\rho d\mu$$

$$+ \frac{\nu}{x} x^{\frac{1-\nu}{2}} t^{\alpha-1} \int_0^\infty \int_0^\infty \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu) \left[\frac{1-\nu}{2x} J_{\frac{1-\nu}{2}}(\mu x) + \mu J'_{\frac{1-\nu}{2}}(\mu x)\right] \rho\mu$$

$$\times\left[\psi(\rho)\ E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right)+\varphi(\rho)\,t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right)+\tau(\rho)\,t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right)\right]d\rho d\mu.$$

After some simplifications, we get

$$-\int\limits_0^\infty\int\limits_0^\infty \rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x)\,\rho\mu^3$$

$$\times\left[\psi(\rho)\ E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right)+\varphi(\rho)\,t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right)+\tau(\rho)\,t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right)\right]d\rho d\mu$$

$$=-\int\limits_0^\infty\int\limits_0^\infty \rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x)\,\rho\mu\mu^2$$

$$\times\left[\psi(\rho)\ E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right)+\varphi(\rho)\,t^{-1}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right)+\tau(\rho)\,t^{-2}E_{\alpha,\alpha-2}\left(-\mu^2t^\alpha\right)\right]d\rho d\mu.$$

We can conclude that the solution (3.18) satisfies equation (3.1).

Let us show that the constructed function (3.18) satisfies the initial conditions (3.2), (3.3) and (3.4). The Riemann-Liouville fractional derivative of order $\alpha-1$ $(2<\alpha<3)$ with respect to the variable from the constructed function (3.18) has the following form.

$${}^{RL}D_{0t}^{\alpha-1}u(x,t)=x^{\frac{1-\nu}{2}}\int\limits_0^\infty\int\limits_0^\infty \rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x)\,\rho\mu$$

$$\times\left[\psi(\rho)\ E_{\alpha,1}\left(-\mu^2t^\alpha\right)-\mu^2\varphi(\rho)\,t^{\alpha-1}E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right)-\mu^2\tau(\rho)\,t^{\alpha-2}E_{\alpha,\alpha-1}\left(-\mu^2t^\alpha\right)\right]d\rho d\mu$$

Using the condition (3.2) and the properties of the Dirac delta function (2.7), (2.8), we obtain

$${}^{RL}D_{0t}^{\alpha-1}u(x,t)\big|_{t=0}=x^{\frac{1-\nu}{2}}\int\limits_0^\infty\int\limits_0^\infty \psi(\rho)\ \rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x)\,\rho\mu d\rho d\mu$$

$$=x^{\frac{1-\nu}{2}}\int\limits_0^\infty \rho^{\frac{\nu-1}{2}}\psi(\rho)\,\delta(x-\rho)\,d\rho=x^{\frac{1-\nu}{2}}x^{\frac{\nu-1}{2}}\psi(x)=\psi(x).$$

The Riemann-Liouville fractional derivative of order $\alpha-2$ $(2<\alpha<3)$ with respect to the variable from the constructed function (3.18) has the following form

$${}^{RL}D_{0t}^{\alpha-2}u(x,t)=x^{\frac{1-\nu}{2}}\int\limits_0^\infty\int\limits_0^\infty \rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x)\,\rho\mu$$

$$\times\left[\psi(\rho)\,t\,E_{\alpha,2}\left(-\mu^2t^\alpha\right)+\varphi(\rho)\,E_{\alpha,1}\left(-\mu^2t^\alpha\right)-\mu^2\tau(\rho)\,t^{\alpha-1}E_{\alpha,\alpha}\left(-\mu^2t^\alpha\right)\right]d\rho d\mu.$$

Using the condition (3.3) and the properties of the Dirac delta function (2.7), (2.8), we obtain

$${}^{RL}D_{0t}^{\alpha-2}u(x,t)\big|_{t=0}=x^{\frac{1-\nu}{2}}\int\limits_0^\infty\int\limits_0^\infty \varphi(\rho)\ \rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x)\,\rho\mu d\rho d\mu$$

$$=x^{\frac{1-\nu}{2}}\int\limits_0^\infty \rho^{\frac{\nu-1}{2}}\varphi(\rho)\,\delta(x-\rho)\,d\rho=x^{\frac{1-\nu}{2}}x^{\frac{\nu-1}{2}}\varphi(x)=\varphi(x).$$

The Riemann-Liouville fractional derivative of order $\alpha-3$ $(2<\alpha<3)$ with respect to the variable from the constructed function (3.18) has the following form

$${}^{RL}D_{0t}^{\alpha-3}u(x,t)=x^{\frac{1-\nu}{2}}\int\limits_0^\infty\int\limits_0^\infty \rho^{\frac{\nu-1}{2}}J_{\frac{1-\nu}{2}}(\rho\mu)\,J_{\frac{1-\nu}{2}}(\mu x)\,\rho\mu$$

$$\times \left[\psi(\rho)\, t^2 E_{\alpha,3}\left(-\mu^2 t^\alpha\right) + \varphi(\rho)\, t E_{\alpha,2}\left(-\mu^2 t^\alpha\right) + \tau(\rho)\, E_{\alpha,1}\left(-\mu^2 t^\alpha\right)\right] d\rho d\mu.$$

Using the condition (3.4) and the properties of the Dirac delta function (2.7), (2.8), we obtain

$${}^{RL}D_{0t}^{\alpha-3} u(x,t)\big|_{t=0} = x^{\frac{1-\nu}{2}} \int\limits_0^\infty \int\limits_0^\infty \tau(\rho)\, \rho^{\frac{\nu-1}{2}} J_{\frac{1-\nu}{2}}(\rho\mu)\, J_{\frac{1-\nu}{2}}(\mu x)\, \rho\mu d\rho d\mu$$

$$= x^{\frac{1-\nu}{2}} \int\limits_0^\infty \rho^{\frac{\nu-1}{2}} \tau(\rho)\, \delta(x-\rho)\, d\rho = x^{\frac{1-\nu}{2}} x^{\frac{\nu-1}{2}} \tau(x) = \tau(x).$$

Consequently, the solution (3.18) satisfies the initial conditions (3.2), (3.3) and (3.4). Now we show that function (3.18) satisfies condition (3.6). For any fixed t, function $T(t)$ is bounded. Further, taking into account the asymptotic representation of the Bessel functions [39] for $z \to \infty$, with $|\arg(z)| < \pi$

$$J_\mu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi\mu}{2} - \frac{\pi}{4}\right),$$

then $X \sim c(\lambda)\, x^{-\frac{\nu}{2}}$ and in addition to this, it can be shown that $u(x,t) \in \mathcal{S}(\Omega)$. And this means that on the basis of (3.19) we are convinced that the condition (3.6) is satisfied. From this we can conclude that the integral (3.18) is convergent.This ends the proof of existence of solution to the Cauchy problem (3.1)-(3.4). Theorem 3.3 has been proved.

□

4. Acknowledgments

We would like to express our sincere gratitude to the anonymous referees for their invaluable comments and insightful feedback that improved the paper. This work does not have any conflicts of interest. There are no funders to report for this submission.

References

[1] Shxanukov M., O nekotorix krayevix zadachax dlya uravneniya tretego poryadka, voznikayushix pri modelirovanii filtratsii jidkosti v poristix sredax. Differens. uravneniya, (1982) Vol. 18, Iss. 4, P. 689–699.

[2] Rijov O., Asimptoticheskaya kartina obtekaniya tel vrasheniya so zvukovim potokom vyazkogo i teploprovodyashego gaza. Prikl. Matem. I mexan. (1965) Vol. 29, Iss. 6, P. 1004–1014.

[3] Napolitano L. D., Ryzhov O. S., An analogy between nonequilibrium and viscous inert flows at transonic velocities. Zh.Vychisl. Mat. Mat. Fiz. (1971) Vol. 11, Iss. 5, P. 1229–1261.

[4] Diyesperov V., O funksii Grina linearizovannogo vyazkogo transzvukovogo uravneniya. Jurnal vichisl. matem. i matem. fiziki, (1972) Vol. 12, Iss. 5, P. 1265–1279.

[5] Block H., Sur les equations lineaires aux derives parielles a carateristiques multiples. Ark. Mat. Astron. Fus. (1912) Vol. 7, Iss. 13, P. 1–34.

[6] Del Vicchio E., Sulleequazioni $z_{xxx} - z_y + \varphi_1(x,y) = 0$, $z_{xxx} - z_{yy} + \varphi_2(x,y) = 0$. Memorie R. Accad. Sci. (1915) Vol. 2, Iss. 66, P. 1–41.

[7] Djurayev T., Ob avtomodelnom reshenii odnogo uravneniya tretego poryadka s kratnimi xarakteristikami. estnik Sam. gos. texn. un-ta. Ser. Fiz.-mat. nauki, (2007) Vol. 2, Iss. 15, P. 18–26.

[8] Djurayev T., Apakov Y., K teorii uravneniya tretego poryadka s kratnimi xarakteristikami, soderjashego vtoruyu proizvodnuyu po vremeni. Ukrainskiy matematicheskiy jurnal, (2010) Vol. 62, Iss. 1, P. 40–51.

[9] Apakov Y. P., Construction of Green's Function for One Problem of Rectangular Region. Malaysian Journal of Mathematical Sciences, (2010) Vol. 4, Iss. 1, P. 1–16.

[10] Apakov Y. P., On a Method for Solving Boundary Problems for Third-order Equation with Multiple Characteristics. Modern Aspects of the Theory of Partial Differential Equations. Operator Theory: Advances and Applications. Springer, (2011), vol. 216, P. 65–78.

[11] Zaitseva N., Mixed problems with integral conditions for hyperbolic equations with the Bessel operator. Differential Equations, (2023) Vol. 59, Iss. 1, P. S1–S72.

[12] Kipriyanov I., Singular Elliptic Boundary Value Problems. Nauka, Moscow, (1997).

[13] Carroll R., Showalter R., Singular and Degenerate Cauchy Problems. Academic Press, New York, (1976).

[14] Matiichuk M., Parabolic Singular Boundary Value Problems. Inst. Mat. NAN Ukraini, Kyiv, (1999).

[15] Kipriyanov I., On boundary value problems for partial differential equations with the Bessel operator. Dokl. Akad. Nauk SSSR, (1964) Vol. 158, Iss. 2, P. 275–278.

[16] Kipriyanov I., Ivanov L., Fundamental solutions of homogeneous B-hyperbolic equations. Sib. Math. J. (1980) Vol. 21, Iss. 4, P. 552–557.

[17] Katrakhov V., General boundary value problems for one class of singular and degenerate equations. Dokl. Akad. Nauk SSSR, (1980) Vol. 251, Iss. 6, P. 1296–1300.

[18] Katrakhov V., General boundary value problems for one class of singular and degenerate elliptic equations. Math. USSR-Sb. (1981) Vol. 40, Iss. 3, P. 325–347.

[19] Katrakhov V., Sitnik S., Composition method for constructing B-elliptic, B-parabolic, and B-hyperbolic transformation operators. Dokl. Math. (1995) Vol. 50, Iss. 1, P. 70–77.

[20] Katrakhov V., Sitnik S., Transformation operator method and boundary value problems for singular elliptic equations. Sovrem. Mat. Fundam. Napravl. (2018) Vol. 64, Iss. 2, P. 211–426.

[21] Kipriyanov I., Kononenko V., Fundamental solutions of B-elliptic equations. Differ. Uravn. (1967) Vol. 3, Iss. 1, P. 114–129.

[22] Lyakhov L., B-Hypersingular Integrals and Applications to the Description of Kipriyanov Function Classes and to Integral Equations with B-Potential Kernels. Izd. LGPU, Lipetsk, (2007).

[23] Muravnik A., Functional differential parabolic equations: Integral transformations and qualitative properties of solutions of the Cauchy problem. J. Math. Sci. (2016) Vol. 216, Iss. 3, P. 345–496.

[24] Lyakhov L., Polovinkin I., Shishkina E., Formulas for the solution of the Cauchy problem for a singular wave equation with Bessel time operator. Dokl. Math. (2014) Vol. 90, Iss. 3, P. 737–742.

[25] Sitnik S., Application of Bushman–Erdelyi transformation operators and their generalizations in the theory of differential equations with singularities in the coefficients. Voronezh State Univ., Voronezh, (2016).

[26] Sitnik S., Shishkina E., Application of Bushman–Erdelyi transformation operators and their generalizations in the theory of differential equations with singularities in the coefficients. Transformation Operator Method for Differential Equations with Bessel Operators. Fizmatlit, Moscow, (2019).

[27] Shishkina E., The general Euler–Poisson–Darboux equation and hyperbolic B-potentials. Sovrem. Mat. Fundam. Napravl. (2019) Vol. 65, Iss. 2, P. 157–338.

[28] Fitouhi A., Jebabli I., Shishkina E., Sitnik S., Applications of the integral transforms composition method to wave-type singular differential equations and index shift transmutations. Electron. J. Differ. Equat. (2018) Vol. 130, P. 1–27.

[29] Lokenath D., Dambaru B., Integral transforms and their application. CRC Press Taylor and Francis Group, New York, (2015).

[30] Davies B., Integral transforms and their applications. Springer-Verlag, New York, (2001).

[31] Piessens R., The Hankel Transform. The Transforms and Applications Handbook: Second Edition. CRC Press LLC, (2000).

[32] Miller K., Ross B., An introduction to the fractional calculus and fractional differential equations. John Wiley and Sons, New York, (1993).

[33] Arena O., On a singular parabolic equation related to axially symmetric heat potentials. Ann. mat. Pura ed appl. (1975) Vol. 105, Iss. 4, P. 347–393.

[34] Arfken G., Weber H., Mathematical Methods for Physicists. Elsevier, Oxford, (2005).

[35] Bracewell R., The Fourier Transform and Its Applications. McGraw-Hill, New York, (2000).

[36] Schwartz L., Théorie des distributions. l'Institut de Mathmatique de l'Universit de Strasbourg, (1950).

[37] Britvina L. E., Generalized Hankel transformation, conditions of its existence, properties, convolutions. Bulletin of NSU, (2002), Iss. 22, P. 5–8.

[38] Kilbas A., Srivastava H., Trujillo J., Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam etc., (2006).

[39] Erdelyi A., Magnus W., Oberhettinger F., Tricomi F., Higher Transcendental Functions. McGraw-Hill, New York, Toronto and London, (1953).

Hasanov A.,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan.
e-mail: anvarhasanv@yahoo.com

Yuldashova H.A.,
National Research University Tashkent Institute of
Irrigation and Agricultural Mechanization Engineers
Tashkent, Uzbekistan.
e-mail: hilolayuldashova77@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 72–81
DOI: 10.29229/uzmj.2026-1-7

Differential game with slow pursuers on the 1-skeleton graph of the dodecahedron

Ibaydullaev T., Ibragimov G., Holboyev A., Muxammadjonov A.

Abstract. This work is devoted to study pursuit problems involving multiple pursuers and a single evader, all moving on the 1-skeleton graph of the dodecahedron. Solutions to the pursuit problems are provided, where the maximum speeds of the pursuers are less than the maximum speed of the evader. Conditions for the maximum speeds of the pursuers are obtained and a strategy for the pursuers to capture the evader is presented.

Keywords: Differential game, pursuers, evader, pursuit problem, 1-skeleton graph of the dodecahedron, slow evaders, number of pursuers
MSC (2020): 05C57; 91A43

1. Introduction

According to the fundamental principles of differential games developed by N.N. Krasovskii, L.S. Pontryagin, A.I. Subbotin, and others, a differential game is considered as a control problem from the perspective of either the pursuer or the evader [1, 2, 3, 4]. From this viewpoint, the game reduces either to a pursuit problem (pursuit) or to an evasion problem (evasion) [5, 6, 7, 8, 9, 10, 11].

Multi-pursuer differential games are of increasing interest (see, for example, [12, 13, 9, 14, 15, 16, 14, 17, 18, 19]).

There are several types of dynamic games on graphs. The first and most studied type is the class of games on abstract graphs. Using such games, well-known mathematicians such as M. Aigner, T. Andreae, A. Bonato, R.J. Nowakowski, M. Fromme, A. Quilliot and others have conducted research and obtained significant results [20, 21, 13, 22, 6, 23, 24, 9, 25]. A relatively narrower, but more interesting class of dynamic games on geometric graphs involves the classical "pursuer-evader" game with moving points moving along the edges of the graph. Dynamic games of this type have been studied by A.A. Azamov, A.Sh. Kuchkarov, G. Ibragimov, T. Ibaydullaev, and A.G. Holboyev, yielding interesting results [21, 14, 17, 18, 19, 26].

When considering a pursuit problem, we assume that the evader Q is initially located on an arbitrary edge of the dodecahedron, and we control the pursuers with the aim of capturing the evader. Therefore, we place the pursuers on the edges of the dodecahedron according to our objective and maneuver them based on a strategy aligned with this goal. The strategy of the evader is assumed to be arbitrary. On the other hand, when considering an evasion problem, we assume that initial positions of the pursuers P_i, $i = 1, 2, \ldots, m$, are arbitrary, and we control the evader with the aim of avoiding encounters with the pursuers. Therefore, we position the evader on the edges of the dodecahedron according to our objective and maneuver it based on a strategy corresponding to this goal. The strategies of the pursuers are assumed to be arbitrary.

1.1. Statement of problem.

Let

$$AA_1A_2A_3A_{11}A_{12}A_{21}A_{22}A_{31}A_{32}BB_1B_2B_3B_{11}B_{12}B_{21}B_{22}B_{31}B_{32}$$

be a dodecahedron with edges of unit length. We consider that a group of pursuers $\mathbf{P} = \{P_1, P_2, ..., P_m\}$ pursues a single evader Q (Fig.1) along these edges. These are typically called players. We assume that the players always see each other. We use the symbols P_i and Q to denote players and their positions on the edges of the dodecahedron. Naturally, the points P_i and Q are moving points that depend on time: $P_i = P_i(t), Q = Q(t)$. The states of the players at time $t = 0$ are called initial states or initial positions of the players. Clearly, these are given by $P_i(0)$, $Q(0)$, $i = 1, 2, ..., m$.

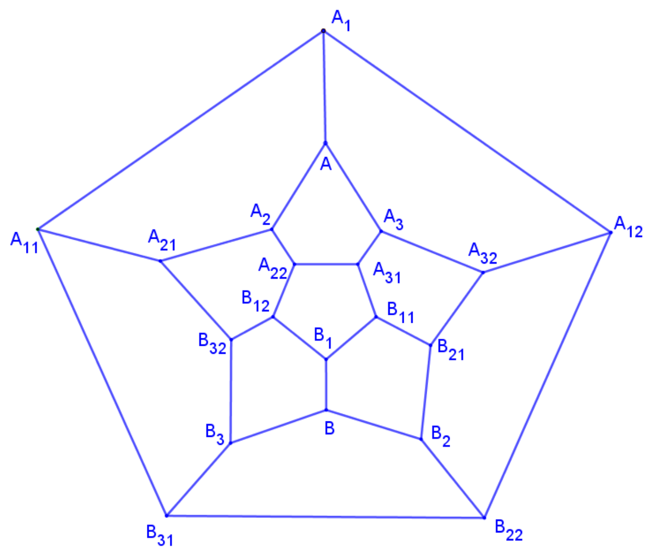

Figure 1. 1-skeleton graph of a dodecahedron.

Definition 1.1. If there exists a strategy for the group of pursuers $\mathbf{P} = \{P_1, P_2, ..., P_m\}$ such that, under this strategy, there is a time $T > 0$ for which, regardless of the evader's strategy and initial position on any edge of the dodecahedron, the equality $P_i(T) = Q(T)$ holds for some $i = 1, 2, ..., m$, then the pursuers are said to win the game (or equivalently, the game can be concluded in favor of the pursuers in finite time, or simply, the game is completed).

Here, T is called the guaranteed pursuit time.

Suppose that the players move simply along edges of the dodecahedron. Let the velocity vector of the evader be denoted by v and its maximum magnitude be denoted by σ. Likewise, let those of the pursuers be denoted by u_i and their respective maximum magnitudes by ρ_i: $|v| \le \sigma$, $|u_i| \le \rho_i$, $\sigma \ge \rho_1 \ge \rho_2 \ge ... \ge \rho_m > 0$.

2. Main result

According to the above, the motions of the players are given by

$$\dot{P}_i(t) = u_i, P_i(0) = P_{i0}, i = 1, 2, ..., m;$$

$$\dot{Q}(t) = v, Q(0) = Q_0.$$

Actually, $P_i, Q, u_i, v \in \mathbb{R}^3$, but since the game is considered on a graph, we can assume that $P_i, Q, u_i, v \in \mathbb{R}^2$.

Theorem 2.1. *If $\sigma = 1, \rho_1 = \frac{2}{3}, \rho_2 = \frac{2}{3}$ and $\rho_3 > 0$, then the group of pursuers $\boldsymbol{P} = \{P_1, P_2, P_3\}$ wins the game.*

Proof. Let the game begin from an arbitrary initial position. Since we control the pursuers, within some time moment $t = t_1$, we can move pursuer P_1 to vertex A, pursuer P_2 to vertex B and during this time interval, the third pursuer P_3 continuously pursues the evader Q (see Fig. 2a).

At this time, the evader Q is located at some vertex or edge of the dodecahedron.

Lemma 2.2. *Pursuer P_1 can guard the edges AA_1, AA_2, AA_3, emanating from vertex A.*

Proof. The pursuer P_1 and the evader Q can be in one of the following three states:

$$a)|AQ| > 2.5, \quad b)|AQ| = 2.5, \quad c)|AQ| < 2.5.$$

In each case, we will show how the pursuer P_1 moves and how it guards the edges AA_1, AA_2, AA_3. a) Let $|AQ| > 2.5$. The pursuer P_1 remains without moving at vertex A until the evader Q approaches within distance $|P_1Q| = 2.5$. If the evader Q reaches distance $|P_1Q| = 2.5$, then we continue with the case b).

b) Let $|AQ| = 2.5$. In this case, the evader Q is located at the midpoint of one of the following edges: $A_{11}B_{31}, A_{11}A_{21}, A_{21}B_{32}, A_{22}B_{12}, A_{22}A_{31}, A_{31}B_{11}, A_{32}B_{21}, A_{32}A_{12}, A_{12}B_{22}$ (Fig. 2b). $\square$

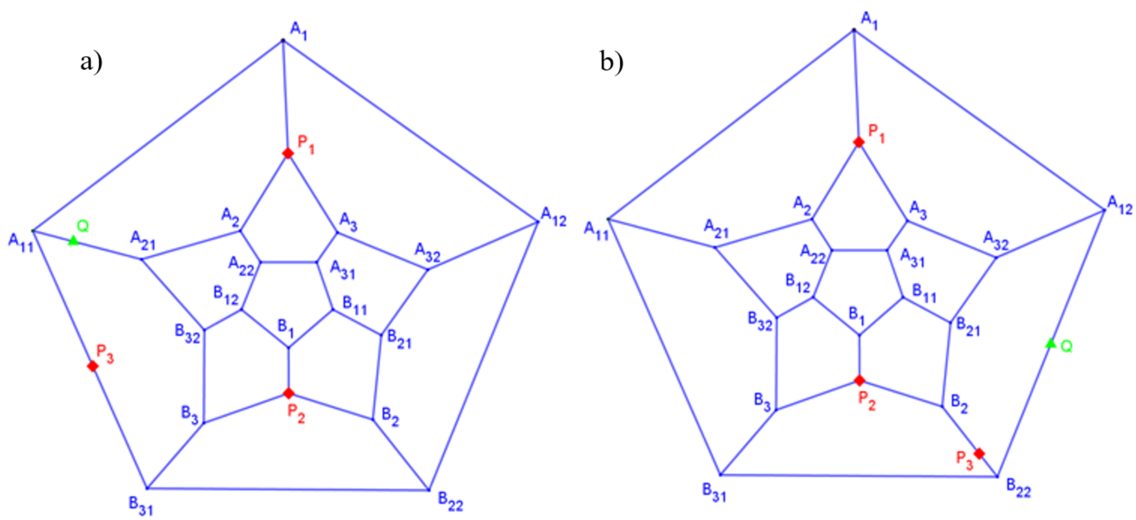

Figure 2. a) a case when $|AQ| > 2.5$; b) a case when $|AQ| = 2.5$.

If the evader Q moves toward a vertex B_{ij}, then the pursuer P_1 remains stationary at vertex A (we return to case a)).

If the evader Q moves toward a vertex A_{ij}, then the pursuer P_1 moves along the edge AA_i, maintaining the relation $3P_1A = 1 - 2QA_{ij}$ with the evader Q. When the evader Q reaches the vertex A_{ij}, the pursuer P_1 reaches a point located at $\frac{1}{3}$ of the edge AA_i from vertex A (see Fig. 3a).

Next, if the evader Q continues moving toward the vertex A_i, then the pursuer P_1 also moves along the edge AA_i toward the vertex A_i, maintaining the relation $3P_1A_i = 2QA_i$ with the evader Q. If the evader Q reaches the vertex A_i, then the pursuer P_1 simultaneously reaches the same vertex and captures the evader Q. If the evader Q chooses to move along a different edge without proceeding from A_{ij} to A_i, then the pursuer P_1 moves back toward vertex A. When the evader Q reaches the midpoint of an edge, the pursuer P_1 returns to vertex A, returning to the case b), where $|P_1Q| = 2.5$. Using the above strategy, the pursuer P_1 can guard edges AA_1, AA_2, AA_3.

c) Suppose that $|AQ| < 2.5$. In this case, the evader Q moves along the path $AA_iA_{ij}C$, i.e., along the broken line $AA_iA_{ij}C$. Here, the point C is the midpoint of an edge which is formed from vertex A_{ij} and satisfying the condition $|AC| = 2.5$ (Fig. 3b).

The pursuer P_1 moves toward the evader Q along the broken path $AA_iA_{ij}C$ until the relation $|QC| = 1.5|AP_1|$ is satisfied. As a result, the pursuer will either capture the evader Q or the condition $|QC| = 1.5|AP_1|$ will be satisfied (see Fig. 4a). Once the relation $|QC| = 1.5|AP_1|$ holds, the pursuer P_1 continues moving symmetrically with respect to the evader Q, just as in case b). Using this strategy, the pursuer P_1 maintains control over the edges AA_1, AA_2, AA_3.

Thus, it has been shown that the pursuer P_1 can guard the edges AA_1, AA_2, AA_3 emanating from from vertex A. Lemma 2.2 is proved.

According to the lemma 2.2, the pursuer P_2 can similarly guard the edges BB_1, BB_2, BB_3 emanating from vertex B.

The pursuer P_3 always pursues the evader Q. Due to the pursuit by P_3, the evader Q will be forced to move sequentially through the vertices of the dodecahedron. Assume that at time $t = t_2$, where $t_2 \geq t_1$, the evader Q is located at a vertex A_{ij} or B_{ij} of the dodecahedron. If the evader Q moves from vertex A_{ij} or B_{ij} to vertex A_i or B_i, respectively, then according to Lemma 2.2, the pursuer P_1 or P_2 will intercept and capture the evader accordingly. To avoid being captured by P_1 or P_2, the evader Q must move only through the vertices A_{ij} and B_{ij}. The path connecting these vertices consists of a single cycle: $A_{11}A_{21}B_{32}B_{12}A_{22}A_{31}B_{11}B_{21}A_{32}A_{12}B_{22}B_{31}A_{11}$ (Fig. 4.b)

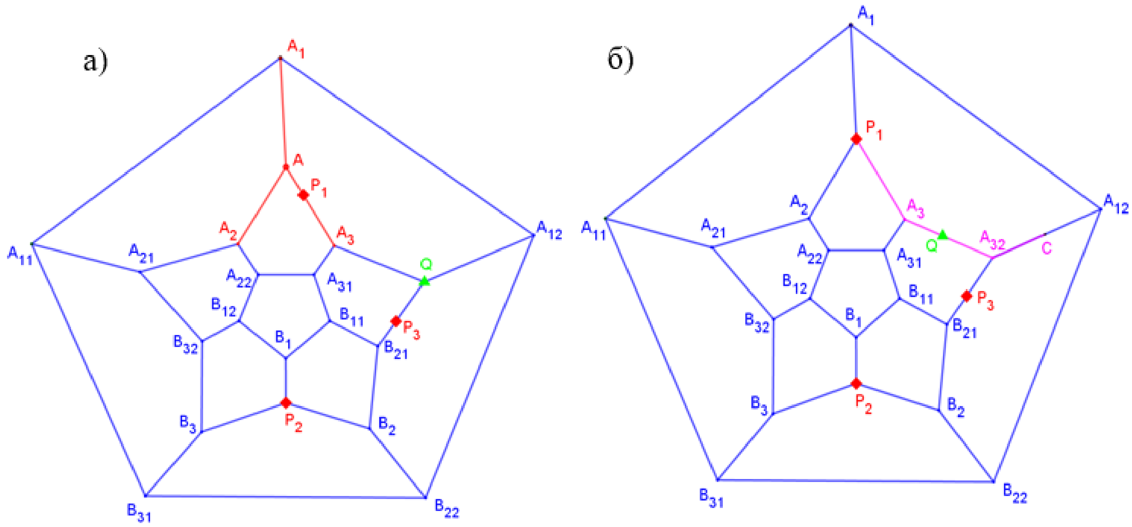

Figure 3. a) guarding the edges AA_1, AA_2, AA_3 by the pursuer P_1; b) a case when $|AQ| < 2.5$.

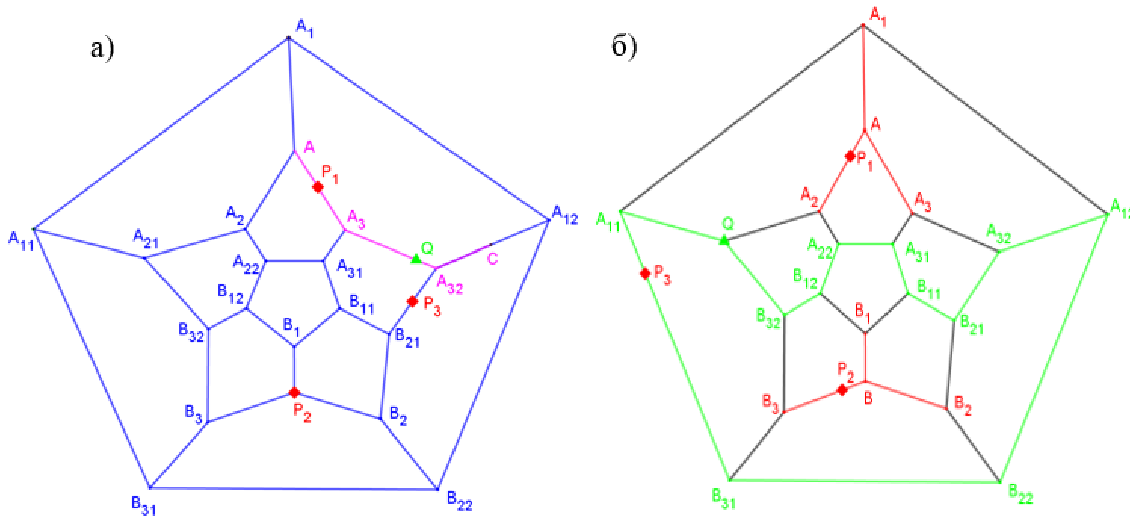

Figure 4. a) the condition $|QC| = 1.5|AP_1|$ will be satisfied; b) the cycle $A_{11}A_{21}B_{32}B_{12}A_{22}A_{31}B_{11}B_{21}A_{32}A_{12}B_{22}B_{31}A_{11}$.

The pursuer P_3 can pursue the evader Q either clockwise or counterclockwise along the cycle

$$A_{11} \to A_{21} \to B_{32} \to B_{12} \to A_{22} \to A_{31} \to B_{11} \to B_{21} \to A_{32} \to A_{12} \to B_{22} \to B_{31} \to A_{11}.$$

Suppose that the pursuer P_3 pursues the evader Q in the clockwise direction.

Let, at time $t = t_3$, where $t_3 \geq t_2$, the evader Q moves to the vertex B_{31} of the dodecahedron. At this moment, the pursuer P_1 is located at a point $\frac{1}{3}$ along the edge AA_1 from vertex A, while the pursuer P_2 is located at a point $\frac{1}{3}$ along the edge BB_3 from vertex B, and the pursuer P_3 is pursuing the evader Q (see Fig. 5a).

Starting from time $t > t_3$, the pursuers will move as follows:

The pursuer P_3 continues to pursue the evader Q along the cycle

$$A_{11} \to A_{21} \to B_{32} \to B_{12} \to A_{22} \to A_{31} \to B_{11} \to B_{21} \to A_{32} \to A_{12} \to B_{22} \to B_{31} \to A_{11}.$$

As a result, the evader Q should move from the vertex B_{31} to an adjacent vertex of the dodecahedron, otherwise the pursuer will catch the evader.

If the evader Q moves along the edge $B_{31}B_{22}$ toward the vertex B_{22}, then it will be captured by pursuer P_3.

If the evader Q moves toward the vertex B_3 along the edge $B_{31}B_3$, then according to the lemma, it will be captured by the pursuer P_2.

If the evader Q moves along the edge $B_{31}A_{11}$ toward the vertex A_{11}, then the pursuer P_1 follows the strategy described in Lemma 2.2, while the pursuer P_2 moves along the edge BB_3 toward the vertex B_3, maintaining the relation $3|P_2B_3| = 2|QA_{11}|$ with the evader Q. At time $t = t_4$, where $t_4 > t_3$, the evader Q reaches the vertex A_{11} of the dodecahedron, the pursuer P_1 is located at a point $\frac{1}{3}$ along the edge AA_1 from vertex A, and the pursuer P_2 is located at vertex B_3 (see Fig. 5b).

The pursuer P_3 continues to pursue the evader Q. The evader Q will again be forced to move to another vertex of the dodecahedron.

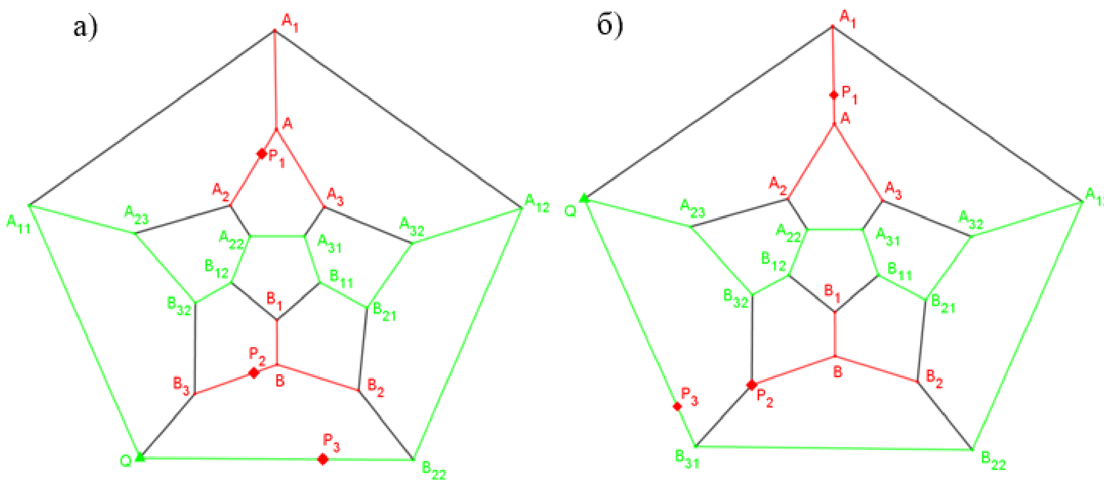

Figure 5. The process of trapping the evader.

If the evader Q moves along the edge $A_{11}B_{31}$ toward the vertex B_{31}, then it will be captured by the pursuer P_3.

If the evader Q moves toward the vertex A_1 along the edge $A_{11}A_1$, then according to the lemma 2.2, it will be captured by the pursuer P_1.

If the evader Q moves toward the vertex A_{21} along the edge $A_{11}A_{21}$, then the pursuer P_1 moves to a point $\frac{1}{3}$ along the edge AA_2 from vertex A, while the pursuer P_2 moves along the edge B_3B_{32} to a point $\frac{2}{3}$ along the edge B_3B_{32} from vertex B_3, maintaining the relation $3|P_2B_{32}| = 1 + 2|QA_{21}|$ with the evader Q. Then, at time $t = t_5$, where $t_5 > t_4$, the evader Q reaches the vertex A_{21} of the dodecahedron, the pursuer P_1 is located at a point $\frac{1}{3}$ along the edge AA_2 from vertex A, and the pursuer P_2 is located at a point $\frac{2}{3}$ along the edge B_3B_{32} from vertex B_3 (see Fig. 6).

At the next step, the evader Q will be captured by the pursuer P_3, if it moves from vertex A_{21} to the adjacent vertex A_{11}, the pursuer P_1, if it moves from vertex A_{21} to the adjacent vertex A_2, the pursuer P_2, if it moves from vertex A_{21} to the adjacent vertex B_{32}.

Theorem 2.1 is proved.

□

Theorem 2.3. *If $\sigma = 1$, $\rho_1 = \frac{1}{3}$, $\rho_2 = \frac{1}{3}$, $\rho_3 = \frac{1}{3}$, and $\rho_4 > 0$, then the group of pursuers $P = \{P_1, P_2, P_3, P_4\}$ wins the game.*

Proof. To prove the theorem, it suffices to construct a strategy for the pursuers such that $P_i(T) = Q(T), i = 1, 2, 3, 4$, holds in finite time $T > 0$, starting from an arbitrary initial position.

Since we control the pursuers, within some time $t = t_1$, we can move the pursuer P_1 to a point $\frac{1}{3}$ along the edge AA_1, measured from vertex A, the pursuer P_2 to a point $\frac{1}{3}$ along the edge B_2B_{21},

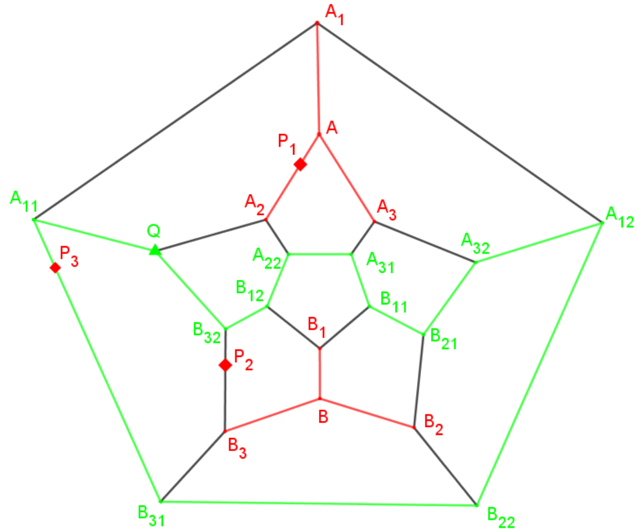

Figure 6. Critical moment in the pursuit with Q trapped between P_1, P_2 and P_3, ensuring eventual capture.

measured from vertex B_{21}, the pursuer P_3 to a point $\frac{1}{3}$ along the edge B_3B_{32}, measured from vertex B_{32}, while the fourth pursuer P_4 continuously pursues the evader Q during this time interval (see Fig. 7a).

From this moment on, the pursuers P_1, P_2, P_3 guard the edges $AA_1, B_2B_{21}, B_3B_{32}$, respectively, where they are positioned.

Lemma 2.4. *The pursuer P_1 can guard the edge AA_1.*

Proof. Let at time $t = t_1$, the pursuer P_1 be located at a point $\frac{1}{3}$ along the edge AA_1, measured from vertex A (see Fig. 7a). As a result of the pursuit by P_4, the evader Q is forced to move through the vertices of the dodecahedron; otherwise, it will be captured by P_4. We now show how the pursuer P_1 moves depending on the motion of the evader Q, such that P_1 maintains control over the edge AA_1.

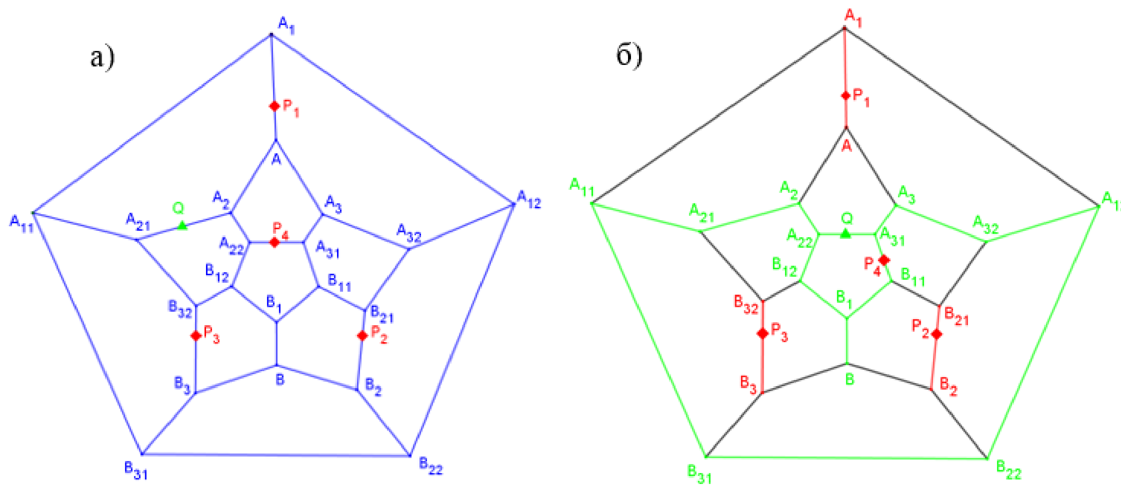

Figure 7. Guarding edges by P_1, P_2, P_3 pursuers.

I. If the evader Q moves from vertex A_2 or A_3 toward vertex A, then the pursuer P_1 also moves toward the same vertex A, maintaining the relation $3|P_1A| = |QA|$ with the evader Q. As a result, if the evader Q reaches vertex A, it will be captured by the pursuer P_1.

II. If the evader Q moves from some vertex toward vertex A_{11} or A_{12}, then the pursuer P_1 moves from the point $\frac{1}{3}$ along the edge AA_1 (measured from vertex A) toward the point $\frac{2}{3}$ along the same edge AA_1. If the evader Q reaches vertex A_{11} or A_{12}, then the pursuer P_1 reaches the point $\frac{2}{3}$ along the edge AA_1, measured from vertex A.

III. If the evader Q moves from some vertex toward vertex A_2 or A_3, then the pursuer P_1 moves from the point $\frac{2}{3}$ along the edge AA_1 (measured from vertex A) back toward the point $\frac{1}{3}$ along the same edge AA_1. If the evader Q reaches vertex A_2 or A_3, then the pursuer P_1 reaches the point $\frac{1}{3}$ along the edge AA_1, measured from vertex A.

IV. If the evader Q moves from vertex A_{11} or A_{12} toward vertex A_1, then the pursuer P_1 also moves toward the same vertex A_1, staying on the same line as the evader Q. As a result, the evader Q will be captured by the pursuer P_1 at vertex A_1.

V. In all other cases, the pursuer P_1 remains stationary.

Thus, it has been shown that if the evader Q reaches either vertex A or A_1 of the edge AA_1, it will be captured by the pursuer P_1 using the strategy described above.

Therefore, the pursuer P_1 can guard the edge AA_1. The edges AA_1, AA_2, AA_3, A_1A_{11}, A_1A_{12}, which share the vertices A and A_1, are highlighted with red lines (see Fig. 7b). □

Applying Lemma 2.4 to the pursuers P_2 and P_3, it can be shown that they can also take control of the edges B_2B_{21}, B_3B_{32} and adjacent edges (highlighted with red lines, see Fig. 7b).

The pursuer P_4 always pursues the evader Q. In order for evasion to occur, the evader Q must not be captured by any of the pursuers within an infinite time interval. As a result of the pursuit by P_4 and the application of Lemma 2.4 to the pursuers P_1, P_2, P_3, the evader Q is forced to move along the green-colored edges of the dodecahedron (see Fig. 7b). The part of the dodecahedron formed by the green edges consists of three cycles. Due to the pursuit by P_4, the evader Q may move either clockwise or counterclockwise along these cycles.

Assume that at time $t = t_2$, where $t_2 \geq t_1$, the evader Q is located at vertex A_{22} of the dodecahedron and the pursuer P_4 is pursuing along the edge $A_{22}A_{31}$ (see Fig. 8a).

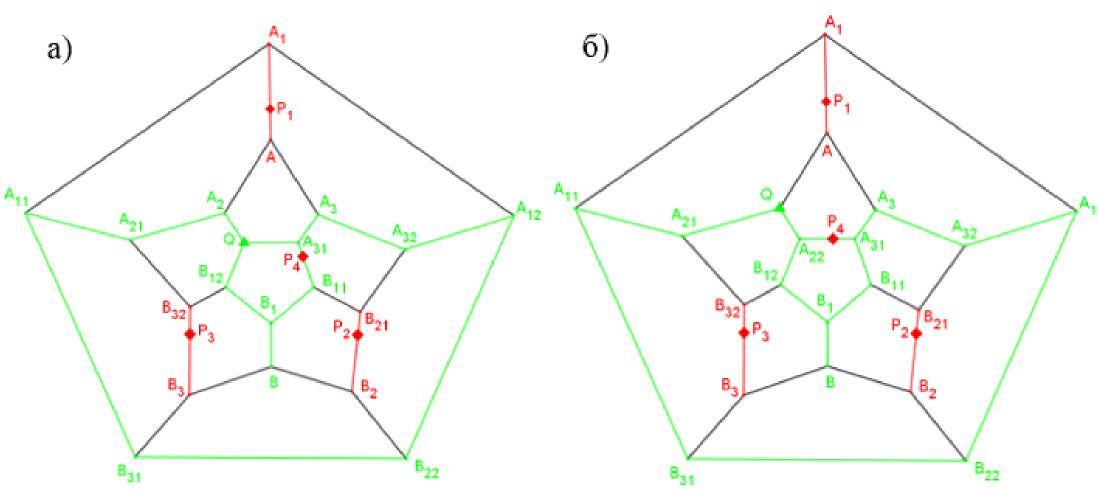

Figure 8. Cycles that the evader Q can move.

Starting from this moment, the evader Q can move along one of two cycles, since other cycles are guarded by the pursuers:

a) in the direction $A_{22} \to A_2 \to A_{21} \to A_{11} \to \ldots$;

b) in the direction $A_{22} \to B_{12} \to B_1 \to B_{11} \to \ldots$.

We will consider these two directions separately.

First, consider the case where the evader Q moves in the direction $A_{22} \to A_2 \to A_{21} \to A_{11} \to \ldots$.

Suppose that starting at time $t = t_2$, the evader Q begins to move toward vertex A_2 along the path $A_{22} \to A_2 \to A_{21} \to A_{11} \to \ldots$ and reaches A_2 at time $t = t_3$, where $t_3 \geq t_2$. At this moment, the pursuer P_1 is located at a point $\frac{1}{3}$ along the edge AA_1, measured from vertex A (see Fig. 8b).

Next, suppose that starting at time $t = t_3$, the evader Q moves toward vertex A_{21} along the edge A_2A_{21} and reaches it at time $t = t_4$, where $t_4 \geq t_3$. At this moment, the pursuer P_1 is located at a point $\frac{2}{3}$ along the edge AA_1, measured from vertex A and the pursuer P_3 is located at a point $\frac{1}{3}$ along the edge B_3B_{32}, measured from vertex B_{32} (see Fig. 9a).

In the same way, the pursuer P_4 continues to pursue the evader Q. If the evader Q moves from vertex A_{21} to vertex A_{11}, then the pursuer P_1 moves toward vertex A_1. As a result, when the evader Q arrives at vertex A_{11} at time $t = t_5$, where $t_5 \geq t_4$, the pursuer P_1 also arrives at vertex A_1 at the same time (see Fig. 9b).

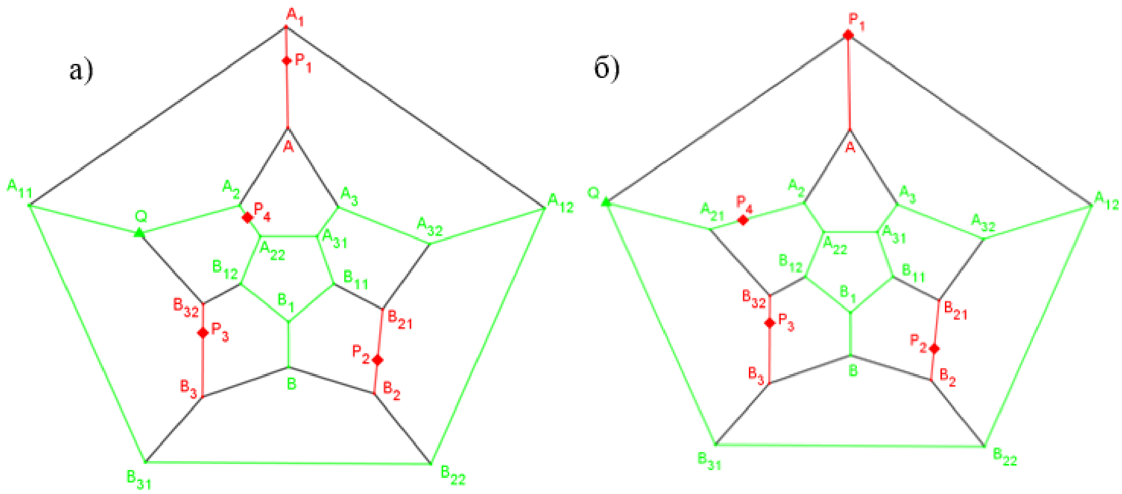

Figure 9. The process of trapping the evader.

If the evader Q moves from vertex A_{11} to vertex B_{31}, then the pursuer P_1 moves along the edge A_1A_{12} and the pursuer P_3 moves along the edge $B_{32}B_3$. If the evader Q reaches vertex B_{31} at time $t = t_6$, where $t_6 \geq t_5$, then the pursuer P_1 will be located at a point $\frac{1}{3}$ along the edge A_1A_{12}, measured from vertex A_1 and the pursuer P_3 will be located at a point $\frac{2}{3}$ along the edge B_3B_{32}, measured from vertex B_{32}, at that moment (see Fig. 10a).

As a result of the pursuit by P_4, if the evader Q continues its motion from vertex B_{31} to vertex B_{22}, then the pursuer P_1 continues moving along the edge A_1A_{12}. If the evader Q arrives at vertex B_{22} at time $t = t_7$, where $t_7 \geq t_6$, then the pursuer P_1 will be located at a point $\frac{2}{3}$ along the edge A_1A_{12}, measured from vertex A_1 and the pursuer P_2 will be located at a point $\frac{1}{3}$ along the edge $B_{21}B_2$, measured from vertex B_2, at that moment (see Fig. 10b).

Thus, as a result of the pursuit by the pursuers P_1, P_2, P_3 and P_4, the evader Q will be captured at time $t = t_8$, where $t_8 \geq t_7$.

If the evader Q moves in the direction $A_{22} \to B_{12} \to B_1 \to B_{11} \to \ldots$, then the pursuers, by using a strategy similar to the one used in the direction $A_{22} \to A_2 \to A_{21} \to A_{11} \to \ldots$, will also capture the evader.

Theorem 2.3 is proved. □

3. Conclusions

This paper investigates pursuit differential games on the 1-skeleton graph of a dodecahedron, focusing on scenarios where multiple pursuers aim to capture a single evader with higher maximum speed. The study provides rigorous strategies and conditions under which the pursuers can guarantee the capture of the evader in finite time, despite the evader's speed advantage. The result of the paper shows that reducing the speed of the pursuers causes the number of pursuers to increase. Moreover, the findings advance the understanding of differential games on polyhedral graphs and demonstrate how slower pursuers can overcome a faster evader through strategic positioning and collaboration. Future research could explore generalizations to other regular polyhedra, dynamic speed adjustments,

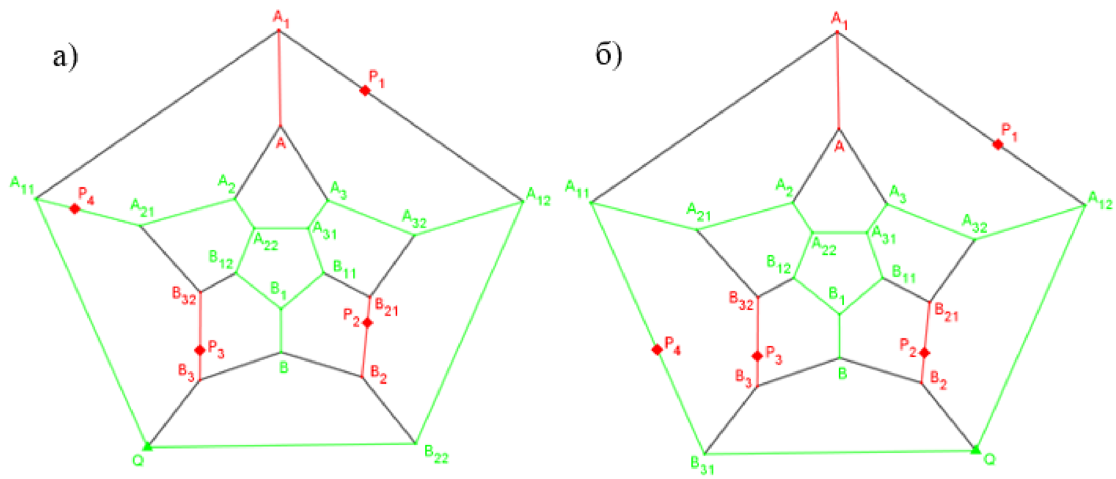

Figure 10. The evader's remaining escape routes are eliminated by pursuers' movements.

or stochastic evader behavior. This work bridges combinatorial game theory and geometric control, offering practical insights for applications in robotics, surveillance, and network security.

References

[1] Bulgakova M., Petrosyan L., Multistage games with pairwise interactions on full 345 graph. Mat. Teor. Igr Pril. (2019) Vol. 11, P. 3–20.

[2] Krasovskii N., Subbotin A., Positional differential games. Nauka, Moscow, (1974).

[3] Pontryagin L., Mishchenko A., The linear differential game of pursuit (analytic theory). Math. sb. (1986) Vol. 131, Iss. 2, P. 131–158.

[4] Petrosyan L., Sedakov A., Multi-step network game with full information. Math. Theory Games Its Appl. (2009) Vol. 1, P. 66–81.

[5] Azamov A., Samatov B., The Π-strategy: Analogies and applications. Contributions to Game Theory and Management, (2011) Vol. 4, P. 33–46.

[6] Bonato A., Nowakowski R., The game of cops and robbers on graphs. American Mathematical Society, (2011).

[7] Chernousko F., Zak V., On differential games of evasion from many pursuers. J. Optim. Theory Appl. (1985) Vol. 46, P. 461–470.

[8] Chikrii A., Prokopovich P., Simple pursuit of one evader by a group. Cybern Syst. Anal. (1992) Vol. 28, P. 438–444.

[9] Nowakowski R., Unsolved problems in combinatorial games. Games of No Chance. (2019), vol. 5, P. 125–168.

[10] Samatov B., Ibragimov G., Khodjibayeva I., Pursuit-evasion differential games with the Gronwall type constraints on controls. Ural Mathematical Journal, (2020) Vol. 6, Iss. 2, P. 95–107.

[11] Samatov B., Juraev B., Pursuit-evasion problems with a constraint for energy expenditure of velocity. Uzbek Mathematical Journal, (2022) Vol. 66, Iss. 4, P. 146–155.

[12] Kuchkarov A., Ibragimov G., Ferrara M., Simple motion pursuit and evasion differential games with many pursuers on manifolds with Euclidean metric. Discrete Dynamics in Nature and Society, (2016) Vol. 2016,

[13] Blagodatskikh A., Petrov N., Conflict interaction of groups of controlled objects. Udmurt State University, Izhevsk, (2009).

[14] Azamov A., Kuchkarov A., Holboyev A., The pursuit-evasion game on the 1-skeleton graph of the regular polyhedron. I. Mat. Teor. Igr Pril. (2015) Vol. 7, P. 3–15.

[15] Ibragimov G., Salleh Y., Alias I., Pansera B., Ferrara M., Evasion from several pursuers in the game with coordinate-wise integral constraints. Dynamic Games and Applications, (2023) Vol. 13, P. 819–842.

[16] Ibragimov G., Evasion Differential Game of One Evader and Many Slow Pursuers. Dynamic games and applications, (2023),

[17] Azamov A., Kuchkarov A., Holboyev A., The pursuit-evasion game on the 1-skeleton graph of the regular polyhedron. II. Mat. Teor. Igr Pril. (2016) Vol. 8, P. 3–13.

[18] Azamov A., Kuchkarov A., Holboyev A., The pursuit-evasion game on the 1-skeleton graph of the regular polyhedron. III. Mat. Teor. Igr Pril. (2019) Vol. 11, P. 5–23.

[19] Azamov A., Kuchkarov A., Holboyev A., Pursuit-evasion game on edge graphs of regular polyhedrons in presence of slow pursuers. Uzb. Math. J. (2017), Iss. 1, P. 140–145.

[20] Andreae T., Hartenstein F., Wolter A., A two-person game on graphs where each player tries to encircle his opponent's men. Theoretical Computer Science, (1999) Vol. 215, Iss. 1–2, P. 305–323.

[21] Azamov A., Ibaydullaev T., Ibragimov G., Alias I., Optimal number of pursuers in the differential games on the 1-skeleton of orthoplex. Symmetry, (2021) Vol. 13, P. 2170.

[22] Bonato A., Golovach P., Hahn G., Kratochvil J., The capture time of a graph. Discrete Mathematics, (2009) Vol. 309, Iss. 18, P. 5588–5595.

[23] Mamadaliev N., Ibaydullaev T., On the Modified Third Method in the Pursuit Problem for Differential-Difference Equations of Neutral Type. Russian Mathematics, (2021) Vol. 65, Iss. 11, P. 18–28.

[24] Mamadaliev N., Ibaydullaev T., Abdualimova G., On Game Problems of Controlling Pencil Trajectories under Integral Constraints on the Controls of Players. Russian Mathematics, (2023) Vol. 67, Iss. 6, P. 34–45.

[25] Sun W., Tsiotras P., Yezzi A., Multiplayer pursuit-evasion games in three-dimensional flow fields. Dyn. Games Appl. (2019) Vol. 9, P. 1188–1207.

[26] Ibragimov G., Holboyev A., Ibaydullaev T., Pansera B., Article pursuit differential game with slow pursuers on the 1-skeleton graph of the icosahedron. Mathematics, (2022) Vol. 10, P. 1435.

Tulanboy Ibaydullaev,
Andijan State University,
Andijan, Uzbekistan.
e-mail: ibaydullayev73@mail.ru

Gafurjan Ibragimov,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan;
University of Public Safety of the Republic
of Uzbekistan, Tashkent, Uzbekistan;
Alfraganus University, Tashkent, Uzbekistan;
e-mail: ibragimov.math@gmail.com

Azamat Holboyev,
National pedagogical university of Uzbekistan
after Nizamii, Tashkent, Uzbekistan.
e-mail: azamatholboyev@gmail.com

Akbarjon Muxammadjonov,
V.I.Romanovskiy Institute of Mathematics Uzbekistan
Academy of Sciences,
Tashkent, Uzbekistan.
e-mail: akbarshohmuhammadjonov6764@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 82–93
DOI: 10.29229/uzmj.2026-1-8

On representations of a given number as the sum of two primes and the square of a third prime in an arithmetic progression

Imamov O.

Abstract. This paper considers the problem of representing a given natural number as combination of two prime numbers and the square of a third prime number taken from an arithmetic progression. It was the first to establish the solvability of the equation under consideration in prime numbers from the arithmetic progression, and prove a lower estimate for the number of solutions to this equation. The results obtained are important in the study of additive problems with prime numbers. The proof of the obtained results uses the Hardy-Littlewood circular method and the Vinogradov method of trigonometric sums.

Keywords: diophantine equation; congruent solution; exceptional zero; Dirichlet L-function; principal characters; Legendre symbol; minor arc; major arc; singular series; singular integral
MSC (2020): 11D61

1. Introduction

Let us consider the following equation:

$$a_1p_1 + a_2p_2 + a_3p_3^2 = b, \tag{1.1}$$

where a_1, a_2, a_3, b are integers and p_1, p_2, p_3 are prime numbers.

If in (1.1) we take $a_1 = a_2 = a_3 = 1$, then we arrive at a classical problem posed in 1938 by Hua Loo-Keng [1]. I. Allakov and N. Muzropova [2] proved that equation (1.1) has solutions in prime numbers under certain conditions and obtained a lower bound for the number of such representations. M. C. Liu and Tao Zhan [3], proved in the linear case, that is, for Goldbach's ternary theorem with prime variables from an arithmetic progression, that if N is a sufficiently large number, then there exists a constant number $\delta > 0$ such that for all positive integers $N \leq D^{\delta}$ and for $\sum\limits_{i=1}^{3} l_i = b \,(\bmod D)$ Diophantine equation

$$\begin{cases} b = p_1 + p_2 + p_3, \\ p_i \equiv l_i \,(\bmod D)\,, i = 1, 2, 3, \end{cases}$$

has a solution, where $(l_i, D) = 1$.

In the works of I. Allakov and O. Sh. Imamov [4]–[5], a lower bound was obtained for the number of representations of a natural number as the sum of the squares of five prime numbers from an arithmetic progression. In [6], we improved the estimate of the exceptional set in the problem of representing a natural number as a sum of squares of four prime numbers, while in [7], the problem of representing a natural number as a sum of squares of four prime numbers from an arithmetic progression was considered.

In this work, we examine the solvability conditions of equation (1.1) in prime numbers p_i from an arithmetic progression, $p_i \equiv l_i (\bmod D)$, $i = 1, 2, 3$, $D \leq N^{\delta}$.

We assume, in the general case, that $a_i \neq 0$, $i = 1, 2, 3$ and

$$\gcd(a_1, a_2, a_3) = 1. \tag{1.2}$$

Furthermore, following the approach in Xua‘s work on the Tarry problem (see [8], p. 162), we consider the solvability condition of equation (1.1) in the sense of congruences. That is, we define the quantity $N(q)$ as follows:

$$N(q) := card\,\{\ (n_1, n_2, n_3)\,|\,1 \leq n_j \leq q,\, (n_j, q) = 1\,,\ \ a_1n_1 + a_2n_2 + a_3n_3^2 \equiv b\,(\bmod q)\} \tag{1.3}$$

and require that the following condition holds:

$$\text{"for all } q \geq 1,\ N(q) \geq 1\text{"}. \tag{1.4}$$

Let us assume that the integers a_1, a_2, a_3, b satisfy conditions (1.2), (1.3) and define B as

$$B := \max\{2, |a_1|, |a_2|, |a_3|\}. \tag{1.5}$$

In the present paper, by combining the methods of [2],[3] the following result has been proved:

Theorem 1.1. *If a_1, a_2, a_3, b satisfy conditions* (1.2) *and* (1.3)*. Then there exists an effective constant $A > 0$ such that the following assertions hold:*

(a) If all of a_1, a_2, a_3 are positive and $b \geq B^A$, then equation (1.1) *has a solution in prime numbers p_1, p_2, p_3 taken from an arithmetic progression, $p_i \equiv l_i (\mathrm{mod} D)$, $i = 1, 2, 3$, $D \leq N^\delta$.*

(b) If not all of a_1, a_2, a_3 have the same sign, then equation (1.1) *has a solution in primes p_1, p_2, p_3 from an arithmetic progression $p_i \equiv l_i (\mathrm{mod} D)$, $i = 1, 2, 3$, $D \leq N^\delta$, satisfying for p_1, p_2, p_3 not exceeding $3|b| + B^A$.*

Corollary 1.2. *If N is sufficiently large, then the number of solutions of equation* (1.1) *in prime numbers $NB^{-1} < p_1, p_2, p_3^2 \leq N$, $p_i \equiv l_i (\mathrm{mod} D)$, $i = 1, 2, 3$, $D \leq N^\delta$ is at least $c_1 N^{3/2} \left(BQ^{83/420} D \ln^3 N\right)^{-1}$, where c_1 is a positive constant and $Q = N^{21\delta}$.*

2. Integral representation of the problem and minor arcs

From now on, we denote a prime number by p (with or without indices). The constants $c_1, c_2, \ldots$ are effective positive absolute constants. The constant δ is a sufficiently small effective positive number and its value may depend on the values of the constants c_j.

Let

$$Q := N^{21\delta},\ T := Q^{\frac{1}{\sqrt{\delta}}},\quad L := NB^{-1},\ L_1 := \sqrt{L},\ N_1 := \sqrt{N}. \tag{2.1}$$

We choose N so that it satisfies

$$B \leq Q^\delta. \tag{2.2}$$

For any integer y and any positive integer q, we define $e(y) = e^{2\pi i y}$ and $e_q(y) = e^{2\pi i \frac{y}{q}} = e(y/q)$.

We define the sums

$$S_i(y) := \sum_{\substack{L < n_i \leq N \\ n_i \equiv l_i (\bmod D)}} \Lambda(n_i)\, e(n_i y)\ ,\quad i = 1, 2;\quad S_3(y) = \sum_{\substack{L < n_3^2 \leq N \\ n_3 \equiv l_3 (\bmod D)}} \Lambda(n_3)\, e\left(n_3^2 y\right), \tag{2.3}$$

where $\Lambda(n)$ is the Mangoldt function.

Let

$$\tau = T^{1/4} N^{-1}. \tag{2.4}$$

We divide the interval $[\tau; 1 + \tau]$ into main and additional subintervals in the usual way (see. [2]).

We define

$$I(N, b, D) = I(N) := \sum_{\substack{L < n_1, n_2, n_3^2 \leq N \\ a_1 n_1 + a_2 n_2 + a_3 n_3^2 = b \\ n_i \equiv l_i (\bmod D)}} \Lambda(n_1)\, \Lambda(n_2)\, \Lambda(n_3). \tag{2.5}$$

Using (2.3) and (2.5), we can express $I(N)$ as

$$I(N) = \int_\tau^{1+\tau} e(-bx)\, S_1(a_1 x)\, S_2(a_2 x)\, S_3(a_3 x)\, dx. \tag{2.6}$$

Noting that $M \cup M' = [\tau,\ 1 + \tau]$, we rewrite (2.6) as the sum of two integrals:

$$I(N) = \left(\int_M + \int_{M'}\right) e(-bx) \prod_{i=1}^{3} S_i(a_i x)\, dx = I_1(N) + I_2(N)\ . \tag{2.7}$$

In (2.7), the integral over the major arcs is denoted by $I_1(N)$, while the integral over the minor arcs is denoted by $I_2(N)$. From (2.7), it follows that $I(N) > I_1(N) - |I_2(N)|$. If we can show that $I(N) > 0$, then we can conclude that equation (1.1) has a solution in prime numbers in arithmetic progression.

First, we analyze the integral $I_2(N)$, for which we prove the following lemma.

Lemma 2.1. *For any $x \in M'$, the following estimate holds: $S_3(a_3x) \ll d^{1/2}D^{-1}N^{\frac{1+\varepsilon}{2}}Q^{-\frac{1}{4}}B^{\frac{1}{4}}$, where $d = (D, q)$.*

Proof. According to corollary 1.2.1 in [9], if $(h, q) = 1,\ 1 \le h \le q,\ \ |\alpha - hq^{-1}| < q^{-2}$ and $D^2 \le N$, then the estimate $S_3(\alpha) \ll D^{-1}N^{1+\varepsilon}\left(d^2q^{-1} + DN^{-1/2} + qDN^{-2}d^{-1}\right)^{1/4}$ holds. According to Dirichlet's theorem on Diophantine approximations, there exists integers h and q satisfying

$$1 \le q \le \tau^{-1},\ \ (h, q) = 1,\ \ \left|a_ix - hq^{-1}\right| < \tau q^{-1}. \tag{2.8}$$

We devide both sides of the inequality in (2.8) by $\left|x - h(|a_i|\,q)^{-1}\right| < \tau(|a_i|\,q)^{-1}$. From this, we get

$$\left|x - h'(q')^{-1}\right| < \tau(q')^{-1}. \tag{2.9}$$

Here, q' is defined as the positive divisor of a_iq, and $\gcd(h',\ q') = 1$. It is not difficult to see that $q' > Q$. In fact, if $q' \le Q$, then from (2.9) and $x \in [\tau,\ 1+\tau]$ it follows that $1 \le h' \le q' \le Q$, that is, $x \in M$, which contradicts the condition $x \in M'$. Therefore,

$$Q|a_j|^{-1} < q \le \tau^{-1}. \tag{2.10}$$

According to (2.10) we obtain, $S_3(a_3x) \ll D^{-1}N^{\frac{1+\varepsilon}{2}}d^{1/2}B^{1/4}Q^{-1/4}$. □

Lemma 2.2. *If $\varepsilon < 0{,}05\delta$, then $|I_2(N)| \ll D^{-1}N^{3/2}d^{1/2}B^{1/4}Q^{-1/5}\ln N$ holds.*

Proof. Using Lemma 2.1 and (2.7), we obtain:

$$|I_2(N)| \ll D^{-1}N^{\frac{1}{2}+\varepsilon}d^{1/2}B^{1/4}Q^{-1/4}\int\limits_{\tau}^{1+\tau}\left\{|S_1(a_1x)|^2 + |S_2(a_2x)|^2\right\}dx. \tag{2.11}$$

Here, $\int\limits_{\tau}^{1+\tau}|S_i(a_ix)|^2dx = \sum\limits_{L<n_i\le N}\Lambda^2(n) \ll \ln N \cdot \sum\limits_{n\le N}\Lambda(n) \ll N\ln N$ [10]. Thus, from (2.11) and the lemma conditions, we obtain: $|I_2(N)| \ll D^{-1}N^{3/2}d^{1/2}B^{1/4}Q^{-1/5}\ln N$. □

3. Simplification of the integral over major arcs

Let $\chi(\operatorname{mod} q)$ and $\chi_0(\operatorname{mod} q)$ denote, respectively, an arbitrary character and the principal Dirichlet character modulo q. It is known (see §2, Chapter III of the work [9]) that there exists a constant c_2 such that the L function $L(s, \chi)$ may have at most one real zero $\tilde{\beta}$ for a given real character $\tilde{\chi}\,(\operatorname{mod}\tilde{r})$ with modulus $\tilde{r} \le T$, in the region $\sigma > 1 - c_2(\ln T)^{-1},\ |t| \le T$. If such a zero exists, it is unique and is called the exceptional zero of the function $L(s, \chi)$, where $s = \sigma + it$. Moreover, this exceptional zero $\tilde{\beta}$ satisfies the inequality

$$c_3\left(\tilde{r}^{\frac{1}{2}}\ln^2\tilde{r}\right)^{-1} \le 1 - \tilde{\beta} \le c_2/\ln T. \tag{3.1}$$

The summation $\sum\limits_{h=1}^{q}{}'$ or $\sum\limits_{(h,q)=1}$ denotes the sum taken over h satisfying the condition $(h, q) = 1$, $1 \le h \le q$. For the character $\chi(\operatorname{mod} Dq/d)$, we define the following functions:

$$S_i(\chi, y) = \sum_{\substack{L<n_i\le N \\ n_i\equiv l_i(\operatorname{mod} D)}}\Lambda(n_i)\,\chi(n_i)\,e(n_iy)\,,\quad S_3(\chi, y) = \sum_{\substack{L<n_3^2\le N \\ n_3\equiv l_3(\operatorname{mod} D)}}\Lambda(n_3)\,\chi(n_3)\,e\left(n_3^2y\right),$$

$$I_i(y)=\int\limits_L^N e(x_iy)\,dx_i,\ I_3(y):=\int\limits_{L_1}^{N_1} e\left(x_3^2y\right)dx_3,\ \tilde I_i(y)=\int\limits_L^N x_i^{\tilde\beta-1}e(x_iy)\,dx_i,\ \tilde I_3(y)=\int\limits_{L_1}^{N_1} x_3^{\tilde\beta-1}e\left(x_3^2y\right)dx_3,$$

$$I_i(\chi,y)=\sum_{|\gamma|\le T}{}'\int\limits_L^N x_i^{\rho-1}e(x_iy)\,dx_i,\qquad I_3(\chi,y)=\sum_{|\gamma|\le T}{}'\int\limits_{L_1}^{N_1} x_3^{\rho-1}e\left(x_3^2y\right)dx_3,\tag{3.2}$$

where $i=1,2$ and the summation $\sum\limits_{|\gamma|\le T}{}'$ denotes the sum over zeros $\rho=\beta+i\gamma$ of the function $L(s,\chi)$ in the region $1/2\le\beta\le 1-c_2(\ln T)^{-1},\ |\gamma|\le T$ with $\tilde\beta$ being the exceptional zero if it exists.

Lemma 3.1. *For any real number y and characters $\chi(\bmod Dq/d)$ with $Dq/d\le T$, the following equalities hold:*

$$S_i(\chi,y)=\delta'_\chi I_i(y)-\delta_\chi\tilde I_i(y)-I_i(\chi,y)+O\left((1+|y|\,N)\,NT^{-1}\ln^2N\right),\quad i=1,2$$

$$S_3(\chi,y)=\delta'_\chi I_3(y)-\delta_\chi\tilde I_3(y)-I_3(\chi,y)+O\left((1+|y|\,N_1)\,N_1T^{-1}\ln^2N\right),\tag{3.3}$$

where

$$\delta'_\chi=\begin{cases}1, & if\ \ \chi\equiv\chi_0(\bmod Dq/d)\\ 0, & otherwise\end{cases},\qquad \delta_\chi=\begin{cases}1, & if\ \ \chi\equiv\tilde\chi\chi_0(\bmod Dq/d),\\ 0, & otherwise.\end{cases}$$

A proof of this lemma can be found in [9]. In the next step, to transform the exponential sum $S_i(\alpha)$ containing the character χ into the integral form above, we need the following new definitions.

$d(q)$ is defined as follows: $D=p_1^{\alpha_1}\cdots p_s^{\alpha_s}D_0,\ q=p_1^{\beta_1}\cdots p_s^{\beta_s}q_0,\ (D_0,\,q_0)=1,\ d(q)=p_1^{\gamma_1}\cdots p_s^{\gamma_s}$, where $\gamma_i=\min(\alpha_i,\,\beta_i)$, for $i=1,...,s$. We define $d_1(q)$ and $d_2(q)$ as follows:

$$d_1(q):=p_1^{\delta_1}\cdots p_s^{\delta_s},\quad \delta_i=\begin{cases}\alpha_i, & \text{if }\ \beta_i>\alpha_i\\ 0, & otherwise,\end{cases},\quad d_2(q):=d(q)/d_1(q)\ .\tag{3.4}$$

For convenience, we write $d=d(q)$, $d_1=d_1(q)$ va $d_2=d_2(q)$. From the above definitions, it follows that $(d_1,\,d_2)=1$ and $(D/d_1,q/d_2)=1$.

Lemma 3.2. *If $y=a_i\left(hq^{-1}+\lambda\right)$, then the equality*

$$S_i(y)=S_i\left(a_i\left(hq^{-1}+\lambda\right)\right)=\frac{1}{\varphi(\frac{D}{d_1})\varphi(\frac{q}{d_2})}\sum_{\zeta(\bmod D/d_1)}\bar\zeta(l_i)\sum_{\eta(\bmod q/d_2)}G_i(h,\bar\eta,q)S_i(\zeta\eta,a_i\lambda)+O\left(\ln^2N\right),$$

holds for $i=1,2,3$. Here,

$$G_i(h,\bar\eta,q)=G(D,l_i,h,\bar\eta,q)=\sum_{\substack{(z_i,q)=1,\\ z_i\equiv l_i(\bmod D)}}e\left(\frac{a_ihz_i}{q}\right)\bar\eta(z_i),\quad i=1,2;$$

$$G_3(h,\bar\eta,q)=G_3(D,l_3,h,\bar\eta,q)=\sum_{\substack{(z_3,q)=1,\\ z_3\equiv l_3(\bmod D)}}e\left(\frac{a_3hz_3^2}{q}\right)\bar\eta(z_3)\tag{3.5}$$

and η, ζ- are characters modulo q/d_2 and D/d_1, respectively.

Proof. By the definition of $S_i(y)$, we have:

$$S_i(y)=\sum_{\substack{L<n_i\le N,\\ n_i\equiv l_i(\bmod D),\ (n,q)=1}}\Lambda(n_i)e(n_iy)+O\left(\sum_{\substack{p^k\le N\\ p|q,\quad k\ge2}}\ln p\,e(p^ky)\right)=$$

$$= \sum_{\substack{(z_i,q)=1 \\ z_i \equiv l_i(\bmod\ d)}} e\left(\frac{a_i h z_i}{q}\right) \sum_{\substack{L<n_i\le N \\ n_i\equiv l_i(\bmod\ D),\ n_i\equiv z_i(\bmod\ q)}} \Lambda(n_i)e(a_i n_i\lambda) + O\left(\ln^2 N\right).$$

If $z_i \equiv l_i(\mathrm{mod} d)$, then the inner sum over the main range is empty, and therefore we can restrict the sum over z_i satisfying the condition $z_i \equiv l_i(\mathrm{mod} d)$. On the other hand, the condition $z_i \equiv l_i(\mathrm{mod} d)$ is equivalent to the conditions $n_i \equiv l_i(\mathrm{mod}\, D)$ and $n_i \equiv z_i(\mathrm{mod} q)$. These, in turn, are equivalent to $n_i \equiv l_i(\mathrm{mod}\, D/d_1)$ and $n_i \equiv c_i(\mathrm{mod}\, q/d_2)$, respectively. In this case, due to the orthogonality property of characters [9] and according to (3.2), the following equality holds for $S_i(y)$:

$$S_i(y) = \frac{1}{\varphi(D/d_1)\varphi(q/d_2)} \sum_{\zeta(\bmod\ D/d_1)} \bar{\zeta}(l_i) \sum_{\eta(\bmod\ q/d_2)} \sum_{\substack{(z_i,q)=1 \\ z_i\equiv l_i(\bmod\ d)}} e\left(\frac{a_i h z_i}{q}\right) \bar{\eta}(z_i) \times$$

$$\times \sum_{\substack{L<n_i\le N \\ n_i\equiv l_i(\bmod\ D)}} \zeta\eta(n_i)\Lambda(n_i)e(a_i n_i\lambda) + O(\ln^2 N).$$

For $S_3(y)$ we have the following:

$$S_3(y) = \varphi^{-1}(D/d_1)\varphi^{-1}(q/d_2) \sum_{\zeta(\bmod\ D/d_1)} \bar{\zeta}(l_3) \sum_{\eta(\bmod\ q/d_2)} G_3(h,\bar{\eta},q)S_3(\zeta\eta,\lambda) + O\left(\ln^2 N\right).$$

Thus proving Lemma 3.2. □

Now, using the lemmas above, we simplify $I_1(N)$. For any $y = (hq^{-1}+\lambda) \in m(h,q)$ satisfying $|\lambda| < \tau/q$ and $q \le Q$, equation (3.3) and Lemma 3.2 imply that $S_i(y)$ can be written as follows:

$$S_i(y) = \varphi^{-1}(D/d_1)\varphi^{-1}(q/d_2)\times$$

$$\times\left\{G_i(h,\bar{\eta}_0,q)I(a_i\lambda) - \delta_q\tilde{\zeta}\zeta_0(l_i)\, G_i(h,\bar{\eta}\eta_0,q)\tilde{I}(a_i\lambda) - \sum_{\substack{\zeta\ (\bmod\ D/d_1) \\ \eta\ (\bmod\ q/d_2)}} \tilde{\zeta}(l_i)\, G_i(h,\bar{\eta},q)\tilde{I}(\zeta\eta,a_i\lambda)\right\}+$$

$$+O(\varphi^{-1}(q/d_2) \sum_{\eta\ (\bmod\ q/d_2)} |G_i(h,\bar{\eta},q)|(1+|a_i\lambda|N)NT^{-1}\ln^2 N) + O(\ln^2 N),$$

where $\tilde{\zeta}\zeta_0\,(\mathrm{mod} D/d_{\,1})\,\tilde{\eta}\eta_0\,(\mathrm{mod} q/d_{\,2}) = \tilde{\chi}\chi_0\,(\mathrm{mod} Dq/d\,)$, $\tilde{\zeta},\tilde{\eta}$ are primitive characters and

$$\delta_q := \begin{cases} 1, & \text{if } \tilde{\chi}\ (\mathrm{mod}\ \tilde{r})\ \text{exists and } \tilde{r} \mid Dq/d, \\ 0, & \textit{otherwise}. \end{cases}$$

From (1.5), (2.4) and (3.5), since $|a_i| \le B$, $|\lambda| < \tau/q$ and $|a_i\lambda|\,N < BT^{1/4}q^{-1}$, we can trivially estimate the following term as: $\sum_{\eta\ (\bmod\ q/d_2)} |G_i(h,\bar{\eta},q)| \ll \varphi(q/d_2)\varphi(q)$. Using this, we can evaluate the term under the symbol O : $\varphi^{-1}(q/d_2) \sum_{\eta\ (\bmod\ q/d_2)} |G_i(h,\bar{\eta},q)|(1+|a_i\lambda|N)NT^{-1}\ln^2 N \le NBT^{-3/4}\ln^2 N$. Therefore, for $y = h/q+\lambda \in m(h,q)$ we obtain the following:

$$S_i(y) = \varphi^{-1}(D/d_1)\,\varphi^{-1}(q/d_2)\,H_i(h,q,\lambda) + O\left(NBT^{-3/4}\ln^2 N\right). \tag{3.6}$$

By similarly reasoning, we estimate $S_3(y)$ as follows:

$$S_3(y) = \varphi^{-1}(D/d_1)\,\varphi^{-1}(q/d_2)\,H_3(h,q,\lambda) + O\left(N^{1/2}BT^{-1}\ln^2 N\right). \tag{3.7}$$

The remaining term is estimated as: $\frac{1}{\varphi(\frac{q}{d_2})}\sum\limits_{\eta \pmod{\frac{q}{d_2}}} |G_3(h,\bar{\eta},q)|(1+|a_3\lambda|N_1)N_1T^{-1}\ln^2 N \ll \frac{B\ln^2 N}{T}$.

Here

$$\begin{cases} H_i\left(h,q,\lambda\right) := G_i(h,\bar{\eta}_0,q)I_i(\lambda) - \delta_q\tilde{\zeta}\zeta_0\left(l_i\right)G_i(h,\bar{\eta}\eta_0,q)\tilde{I}_i(\lambda) - F_i\left(h,q,\lambda\right), \\ F_i\left(h,q,\lambda\right) := \sum\limits_{\zeta \pmod{D/d_1}\,\eta \pmod{q/d_2}} \tilde{\zeta}\left(l_i\right)G_i(h,\bar{\eta},q)\tilde{I}_i(\zeta\eta,\lambda). \end{cases} \tag{3.8}$$

To estimate $H_i\left(h,q,\lambda\right)$ we use Lemma 3.3 from [11] and Lemma 3 from [2]. Then we have: $\varphi^{-1}\left(D/d_1\right)\varphi^{-1}\left(q/d_2\right)H_i\left(h,q,\lambda\right) \ll \varphi\left(q\right)N$, $\varphi^{-1}\left(D/d_1\right)\varphi^{-1}\left(q/d_2\right)H_3\left(h,q,\lambda\right) \ll \varphi\left(q\right)N^{1/2}$.
So, from (3.6) and (3.7), we have: $S_1\left(y\right) = \varphi^{-1}\left(D/d_1\right)\varphi^{-1}\left(q/d_2\right)\{H_1+R\}$,
$S_2\left(y\right) = \varphi^{-1}\left(D/d_1\right)\varphi^{-1}\left(q/d_2\right)\{H_2+R\}$, $S_3\left(y\right) = \varphi^{-1}\left(D/d_1\right)\varphi^{-1}\left(q/d_2\right)\{H_3+R_1\}$.

Thus

$$I_1\left(N\right) = \sum_{q\le Q}\varphi^{-3}\left(\frac{D}{d_1}\right)\varphi^{-3}\left(\frac{q}{d_2}\right)\sum_{(h,q)=1}\int\limits_{-\tau/q}^{\tau/q} e\left(-b\left(\frac{h}{q}+\lambda\right)\right)\prod_{i=1}^{3}H_i\left(h,q,\lambda\right)d\lambda +$$

$$+O\left(\sum_{q\le Q}\sum_{(h,q)=1}\frac{\tau}{q}\varphi^2\left(q\right)N^{\frac{5}{2}}B^3T^{-\frac{3}{4}}\ln^6 N\right).$$

We estimate the remainder term in the final expression as follows: $\ll N^{\frac{3}{2}}T^{-1/2}Q^3B^3\ln^6 N \ll N^{\frac{3}{2}}Q^{-1}$.
As a result, we obtain the following expression for $I_1\left(N\right)$:

$$I_1\left(N\right) = \sum_{q\le Q}\varphi^{-3}\left(D/d_1\right)\varphi^{-3}\left(q/d_2\right)\sum_{(h,q)=1}\int\limits_{-\tau/q}^{\tau/q} e\left(-b\left(h/q+\lambda\right)\right)\prod_{i=1}^{3}H_i\left(h,q,\lambda\right)d\lambda + O\left(N^{\frac{3}{2}}Q^{-1}\right).$$

Now we extend the integration interval $\left[-\frac{\tau}{q},\frac{\tau}{q}\right]$ to $(-\infty;\infty)$. Arguing as at the end of §3 of paper [2], we obtain

$$I_1\left(N\right) = \sum_{q\le Q}\frac{1}{\varphi^3\left(\frac{D}{d_1}\right)\varphi^3\left(\frac{q}{d_2}\right)}\sum_{(h,q)=1}e_q\left(-bh\right)\int\limits_{-\infty}^{\infty} e\left(-b\lambda\right)\prod_{i=1}^{3}H_i\left(h,q,a_i\lambda\right)d\lambda + O\left(N^{3/2}Q^{-1}\right). \tag{3.9}$$

4. Singular Series and Singular Integral of the Problem

Regarding the special series and the special integral, taking into account the specifics of our problem, we formulate several lemmas, the proof of which is, in principle, similar to the proof of analogous results given in the works [7], [12].

Lemma 4.1. *If $\chi\left(\mathrm{mod}\,p^\beta/d_2\right)$ is an arbitrary character such that $\beta\ge 0$, $d_2 = d_2\left(p^\beta\right)$ is defined as in (3.4) and α is defined from the relation $p^\alpha\| D$. Then the following holds:*
(a) if $\chi\left(\mathrm{mod}\,p^\beta\right)$ is a primitive character and $p|\,h$, $\beta>\alpha$, then $G_i\left(h,\chi,p^\beta\right)=0$, $i=1,2,3$.
(b) if η_0 is a character modulo $p^t/d_2\left(p^t\right)$ such that, $p\nmid h$ and $t\ge\theta+\max\{\theta,\alpha,\beta\}$, then $G_i\left(h,\chi\eta_0,p^t\right)=0$. Here $\theta = 1+[2/p]$, $i=1,2,3$.
(c) if $p\nmid h$, then $G_i\left(h,\chi,p^\beta\right)\le(2,p)\left(h,p^\beta\right)^{1/2}p^{\beta/2}$, $i=1,2$, $G_3\left(h,\chi,p^\beta\right)\le 2\left(2,p\right)\left(h,p^\beta\right)^{1/2}p^{\beta/2}$.

In our subsequent analysis, we encounter the following sums:

$$Z\left(q\right) := Z\left(q,\eta_1,\eta_2,\eta_3\right) := \sum_{(h,q)=1}e_q\left(-bh\right)\prod_{i=1}^{3}G_i\left(a_ih,\eta_i,q\right) \tag{4.1}$$

and

$$Y\left(q\right) := Y\left(q,\eta_1,\eta_2,\eta_3\right) := \sum_{h=1}^{q}e_q\left(-bh\right)\prod_{i=1}^{3}G_i\left(a_ih,\eta_i,q\right), \tag{4.2}$$

where η_i is a character modulo $q/d_2(q)$. The expression $Y(q)$ can be written in the following form:

$$Y(q,\eta_1,\eta_2,\eta_3) = q\sum_{(q)} \eta_1(z_1)\,\eta_2(z_2)\,\eta_3(z_3). \tag{4.3}$$

The sum $\sum\limits_{(q)}$ is taken over all triples $z_1,\ z_2,z_3$ such that (1.3) and (1.4). In expression (4.3) if all the characters η_i are principal, then it can be seen that:

$$Y(q,\eta_0,\eta_0,\eta_0) = qN(q), \tag{4.4}$$

this equality holds. Moreover, we define

$$A(q) := \varphi^{-3}\left(q(D,q)^{\odot}/d\right) Z(q,\eta_0,\eta_0,\eta_0), \tag{4.5}$$

where $(D,q)^{\odot}$ denotes the product of the common prime divisors of D and q. $(D,q)^{\odot}\Big\| D$ indicates that it is the largest divisor of D such that if $p^{\alpha}\|(D,q)^{\odot}$, then necessarily $p^{\alpha}\| D$.

Lemma 4.2. *The functions $Z(q)$ and $Y(q)$ are multiplicative in the following sense: that is, if $q = q_1\cdots q_t$ and $(q_i,\ q_j) = 1$ for $i\neq j$ then for each $i = 1,2,3$, $\eta_i\,(\operatorname{mod} q/d_2(q)\,) = \prod\limits_{j=1}^{t}\eta_{ij}\,(\operatorname{mod} q_j/d_2(q_j))$ is a valid decomposition. In this case, we have: $Z(q,\eta_1,\eta_2,\eta_3) := \prod\limits_{j=1}^{t} Z(q_j,\eta_{1j},\eta_{2j},\eta_{3j})$ and $Y(q,\eta_1,\eta_2,\eta_3) := \prod\limits_{j=1}^{t} Y(q_j,\eta_{1j},\eta_{2j},\eta_{3j})$, also hold. In particular, this implies that $N(q)$ and $A(q)$ are also multiplicative functions with respect to q.*

Lemma 4.3. *For any positive integer q the following estimate holds:*
$\varphi^{-3}\left(\frac{Dq}{d(q)}\right) Z(q) \ll \frac{d^3(q)}{D^3} q^{-1/2}\mathcal{L}^{-3}$, *where* $\mathcal{L} = \left(\ln\ln\frac{Dq}{d(q)}\right)$.

Lemma 4.4. *Suppose that $\chi_i\left(\operatorname{mod} p^{\beta_i}\right)$ for $i = 1,2,3$ are primitive characters or that $\beta = \max\{\beta_1,\ \beta_2,\beta_3\}$ and we choose $\alpha := \alpha(p)$ so that the condition $p^{\alpha(p)}\big| D$ is satisfied. For simplicity, in the notation we write $Z(p^t) = Z(p^t;\ \chi_1\chi_0,\ \chi_2\chi_0,\ \chi_3\chi_0)$, where χ_0, is the principal character modulo p^t. Then the following statements hold:*
(a) If $\beta>\alpha$, then $Z\left(p^{\beta}\right) = Y\left(p^{\beta}\right)$.
(b) If $t\geq\theta+\max\{\theta,\ \beta,\ \alpha\}$, then $Z(p^t) = 0$. Where $\theta = 1+[2/p\,]$.
(c) If $\beta>\alpha$, then $\sum\limits_{v=\beta}^{t}\varphi^{-3}(p^v)\,Z(p^v) = \varphi^{-3}(p^t)\,Y(p^t)$ or $\beta = 0$ and $t>\alpha$, then
$\sum\limits_{v=0}^{\alpha}\varphi^{-3}(p^a)\,Z(p^v) + \sum\limits_{v=\alpha+1}^{t}\varphi^{-3}(p^v)\,Z(p^v) = \varphi^{-3}(p^t)\,Y(p^t)$.

In Lemma 4.4, let $\chi_1 = \chi_2 = \chi_3 = \chi_0$ and $\beta = 0$ then the following result is obtained.

Corollary 4.5. *Suppose $N(q)$, $A(q)$ and $\alpha = \alpha(p)$ are defined respectively as in (4.6), (4.5) and similarly as in Lemma 4.4. Then the following statements hold:*
(a) if $p\geq 3$, $t\geq 1+\alpha$, then $A(p^t) = 0$, if $t\geq 2+\max\{2,\alpha\}$, then $A(2^t) = 0$.
(b) if $p\geq 3$, $t\geq\alpha$, then $p^t\varphi^{-3}(p^t)\,N(p^t) = p^{\alpha}\varphi^{-3}(p^{\alpha})\,N(p^{\alpha})$.
(c) if $t\geq\alpha'$, where $a' = 1+\max\{2,\alpha\}$, then $2^t\varphi^{-3}(2^t)\,N(2^t) = 2^{\alpha'}\varphi^{-3}\left(p^{\alpha'}\right)N\left(p^{\alpha'}\right)$.

Taking the above result into account, we introduce the following notation:

$$s(p) := \sum_{0\leq t<\theta+\max\{\theta,\alpha(p)\}} A\left(p^t\right) = \varphi^{-3}\left(\sigma\left(p^{\alpha(p)}\right)p^{\alpha(p)}\right)N\left(\sigma\left(p^{\alpha(p)}\right)p^{\alpha(p)}\right)\sigma\left(p^{\alpha(p)}\right)p^{\alpha(p)}. \tag{4.6}$$

Here $\sigma(q) = 1,4,2$ corresponds respectively to $2\nmid q$, $2\,\|q$ and $4\,|q$.

Lemma 4.6. *For $s(p)$ the following assertions hold:*
(a) if $p \neq 2$, $\alpha = \alpha(p) \geq 1$, then $s(p) = \varphi^{-3}(p^\alpha) p^\alpha$.
(b)

$$s(2) = \begin{cases} 2^3, & if\ \alpha(2) = 1, \\ \varphi^{-3}\left(2^{\alpha(2)}\right) 2^{\alpha(2)+1}, & if\ \alpha(2) \geq 2. \end{cases}$$

Therefore, we also have: $s(2) = \varphi^{-3}(2^\alpha) 2^\alpha \sigma(D)$.
(c) if $p \neq 2$ and $p \nmid D$, then $s(p) = 1 + A(p)$, and if $2 \nmid D$, then $s(2) = 1 + A(2) + A(2^2) + A(2^3)$.

Lemma 4.7. *The following statements hold:*
(a) if $p \nmid D$, then $|A(p)| < 10p^{-2}$.
(b) The product $\prod_p s(p)$ converges absolutely, and $\prod_p s(p) \gg \varphi^{-3}(D) D\sigma(D)$.
(c) $\sum_{q=1,\ (q,r)=1}^{\infty} \varphi^{-3}(Dq/d) Z(q; \eta_0, \eta_0, \eta_0) = \prod_{p\nmid r} s(p) = \frac{\sigma(D/(D,r))D/(D,r)}{\varphi^3(D/(D,r))} \prod_{p\nmid r,\ p\nmid D} s(p)$.
(d) $\sum_{q\geq y} \varphi^{-3}(Dq/d) Z(q; \eta_0, \eta_0, \eta_0) \ll y^{-1} D^{-1} \ln^{10}(y+1)$.

Lemma 4.8. *Suppose $r_i |\, Dq/d$, for $i = 1, 2, 3$ and $\chi_i\,(\mathrm{mod} r_i) = \zeta_i\left(\mathrm{mod}\left(r_i, \frac{D}{d_1}\right)\right)\eta_i\left(\mathrm{mod}\left(r_i, \frac{q}{d_2}\right)\right)$. All characters are primitive and if $r = [r_1, r_2, r_3]$, then the following estimate holds:*
(a) $\sum_{q\leq Q,\ r|Dq/d} \left|\varphi^{-3}\left(\frac{Dq}{d}\right) Z(q, \eta_1\eta_0, \eta_2\eta_0, \eta_3\eta_0) \prod_{i=1}^{3} \zeta_i\zeta_0(l_i)\right| \ll r^{-1/2} L^{-3}$.
(b) Suppose $\alpha(p)$ is defined as in Lemma 4.4, and furthermore let $r_i = r_i^{(1)} r_i^{(2)}$, $r^{(j)} = \left[r_1^{(j)}, r_2^{(j)}, r_3^{(j)}\right]$, $\chi_i\,(\mathrm{mod} r_i) = \chi_i^{(1)}\left(\mathrm{mod} r_i^{(1)}\right)\chi_i^{(2)}\left(\mathrm{mod} r_i^{(2)}\right)$, $\left(r_i^{(1)}, r_i^{(2)}\right) = 1$, $i = 1, 2, 3$, $j = 1, 2$, if $p^\beta \| r_i^{(1)}$, then $\beta > \alpha(p)$ and if $p^\beta \| r_i^{(2)}$, then $\beta \leq \alpha(p)$. If $D = D_1 D_2$, $(D_1, D_2) = 1$, then $p^\beta \| r$ and if $p |\, D_1$, then $\beta \leq \alpha(p)$, if $p^\beta \| r$ and $p |\, D_2$, then $\beta > \alpha(p)$. In this case, we have

$$E := \sum_{q\leq Q,\ r|dq/h} \varphi^{-3}\left(\frac{Dq}{d}\right) Z(q, \eta_1\eta_0, \eta_2\eta_0, \eta_3\eta_0) \prod_{i=1}^{3} \zeta_i\zeta_0(l_i) =$$

$$= \prod_{i=1}^{3} \chi_i^{(2)}(l_i)\,(\mathrm{mod} r_2) \frac{\sigma(D_1) D_1}{\varphi^3(D_1)} \cdot \frac{Y\left(\sigma\left(r^{(1)}\right) r^{(1)}\right)}{\varphi^3\left(\sigma\left(r^{(1)}\right) r^{(1)}\right)} \prod_{p\nmid D,\ p\nmid r} s(p) + O\left(Q^{-1/2} \ln^4 Q\right).$$

Lemma 4.9. *For arbitrary complex numbers, $0 < \mathrm{Re}\,\rho_i \leq 1$, $i = 1, 2, 3$ the following equality holds:*

$$\int_{-\infty}^{\infty} e(-n\lambda) \prod_{i=1}^{2} \left(\int_L^N {x_i}^{\rho_i - 1} e(a_i x_i \lambda)\, dx_i\right) \int_{L^{1/2}}^{N^{1/2}} {x_3}^{\rho_3 - 1} e(a_3 x_3 \lambda)\, dx_3 d\lambda =$$

$$= \frac{N^{3/2}}{2(|a_3|)} \int_{\mathcal{D}} (Nx_1)^{(\rho_1 - 1)} (Nx_2)^{(\rho_2 - 1)} (Nx_3)^{(\rho_3 - 1)/2} x_3^{-1/2} dx_1 dx_2, \tag{4.7}$$

where $x_3 := (bN^{-1} - a_1 x_1 - a_2 x_2)\, a_3^{-1}$ and

$$\mathcal{D} := \left\{(x_1, x_2) : LN^{-1} \leq x_1, x_2 \leq 1,\ \left(LN^{-1}\right)^{1/2} \leq x_3 \leq 1\right\}. \tag{4.8}$$

Furthermore

$$\int_{\mathcal{D}} x_3^{-1/2} dx_1 dx_2 \gg 1. \tag{4.9}$$

The proofs of Lemmas 4.1-4.9 are essentially analogous to the arguments presented in [7], [12], [13] and [14]. Therefore, given the limited space of the article, we omit the evidence. These lemmas take into account the specifics of the present problem and for convenience of reference, we have stated them without proofs.

5. Estimation of the Integral $I_1(N)$ and Completion of the Proof of the Theorem 1

We now aim to obtain the required lower bound for $I_1(N)$. From equality (3.8) it follows that the product $\prod_{i=1}^{3} H_i(h,q,\lambda)$ consists of a sum of $3^3 = 27$ terms. We will divide these terms into the following three categories.

(C1): the term $\prod_{i=1}^{3} G_i(h,\bar{\eta}_0,q)I(\lambda)$.

(C2): the 19 terms each of which has at least one $F_i(h,q,\lambda)$as factor.

(C3): the 7 remaining terms.

For convenience, we write, for $i = 1,2,3$,

$$M_i = \sum_{q\le Q}\varphi^{-3}(Dq/d)\sum_{(h,q)=1} e_q(-bh)\int_{-\infty}^{\infty} e(-b\lambda)\{sum\ of\ the\ terms\ in\ (C_i)\}\,d\lambda. \tag{5.1}$$

In view of (3.9), we have

$$I_1(N) = M_1 + M_2 + M_3 + O\left(N^{3/2}Q^{-1}\right). \tag{5.2}$$

For distinct integers $m_1, m_2, ...$ taken from the set $\{1,\ 2, 3\}$, let:

$$P(m_1,m_2,...) := \frac{N^{3/2}}{2(|a_3|)}\int_{\mathcal{D}} (Nx_{m_1})^{(\rho_1-1)}(Nx_{m_2})^{(\rho_2-1)}(Nx_{m_3})^{(\rho_3-1)/2}x_3^{-1/2}dx_1dx_2 \tag{5.3}$$

and

$$\Delta(m_1,m_2,...) := \tilde{\chi}(n_{m_1})\,\tilde{\chi}(n_{m_2})\,..., \tag{5.4}$$

where the region $\mathcal{D}$ is defined in (4.8), $\tilde{\chi}$ and $\tilde{\beta}$ are the exceptional character and exceptional zero respectively. Let

$$P_0 := \frac{N^{3/2}}{2(|a_3|)}\int_{\mathcal{D}} x_3^{-1/2}dx_1dx_2. \tag{5.5}$$

Clearly, from (4.9) we have

$$|P(m_1,m_2,...)| \le P_0 \ll N^{3/2}B^{-1}. \tag{5.6}$$

Lemma 5.1. *The following equality holds.* $M_1 = \frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid d} s(p)\,P_0 + O\left(N^{3/2}B^{-1}D^{-2}Q^{-1}\ln^{10}Q\right)$.

Proof. Based on (5.1)

$$M_1 = \sum_{q\le Q}\varphi^{-3}(Dq/d)\sum_{(h,q)=1} e_q(-bh)G_1(h,\bar{\eta}_0,q)G_2(h,\bar{\eta}_0,q)G_3(h,\bar{\eta}_0,q)\times$$

$$\times\int_{-\infty}^{\infty} e(-b\lambda)\int_L^N e(a_1x_1\lambda)\,dx_1\int_L^N e(a_2x_2\lambda)\,dx_2\int_L^N e\left(a_3x_3^2\lambda\right)dx_3d\lambda.$$

If in (5.3) we set $\rho_1 = \rho_2 = \rho_3 = 1$, then the above integral is equals P_0. In view of (4.1), the above double sum is $\sum_{q\le Q}\varphi^{-3}(Dq/d)Z(q)$. By Lemma 4.7 (c), (d), this can be written as:

$$\frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D} s(p) + O\left(\sum_{q>Q}|A(q)|\right) = \frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D} s(p) + O\left(Q^{-1}D^{-2}\ln^{10}Q\right).$$

From this expression and equality (5.6), the proof of Lemma 5.1 follows. Namely,

$$M_1 = \left(\frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D} s(p) + O\left(\frac{\ln^{10}Q}{QD^2}\right)\right)P_0 = \frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D} s(p)\,P_0 + O\left(N^{3/2}\{BD^2Q\}^{-1}\ln^{10}Q\right). \qquad \square$$

Lemma 5.2. *If the exceptional zero $\tilde{\beta}$ exists and $\tilde{r}_1$, d_1 are defined as in Lemma 4.8 (b), by taking $r^{(1)} = \tilde{r}_1$, then*

(a) $$M_3 = \frac{\sigma(D_1)D_1}{\varphi^3(D_1)} \cdot \frac{\sigma(\tilde{r}_1)\tilde{r}_1}{\varphi^3(\sigma(\tilde{r}_1)\tilde{r}_1)} \prod_{p\nmid D,\ p\nmid\tilde{r}_1} s(p) \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \Big\{ -\sum_{i=1}^{3} \Delta(i)\, P(i) + \sum_{1\le i<j\le 3} \Delta(i,j)\, P(i,j) -$$
$$-\Delta(1,2,3)\, P(1,2,3)\Big\} + O\left(N^{3/2}B^{7/2}Q^{-1/2}\ln^4 Q\right).$$

(b) $M_3 \ll d^3 D^{-3} N^{3/2} B \tilde{r}_1^{-1/2} \mathcal{L}^{-3}$.

Proof. Part (a) is proved by reasoning similar to that in the proof of part (a) of Lemma 13 in [2]. The bound in (b) can be deduced directly from Lemma 4.3. □

Define
$$\Omega = \begin{cases} \left(1-\tilde{\beta}\right)\ln T, & if\ \tilde{\beta}\,\text{exists}, \\ 1, & \text{otherwise}. \end{cases} \tag{5.7}$$

In view of Corollary 4.5, Lemma 4.6 and (4.6), we have
$$\prod_{p\nmid D,\ p\nmid\tilde{r}_1} s(p) = \sigma(r'')\, r'' \varphi^{-3}\left(\sigma(r'')\, {r''}_1\right) N\left(\sigma(r'')\, {r''}_1\right), \tag{5.8}$$

$$\frac{\sigma({r'}_1)\, {r'}_1}{\varphi^3\left(\sigma({r'}_1)\, {r'}_1\right)} N\left(\sigma({r'}_1)\, r'\right) = \frac{\sigma(D_2)\, D_2}{\varphi^3\left(\sigma(D_2)\, D_2\right)} N\left(\sigma(D_2)\, D_2\right) = \frac{\sigma(D_2)\, D_2}{\varphi^3(D_2)}, \tag{5.9}$$

where $r''r' = r$, $(r'', r') = 1$, $(r'', D) = 1$, $r' \mid D^{\odot}$, ${r'}_1$, $D \mid (r, D)$ have the same prime factors and the exponent of each prime factor of D_2 is less than in ${r'}_1$. Hence we can write M_1 in the form.
$$M_1 = \frac{\sigma(D_1)\, D_1}{\varphi^3(D_1)} \cdot \frac{\sigma(\tilde{r}_1)\, \tilde{r}_1}{\varphi^3\left(\sigma(\tilde{r}_1)\, \tilde{r}_1\right)} \prod_{p\nmid D,\ p\nmid\tilde{r}_1} s(p) \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} P_0 + O\left(N^{3/2}B^{-1}D^{-2}Q^{-1}\ln^{10} Q\right).$$

Comparing it with the form of M_3 in Lemma 5.2 (a), we have
$$M_1 + M_3 = \frac{\sigma(D_1)D_1}{\varphi^3(D_1)} \cdot \frac{\sigma(\tilde{r}_1)\tilde{r}_1}{\varphi^3(\sigma(\tilde{r}_1)\tilde{r}_1)} \prod_{p\nmid D,\ p\nmid\tilde{r}_1} s(p) \left(\sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} P_0 + \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \Big\{ -\sum_{i=1}^{3} \Delta(i)\, P(i) + \right.$$
$$\left. + \sum_{1\le i<j\le 3} \Delta(i,j)\, P(i,j) - \Delta(1,2,3)\, P(1,2,3) \Big\} \right) + O\left(N^{3/2}B^{7/2}Q^{-1/2}\ln^4 Q\right).$$

Applying formulas (5.3), (5.4) and (5.5) to the right-hand side of the last equality, we obtain:
$$M_1 + M_3 = \frac{\sigma(D_1)\, D_1}{\varphi^3(D_1)} \cdot \frac{\sigma(\tilde{r}_1)\, \tilde{r}_1}{\varphi^3\left(\sigma(\tilde{r}_1)\, \tilde{r}_1\right)} \prod_{p\nmid D,\ p\nmid\tilde{r}_1} s(p) \left\{ \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \frac{N^{3/2}}{2(|a_3|)} \int_{\mathcal{D}} x_3^{-1/2} dx_1 dx_2 - \right.$$
$$-\sum_{i=1}^{2} \left(\sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \tilde{\chi}(n_i) \left(\frac{N^{3/2}}{2(|a_3|)} \int_{\mathcal{D}} \prod_{i=1}^{2} (Nx_i)^{(\rho_i - 1)} dx_1 dx_2 \right) \right) -$$
$$- \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \tilde{\chi}(n_3) \left(\frac{N^{3/2}}{2(|a_3|)} \int_{\mathcal{D}} (Nx_3)^{\frac{(\rho_3-1)}{2}} dx_1 dx_2 \right) +$$
$$+ \sum_{(\sigma(\tilde{r}_1)\tilde{r}_1)} \tilde{\chi}(n_1)\, \tilde{\chi}(n_2) \left(\frac{N^{3/2}}{2(|a_3|)} \int_{\mathcal{D}} (Nx_1)^{(\rho_1-1)} (Nx_2)^{(\rho_2-1)} dx_1 dx_2 \right) +$$

$$+\sum_{(\sigma(\tilde r_1)\tilde r_1)}\tilde\chi(n_2)\tilde\chi(n_3)\left(\frac{N^{3/2}}{2(|a_3|)}\int_{\mathcal D}(Nx_2)^{(\rho_2-1)}(Nx_3)^{\frac{(\rho_3-1)}{2}}dx_1dx_2\right)+$$

$$+\sum_{(\sigma(\tilde r_1)\tilde r_1)}\tilde\chi(n_1)\tilde\chi(n_3)\left(\frac{N^{3/2}}{2(|a_3|)}\int_{\mathcal D}(Nx_1)^{(\rho_1-1)}(Nx_3)^{\frac{(\rho_3-1)}{2}}dx_1dx_2\right)-\sum_{(\sigma(\tilde r_1)\tilde r_1)}\tilde\chi(n_1)\tilde\chi(n_2)\tilde\chi(n_3)\times$$

$$\times\left(\frac{N^{3/2}}{2(|a_3|)}\int_{\mathcal D}(Nx_1)^{(\rho_1-1)}(Nx_2)^{(\rho_2-1)}(Nx_3)^{\frac{(\rho_3-1)}{2}}dx_1dx_2\right)\Bigg\}+O\left(N^{3/2}B^{7/2}Q^{-1/2}\ln^4Q\right). \qquad (5.10)$$

The expression inside the curly braces is not less than

$$\geq\sum_{(\sigma(\tilde r_1)\tilde r_1)}\int_{\mathcal D}x_3^{-1/2}\prod_{i=1}^{2}\left(1-\tilde\chi(n_i)(Nx_i)^{(\rho_i-1)}\right)\left(1-\tilde\chi(n_3)(Nx_3)^{\frac{(\rho_3-1)}{2}}\right)dx_1dx_2.$$

It remains to estimate the integral. Since $LN^{-1}\leq x_1,x_2,x_3\leq 1$ and $\left(1-\tilde\chi(n_i)(Nx_i)^{(\rho_i-1)}\right)\geq 1-L^{\tilde\beta-1}$, $\left(1-\tilde\chi(n_3)(Nx_3)^{\frac{(\rho_3-1)}{2}}\right)\geq 1-L^{\tilde\beta-1}$ are present in the domain $\mathcal D$ we obtain the following: $\prod_{i=1}^{2}\left(1-\tilde\chi(n_i)(Nx_i)^{(\rho_i-1)}\right)\left(1-\tilde\chi(n_3)(Nx_3)^{\frac{(\rho_3-1)}{2}}\right)\geq\left(1-L^{\tilde\beta-1}\right)^3$. From equalities (2.1) and (2.2), it follows that: $L=NB^{-1}\geq NQ^{-\delta}=N^{1-\delta^2}>\sqrt{N}$. Thus, using formulas (2.1) and (5.7), we obtain the following: $1-L^{(\tilde\beta-1)}\geq 1-\exp\left(-1/2\left(1-\tilde\beta\right)\ln N\right)\geq\min\left\{1/2,\,1/4\left(\left(1-\tilde\beta\right)\ln N\right)\right\}\geq\Omega$. Then the main term in (5.10) is $\gg\Omega^3\frac{\sigma(D_1)D_1}{\varphi^3(D_1)}\cdot\frac{\sigma(\tilde r_1)\tilde r_1}{\varphi^3(\sigma(\tilde r_1)\tilde r_1)}\prod_{p\nmid D,\ p\nmid\tilde r_1}s(p)\sum_{(\sigma(\tilde r_1)\tilde r_1)}P_0$. Hence, by (5.8) and (5.9), we have.

Lemma 5.3. $M_1+M_3\geq\Omega^3\frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D}s(p)P_0+O\left(N^{3/2}B^{7/2}Q^{-1/2}\ln^4Q\right).$

We need to estimate M_2 By the Deuring–Heilbronn phenomenon, we have.

Lemma 5.4. $M_2\ll\Omega^3\exp\left(-c/\sqrt{\delta}\right)\frac{\sigma(D)D}{\varphi^3(D)}P_0\prod_{p\nmid D,\,p\nmid r}s(p)+O\left(N^{3/2}Q^{-1/2}B^{7/2}\ln^4Q\right).$

Proof. It is proved by reasoning similar to that used in the proof of Lemma 15 in [15]. □

Combining the results obtained above, we can derive the following estimate for $I_1(N)$. To this end, let us consider two cases.

Case 1. If there is no $\tilde\beta$ exceptional zero of the Dirichlet L-function, or if it exists and the modulus of the corresponding exceptional character $\tilde r>Q^{1/10}$. From Lemma 5.1 and part (b) of Lemma 5.2, as well as from Lemma 5.4 for sufficiently small δ we obtain:

$$I_1(N)\geq\frac{\sigma(D)D}{\varphi^3(D)}P_0\prod_{p\nmid D,\ p\nmid r}s(p)+O\left(\frac{d^3}{D^3}N^{3/2}B^{-1}Q^{-1/20}L^{-3}\right).$$

Then, according to Lemma 4.7 (a), we have: $I_1(N)\gg N^{3/2}\left(BQ^{1/21}D\right)^{-1}$. Here, $D\leq Q^{1/21}$.

Case 2. If there exists a $\tilde\beta$ exceptional zero of the Dirichlet L-function and the modulus of the corresponding exceptional character $\tilde r\leq Q^{1/10}$. Then, using Lemmas 5.3 and 5.4, and for sufficiently small δ, we obtain:

$$I_1(N)\geq\Omega^3\frac{\sigma(D)D}{\varphi^3(D)}\prod_{p\nmid D}s(p)P_0+O\left(N^{3/2}B^{7/2}Q^{-1/2}\ln^4Q\right).$$

Here, taking into account that $\Omega\gg\left(\tilde r^{1/2}\ln^2\tilde r\right)^{-1}\ln T\gg Q^{-1/20}\ln^{-1}Q$, is as in (3.1), we consider $I_1(N)\gg N^{3/2}\left(BQ^{83/420}D\right)^{-1}$ and finally, by comparing the estimates in both cases with Lemma 2.3, it follows that for sufficiently large N, $I_1(N)>|I_2(N)|$ holds. Thus, our theorem is proved.

5.1. **Proof of Corollary 1.1.** Now let us estimate the number of solutions of equation (1.1) in prime numbers:

$$R(b) = R(b, N, D) = \sum_{\substack{L<p_1,p_2,p_3^2\leq N\\ b=a_1p_1+a_2p_2+a_3p_3^2}} 1$$

Based on (2.8) we have:

$$I(N) = R(b)\ln^3 N + O(N\ln N). \tag{5.11}$$

Since $I(N) > I_1(N) - |I_2(N)|$, it follows from (5.11) that $R(b) \gg (I_1(N) - |I_2(N)|)(\ln N)^{-3} + O\left(N(\ln N)^{-2}\right)$. Hence we obtain an estimate $R(b) \gg N^{3/2}\left(BQ^{83/420}D\ln^3 N\right)^{-1}$.

I would like to take this opportunity to thank my supervisor Professor I. Allakov for his advice and support during the preparation of the article.

References

[1] Hua L.-K., Some results in the additive prime number theory. Quart. J. Math. Oxford, (1938), P. 68–80.

[2] Allakov I., Muzropova N., The solution of some equation in primes. Chebyshevskii sbornik, (2024) Vol. 25, Iss. 4, P. 5–26.

[3] Liu M. C., Zhan T., The Goldbach problem with primes in arithmetic progressions. Analytic Number Theory. Cambridge Univ. Press, (1997), vol. 247, P. 227–251.

[4] Allakov I., Imamov O., About numbers representable as the sum of fives of square of primes from arithmetic progression. Samarkand University Scientific Bulletin, (2024), Iss. 3, P. 113–119.

[5] Allakov I., Imamov O., A lower estimate for the quantity of a natural number represented as a sum of five squared prime numbers from an arithmetic progression. Bull. Inst. Math. (2024) Vol. 7, Iss. 4, P. 86–93.

[6] Imamov O., On numbers representable as the sum of four squares of prime numbers. Samarkand University Scientific Bulletin, (2025), Iss. 1, P. 106–110.

[7] Allakov I., Imamov O., The exceptional set for the sum of the squares of four prime numbers in an arithmetic progression. Bull. Inst. Math. (2025) Vol. 8, Iss. 5, P. 1–17.

[8] Hua L.-K., Additive prime number theory. Tr. Math. Institute named after V.A. Steklova, (1947), Iss. 22, P. 3–179.

[9] Allakov I., Estimation of Trigonometric Sums and Their Applications to Solving Some Additive Problems of Number Theory. Surkhan Nashr Publishing House, Termez, (2021).

[10] Montgomery H., Vaughan R., Multiplicative number theory: I. Classical theory. Cambridge, (2006).

[11] Wang Y., Numbers representable by five prime squares with primes in an arithmetic progression. Acta Arithmetica, (1999) Vol. 90, Iss. 3, P. 217–244.

[12] Allakov I., Muzropova N., Investigation of a Special Series in the Problem of Representing Numbers as the Sum of Two Primes and the Square of a Third Prime. Scientific Bulletin of Namangan State University, (2024), Iss. 5, P. 38–46.

[13] Liu M. C., Tsang K. M., Small prime solutions of linear equations. Proc. Intern. Number. Th. Conf. 1987. Laval University. Cand. Math. Soc. (1989), P. 595–624.

[14] Liu M. C., Tsang K. M., Small prime solutions of same additive equations. Mh. Math. (1991) Vol. 111, P. 147–169.

[15] Allakov I., Muzrapova N., About the presentation of numbers by the sum of two prime numbers and the square of the third prime number. Bull. Inst. Math. (2024) Vol. 7, Iss. 6, P. 108–131.

Imamov O. Sh.,
Termez State University,, Termez, Uzbekistan.
e-mail: oybekimamov000@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 94–99
DOI: 10.29229/uzmj.2026-1-9

Extension of separately analytic functions with thin singularities on one-dimensional parallel sections

Imomkulov S., Rasulov K.

Abstract. Let two bounded simply connected domains $D \subset \mathbb{C}_z$, $G \subset \mathbb{C}_w$, and two locally regular compact subsets $E \subset D$, $F \subset G$ be given. If $f(z,w)$ is a separately analytic function with finitely many singular points in each section of $D \times \{w^0\}$ and $\{z^0\} \times G$ for any $(z^0, w^0) \in E \times F$ on the set $X = (D \times F) \cup (E \times G)$, then it extends holomorphically to the domain

$$\hat{X} = \{(z,w) \in D \times G : \ \omega^*(z,E,D) + \omega^*(w,F,G) < -1\},$$

except on an analytic set S. Where $\omega^*(z,E,D)$ is the harmonic measure of the set $E \subset D \subset \mathbb{C}$ relative to the domain D.

Keywords: Hartogs's phenomena, separately analytic functions, harmonic measure, singular set
MSC (2020): 32A10, 32D10

1. Introduction

A function of several complex variables $w = f(z_1, z_2, ..., z_n)$, $z = (z_1, z_2, ..., z_n) \in D \subset \mathbb{C}_z^n$ is called holomorphic in the domain D if it is real differentiable with respect to the set of all variables and satisfies the following equalities

$$\frac{\partial f}{\partial \bar{z}_1} = \frac{\partial f}{\partial \bar{z}_2} = ... = \frac{\partial f}{\partial \bar{z}_n} = 0$$

for any $z \in D$.

The following question naturally arises: if a function $w = f(z_1, z_2, ..., z_n)$ of several complex variables is holomorphic in each variable when the other variables are fixed, then is it holomorphic with respect to the set of all variables? Hartogs [1] gives a positive answer to this question: *if a function $w = f(z_1, z_2, ..., z_n)$ is holomorphic in the domain $D \subset \mathbb{C}^n$ with respect to each of the variables z_j, $j = \overline{1.n}$, then it is holomorphic in D with respect to the set of all variables.*

The proof of this theorem is based on the important, so-called Hartogs lemma "On the extension of functions of several complex variables along a fixed direction": *let a function $f(z,w)$ be holomorphic in a polydisk $U \times V_r = \{z \in \mathbb{C}^n : \ |z| < 1\} \times \{w \in \mathbb{C} : \ |w| < r\}$, $r > 0$, and for each fixed $z^0 \in U$, the function $f(z^0, w)$ extends holomorphically to a larger disk $V_R = \{w \in \mathbb{C} : \ |w| < R\}$, $R > r$ with respect to w. Then the function $f(z,w)$ extends holomorphically to a larger polydisk $U \times V_R$ with respect to the set of all variables.*

These results of Hartogs play a fundamental role in the theory of analytic continuation of functions of several complex variables.

The following problem was posed by M.Hukuhara [2]: let two domains $D \subset \mathbb{C}_z^n$, $G \subset \mathbb{C}_w^m$, and two sets $E \subset D$, $F \subset G$ be given. Suppose that the function $f(z,w)$ originally defined on the set $E \times F$ has the following properties:

1) for each fixed $w^0 \in F$, the function $f(z, w^0)$ extends holomorphically to D,

2) for each fixed $z^0 \in E$, the function $f(z^0, w)$ extends holomorphically to G.

Then we say that the function $f(z,w)$ defines some separately analytic function on

$$X = (D \times F) \cup (E \times G).$$

The task is to determine the domain $\hat{X}$, $\left(\hat{X} \supset X\right)$ to which the function $f(z,w)$ admits a holomorphic extension with respect to the set of all variables.

In case $n = m = 1$, this problem was solved by J.Siciak [3], and in the general case by V.P.Zakharyuta [4]: *let $D \subset \mathbb{C}_z^n$ and $G \subset \mathbb{C}_w^m$ be strongly pseudoconvex domains, and let E and F be closed subsets of D and G respectively. If $f(z,w)$ is a separately analytic function on the set*

$$X = (D \times F) \cup (E \times G)\,,$$

then it extends holomorphically to the domain

$$\hat{X} = \{(z,w) \in D \times G : \ \omega^*(z,E,D) + \omega^*(w,F,G) < -1\}\,.$$

Here, $\omega^*(z,E,D)$ denotes the basic quantity of complex potential theory, known as the P-measure of the set $E \subset D \subset \mathbb{C}^n$ relative to the domain D (see. [5], [6]): let $\mathcal{U}(E,D)$ be a class of functions $u(z) \in psh(D)$, such that $u|_E \le -1,\ u|_D \le 0$. We put

$$\omega\,(z,E,D) = \sup\{u(z) : \ u \in \mathcal{U}(E,D)\}\,,$$

then $\omega^*(z,E,D) = \overline{\lim_{\xi \to z}}\,\omega\,(\xi,E,D)$ is a maximal plurisubharmonic function, and it is called the P-measure of the set E relative to the domain D. In the one-dimensional case, $\omega^*(z,E,D)$ is called a harmonic measure.

An analogue of Hartogs' theorem in the class of meromorphic functions was given by E.Sakai [7]: *let $S \subset D \times G$ be a relatively compact set such that $intS = \emptyset$ and that it does not split the domain $D \times G$. Let A (respectively B) be the set of all points $z \in D$ ($w \in G$) such that $int\{w \in G : \ (z,w) \in S\} = \emptyset$ ($int\{z \in D : \ (z,w) \in S\} = \emptyset$). Let the function $f(z,w)$ be defined on $(D \times G) \backslash S$ and be separately meromorphic on $X = (A \times G) \cup (D \times B)$. Then there exists a function $\hat{f}$ meromorphic in the domain $D \times G$, such that $\hat{f}\Big|_{X \backslash S} = f$.*

The works of V.Rothstein [8], M.V.Kazaryan [9, 10], S.M.Ivashkovich [11], J.Ruppenthal [12], A.S.Sadullaev and E.M.Chirka [13], A.S.Sadullaev and S.A.Imomkulov [14], S.A.Imomkulov [**6**], T.T.Tuichiev [15], M.Yarnicki and P.Pflug [16, 17], A.A.Gonchar [18], P.Pflug and V.A.Nguyen [19], A.A.Atamuratov [20] are also directly related to the continuation of this topic.

The following theorem of A.A.Atamuratov [20] on the continuation of separately meromorphic functions is distinctive in that there is no condition on the set of poles of the given separately meromorphic function: *let $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$ be strongly pseudoconvex domains, and $E \subset D$, $F \subset G$ be nonpluripolar compact sets. Suppose that the function $f(z,w)$ is continuous on the set $E \times F$ and has the following properties:*

1) *for each $w^0 \in F$, there exists a function $\phi_{w^0}(z)$, meromorphic in D, such that for every $\xi \in E$,*

$$\phi_{w^0}(\xi) = f(\xi, w^0);$$

2) *for each $z^0 \in E$, there exists a function $\psi_{z^0}(w)$, meromorphic in G, such that for every $\eta \in F$,*

$$\psi_{z^0}(\eta) = f(z^0, \eta).$$

Then $f(z,w)$ defines a separately meromorphic function on the set $X = (D \times F) \cup (E \times G)$ and it extends meromorphically to the domain

$$\hat{X} = \{(z,w) \in D \times G : \ \omega^*(z,E,D) + \omega^*(w,F,G) < -1\}\,.$$

In this paper, we continue our study of holomorphic extension with singularities on sections and establish an important theorem concerning separately holomorphic functions.

2. Main result

The main theorem of the present paper is the following:

Theorem 2.1. *Let two bounded simply connected domains $D \subset \mathbb{C}_z$, $G \subset \mathbb{C}_w$ and two locally regular compact subsets $E \subset D$, $F \subset G$ be given. Suppose that a function $f(z,w)$, originally defined on the set $E \times F$, satisfies the following conditions of separate analyticity:*

1) for each fixed $w^0 \in F$, the function $f(z,w^0)$, defined on E, extends holomorphically to D, except on a finite set of singular points $S_{w^0} \subset D\backslash E$;

2) for each fixed $z^0 \in E$, the function $f(z^0,w)$, defined on F, extends holomorphically to G, except on a finite set of singular points $S_{z^0} \subset G\backslash F$.

Then the function $f(z,w)$ extends holomorphically to the domain

$$\hat{X} = \{(z,w) \in D \times G : \ \omega^*(z,E,D) + \omega^*(w,F,G) < -1\},$$

except on an analytic set of singularities $S \subset \hat{X}$.

A set $E \subset D \subset \mathbb{C}_z$ is called (globally) regular at a point $z^0 \in \bar{E}$ relative to the domain D if $\omega^*(z^0,E,D) = -1$. It is called locally regular at the point z^0 if for any circle $U(z^0,r)$, $r > 0$, the intersection $\bar{U}(z^0,r) \cap \bar{E}$ is regular at the point z^0.

A set $E \subset D$ is called locally regular if it is locally regular at every point of $z \in E$.

3. The radius of the maximal circle of holomorphy of the functions, except on a discrete set of singularities

In the work of A.S.Sadullaev [21], the following characteristic of holomorphic functions was introduced: let $f(z,w)$ be a holomorphic function in the polydisk $U \times V_r = \{z \in \mathbb{C} : \ |z| < 1\} \times \{w \in \mathbb{C} : \ |w| < r\}$, and let $\mathcal{R}(z)$ denote the radius of the maximal circle on which the function $f(z,w)$, with respect to w, extends as a holomorphic function with discrete singularities.

The following holds:

Lemma 3.1. *[21] A function $-\ln \mathcal{R}_*(z)$ is subharmonic in $U \subset \mathbb{C}$, where $\mathcal{R}_*(z) = \underline{\lim}_{\zeta \to z} \mathcal{R}(\zeta)$ is the lower regularization. Moreover, the set $\{z \in U : \ \mathcal{R}_*(z) < \mathcal{R}(z)\}$ is polar in U and the function $f(z,w)$ extends holomorphically to $\Omega\backslash S$, where $\Omega = \{(z,w) : \ |w| < \mathcal{R}_*(z), \ z \in U\}$ and S is an analytic subset of Ω.*

Remark 3.2. *The analiticity of the exceptional set S in Lemma 3.1 is established using methods developed by A.S.Sadullaev and E.M.Chirka for extending holomorphic functions with fine singularities along a given direction, as detailed in* [13].

Lemma 3.3. *Let the function $f(z,w)$ be holomorphic in the domain $U \times V_r$. For each fixed point z^0 from some nonpolar compact set $E \subset U$, the function $f(z^0,w)$, with respect to w, extends holomorphically to the larger circle $V_R = \{w \in \mathbb{C} : \ |w| < R\}$, $R > r$, except on finitely many singular points. Then the function $f(z,w)$ extends holomorphically to the domain*

$$\left\{(z,w) : \ |w| < R^{-\omega^*(z,E,U)} \cdot r^{1+\omega^*(z,E,U)}, \ z \in U\right\},$$

except on an analytic set of singularities.

Proof. In the proof, we use Lemma 3.1. Let the function $f(z,w)$ satisfy all the conditions of Lemma 3.3; that is, the function $f(z,w)$ is holomorphic in $U \times V_r$ and for each $z^0 \in E \subset U$, the function $f(z^0,w)$, with respect to w, extends holomorphically to $V_R = \{w : \ |w| < R\}$, except on a finite set of points S_{z^0}.

Hence, by Lemma 3.1, it follows that the function $f(z,w)$ extends holomorphically to the domain

$$\{(z,w) : \ |w| < \mathcal{R}_*(z), \ z \in U\},$$

except on some analytic set S, where $\mathcal{R}_*(z)|_E \geq R$, $\mathcal{R}_*(z)|_U \geq r$ holds.

It follows that the subharmonic function

$$u(z) = -\frac{\ln \mathcal{R}_*(z) - \ln r}{\ln R - \ln r}$$

belongs to the class $\mathcal{U}(E,U)$, i.e., $u|_E \leq -1$, $u|_U \leq 0$ and by definition $u(z) \leq \omega^*(z,E,U)$. Hence, $\mathcal{R}_*(z) \geq R^{-\omega^*(z,E,U)} \cdot r^{1+\omega^*(z,E,U)}$, $z \in U$, i.e.,

$$\{(z,w) : \ |w| < \mathcal{R}_*(z), \ z \in U\} \supset \left\{(z,w) : \ |w| < R^{-\omega^*(z,E,U)} \cdot r^{1+\omega^*(z,E,U)}, \ z \in U\right\}.$$

Thus, any function $f(z,w)$ satisfying the conditions of Lemma 3.3 can be holomorphically continued to the domain $\left\{(z,w): \ |w| < R^{-\omega^*(z,E,U)} \cdot r^{1+\omega^*(z,E,U)}\right\}$, except on the analytic set S.

□

Corollary 3.4. *Let $U \subset \mathbb{C}_z$ be a bounded open set, $G \subset \mathbb{C}_w$ a bounded simply connected domain, $E \subset U$ a nonpolar set and $W \subset G$ an open set. If the function $f(z,w)$ is holomorphic in $U \times W$, and for every fixed $z^0 \in E$, the function $f(z^0,w)$, with respect to w, extends holomorphically to G, except on finitly many singular points, then the function $f(z,w)$ extends holomorphically to the domain*

$$\{(z,w): \ \omega^*(z,E,U) + \omega^*(w,W,G) < -1\},$$

except on an analytic set S.

Lemma 3.5. *[22] Let $D \subset \mathbb{C}^n$, $(n > 1)$ be a bounded domain, and let $\hat{D}$ be the holomorphic hull of D. Then every holomorphic function defined in the domain $D \backslash A$, where $A \subset D$ is an analytic set of full dimension $(n-1)$, exdends holomorphically to the domain $\hat{D}$, except on an analytic set $\hat{A} \subset \hat{D}$, satisfying $\hat{A} \cap D = A$.*

4. Proof of Theorem 2.1

Let $D \subset \mathbb{C}_z$, $G \subset \mathbb{C}_w$ be bounded simply connected domains and $E \subset D$, $F \subset G$ are locally regular compact sets. Let the function $f(z,w)$ satisfy all the conditions of Theorem 2.1. We consider the following open neighborhoods of the regular compact sets

$$U_j = \left\{z \in D: \ \omega^*(z,E,D) < -1 + \frac{1}{j}\right\}, W_j = \left\{w \in G: \ \omega^*(w,F,G) < -1 + \frac{1}{j}\right\}$$

and

$$F_j = \left\{w^0 \in F: \ U_j \cap S_{w^0} = \emptyset\right\}, E_j = \left\{z^0 \in E: \ W_j \cap S_{z^0} = \emptyset\right\}, j = 1, 2, ...$$

Note that the open sets U_j and W_j consist of the union of a finite number of simply connected domains. It is clear that $E_j \subset E_{j+1}$, $F_j \subset F_{j+1}$ and $E = \bigcup_{j=1}^{\infty} E_j$, $F = \bigcup_{j=1}^{\infty} F_j$. Since E and F are nonpolar, there exists a number $j_0 \in \mathbb{N}$ such that the sets E_j and F_j are nonpolar for all $j \geq j_0$. According to the Sichak-Zakharyuta theorem, the function $f(z,w)$ extends holomorphically to the open set

$$\hat{\Omega}_j = \left\{(z,w) \in U_j \times W_j: \ \omega^*(z,E_j^0,U_j) + \omega^*(w,F_j^0,W_j) < -1\right\}.$$

Now, we consider the following open sets:

$$U_j^{(\alpha)} = \left\{z \in U_j: \ \omega^*(z,E_j^0,U_j) < -\alpha\right\}$$

$$W_j^{(\alpha)} = \left\{w \in W_j: \ \omega^*(w,F_j^0,W_j) < -1 + \alpha\right\}$$

for which $\alpha \in (0,1)$ is a fixed number. It is clear that $U_j^{(\alpha)} \times W_j^{(\alpha)} \subset \hat{\Omega}_j$.

It follows that the function $f(z,w)$ is holomorphic in $U_j^{(\alpha)} \times W_j^{(\alpha)}$. For each fixed $z^0 \in E_j^0$, the function $f(z^0,w)$, with respect to w, is holomorphic in $G \backslash S_{z^0}$. Hence, the restriction of the function $f(z,w)$ to the set $Y_j^{(\alpha)} = \left(U_j^{(\alpha)} \times W_j^{(\alpha)}\right) \cup \left(E_j^0 \times G\right)$ satisfies all the conditions of Corollary 3.4 and admits a holomorphic continuation to the open set

$$\hat{Y}_j^{(\alpha)} = \left\{(z,w) \in U_j^{(\alpha)} \times G: \ \omega^*(z,E_j^0,U_j^{(\alpha)}) + \omega^*(w,W_j^{(\alpha)},G) < -1\right\},$$

except on an analytic set $M_j^{(\alpha)} : M_j^{(\alpha)} \cap (\{z^0\} \cap G) = S_{z^0}$, for almost all $z^0 \in E_j$ with respect to the harmonic measure.

Similarly, the restriction of the function $f(z,w)$ to the set $Z_j^{(\alpha)} = \left(D \times F_j^0\right) \cup \left(U_j^{(\alpha)} \times W_j^{(\alpha)}\right)$ satisfies all the conditions of Corollary 3.4 and admits holomorphic continuation to the open set

$$\hat{Z}_j^{(\alpha)} = \left((z,w) \in D \times W_j^{(\alpha)}: \ \omega^*(z,U_j^{(\alpha)},D) + \omega^*(w,F_j^0,W_j^{(\alpha)}) < -1\right),$$

except on an analytic set $L_j^{(\alpha)} : L_j^{(\alpha)} \cap (D \cap \{w^0\}) = S_{w^0}$, for almost all $w^0 \in F_j$ with respect to the harmonic measure.

On the other hand, $Y_j^{(\alpha)} \subset \hat{X}_j$ and $Z_j^{(\alpha)} \subset \hat{X}_j$, where

$$\hat{X}_j = \left\{(z,w) \in D \times G : \ \omega^*(z, E_j^0, D) + \omega^*(w, F_j^0, G) < -1\right\}.$$

It follows that the holomorphic hulls $\hat{Y}_j^{(\alpha)}$ and $\hat{Z}_j^{(\alpha)}$ also belong to $\hat{X}_j$. Moreover $(D \times F_j^0) \cup (E_j^0 \times G) \subset \hat{Y}_j^{(\alpha)} \cup \hat{Z}_j^{(\alpha)} \subset \hat{X}_j$ holds.

Consequently, by Lemma 3.5, the function $f(z,w)$ extends holomorphically to the domain $\hat{X}_j$, except on some analytic set S_j satisfying $\hat{Y}_j^{(\alpha)} \cap S_j = M_j^{(\alpha)}$, $\hat{Z}_j^{(\alpha)} \cap S_j = L_j^{(\alpha)}$.

Finally, tending to $j \to +\infty$, and using the continuity of the harmonic measure, we obtain

$$\lim_{j \to +\infty} \hat{X}_j = \hat{X},$$

where

$$\hat{X} = \{(z,w) \in D \times G : \ \omega^*(z, E, D) + \omega^*(w, F, G) < -1\}.$$

Once again, by applying Lemma 3.5, we obtain that the function $f(z,w)$ extends holomorphically to the domain $\hat{X}$, except on an analytic set S satisfying $\hat{X}_j \cap S = S_j$. *This completes the proof of Theorem 2.1.*

The following example shows that this theorem does not exclude the case $S \cap (E \times F) \neq \emptyset$.

Example. Let us consider the set $X = (U \times F) \cup (E \times V)$, where $U = \{z \in \mathbb{C} : \ |z| < 1\}$, $V = \{w \in \mathbb{C} : \ |w| < 1\}$, $E = \left[0, \ \frac{1}{2}\right]$ and $F = \left[0, \ \frac{1}{2}\right]$. We put

$$f(z,w) = \begin{cases} \frac{z \cdot w}{z+w}, & z \cdot w \neq 0 \\ 0, & z \cdot w = 0. \end{cases}$$

Then the function $f(z,w)$ satisfies all the conditions of Theorem 2.1. The extension of the function has the set of singularities $S = \{(z,w) : \ z + w = 0\}$, and the intersection $S \cap (E \times F) = \{(0,0)\}$ is nonempty.

Acknowledgement. The authors would like to express their profound gratitude to Academician Azimbay Sadullaev for his exceptional guidance, thoughtful insights, and invaluable support throughout the development of this work.

References

[1] Hartogs F., Zur Theorie der analytischen Funktionen mehrerer unabhdngiger Verdnderlichen, insbesondere ber die Darstellung derselben durch Reihen, welche nach Potenzen einer Verdnderlichen fortschreiten. Math. Ann. (1906) Vol. 62, P. 1–88.

[2] Hukuhara M., L'extensions du theoreme d'Osgood et de Hartogs. Kansu-hoteisiki ogobi Oyo-Kaiseki, (1930) Vol. 48, Iss. 8, P. 48–49.

[3] Siciak J., Separately analytic functions and envelopes of holomorphy of some lower dimensional subsets of $\mathbb{C}^n$. Annales Polonici Mathematici, (1969) Vol. 22, Iss. 2, P. 145–171.

[4] Zaharjuta V., Separately analytic functions, generalizations of Hartogs' theorem, and envelopes of holomorphy. Mathematics of the USSR-Sbornik, (1976) Vol. 30, Iss. 1, P. 51–67.

[5] Sadullaev A. S., Plurisubharmonic measure and capacities on complex manifolds. Russian Mathematical Surveys, (1981) Vol. 36, P. 61–119.

[6] Sadullaev A. S., Plyurisubharmonic functions. Itogi Nauki I Tekhniki. Sovrem. Probl. Mat. Fund Napr. (1985) Vol. 8, P. 65–111.

[7] Sakai E., A note on meromorphic functions in several complex variables. Mem. Fac. Sci. Kyushu Univ. ser. A, (1957), Iss. 1, P. 75–80.

[8] Rothstein W., Ein neuer Beweis des Hartogschen Hauptsatzes und seine Ausdehnung auf meromorphe funktionen. Math. Z. (1950) Vol. 53, P. 84–95.

[9] Kazaryan M., On functions of several complex variables that are separately meromorphic. Mathematics of the USST-Sbornik, (1976) Vol. 4, P. 481–489.

[10] Kazaryan M., Holomorphic continuation of functions with special singularities in $\mathbb{C}^n$. Dokl. Akad. Nauk Arm. SSR, (1988) Vol. 76, P. 13–17.

[11] Ivashkovich S., The Hartogs-type extension theorem for meromorphic maps into compact Kehler manifolds. Inv. Math. (1992) Vol. 109, P. 47–54.

[12] Ruppenthal J., A $\bar{\partial}-$theoretical proof of Hartogs' extension theorem on Stein spaces with isolated singularities. Journal of Geometric Analysis, (2008) Vol. 18, P. 1127–1132.

[13] Sadullaev A. S., Chirka E. M., On continuation of functions with polar singularities. Mathematics of the USSR-Sbornik, (1988) Vol. 60, Iss. 2, P. 377–384.

[14] Sadullaev A. S., Imomkulov S. A., Extension of Holomorphic and Pluriharmonic Functions with Thin Singularities on Parallel Sections. Proc. Steklov Inst. Math. (2006) Vol. 253, P. 144–159.

[15] Tuichiev T. T., Continuation of functions in a fixed direction. Siberian Mathematical Journal, (1988) Vol. 29, Iss. 3, P. 142–147.

[16] Jarnicki M., Pflug P., An extension theorem for separately holomorphic functions with analytic singularities. Ann. Pol. Math. (2003) Vol. 80, P. 143–161.

[17] Jarnicki M., Pflug P., An extension theorem for separately meromorphic functions with pluripolar singularities. Kyushu J. Math. (2003) Vol. 57, P. 291–302.

[18] Gonchar A., On analytic continuation from the "edge of the wedge". Annales Academiae Scientiarum Fennicae. Series A I. Mathematica, (1985) Vol. 10, P. 221–225.

[19] Pflug P., Viet-Ahn N., A boundary cross theorem for separately holomorphic functions. Ann. Pol. Math. (2004) Vol. 84, P. 237–271.

[20] Atamuratov A., Meromorphic continuation of separately meromorphic functions defined on part of a domain boundary. Uz. Math. Journal, (2009), Iss. 3, P. 18–26.

[21] Sadullaev A. S., A note on meromorphic functions with discrete singularities oz_n. Uz. Math. Journal, (2010), Iss. 1, P. 130–134.

[22] Dloussky G., Envelopes d'holomorphie et prolongement d'hypersurfaces. Seminaire Pierre Lelong (Analyse). Springer-Verlag, (1977), vol. 578, P. 217–235.

Imomkulov Sevdiyor,
National University of Uzbekistan,
Tashkent, Uzbekistan.
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences
Tashkent, Uzbekistan.
e-mail: sevdiyor_i@mail.ru

Rasulov Kamol,
Tashkent University of Social Innovation,
Tashkent, Uzbekistan.
e-mail: rasulov.kamol@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 100–106
DOI: 10.29229/uzmj.2026-1-10

Spectral properties of the one-dimensional Schrödinger Hamiltonian with non-local $\delta'(x \pm y)$ potentials

Ismoilov G.

Abstract. We consider a model of one-dimensional Schrödinger Hamiltonian perturbed by two identical non-local interactions of the form $\delta'(x \pm y)$, symmetrically located at the points $\pm y$ from the origin. The Schrödinger operator under consideration is constructed as a self-adjoint extension of the symmetric Laplacian. The essential spectrum is described, and the condition for the existence of the eigenvalue of the Schrödinger operator is investigated. The main results are based on the analysis of the spectral analysis of self-adjoint extension of the operator $\mathbf{h}$.

Keywords: Schrödinger operators, non-local delta prime interactions, eigenvalues, eigenfunctions
MSC (2020): 47A10, 47B25, 81Q10

1. Introduction

The problem of point interactions among two or three identical quantum particles interacting via point potentials (also referred to as contact or singular potentials) has been studied extensively in the literature. Early foundational work was carried out by Berezin and Faddeev [1], as well as R.A. Minlos and L.D. Faddeev [2, 3], who proposed a rigorous mathematical framework for describing point interactions between two and three particles, respectively.

In [2, 3], the Hamiltonian of such systems was analyzed using the theory of self-adjoint extensions of symmetric operators. It was represented as a self-adjoint extension of the symmetric Laplace operator, defined on the domain of functions of three variables $x_1, x_2, x_3 \in \mathbb{R}$ that vanish whenever two coordinates coincide, i.e., $x_j = x_k$ for $j \neq k$, $j, k = 1, 2, 3$.

This extension is known as the Skornyakov–Ter-Martirosyan extension. In [4], building upon the results of [1, 2], the Hamiltonian for a three-particle system (two identical fermions and a third particle of different type, all with equal masses) interacting via point potentials was investigated. It was shown that the Skornyakov–Ter-Martirosyan extensions are self-adjoint and semi-bounded.

In the work, we consider a one-dimensional quantum particle interacting with external fields through non-local Dirac delta prime potentials of the form $\delta'(x \pm y)$, located symmetrically at $\pm y$, with $y > 0$. Formally, the corresponding Schrödinger operator is given by

$$\widehat{\mathbf{h}} := \widehat{\mathbf{h}}_0 - \delta'(x + y) - \delta'(x - y), \tag{1.1}$$

where $\widehat{\mathbf{h}}_0 := -\Delta$ is the Laplace operator, and $\delta'(x)$ denotes the derivative of the Dirac delta function. However, due to the singular nature of the delta function's derivative, the expression in (1.1) does not define a proper operator on the Hilbert space $L^2(\mathbb{R})$.

To give a rigorous meaning to (1.1), we construct the operator as a self-adjoint extension of the symmetric Laplace operator with a suitably restricted domain. This construction ensures that the resulting operator is mathematically well-defined and suitable for physical analysis. The theory of self-adjoint extensions provides the necessary tools for this rigorous formulation.

In the work [5], a non-local Schrödinger operator of the form (1.1) is investigated by introducing a regularized version of the singular interaction and applying a renormalization procedure to the coupling constant λ. This approach leads to a self-adjoint extension of the operator, which is shown to possess two negative eigenvalues. The dependence of these eigenvalues on the coupling constant and the separation distance y is analyzed. Furthermore, the resonance behavior of the corresponding operator is also investigated.

One-dimensional models with point perturbations are particularly advantageous, as they allow the exploration of various qualitative features of quantum systems. For related studies involving both local and non-local delta potentials in one-body problems, see [6, 7, 8, 9, 10, 11, 12].

In the momentum representation, after reduction of variables, we construct the Krein-Višik-Birman extension $\mathbf{h}$ of the associated Hamiltonian. It is proven that the essential spectrum of this extension coincides with the non-negative real axis. We also study the conditions under which the operator admits eigenvalues. The main results of this work are based on the spectral analysis of the operator $\mathbf{h}$. In particular, we describe the essential spectrum (see Theorem 2.1) and explicitly establish the existence of eigenvalues of the operator $\mathbf{h}$ (see Theorem 2.3).

1.1. **Preliminaries.** Now, in order to give meaning to a Schrödinger operator with particle interactions involving non-local delta functions, as defined in (1.1), we define it on the set of functions in the $L^2(\mathbb{R})$ space that satisfy the condition $\phi'(\pm y) = 0$. Thus, the singular contributions arising from the action of the Laplace operator are canceled out by the delta functions in (1.1). It should be emphasized that any operator defined in this way is an extension of the symmetric operator $\widehat{\mathbf{h}}_0$, which is defined on the following manifold:

$$D(\widehat{\mathbf{h}}_0) = \left\{\phi \in L^2(\mathbb{R}) : \Delta\phi \in L^2(\mathbb{R}),\ \phi'(\pm y) = 0,\ y > 0\right\}, \tag{1.2}$$

where the singular contributions related to the delta functions in (1.1) disappear.

After applying the Fourier transform, the operator $\widehat{\mathbf{h}}_0$ transforms into the operator

$$(\mathbf{h}_0 f)(p) = p^2 f(p)$$

defined on the set $D(\mathbf{h}_0) \subset L^2(\mathbb{R})$ consisting of functions $f(p)$ that satisfy:

$$\int\limits_{\mathbb{R}} (1+p^2)^2 |f(p)|^2\, dp < \infty,$$

$$\int\limits_{\mathbb{R}} p e^{\pm iyp} f(p)\, dp = 0. \tag{1.3}$$

According to [13], the defeciency subspace $\mathfrak{R}_z$ of the operator $\mathbf{h}_0$, is defined by

$$\mathfrak{R}_z = (\mathrm{Ran}(\mathbf{h}_0 - zI))^{\perp} = \left\{\hat{\varphi} \in L^2(\mathbb{R}) : (\mathbf{h}_0 - zI) f \perp \hat{\varphi},\quad f \in D(\mathbf{h}_0)\right\}. \tag{1.4}$$

Moreover, from (1.3) and (1.4), for any $z \in \Pi_0 = \mathbb{C} \setminus [0, \infty)$, the deficiency subspace $\mathfrak{R}_z \subset L^2(\mathbb{R})$ of $\mathbf{h}_0$ consists of functions of the form

$$\hat{\varphi}(p) = \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{p^2 - \bar{z}}, \quad \hat{c}_1, \hat{c}_2 \in \mathbb{C}. \tag{1.5}$$

It follows from (1.5) that for any $z \in \Pi_0$, the deficiency subspace $\mathfrak{R}_z$ is two-dimensional. Therefore, $\mathbf{h}_0$ is a symmetric operator with deficiency indices (2,2). Using the general extension theory [14], we conclude that the operator $\mathbf{h}_0$ admits two-parameter family of self-adjoint extensions.

Since the operator $\mathbf{h}_0$ is non-negative, we use the theory of extensions of semibounded operators, as developed in [2, 3]. Furthermore, the deficiency indices of $\mathbf{h}_0$ remain constant for all $z \in \Pi_0$. Therefore, it suffices to analyze the case $z = -1$. The deficiency subspace $\mathfrak{R}_{-1}$ of $\mathbf{h}_0$ consists of functions of the form

$$\hat{\varphi}_{-1}(p) = \frac{\hat{c}_1 p e^{ix_0 p} + \hat{c}_2 p e^{-ix_0 p}}{p^2 + 1},\ (\hat{c}_1, \hat{c}_2) \in \mathbb{C}^2.$$

Following the approach in [2, 3] and [4], the adjoint operator $\mathbf{h}_0^*$ is characterized by the following lemma.

Lemma 1.1. *The domain $D(\mathbf{h}_0^*)$ of the adjoint operator $\mathbf{h}_0^*$ consists of functions of the form*

$$g(p) = f(p) + \frac{\hat{d}_1 p e^{iyp} + \hat{d}_2 p e^{-iyp}}{p^2 + 1} + \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{(p^2 + 1)^2} \tag{1.6}$$

where $f \in D(\mathbf{h}_0), \hat{c}_1, \hat{c}_2, \hat{d}_1, \hat{d}_2 \in \mathbb{C}$. The operator $\mathbf{h}_0^*$ acts on functions of the form (1.6) *via the rule*

$$\mathbf{h}_0^* g(p) = p^2 g(p) - \hat{d}_1 p e^{ix_0 p} - \hat{d}_2 p e^{-ix_0 p},$$

with the constants $\hat{d}_1, \hat{d}_2$ taken from the decomposition (1.6).

We now construct the extension of the operator $\mathbf{h}_0$. Define the set $D(\mathbf{h}), D(\mathbf{h}_0) \subset D(\mathbf{h}) \subset D(\mathbf{h}_0^*)$, as follows:

$$D(\mathbf{h}) = \left\{ g \in D(\mathbf{h}_0^*) : g(p) = f(p) + \frac{\hat{c}_1 pe^{iyp} + \hat{c}_2 pe^{-iyp}}{p^2+1} + \frac{\hat{c}_1 pe^{iyp} + \hat{c}_2 pe^{-iyp}}{(p^2+1)^2}, f \in D(\mathbf{h}_0) \right\}. \tag{1.7}$$

The restriction of the operator $\mathbf{h}_0^*$ to the domain $D(\mathbf{h})$ is denoted by $\mathbf{h}$, and it acts as

$$(\mathbf{h}g)(p) = p^2 g(p) - \hat{c}_1 pe^{iyp} - \hat{c}_2 pe^{-iyp}, \tag{1.8}$$

where the constants $\hat{c}_1$ and $\hat{c}_2$ come from the decomposition (1.7) of the function g.
By construction, $\mathbf{h}$ is an extension of the operator $\mathbf{h}_0$.

Theorem 1.2. *The extension* $\mathbf{h}$ *is a self-adjoint operator.*

Proof. It is straightforward to check that for any $g_1, g_2 \in D(\mathbf{h})$, the following relation

$$(\mathbf{h}g_1, g_2) = (g_1, \mathbf{h}g_2),$$

holds, showing that $\mathbf{h}$ is symmetric operator. To prove self-adjointness, it suffices to show that the deficiency indices of $\mathbf{h}$ are $(0,0)$.
Let $\hat{\varphi}_b \in \mathfrak{N}_{-1}(\mathbf{h}_0)$. Then

$$\hat{\varphi}_b(p) = \frac{\hat{b}_1 pe^{iyp} + \hat{b}_2 pe^{-iyp}}{p^2+1}, \quad \hat{b}_1, \hat{b}_2 \in \mathbb{C}.$$

For any $g \in D(\mathbf{h})$, the equality

$$((\mathbf{h}+I)g, \hat{\varphi}_b) = 0$$

holds. Considering (1.7), we obtain

$$((\mathbf{h}+I)g, \hat{\varphi}_b) = ((\mathbf{h}_0+I)f, \hat{\varphi}_b) + \int\limits_{\mathbb{R}} \frac{(\hat{c}_1 pe^{iyp} + \hat{c}_2 pe^{-iyp})\overline{(\hat{b}_1 pe^{iyp} + \hat{b}_2 pe^{-iyp})}}{(p^2+1)^2}\, dp.$$

From the relation

$$\int_{\mathbb{R}^3} (p^2+1) f(p) \overline{\hat{\varphi}_b(p)}\, dp = 0,$$

and choosing $\hat{c}_1 = \hat{b}_1, \hat{c}_2 = \hat{b}_2$, we get

$$\int_{\mathbb{R}} \frac{|\hat{b}_1 pe^{iyp} + \hat{b}_2 pe^{-iyp}|^2}{(p^2+1)^2}\, dp = 0,$$

which implies

$$\hat{b}_1 pe^{iyp} + \hat{b}_2 pe^{-iyp} = 0.$$

Hence $\hat{b}_1 = \hat{b}_2 = 0$, so $\hat{\varphi}_b(p) = 0$, and the deficiency indices of $\mathbf{h}$ are $(0,0)$.

2. Spectral properties of the operator $\mathbf{h}$

The main results of the paper are stated in the following theorems.

Theorem 2.1. *The essential spectrum of the operator* $\mathbf{h}$ *coincides with the nonnegative real semiaxis* $[0, \infty)$.

Proof. For each $z \geq 0$ consider the sequence of cut-off layers:

$$\mathcal{G}_n(z) = \left\{ p \in \mathbb{R} : \sqrt{z} + \frac{1}{n+1} < |p| < \sqrt{z} + \frac{1}{n} \right\}, \quad n = 1, 2, 3, \ldots$$

Each layer $\mathcal{G}_n(z)$ is divided into two symmetric parts:

$$\mathcal{G}_n^+(z) = \{p \in \mathcal{G}_n(z) : p \geq 0\}, \quad \mathcal{G}_n^-(z) = \{p \in \mathcal{G}_n(z) : p < 0\}.$$

By construction, these parts are equal in measure:

$$\mu(\mathcal{G}_n^+(z)) = \mu(\mathcal{G}_n^-(z)) = \frac{1}{2}\mu(\mathcal{G}_n(z)).$$

A simple calculation shows that

$$V_n = \mu\left(\mathcal{G}_n(z)\right) = \frac{2}{n\,(n+1)}.$$

Define a sequence of test functions $f_n^{(z)}$, $n \in \mathbb{N}$, as follows:

$$f_n^{(z)}(p) = \begin{cases} \dfrac{p\cos{(yp)}}{\sqrt{V_n}}, \text{if } p \in \mathcal{G}_n^+(z) \\ -\dfrac{p\cos{(yp)}}{\sqrt{V_n}}, \text{if } p \in \mathcal{G}_n^-(z) \\ 0, \text{if } p \in \mathbb{R} \setminus \mathcal{G}_n(z). \end{cases}$$

It is easy to verify that $f_n^{(z)} \in L^2(\mathbb{R})$, $\|f_n^{(z)}\| = 1$, and $f_n^{(z)} \perp f_m^{(z)}$ for $n \neq m$. Moreover, $f_n^{(z)} \in D(\mathbf{h}_0)$, i.e.,

$$\int\limits_{\mathbb{R}} pe^{\pm iyp} f_n^{(z)}(p)dp = 0, \quad \forall n \in \mathbb{N}.$$

Furthermore,

$$\left\|(\mathbf{h} - zI)f_n^{(z)}\right\|^2 = \int\limits_{\mathbb{R}} |(p^2 - z)f_n^{(z)}(p)|^2 dp = \frac{1}{V_n} \int\limits_{\mathcal{G}_n(z)} |(p^2 - z)p\cos(yp)|^2\, dp \leq$$

$$\leq \frac{1}{V_n} \int\limits_{\mathcal{G}_n(z)} |(p^2 - z)p|^2 dp = \frac{2}{V_n} \int\limits_{\sqrt{z}+\frac{1}{n+1}}^{\sqrt{z}+\frac{1}{n}} (p^2 - z)^2 p^2 dp <$$

$$< \frac{2}{V_n}\left(2\sqrt{z} + \frac{1}{n}\right)^2 \left(\sqrt{z} + \frac{1}{n}\right)^2 \frac{1}{n^2} \cdot \frac{1}{n(n+1)} = \frac{1}{n^2}\left(2\sqrt{z} + \frac{1}{n}\right)^2 \left(\sqrt{z} + \frac{1}{n}\right)^2.$$

Thus,

$$\lim_{n\to\infty} \|(\mathbf{h} - zI)f_n^{(z)}\| = 0.$$

This implies that for every $z \geq 0$, we have $z \in \sigma_{ess}(\mathbf{h})$, so $[0;\infty) \subset \sigma_{ess}(\mathbf{h})$. To prove the reverse inclusion $\sigma_{ess}(\mathbf{h}) \subset [0;\infty)$, we construct the resolvent of $\mathbf{h}$.

Let $\psi \in L^2(\mathbb{R})$. Then, $(\mathbf{h} - zI)g = \psi$. Moreover,

$$(p^2 - z)g(p) - \hat{c}_1 pe^{iyp} - \hat{c}_2 pe^{-iyp} = \psi(p)$$

or

$$g(p) = \frac{\psi(p)}{p^2 - z} + \frac{\hat{c}_1 pe^{iyp} + \hat{c}_2 pe^{-iyp}}{p^2 - z}. \tag{2.1}$$

Comparing this with the know extension of functions in the extended domain (e.g.(1.7)), (2.1) we derive an equation for the unknown coefficients $\hat{c}_1$ and $\hat{c}_2$.

Multiplying both sides by $pe^{\pm iyp}$ and integrating over $\mathbb{R}$ gives:

$$\begin{cases} \hat{a}(z)c_1 + \hat{b}(z)c_2 = \dfrac{2}{\pi} \displaystyle\int\limits_{\mathbb{R}} \frac{se^{iys}\psi(s)}{s^2 - z}\, ds, \\ \hat{b}(z)c_1 + \hat{a}(z)c_2 = \dfrac{2}{\pi} \displaystyle\int\limits_{\mathbb{R}} \frac{se^{-iys}\psi(s)}{s^2 - z}\, ds, \end{cases} \tag{2.2}$$

where

$$\hat{a}(z) = 2\sqrt{-z}e^{-2y\sqrt{-z}} - (1+2y)e^{-2y},\ \hat{b}(z) = 2\sqrt{-z} - 1. \tag{2.3}$$

Here the following elementary integrals are used

$$\int\limits_{\mathbb{R}} \left(\frac{p^2 e^{iyp}}{p^2+1} - \frac{p^2 e^{iyp}}{p^2 - z} \right) dp = -\pi e^{-y} + \pi\sqrt{-z}e^{-y\sqrt{-z}},\ z<0,$$

$$\int\limits_{\mathbb{R}} \frac{p^2 e^{iyp}}{(p^2+1)^2} dp = \frac{\pi}{2}(1-y)e^{-y}.$$

Using these equations, we solve for $\hat{c}_1$ and $\hat{c}_2$ and obtain the following representation of the resolvent $R_z(\mathbf{h})$:

$$(R_z\psi)(p) = \frac{\psi(p)}{p^2 - z} + \frac{4}{\pi(p^2-z)} \left(\frac{p\cos(yp)}{\hat{u}(z)} \int\limits_{\mathbb{R}} \frac{s\cos(ys)\psi(s)}{s^2-z} ds - \frac{p\sin(yp)}{\hat{v}(z)} \int\limits_{\mathbb{R}} \frac{s\sin(ys)\psi(s)}{s^2-z} ds \right).$$

Here

$$\hat{u}(z) := \hat{u}(y;z) = 2\sqrt{-z}(e^{-2y\sqrt{-z}} + 1) - (1+2y)e^{-2y} - 1,$$
$$\hat{v}(z) := \hat{v}(y;z) = 2\sqrt{-z}(e^{-2y\sqrt{-z}} - 1) - (1+2y)e^{-2y} + 1.$$

The condition $p^2 - z \neq 0$ for $z < 0$ ensures that the resolvent of the operator $\mathbf{h}$ exists and is bounded within the domain $\mathbb{C} \setminus ([0,\infty) \cup \{z \in (-\infty, 0) : \hat{u}(z) = 0 \,\text{or}\, \hat{v}(z) = 0\})$.

In light of Lemma 2.2, which states that $\hat{u}(z)$ and $\hat{v}(z)$ possess at most one simple zero, we conclude that $\sigma_d(h) = \{z \in (-\infty,0) : \hat{u}(z) = 0 \,\text{or}\, \hat{v}(z) = 0\}$ and $\sigma_{ess}(\mathbf{h}) = [0,\infty)$. □

The number $z, z < 0$ is an eigenvalue of the operator $\mathbf{h}$ if and only if the number z is the zeros of the function $\hat{u}(z)$ or $\hat{v}(z)$.

Lemma 2.2. *For any $y \in (0,\infty)$, the functions $\hat{u}(z)$ and $\hat{v}(z)$ have only simple zeros in $(-\infty, 0)$.*

Proof. First, we show the monotonicity of the functions $\hat{u}(z)$ and $\hat{v}(z)$. Evaluating the derivatives, we obtain:

$$\hat{u}'(z) = \frac{1 - (1 - 2y\sqrt{-z})e^{-2y\sqrt{-z}}}{\sqrt{-z}},\ \hat{v}'(z) = -\frac{1 + e^{-2y\sqrt{-z}}(1 - 2y\sqrt{-z})}{\sqrt{-z}}.$$

Note that for all $y \neq 0$

$$e^{2y} > 1 + 2y. \tag{2.4}$$

According to (2.4), the function $\hat{u}(z)$ is strictly increasing and the function $\hat{v}(z)$ is strictly decreasing.

Next, to examine the zeros of the functions $\hat{u}(z)$ and $\hat{v}(z)$, we establish the following limits:

$$\hat{u}(-0) = \lim_{z \nearrow 0} \hat{u}(z) = -\left(1 + (1+2y)e^{-2y}\right),$$
$$\hat{u}(-\infty) = \lim_{z \to -\infty} \hat{u}(z) = +\infty$$
$$\hat{v}(-0) = \lim_{z \nearrow 0} v(z) = 1 - (1+2y)e^{-2y},$$
$$\hat{v}(-\infty) = \lim_{z \to -\infty} \hat{v}(z) = -\infty.$$

Thus, for any $y > 0$, the inequality $\hat{u}(-0) < 0 < \hat{u}(-\infty)$ holds. Additionally, since $\hat{u}$ is strictly increasing, for each y there exists a value $z_u(y)$ such that $\hat{u}(z_u(y)) = 0$. Similarly, for each y, we can show that $\hat{v}(z)$ $(-\infty, 0)$ has a unique simple zero $z_v(y)$ in the interval.

Moreover, the following graph illustrates the behavior of the functions $\hat{u}(z)$ and $\hat{v}(z)$ for a fixed value of y.

□

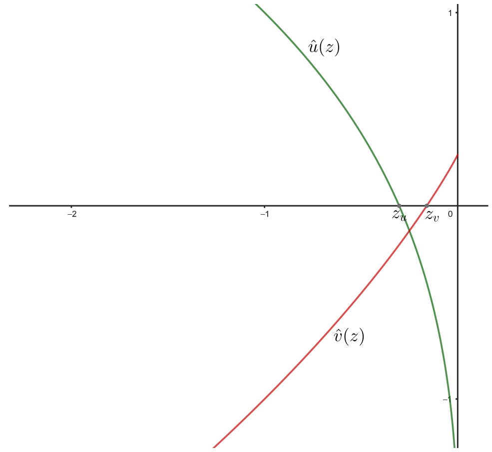

FIGURE 1. Graphs of the functions $\hat{u}(z)$ and $\hat{v}(z)$.

Theorem 2.3. *For any $y \in (0, \infty)$, the operator $\mathbf{h}$ has exactly two negative eigenvalues, z_u and z_v, corresponding to the eigenfunctions (up to a constant factor) of the form*

$$g_1(y;p) = \frac{p\cos(yp)}{p^2 - z_u} \quad \text{and} \quad g_2(y;p) = \frac{p\sin(yp)}{p^2 - z_v},$$

where z_u and z_v are zeros of the functions $\hat{u}(z)$ and $\hat{v}(z)$, respectively.

Proof. The proof of the existence of the eigenvalues of $\mathbf{h}$ follows from Lemma 2.2.

Now we prove that the eigenfunctions of the operator $\mathbf{h}$ have the form

$$g_1(p) = \frac{p\cos(yp)}{p^2 - z_1(y)} \quad \text{and} \quad g_2(p) = \frac{p\sin{(yp)}}{p^2 - z_2(y)}.$$

From equation $(\mathbf{h} - zI)g(p) = 0$ we receive

$$g(p) = \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{p^2 - z}. \tag{2.5}$$

Comparing (1.7) and (2.5) we take the equality

$$f(p) + \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{p^2+1} + \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{(p^2+1)^2} = \frac{\hat{c}_1 p e^{iyp} + \hat{c}_2 p e^{-iyp}}{p^2 - z}.$$

Multiplying by the factors $pe^{\pm iyp}$, we have

$$f(p)pe^{iyp} + \frac{\hat{c}_1 p^2 e^{2iyp} + \hat{c}_2 p^2}{p^2+1} + \frac{\hat{c}_1 p^2 e^{2iyp} + \hat{c}_2 p^2}{(p^2+1)^2} = \frac{\hat{c}_1 p^2 e^{2iyp} + \hat{c}_2 p^2}{p^2 - z},$$
$$f(p)pe^{-iyp} + \frac{\hat{c}_1 p^2 + \hat{c}_2 p^2 e^{-2iyp}}{p^2+1} + \frac{\hat{c}_1 p^2 + \hat{c}_2 p^2 e^{-2iyp}}{(p^2+1)^2} = \frac{\hat{c}_1 p^2 + \hat{c}_2 p^2 e^{-2iyp}}{p^2 - z}.$$

Integrating the last equalities over $\mathbb{R}$ we obtain a system of equations for determining $\hat{c}_1$ and $\hat{c}_2$,

$$\begin{cases} \hat{a}(z)\hat{c}_1 + \hat{b}(z)\hat{c}_2 = 0, \\ \hat{b}(z)\hat{c}_1 + \hat{a}(z)\hat{c}_2 = 0, \end{cases}$$

where $\hat{a}(z)$ and $\hat{b}(z)$ are defined as in equation (2.3). Hence

$$\left(\hat{a}^2(z) - \hat{b}^2(z)\right) c_i = 0, \quad i = 1, 2.$$

If $\hat{a}(z) = \hat{b}(z)$ (resp. $\hat{a}(z) = -\hat{b}(z)$), then as c_i we can take any number, in particular, $c_i = 1$.

Thus, if the number z satisfies the equation $\hat{u}(z) = \hat{a}(z) + \hat{b}(z) = 0$ (resp. $\hat{v}(z) = \hat{a}(z) - \hat{b}(z) = 0$), then z is an eigenvalue of the operator $\mathbf{h}$ and of the functions

$$g_1(p) = \frac{p\cos(yp)}{p^2 - z} \quad \left(\text{resp.} \quad g_2(p) = \frac{p\sin(yp)}{p^2 - z}\right)$$

corresponding eigenfunctions of the operator $\mathbf{h}$. □

References

[1] Berezin F., Faddeev L., Remark on the Schrödinger equation with singular potential. Dokl. Akad. Nauk SSSR, (1961) Vol. 137, Iss. 5, P. 1011–1014.

[2] Minlos R., Faddeev L., Point interaction for a three-particle system in quantum mechanics. Dokl. Akad. Nauk SSSR, (1961) Vol. 141, Iss. 6, P. 1335–1338.

[3] Minlos R., Faddeev L., Comment on the problem of three particles with point interactions. Sov. Phys. JETP, (1962) Vol. 14, Iss. 6, P. 1315–1316.

[4] Minlos R., Shermatov M., On point-like interactions of three quantum particles. Vestnik Moskovkogo Universiteta. Seriya 1. Matematika. Mekhanika, (1989) Vol. 6, P. 7–14.

[5] Fassari S., Gadella M., Nieto L., Rinaldi F., Analysis of a one-dimensional Hamiltonian with a singular double well consisting of two nonlocal δ' interactions. Eur. Phys. J. Plus, (2024) Vol. 139, P. 132.

[6] Albeverio S., Kurasov P., Singular Perturbations of Differential Operators. London Mathematical Society, Cambridge, U.K., (2000).

[7] Gadella M., Glasser M., Nieto L., One Dimensional Models with a Singular Potential of the Type -$\alpha\delta(x) + \beta\delta^{'}(x)$. Int. J. Theor. Phys. (2011) Vol. 50, P. 2144–2152.

[8] Gadella M., Glasser M., Nieto L., The infinite square well with a singular perturbation. Int. J. Theor. Phys. (2011) Vol. 50, P. 2191–2200.

[9] Fassari S., Rinaldi F., On the Spectrum of the Schrödinger Hamiltonian of the One-Dimensional Harmonic Oscillator Perturbed by Two Identical Attractive Point Interactions. Rep. Math. Phys. (2012) Vol. 69, P. 353–370.

[10] Albeverio S., Fassari S., Rinaldi F., The discrete spectrum of the spinless one-dimensional Salpeter Hamiltonian perturbed by δ-interactions. J. Phys. A, (2015) Vol. 48, P. 185301.

[11] Albeverio S., Fassari S., Rinaldi F., The Hamiltonian of the harmonic oscillator with an attractive δ-interaction centred at the origin as approximated by the one with a triple of attractive δ-interactions. J. Phys. A, (2016) Vol. 49, P. 025302.

[12] Erman F., Gadella M., Tunal S., Uncu H., A singular one-dimensional bound state problem and its degeneracies. EPJ Plus, (2017) Vol. 132, P. 352.

[13] Akhiezer N., Glazman I., Theory of Linear Operators in Hilbert Space. Dover Books on Mathematics, (1993).

[14] Birman M., Solomjak M., Spectral theory of selfadjoint operators in Hilbert space. D. Reidel Publishing Co., Dordrecht, (1987).

Ismoilov Golibjon,
Samarkand State University, named after Sharof Rashidov,
Samarkand, Uzbekistan
e-mail: golibjon.ismoilov.tdtu@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 107–115
DOI: 10.29229/uzmj.2026-1-11

Essential and discrete spectrum of the Schrödinger operator of a system of two particles on a lattice

Khalkhuzhaev A., Makhmudov Kh., Miyassarov A.

Abstract. We consider the Hamiltonian of a system of two fermions on a two-dimensional lattice $\mathbb{Z}^2$ with a certain type potential. It is proved that the subspace of odd functions $L_2^o(\mathbb{T}^2)$ is represented as a direct sum of the subspaces $L_2^{eo}(\mathbb{T}^2)$ and $L_2^{oe}(\mathbb{T}^2)$, which are invariant under the operator $H(\mathbf{k})$, $\mathbf{k}=(k_1,k_2)\in\mathbb{T}^2$, associated with this Hamiltonian. For any $k_1\in(-\pi,\pi]$, it is proved that the operator $H^{eo}(k_1,\pi)=H(k_1,\pi)\big|_{L_2^{eo}(\mathbb{T}^2)}$ has an infinite number of eigenvalues and for any $k_1\in(-\pi,\pi)$, the operator $H^{oe}(k_1,\pi)=H(k_1,\pi)\big|_{L_2^{oe}(\mathbb{T}^2)}$ has a finite number eigenvalues lying to the left of the essential spectrum. An asymptotic formula is obtained for the number of eigenvalues of the operator $H^{oe}(k_1,\pi)$ as $k_1\to\pi$.

Keywords: Schrödinger operator, lattice, fermion, quasi-momentum, invariant subspaces, essential spectrum, eigenvalue

1. Introduction and formulation of main results

The discrete spectrum of the two-particle continuous Schrödinger operator $h_\lambda=-\Delta+\lambda V$ has been studied with various assumptions imposed on the potential V. Conditions ensuring the finiteness of the negative spectrum and the absence of positive eigenvalues of h_λ are presented in [1]. When $V\leq 0$, the number of negative eigenvalues $N(\lambda)$ is a non-decreasing function of $\lambda\in(0,\infty)$, and each eigenvalue $z_n(\lambda)$ is monotonically decreasing on $(0,\infty)$. As the coupling constant λ decreases, the bound-state energies of h_λ approach the edge of the continuous spectrum [1], and for certain finite values of λ, they may lie exactly on the boundary. These considerations naturally lead to two questions. Does a threshold state correspond to a bound or virtual state, meaning, is its wave function square-integrable? And as λ decreases further, where do the bound states "disappear to"? The first of these questions has been studied in [2], [3], and [4].

The Hamiltonian of a two-particle system on a lattice is expanded in the momentum representation into the following direct von Neumann integral [5]:

$$H\simeq\int_{\mathbf{k}\in\mathbb{T}^2}\oplus H(\mathbf{k})\ \mathrm{d}\mathbf{k},$$

where $\mathbb{T}^2$ is a two-dimensional torus.

It turns out that the spectrum of the fiber operator is $H(\mathbf{k})$ quite sensitive to variations in the quasi-momentum $\mathbf{k}\in\mathbb{T}^d$. The two-particle Schrödinger operator $H(\mathbf{k})$, $\mathbf{k}\in\mathbb{T}^3$, corresponding to the Hamiltonian of a two-particle system on the three-dimensional lattice $\mathbb{Z}^3$, was considered in [6]. It was shown that the function $N(\mathbf{k})=N(k_1,k_2,k_3)$, representing the number of eigenvalues lying below the essential spectrum of $H(\mathbf{k})$, is non-decreasing with respect to each component $k_i\in[0,\pi]$, for $i=1,2,3$.

In the work [7], the spectral properties of the two-particle Schrödinger operator $H(\mathbf{k})$, acting in the ν-dimensional lattice space $\mathbb{Z}^\nu$, were studied. It was shown that the operator has only a finite number of negative eigenvalues under rather general assumptions on the interaction potential $\hat{v}$.

In [8], was considered the Hamiltonian of a two-particle bosonic system on the two-dimensional lattice $\mathbb{Z}^2$ under a specific type of interaction potential. The associated Schrödinger operator $H(\mathbf{k})$, with total quasi-momentum $\mathbf{k}\in\mathbb{T}^2$, was shown to possess an infinite number of eigenvalues when $\mathbf{k}=\boldsymbol{\pi}=(\pi,\pi)$. It has been established that the eigenvalue $z_0(\pi)=4-\bar{v}(0)$ is non-degenerate, $z_1(\pi)=4-\bar{v}(1)$ has multiplicity two, $z_2(\pi)=4-\bar{v}(2)$ appears with multiplicity four, while for all $n\geq 3$, the eigenvalues $z_n(\pi)=4-\bar{v}(n)$ each have multiplicity five. It was also shown that each multiple eigenvalue of the operator $H(\pi)$ becomes a simple eigenvalue under perturbation. In addition, asymptotic expressions for the eigenvalues of $H(\pi-2\beta,\pi)$ have been obtained with an accuracy up to terms of order β^2.

Let $\mathbb{Z}^2$ be the two-dimensional lattice, and let $(\mathbb{Z}^2)^2$ denote the Cartesian product of $\mathbb{Z}^2$. We denote by $\ell_2((\mathbb{Z}^2)^2)$ the Hilbert space of square-summable functions defined on $(\mathbb{Z}^2)^2$, and by $\ell_2^{as}((\mathbb{Z}^2)^2) \subset \ell_2((\mathbb{Z}^2)^2)$ denote the subspace consisting of antisymmetric functions.

The free Hamiltonian $\hat{H}_0$ for a pair of fermions with mass equal to one on the two-dimensional lattice $\mathbb{Z}^2$ acts as a bounded, self-adjoint operator in the space $\ell_2^{as}((\mathbb{Z}^2)^2)$, and is explicitly defined by

$$\hat{H}_0 = -\frac{1}{2}\Delta_1 - \frac{1}{2}\Delta_2.$$

Here $\Delta_1 = \Delta \otimes I$ and $\Delta_2 = I \otimes \Delta$, where I is the identity operator, the lattice Laplacian Δ is a difference operator describing the transfer of a particle from one site to a nearest-neighbor site:

$$(\Delta\hat{\psi})(\mathbf{x}) = \sum_{j=1}^{2}\left[\hat{\psi}(\mathbf{x}+\mathbf{e}_j) + \hat{\psi}(\mathbf{x}-\mathbf{e}_j) - 2\hat{\psi}(\mathbf{x})\right], \quad \mathbf{x} \in \mathbb{Z}^2, \quad \hat{\psi} \in \ell_2(\mathbb{Z}^2),$$

where the vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ form the standard basis in $\mathbb{Z}^2$.

The total Hamiltonian $\hat{H}$ of the two-fermion system acts in the $\ell_2^{as}((\mathbb{Z}^2)^2)$ and is defined as the difference between the free Hamiltonian $\hat{H}_0$ and the interaction operator $\hat{V}$, as described in [9]:

$$\hat{H} = \hat{H}_0 - \hat{V}.$$

The interaction potential $\hat{V}$ is a multiplication operator acting pointwise as

$$(\hat{V}\hat{\psi})(\mathbf{x},\mathbf{y}) = \hat{v}(\mathbf{x}-\mathbf{y})\hat{\psi}(\mathbf{x},\mathbf{y}), \quad \hat{\psi} \in \ell_2^{as}((\mathbb{Z}^2)^2),$$

where $\hat{v}$ is a potential depending on the relative position of the two particles.

Let us assume that the function $\hat{v}(\mathbf{n})$ is defined as:

$$\hat{v}(\mathbf{n}) = \hat{v}(n_1,n_2) = \begin{cases} 10^{-|\mathbf{n}|}, & \text{if} \quad |n_1| \le 1 \\ 0, \quad \text{if} & |n_1| \ge 2, \end{cases} \tag{1.1}$$

where $|\mathbf{n}| = |n_1| + |n_2|$.

Let $\mathbb{T}^2$ be a two-dimensional torus and $L_2(\mathbb{T}^2 \times \mathbb{T}^2)$ be the Hilbert space of square-integrable functions defined on $\mathbb{T}^2 \times \mathbb{T}^2$, $L_2^{as}(\mathbb{T}^2 \times \mathbb{T}^2) \subset L_2(\mathbb{T}^2 \times \mathbb{T}^2)$ be the subspace of antisymmetric functions with respect to a permutation of variables. Let $F : \ell_2(\mathbb{Z}^2 \times \mathbb{Z}^2) \to L_2(\mathbb{T}^2 \times \mathbb{T}^2)$ be the standard Fourier transform. Let us denote by $\hat{F} : \ell_2^{as}(\mathbb{Z}^2 \times \mathbb{Z}^2) \to L_2^{as}(\mathbb{T}^2 \times \mathbb{T}^2)$ the restriction F in $\ell_2^{as}(\mathbb{Z}^2 \times \mathbb{Z}^2)$. The Hamiltonian $H = H_0 - V = \hat{F}\hat{H}\hat{F}^{-1}$ in the momentum representation commutes with the unitary operators $U_{\mathbf{s}}, \quad \mathbf{s} \in \mathbb{Z}^2$:

$$(U_{\mathbf{s}}f)(\mathbf{k}_1,\mathbf{k}_2) = e^{-i(\mathbf{s},\mathbf{k}_1+\mathbf{k}_2)}f(\mathbf{k}_1,\mathbf{k}_2), \quad f \in L_2^{as}((\mathbb{T}^2)^2).$$

This implies [1] that there are decompositions of the space $L_2^{as}(\mathbb{T}^2 \times \mathbb{T}^2)$ and the operators $U_{\mathbf{s}},$ and H into direct integrals:

$$L_2^{as}((\mathbb{T}^2)^2) = \int_{\mathbb{T}^2} \oplus L_2(F_{\mathbf{k}})\,\mathrm{d}\mathbf{k}, \quad U_{\mathbf{s}} = \int_{\mathbb{T}^2} \oplus U_{\mathbf{s}}(\mathbf{k})\,\mathrm{d}\mathbf{k}, \quad H = \int_{\mathbb{T}^2} \oplus \tilde{H}(\mathbf{k})\,\mathrm{d}\mathbf{k},$$

where

$$F_{\mathbf{k}} = \{(\mathbf{k}_1,\mathbf{k}_2) \in (\mathbb{T}^2)^2 : \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}\}, \quad \mathbf{k} = (k_1,k_2) \in \mathbb{T}^2.$$

The fiber operator $\tilde{H}(\mathbf{k})$, associated with H, acts in the space $L_2(F_{\mathbf{k}})$ and is unitarily equivalent to the two-particle discrete Schrödinger operator $H(\mathbf{k}) := H_0(\mathbf{k}) - V$ that acts in the Hilbert space

$$L_2^{o}(\mathbb{T}^2) := \left\{f \in L_2(\mathbb{T}^2) \,:\, f(-\mathbf{q}) = -f(\mathbf{q})\right\}.$$

The unperturbed operator $H_0(\mathbf{k})$ is an operator of multiplication by the function

$$\varepsilon_{\mathbf{k}}(\mathbf{q}) = \varepsilon\left(\frac{\mathbf{k}}{2}+\mathbf{q}\right) + \varepsilon\left(\frac{\mathbf{k}}{2}-\mathbf{q}\right) = 4 - 2\cos\frac{k_1}{2}\cos q_1 - 2\cos\frac{k_2}{2}\cos q_2, \tag{1.2}$$

$$\varepsilon(\mathbf{q}) = \sum_{i=1}^{2}(1-\cos q_i), \quad \mathbf{k} = (k_1,k_2) \in \mathbb{T}^2.$$

The integral operator V is defined by the kernel $\frac{1}{2\pi}v(\mathbf{q}-\mathbf{s})$, which admits the representation

$$v(\mathbf{q}-\mathbf{s}) = \frac{1}{2\pi}\sum_{\mathbf{n}\in\mathbb{Z}^2}\hat{v}(\mathbf{n})e^{i(\mathbf{q}-\mathbf{s},\mathbf{n})},$$

where the potential function $\hat{v}$ is defined by the equality (1.1).

Note that due to the specific form of $\hat{v}(\mathbf{n})$, the kernel of the integral operator V can be expressed in a separable form as $v(\mathbf{q}) = v_1(q_1)v_2(q_2)$, where the component functions are given by

$$v_1(q_1) = \frac{1}{\sqrt{2\pi}}\{1+\frac{2}{10}\cos q_1\}, \qquad v_2(q_2) = \frac{1}{\sqrt{2\pi}}\{1+2\sum_{m=1}^{\infty}10^{-m}\cos mq_2\}. \tag{1.3}$$

Let $L_2^{oe}(\mathbb{T}^2) = L_2^{o}(\mathbb{T}) \otimes L_2^{e}(\mathbb{T})$ and $L_2^{eo}(\mathbb{T}^2) = L_2^{e}(\mathbb{T}) \otimes L_2^{o}(\mathbb{T})$, where $L_2^{o}(\mathbb{T})$ and $L_2^{e}(\mathbb{T})$ denote the spaces of odd and even functions, respectively (see [10]). Then the space $L_2^{o}(\mathbb{T}^2)$ can be express as a direct sum $L_2^{o}(\mathbb{T}^2) = L_2^{oe}(\mathbb{T}^2) \oplus L_2^{eo}(\mathbb{T}^2)$.

Observe that the subspace $L_2^{eo}(\mathbb{T}^2)$ and $L_2^{oe}(\mathbb{T}^2)$ are invariant with respect to the operator $H(\mathbf{k})$ (see Lemma 3.1). By $H^{eo}(\mathbf{k})$ and $H^{oe}(\mathbf{k})$ we denote the restrictions of the operator $H(\mathbf{k})$ in the subspaces $L_2^{eo}(\mathbb{T}^2)$ and $L_2^{oe}(\mathbb{T}^2)$, respectively.

Theorem 1.1. *For any $k_1 \in (-\pi,\pi]$ the operator $H^{eo}(k_1,\pi)$ has an infinite number of eigenvalues lying to the left of the essential spectrum.*

Let $\mathcal{N}(k_1)$ be the number of the eigenvalues of the operator $H^{oe}(k_1,\pi)$ lying to the left of the essential spectrum. Then the following statement holds for the number $\mathcal{N}(k_1)$ as $k_1 \to \pi$:

Theorem 1.2. *For any $k_1 \in (-\pi,\pi)$, the operator $H^{oe}(k_1,\pi)$ has a finite number of eigenvalues lying to the left of the essential spectrum. The number of eigenvalues $\mathcal{N}(k_1)$ increases as $k_1 \to \pi$ and the following asymptotic formula holds*

$$\lim_{k_1\to\pi}\frac{\mathcal{N}(k_1)}{|\lg\cos\frac{k_1}{2}|} = 1. \tag{1.4}$$

2. Essential spectrum of the operator $H(\mathbf{k})$

As is known, the spectrum of the operator $H_0(\mathbf{k})$ consists of the range of values of the function $\varepsilon_{\mathbf{k}}$:

$$\sigma(H_0(\mathbf{k})) = [m(\mathbf{k}), M(\mathbf{k})],$$

where

$$m(\mathbf{k}) = \min_{\mathbf{q}\in\mathbb{T}^2}\varepsilon_{\mathbf{k}}(\mathbf{q}) = \varepsilon_{\mathbf{k}}(\mathbf{0}) = 2\varepsilon(\frac{\mathbf{k}}{2}), \qquad M(\mathbf{k}) = \max_{\mathbf{q}\in\mathbb{T}^2}\varepsilon_{\mathbf{k}}(\mathbf{q}) = \varepsilon_{\mathbf{k}}(\boldsymbol{\pi}) = 8 - 2\varepsilon(\frac{\mathbf{k}}{2}).$$

The spectrum of the operator V consists of the set $\{0,\ \frac{1}{10^n},\ n\in\mathbb{N}\}$ where $\frac{1}{10^n}$ is the eigenvalue of the operator V. Condition (1.1) implies that the operator V is Hilbert-Schmidt. By Weyl's theorem, since V is compact, the essential spectrum of $H(\mathbf{k})$ coincides with the spectrum of the operator $H_0(\mathbf{k})$, (see [1]), i.e.

$$\sigma_{ess}(H(\mathbf{k})) = [m(\mathbf{k}),\ M(\mathbf{k})].$$

We define $w(\mathbf{k})$ as the width of the essential spectrum of $H(\mathbf{k})$. Accordingly, we obtain:

$$w(\mathbf{k}) = M(\mathbf{k}) - m(\mathbf{k}) = 8 - 4\varepsilon(\frac{\mathbf{k}}{\mathbf{2}}) = 4\cos\frac{k_1}{2} + 4\cos\frac{k_2}{2} \tag{2.1}$$

and

$$\min_{\mathbf{k}\in\mathbb{T}^2} w(\mathbf{k}) = w(\boldsymbol{\pi}) = 0, \qquad \max_{\mathbf{k}\in\mathbb{T}^2} w(\mathbf{k}) = w(\mathbf{0}) = 8,$$

where $\boldsymbol{\pi}= (\pi,\pi)$, $\mathbf{0}= (0,0)$.

It follows from (2.1) that if $k_j \in [0,\pi]$, $j= 1,2$, increases, then the width of the essential spectrum $w(\mathbf{k})$ decreases.

Let us determine the width of the essential spectrum $H(\mathbf{k})$ in the direction $\mathbf{e}_j$, $j = 1,2$, as:

$$w_j(\mathbf{k}) = \max_{p_j\in[-\pi,\pi]} \varepsilon_{\mathbf{k}}(\mathbf{p}) - \min_{p_j\in[-\pi,\pi]} \varepsilon_{\mathbf{k}}(\mathbf{p}) = 4\cos\frac{k_j}{2},\ j= 1,2.$$

Then we have

$$w(\mathbf{k}) = w_1(\mathbf{k}) + w_2(\mathbf{k}).$$

If $\mathbf{k} = \boldsymbol{\pi}$, then the essential spectrum is concentrates at the point $\{4\}$, i.e. $w(\boldsymbol{\pi}) = 0$. The spectrum of the operator $H(\boldsymbol{\pi}) = 4I - V$ consists of eigenvalues of the form $4 - \frac{1}{10^n}$, $n \in \mathbb{N}$. The narrowing of the essential spectrum results in an increased number of eigenvalues for the Schrödinger operator $H(\mathbf{k})$.

From the self-adjointness of the operator $H(\mathbf{k}) = H_0(\mathbf{k}) - V$ and the positivity of V it follows that

$$\sigma(H(\mathbf{k})) \cap (M(\mathbf{k}),\infty) = \emptyset$$

i.e. $\sigma_{disc}(H(\mathbf{k})) \subset (-\infty, m(\mathbf{k}))$.

3. Invariant subspaces under the operator $H(\mathbf{k})$

We state the lemma on the invariant subspaces with respect to the operator $H(\mathbf{k})$.

Lemma 3.1. *The subspaces $L_2^{eo}(\mathbb{T}^2)$ and $L_2^{oe}(\mathbb{T}^2)$ invariant under the operator $H(\mathbf{k})$.*

Proof. Let us prove the invariance of the subspace $L_2^{eo}(\mathbb{T}^2)$ with respect to $H(\mathbf{k})$. From (1) it is clear that the function $\varepsilon_{\mathbf{k}}$ belongs to $L_2^{ee}(\mathbb{T}^2)$, where $L_2^{ee}(\mathbb{T}^2) = L_2^{e}(\mathbb{T}) \otimes L_2^{e}(\mathbb{T})$. From here we conclude that if $f \in L_2^{eo}(\mathbb{T}^2)$, then $\varepsilon_{\mathbf{k}} f \in L_2^{eo}(\mathbb{T}^2)$. This relation proves the invariance of the subspace $L_2^{eo}(\mathbb{T}^2)$ under the operator $H_0(\mathbf{k})$. According to condition (1.1), the kernel $v(p_1,p_2)$ of the operator V belongs to the subspace $L_2^{ee}(\mathbb{T}^2)$. It follows that $g = Vf \in L_2^{oe}(\mathbb{T}^2)$ for any $f \in L_2^{eo}(\mathbb{T}^2)$, i.e.:

$$(Vf)(p_1,p_2) = \frac{1}{2\pi}\int_{\mathbb{T}^2} v(p_1 - s_1, p_2 - s_2) f(s_1,s_2) ds_1 ds_2 \in L_2^{eo}(\mathbb{T}^2).$$

From the above relations, we obtain the invariance of the subspace $L_2^{eo}(\mathbb{T}^2)$ under the operator $H(\mathbf{k}) = H_0(\mathbf{k}) - V$.

According to (1.1) and (1.3), the operator $V^{eo} = V|_{L_2^{eo}(\mathbb{T}^2)}$ has following explicit form:

$$(V^{eo}f)(\mathbf{p}) = \frac{1}{2\pi^2}\int_{\mathbb{T}^2} \{\frac{1}{10}\sin p_2 \sin q_2 + 2\sum_{n=2}^{\infty}\frac{1}{10^n}[\sin np_2 \sin nq_2 +$$

$$+2\cos p_1 \sin(n-1)p_2 \cos q_1 \sin(n-1)q_2]\} f(\mathbf{q}) d\mathbf{q}, \quad f \in L_2^{eo}(\mathbb{T}^2).$$

The invariance of the subspace $L_2^{oe}(\mathbb{T}^2)$ with respect to the operator $H(\mathbf{k})$ is also proved in this way, there operator $V^{oe} = V|_{L_2^{oe}(\mathbb{T}^2)}$ acts to $f \in L_2^{oe}(\mathbb{T}^2)$ as:

$$(V^{oe}f)(\mathbf{p}) = \frac{1}{2\pi^2}\int_{\mathbb{T}^2} \Big\{\frac{1}{10}\sin p_1 \sin q_1 + 2\sum_{n=2}^{\infty}\frac{1}{10^n}\left[\sin p_1 \cos(n-1)p_2 \sin q_1 \cos(n-1)q_2\right]\Big\} f(\mathbf{q})\, d\mathbf{q}.$$

□

Note that the systems $\{\psi_n^o(q)=\frac{1}{\sqrt{\pi}}\sin nq\}_{n\in\mathbb{N}}$ and $\psi_0^e(q)=\dfrac{1}{\sqrt{2\pi}}$, $\{\psi_n^e(q)=\dfrac{1}{\sqrt{\pi}}\cos nq\}_{n\in\mathbb{N}}$ are an orthonormal basises in the subspaces $L_2^o(\mathbb{T})$ and $L_2^e(\mathbb{T})$, respectively. Let $L^o(n)$ and $L^e(n)$ be one-dimensional subspaces spanned by the vectors $\{\psi_n^o(q)\}, n\in\mathbb{N}$ and $\psi_0^e(q)=\dfrac{1}{\sqrt{2\pi}}$, $\{\psi_n^e(q)=\dfrac{1}{\sqrt{\pi}}\cos nq\}_{n\in\mathbb{N}}$. Then we have the following equalities

$$L_2^o(\mathbb{T})=\sum_{n=1}^{\infty}\oplus L^o(n),\quad L_2^e(\mathbb{T})=\sum_{n=0}^{\infty}\oplus L^e(n)$$

From here we obtain

$$L_2^e(\mathbb{T})\otimes L_2^o(\mathbb{T})=\sum_{n=1}^{\infty}\oplus\{L_2^e(\mathbb{T})\otimes L^o(n)\}=\sum_{n=1}^{\infty}\oplus\mathfrak{B}_n^{eo},$$

$$L_2^o(\mathbb{T})\otimes L_2^o(\mathbb{T})=\sum_{n=0}^{\infty}\oplus\{L_2^o(\mathbb{T})\otimes L^e(n)\}=\sum_{n=0}^{\infty}\oplus\mathfrak{B}_n^{oe},$$

where $\mathfrak{B}_n^{eo}:=L_2^e(\mathbb{T})\otimes L^o(n)$ and $\mathfrak{B}_n^{oe}:=L_2^o(\mathbb{T})\otimes L^e o(n)$.

Lemma 3.2. *For each $n\in\mathbb{N}$, the subspace $\mathfrak{B}_n^{eo}$ is invariant under the operator $H^{eo}(k_1,\pi)$.*

Proof. We choose an arbitrary element of $\mathfrak{B}_n^{eo}$ in the form $(f\psi_n^o)(p_1,p_2):=f(p_1)\psi_n^o(p_2)$, where $f\in L_2^e(\mathbb{T})$, then the action of $H^{eo}(k_1,\pi)=H_0(k_1,\pi)-V^{eo}$ on the space $\mathfrak{B}_n^{eo}$ is as follows:

$$(H_0(k_1,\pi)f\psi_n^o)(p_1,p_2)=\Big[\varepsilon_{k_1}(p_1)f(p_1)\Big]\psi_n^o(p_2)\in\mathfrak{B}_n^{eo},\tag{3.1}$$

here $\varepsilon_{k_1}(p)=4-2\cos\frac{k_1}{2}\cos p$,

$$(V^{eo}f\psi_n^o)(p_1,p_2)=\frac{1}{2\pi^2}\int_{\mathbb{T}^2}\Big\{\frac{1}{10}\sin p_2\sin q_2+$$

$$+\sum_{n=2}^{\infty}\frac{1}{10^n}[\sin np_2\sin nq_2+2\cos p_1\sin(n-1)p_2\cos q_1\sin(n-1)q_2]\Big\}(f\psi_n^o)(q_1,q_2)dq_1dq_2=$$

$$=\left[\frac{1}{2\pi}\int_{\mathbb{T}}(\frac{1}{10^n}+\frac{2}{10^{n+1}}\cos p_1\cos q_1)f(q_1)dq_1\right]\psi_n^o(p_2)\in\mathfrak{B}_n^{eo}.\tag{3.2}$$

In obtaining (3.2) we used the orthogonality of the system $\{\psi_n^o\}_{n\in\mathbb{N}}$. According to (3.1) and (3.2) we obtain the relation

$$(H^{eo}(k_1,\pi)f\psi_n^o)(p_1,p_2)=\left[\varepsilon_{k_1}(p_1)f(p_1)-\frac{1}{2\pi}\int\limits_{\mathbb{T}}\Big\{\frac{1}{10^n}+\frac{2}{10^{n+1}}\cos p_1\cos q_1\Big\}f(q_1)dq_1\right]\psi_n^e(p_2)\in\mathfrak{B}_n^{eo}$$

which proves the lemma. □

From expressions (3.1) and (3.2) it can be seen that the restriction $H_n^{eo}(k_1,\pi)$ of the operator $H^{eo}(k_1,\pi)$ in the subspace $\mathfrak{B}_n^{eo}:=L_2^e(\mathbb{T})\otimes L^o(n)$ is of the form:

$$H_n^{eo}(k_1,\pi)=[H_0(k_1)-V_n^e]\otimes I_n,\tag{3.3}$$

where I_n is the identity operator in $L^o(n)$, and $H_n^e(k_1):=H_0(k_1)-V_n^e$ acts in the space $L_2^e(\mathbb{T})$ as follows:

$$(H_n^e(k_1)f)(p)=\varepsilon_{k_1}(p)f(p)-\frac{1}{2\pi}\int_{\mathbb{T}}\Big\{\frac{1}{10^n}+\frac{2}{10^{n+1}}\cos p\cos q\Big\}f(q)dq.\tag{3.4}$$

Lemma 3.3. *For each $n\in\mathbb{Z}_+$, the subspace $\mathfrak{B}_n^{oe}$ is invariant under the operator $H^{oe}(k_1,\pi)$.*

Proof. We choose an arbitrary element of $\mathfrak{B}_n^{oe}$ in the form $\big(f\,\psi_n^e\big)(p_1,p_2):=f(p_1)\,\psi_n^e(p_2)$, where $f\in L_2^o(\mathbb{T})$, then the action of $H^{oe}(k_1,\pi)=H_0(k_1,\pi)-V^{oe}$ on $\mathfrak{B}_n^{oe}$ is as

$$(H_0(k_1,\pi)f\psi_n^e)(p_1,p_2)=\Big[\varepsilon_{k_1}(p_1)f(p_1)\Big]\psi_n^e(p_2)\in\mathfrak{B}_n^{oe}, \tag{3.5}$$

$$(V^{oe}f\psi_n^e)(p_1,p_2)=\frac{1}{(2\pi)^2}\int\limits_{\mathbb{T}^2}\Big\{4\sum_{m=2}^{\infty}\frac{1}{10^m}\sin p_1\cos(m-1)p_2\sin q_1\cos(m-1)q_2\Big\}(f\psi_n^e)(q_1,q_2)\,dq_1dq_2=$$

$$=\Big[\frac{1}{\pi}\int\limits_{\mathbb{T}}\frac{1}{10^{n+1}}\sin p_1\sin q_1 f(q_1)dq_1\Big]\psi_n^e(p_2)\in\mathfrak{B}_n^{oe}. \tag{3.6}$$

(3.6) is obtained based on the orthogonality of the system $\{\psi_n^e\}_{n\in\mathbb{Z}_+}$. □

From (3.5) and (3.6), it can be seen that the restriction of the operator $H^{oe}(k_1,\pi)$ in the subspace $\mathfrak{B}_n^{oe}:=L_2^o(\mathbb{T})\otimes L^e(n)$ has the form

$$H_n^{oe}(k_1,\pi)=[H_0(k_1)-V_n^o]\otimes I_n, \tag{3.7}$$

where I_n is the identity operator in $L^e(n)$. Then the action of $H_n^o(k_1):=H_0(k_1)-V_n^o$ on $L_2^o(\mathbb{T})$:

$$(H_n^o(k_1)f)(p)=\varepsilon_{k_1}(p)f(p)-\frac{1}{\pi}\int\limits_{\mathbb{T}}\frac{1}{10^{n+1}}\sin p\sin q f(q)dq. \tag{3.8}$$

Thus, according to (3.3) and (3.7), we have the following representation for the operators $H^{eo}(k_1,\pi)$ and $H^{oe}(k_1,\pi)$:

$$H^{eo}(k_1,\pi)=\sum_{n=1}^{\infty}\oplus H_n^{eo}(k_1,\pi),\quad H^{oe}(k_1,\pi)=\sum_{n=0}^{\infty}\oplus H_n^{oe}(k_1,\pi). \tag{3.9}$$

4. On the spectrum of the operator $H_n^e(k_1)$

This section is devoted to the analysis of the discrete spectrum of the operator $H^{eo}(k_1,\pi)$. According to (3.3) and (3.4) it is sufficient to study the eigenvalues of operator $H_n^e(k_1)$, $n\in\mathbb{N}$ defined by (3.9). It should be noted that the width $w(k_1):=w(k_1,\pi)$ of the essential spectrum of the operator $H_n^e(k_1)$ is independent of n and is given by

$$w(k_1)=w_1(k_1)=4\cos\frac{k_1}{2}.$$

It is known that the study of the discrete spectrum of the operator $H_n^e(k_1)$ lying to the left of the essential spectrum reduces to the study of the eigenvalues of the self-adjoint, compact and positive operator $T_n^e(k_1,z)=r_0^{\frac12}(k_1,z)V_n^e r_0^{\frac12}(k_1,z)$, $z\in(-\infty,m(k_1))$, where $r_0(k_1,z)$ is the resolvent of the unperturbed operator $H_0(k_1)$ (see [7]). The operator $T_n^e(k_1,z)$ acts on the space $L_2^e(\mathbb{T})$ according to the following formula:

$$(T_n^e(k_1,z)g)(p)=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{[10^{-n}+2\cdot10^{-n-1}\cos p\cos q]g(q)dq}{\sqrt{\varepsilon_{k_1}(p)-z}\sqrt{\varepsilon_{k_1}(q)-z}},\quad z\in(-\infty,m(k_1)).$$

By definition of the operator $T_n^e(k_1,z)$, it has rank two for each $n\in\mathbb{N}$.

Lemma 4.1. *The number $z\in(-\infty,m(k_1))$, is an eigenvalue of the operator $H_n^e(k_1)$ if and only if 1 is the eigenvalue of the operator $T_n^e(k_1,z)$.*

The proof proceeds analogously to that of Theorem 1 in [5].

Let $\mathfrak{m}[\mu,B]$ denote the number of eigenvalues of the self-adjoint operator B that lie above μ, where $\mu>sup\,\sigma_{ess}(B)$ and B acts in a Hilbert space $\mathfrak{H}$.

Lemma 4.2. *If $z < m(k_1)$, then the number of eigenvalues of $H_n^e(k_1)$ below z is equal to the number of eigenvalues of $T_n^e(k_1, z)$ that are greater than 1:*

$$\mathfrak{m}\left[1,\, T_n^e(k_1, z)\right] = \mathfrak{m}\left[-z,\, -H_n^e(k_1)\right].$$

The proof proceeds analogously to that of Lemma 2 in [5].

Theorem 4.3. *For each $n \in \mathbb{N}$ there exists at least one eigenvalue of the operator $H_n^e(k_1)$ lying to the left of the essential spectrum.*

Proof. By Lemma 4.2, we show that for some $z_0 < m(k_1)$ the operator $T_n^e(k_1, z)$ has at least one eigenvalue greater than 1.

Let $g_0(q) = \frac{\sqrt{C_0(z)}}{\sqrt{2\pi}\sqrt{\varepsilon_{k_1}(q)-z}} \in L_2^e(\mathbb{T})$ and $\|g_0\|= 1$, where $C_0(z)$ is the normalizing multiplier. Observe that the result below is valid for all $z<m(k_1)$:

$$C_0(z)=\left(\frac{1}{2\pi}\int_{\mathbb{T}}\frac{dq}{\varepsilon_{k_1}(q)-z}\right)^{-1}=\sqrt{(4-z)^2-4\cos^2\frac{k_1}{2}} \tag{4.1}$$

and

$$\lim_{z\to m(k_1)} C_0(z)= 0. \tag{4.2}$$

Consequently, we arrive at the following expression for the inner product $(T_n^e(k_1, z)g_0, g_0)$:

$$(T_n^e(k_1, z)g_0, g_0) =\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} T_n^e(p, q;k_1, z)g_0(q)dq\overline{g_0(p)}dp=$$

$$=\frac{10^{-n}}{2\pi}\left|\int_{\mathbb{T}}\frac{g_0(q)dq}{\sqrt{\varepsilon_{k_1}(q)-z}}\right|^2+\frac{10^{-n-1}}{\pi}\left|\int_{\mathbb{T}}\frac{\cos q g_0(q)dq}{\sqrt{\varepsilon_{k_1}(q)-z}}\right|^2.$$

Both terms of this sum are nonnegative, so according to (4.1) we have

$$(T_n^e(k_1, z)g_0, g_0)\geq\frac{10^{-n}C_0(z)}{(2\pi)^2}\int_{\mathbb{T}}\frac{dp}{\varepsilon_{k_1}(p)-z}\int_{\mathbb{T}}\frac{dq}{\varepsilon_{k_1}(q)-z}=\frac{10^{-n}}{C_0(z)}.$$

From here and from (4.2) there exists $z=z_0(n) <m(k_1)$ such that for any $z \in (z_0(n), m(k_1))$ the relation $\frac{10^{-n}}{C_0(z)}> 1$ holds. Then, according to the Birman-Schwinger principle (see Lemma 4.2), it follows that $\mathfrak{m}[-m(k_1), -H_n^e(k_1)]>1$. □

Proof of Theorem 1.1. The proof of this theorem follows immediately from Theorem 4.3 and (3.9).

5. On the spectrum of the operator $H_n^o(k_1)$

In this section we give some statements about the spectrum of the operator $H_n^o(k_1), n \in \mathbb{Z}_+$ defined by the formula (3.8), and then give results for the operator $H^{oe}(k_1, \pi)$ using the representations (3.7) and (3.9).

Theorem 5.1. *Let $n \in \mathbb{Z}_+$ and $k_1 \in (-\pi, \pi)$. a) If $\frac{1}{10^{n+1}} > \cos\frac{k_1}{2}$, then the operator $H_n^o(k_1)$ has a simple eigenvalue lying below the essential spectrum;*
b) if $\frac{1}{10^{n+1}} \leq \cos\frac{k_1}{2}$, then the operator $H_n^o(k_1)$ has no eigenvalues lying outside of essential spectrum.

Proof. *a*) Suppose that the equation

$$(H_n^o(k_1)f)(p) = zf(p) \tag{5.1}$$

has a nonzero solution $f \in L_2^o(\mathbb{T})$. We express this equation in the form

$$(\varepsilon_{k_1}(p) - z)\, f(p) = \frac{1}{\pi} \int\limits_{\mathbb{T}} \frac{1}{10^{n+1}} \sin p \sin q f(q) dq.$$

Denoting

$$a = \frac{1}{\pi} \int\limits_{\mathbb{T}} \sin q f(q) dq \tag{5.2}$$

we arrive at the following expression for the eigenfunction:

$$f(p) = \frac{10^{-n-1} a \sin p}{4 - z - 2\cos\frac{k_1}{2}\cos p.} \tag{5.3}$$

From here, by setting (5.3) to (5.2) and after some simplifications, we obtain the following Fredholm determinant $\Delta(k_1, z)$ for the operator $H_n^o(k_1)$:

$$\Delta(k_1, z) = 1 - \frac{10^{-n-1}}{\pi} \int\limits_{\mathbb{T}} \frac{\sin^2 q dq}{4 - z - 2\cos\frac{k_1}{2}\cos q} \tag{5.4}$$

This function is a continuous and monotonically decreasing function of z on the interval $(-\infty, m(k_1))$. Hence we have the following relationship:

$$\lim_{z \to -\infty} \Delta(k_1, z) = 1, \qquad \lim_{z \to m(k_1)} \Delta(k_1, z) = \Delta(k_1, m(k_1)) < 0$$

and

$$\Delta(k_1, m(k_1)) = 1 - \frac{10^{-n-1}}{2\pi \cos\frac{k_1}{2}} \int\limits_{\mathbb{T}} \frac{\sin^2 q dq}{1 - \cos q} = 1 - \frac{10^{-n-1}}{\cos\frac{k_1}{2}}.$$

Since $10^{-n-1} > \cos\frac{k_1}{2}$, then the function $\Delta(k_1, z)$ has zero on the interval $(-\infty, m(k_1))$, i.e. the operator $H_n^o(k_1)$ has unique simple eigenvalue lying to the left of essential spectrum;
$b)$ if $10^{-n-1} < \cos\frac{k_1}{2}$, then the determinant $\Delta(k_1, z)$ does not have zeros in the interval $(-\infty, m(k_1))$.
If $10^{-n-1} = \cos\frac{k_1}{2}$, then the solution of the equation $(H_n^o(k_1)f)(p) = m(k_1)f(p)$ is $f(p) = \dfrac{\sin p}{1 - \cos p} \notin L_2^o(\mathbb{T})$. It follows that the number $z = m(k_1)$ is not an eigenvalue for the operator $H_n^o(k_1)$.

□

Lemma 5.2. *Suppose that the condition a) of the theorem 5.1 holds. Then the operator $H_n^o(k_1)$ has a unique simple eigenvalue lying in a some neighborhood of the eigenvalue $z_n(\pi) = 4 - \frac{1}{10^{n+1}}$ of the operator $H(\boldsymbol{\pi})$ to the left of the essential spectrum:*

$$z_n(k_1) = 4 - \frac{1}{10^{n+1}} - 10^{n+1}\cos^2\frac{k_1}{2}.$$

The proof of this and the following lemma is omitted due to its triviality.

Lemma 5.3. *Let $k_1 \in (-\pi, \pi)$. If for some number $n \in \mathbb{N}$ the inequality $\frac{1}{10^{n+2}} < \cos\frac{k_1}{2} < \frac{1}{10^{n+1}}$ holds, then the operator $H^{oe}(k_1, \pi)$ has exactly n eigenvalues to the left of the essential spectrum, i.e.*

$$\mathcal{N}(k_1) = n \tag{5.5}$$

Proof of theorem 1.2. Suppose that equality (5.5) holds, i.e. implies the relation

$$\frac{1}{10^{n+2}} < \cos\frac{k_1}{2} < \frac{1}{10^{n+1}}.$$

From here we have

$$\lg 10^{-n-2} < \lg\cos\frac{k_1}{2} < \lg 10^{-n-1} \text{ or } n < -1 - \lg\cos\frac{k_1}{2} < n+1.$$

Considering (5.5), we deduce

$$1 + \frac{1}{\mathcal{N}(k_1)} < -\frac{\lg\cos\frac{k_1}{2}}{\mathcal{N}(k_1)} < 1 + \frac{2}{\mathcal{N}(k_1)}.$$

Thus, from here we obtain

$$\lim_{k_1\to\pi} \frac{\mathcal{N}(k_1)}{|\lg\cos\frac{k_1}{2}|} = 1.$$

References

[1] Reed M., Simon B., Methods of Modern Mathematical Physics IV: Analysis of Operators. Academic, New York, (1979).

[2] Rauch J., Perturbation theory for eigenvalues and resonances of Schrödinger Hamiltonians. J. Funct. Anal. (1980) Vol. 35, P. 304–315.

[3] Simon B., The bound state of weakly coupled Schrödinger operators in one and two dimensions. Ann. Phys. (1976) Vol. 97, P. 279–288.

[4] Klaus M., On the Bound state of Schrödinger operators in one dimension. Ann. Phys. (1977) Vol. 108, P. 288–300.

[5] Lakaev S., Khalkhuzhaev A., Spectrum of the two-particle Schrödinger operator on a lattice. Theoret. and Math. Phys. (2008) Vol. 155, Iss. 2, P. 754–765.

[6] Abdullaev J., Khalkhuzhaev A., Usmonov L., Monotonicity of the eigenvalues of the two-particle Schrödinger operatoron a lattice. Nanosystems: Phys. Chem. Math. (2021) Vol. 12, Iss. 6, P. 657–663.

[7] Abdullaev J., Ikromov I., Finiteness of the number of eigenvalues of the two-particle Schrödinger operator on a lattice. Theor. Math. Phys. (2007) Vol. 152, P. 1299–1312.

[8] Abdullaev J., Khalkhuzhaev A., Makhmudov K., Discrete spectrum asymptotics for the two-particle Schrödinger operator on a lattice. Lobachevskii Journal of Mathematics, (2024) Vol. 45, Iss. 10, P. 4828–4845.

[9] Abdullaev J., Perturbation theory for the two-particle Schrödinger operator on a one-dimensional lattice. Theor. Math. Phys. (2005) Vol. 145, Iss. 2, P. 1551–1558.

[10] Reed M., Simon B., Methods of Modern Mathematical Physics IV: Analysis of Operators. Academic, New York, (1980).

Khalkhuzhaev A.M.,
Samarkand State University named after Sharaf Rashidov,
Samarkand, Uzbekistan;
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Samarkand, Uzbekistan;
e-mail: ahmad_x@mail.ru

Makhmudov Kh.Sh.,
Samarkand State University named after Sharaf Rashidov,
Samarkand, Uzbekistan;
e-mail: mahmudovh276@gmail.com

Miyassarov A.A.,
Samarkand State University named after Sharaf Rashidov,
Samarkand, Uzbekistan;
e-mail: azizkhuja_miyassarov@mail.ru

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 116–125
DOI: 10.29229/uzmj.2026-1-12

Classification of three-dimensional complex Leibniz dialgebras with one-dimensional annihilator

Kurbanbaev T., Uzakbaev N.

Abstract. This work focuses on classifying three-dimensional complex Leibniz dialgebras. We present a complete list of these algebras when the annihilator of the associated Leibniz algebra is one-dimensional.

Keywords: Associative algebra, Lie algebra, Leibniz algebra, dialgebra, Leibniz dialgebra, isomorphism, classification
MSC (2020): 17A30, 17A32, 17A60

1. Introduction

In 1990s, Loday introduced the notion of Leibniz algebra, that is a generalization of Lie algebra, where the skew-symmetricity of the bracket is dropped and the Jacobi identity is replaced by the Leibniz identity (the identity has been called Leibniz identity by Loday due to its similarity to Leibniz rule, this is the reason for the class to be called by the name of Leibniz). In right (left) Leibniz algebras, the defining identity requires that each right (left) multiplication operator behaves as a derivation, i.e.,

$$[[x, y], z] = [[x, z], y] + [x, [y, z] \quad \text{or} \quad [x, [y, z]] = [[x, y], z] + [y, [x, z]],$$

respectively. He also introduced several new classes of algebras, including Leibniz algebras, diassociative algebras, dendriform algebras, Zinbiel algebras [1, 2] and showed that the link between them, in particular, Lie and associative algebras can be extended to analogous link between Leibniz algebras and so-called associative dialgebras which are a generalization of associative algebras possessing two composition laws.

By the definition [3], dialgebras are vector spaces with two bilinear products. A diassociative algebra is a vector space with two bilinear associative operations $\vdash$, $\dashv$, satisfying certain conditions [1] required for a dialgebra A to be associative are chosen in such a way that the new operation

$$ab = a \vdash b - b \dashv a \quad \text{or} \quad ab = a \dashv b - b \vdash a$$

turns A into a left (right) Leibniz algebra. Moreover, associative algebras are a particular case of diassociative algebras when two operations coincide. Some examples and applications of dialgebras are given in [4, 5, 6, 1].

The problem of classifying algebras of a particular type lies at the core of algebraic research. It provides the foundation for a deeper understanding of the structural behavior of the algebras involved. The study of structural properties of Leibniz algebras has been initiated by Ayupov and Omirov [7, 8]. Casas gave the list of isomorphism classes of three-dimensional complex Leibniz algebras [9] (two-dimensional case was given by Loday himself). There are classification results of low-dimensional dialgebras [10, 11, 12, 13, 14, 15].

By the motivation of the relation of dialgebras and conformal algebras, in 2008, Kolesnikov [16] gave the technique of how to define the notion of Var-dialgebra for a given variety of algebra Var. In this paper, first, we give the concept of "0-dialgebra" for introducing dialgebras. Then, we deal with the classification problem of 3-dimensional complex Leibniz dialgebras, particularly, we focus on the case where the annihilator

$$\mathcal{L}^{\text{ann}} = \text{ideal}\langle [x, x] \mid x \in \mathcal{L} \rangle$$

is one-dimensional, that is, $\dim(\mathcal{L}^{\text{ann}}) = 1$.

1.1. **Preliminaries.**

Definition 1.1. A vector space $\mathcal{D}$ with two multiplication operators $\dashv$ and $\vdash$ is called a "0-*dialgebra*" if

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z,\ x \dashv (y \vdash z) = x \dashv (y \dashv z)\,,$$

for all $x, y, z \in \mathcal{D}$.

Definition 1.2. A 0-dialgebra $(\mathcal{D}, \dashv, \vdash)$ is called a *diassociative algebra* if

$$(x \vdash y) \vdash z = x \vdash (y \vdash z)\,,\ (x \dashv y) \dashv z = x \dashv (y \dashv z)\,,\ (x \vdash y) \dashv z = x \vdash (y \dashv z)\,,$$

for all $x, y, z \in \mathcal{D}$.

Definition 1.3. A 0-dialgebra $(\mathcal{D}, \dashv, \vdash)$ is called a *Leibniz dialgebra* if

$$\begin{gathered}(x \vdash y) \dashv z = x \vdash (y \dashv z) + (x \dashv z) \vdash y,\\ (x \dashv y) \vdash z = x \vdash (y \vdash z) + (x \vdash z) \dashv y,\\ (x \dashv y) \dashv z = x \dashv (y \dashv z) + (x \dashv z) \dashv y,\end{gathered}$$

for all $x, y, z \in \mathcal{D}$.

Let us consider an example of Lie dialgebras. Suppose $\sum = \{(x_1, x_2, x_3 - x_2\,(x_1, x_3)\,, x_1x_2 + x_2x_1)\}$ then the corresponding dialgebra identities include

$$x_1 \dashv x_2 + x_2 \vdash x_1.$$

A Lie dialgebra $\mathcal{D}$ considered as an ordinary algebra with respect to $[a, b] = a \vdash b,\ a, b \in \mathcal{D}$, is just a left Leibniz algebra. Conversely, every left Leibniz algebra $\mathcal{L}$ is a Lie dialgebra with respect to $a \vdash b = [a, b]\,,\ a \dashv b = -\,[b, a]\,.$ Therefore, a Lie dialgebra is just the same as a Leibniz algebra.

Theorem 1.4. *[9] Any 3-dimensional complex non-Lie Leibniz algebra $\mathcal{L}$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

Algebra	*Table of multiplication*	*Automorphisms*
$\mathcal{L}_1(\alpha)$, $\alpha \in \mathbb{C}$	$[e_2, e_2] = \alpha e_1,\ [e_3, e_2] = e_1,\ [e_3, e_3] = e_1.$	$\varphi(e_1) = (a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2)e_1,$ $\varphi(e_2) = a_{21}e_1 + a_{22}e_2 + a_{23}e_3,$ $\varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3.$
$\mathcal{L}_2$	$[e_3, e_3] = e_1.$	$\varphi(e_1) = a_{33}^2e_1,$ $\varphi(e_2) = a_{21}e_1 + a_{22}e_2,$ $\varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3.$
$\mathcal{L}_3$	$[e_2, e_2] = e_1,\ [e_3, e_3] = e_1.$	$\varphi(e_1) = (a_{32}^2 + a_{33}^2)e_1,$ $\varphi(e_2) = a_{21}e_1 - a_{33}e_2 + a_{32}e_3,$ $\varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3.$
$\mathcal{L}_4$	$[e_1, e_3] = e_1.$	$\varphi(e_1) = a_{11}e_1,$ $\varphi(e_2) = a_{22}e_2,$ $\varphi(e_3) = a_{32}e_2 + e_3.$
$\mathcal{L}_5(\alpha)$, $\alpha \in \mathbb{C}\backslash\{0\}$	$[e_1, e_3] = \alpha e_1,\ [e_2, e_3] = e_2,\ [e_3, e_2] = -e_2.$	$\varphi(e_1) = a_{11}e_1,$ $\varphi(e_2) = a_{22}e_2,$ $\varphi(e_3) = a_{32}e_2 + e_3.$
$\mathcal{L}_6$	$[e_2, e_3] = e_2,\ [e_3, e_2] = -e_2,\ [e_3, e_3] = e_1.$	$\varphi(e_1) = e_1,$ $\varphi(e_2) = a_{22}e_2,$ $\varphi(e_3) = a_{31}e_1 + a_{32}e_2 + e_3.$
$\mathcal{L}_7$	$[e_1, e_3] = 2e_1,\ [e_2, e_2] = e_1,\ [e_2, e_3] = e_2,$ $[e_3, e_2] = -e_2,\ [e_3, e_3] = e_1.$	$\varphi(e_1) = a_{22}^2e_1,$ $\varphi(e_2) = -a_{22}a_{32}e_1 + a_{22}e_2,$ $\varphi(e_3) = \frac{1}{2}(a_{22}^2 - a_{32}^2 - 1)e_1 + a_{32}e_2 + e_3.$

$\mathcal{L}_8(\alpha)$, $\alpha\in\mathbb{C}\backslash\{0\}$	$[e_1,e_3]=\alpha e_1,\ [e_2,e_3]=e_2.$	$\varphi_1(e_1)=a_{12}e_2$, $\varphi_1(e_2)=a_{21}e_1$, $\varphi_1(e_3)=-e_3,$ *where* $\alpha=-1.$ $\varphi_2(e_1)=a_{11}e_1$, $\varphi_2(e_2)=a_{22}e_2$, $\varphi_2(e_3)=a_{33}e_3,$ *where* $\alpha=-1.$ $\varphi_3(e_1)=a_{11}e_1+a_{12}e_2$, $\varphi_3(e_2)=a_{21}e_1+a_{22}e_2$, $\varphi_3(e_3)=e_3,$ *where* $\alpha=1$, $\varphi_4(e_2)=a_{22}e_2$, $\varphi_4(e_3)=e_3,$ *where* $\alpha\neq\pm1.$
$\mathcal{L}_9$	$[e_1,e_3]=e_1+e_2,\ [e_2,e_3]=e_2.$	$\varphi(e_1)=a_{11}e_1+a_{12}e_2$, $\varphi(e_2)=a_{11}e_2$, $\varphi(e_3)=e_3.$
$\mathcal{L}_{10}$	$[e_1,e_3]=e_2,\ [e_3,e_3]=e_1.$	$\varphi(e_1)=a_{33}^2e_1+a_{31}a_{33}e_2$, $\varphi(e_2)=a_{33}^3e_2$, $\varphi(e_3)=a_{31}e_1+a_{32}e_2+a_{33}e_3.$
$\mathcal{L}_{11}$	$[e_1,e_3]=e_2,\ [e_2,e_3]=e_2,\ [e_3,e_3]=e_1.$	$\varphi(e_1)=e_1+(a_{22}-1)e_2$, $\varphi(e_2)=a_{22}e_2$, $\varphi(e_3)=(a_{22}-a_{32}-1)e_1+a_{32}e_2+e_3.$

2. Main result

In this section, we give lists of three-dimensional nontrivial complex Leibniz dialgebras. The idea is as follows. We choose the first part $\mathcal{A}_1=(\mathcal{DL},\dashv)$ of the Leibniz dialgebra from Theorem 1.4, restricting our consideration to the case where the annihilator of the corresponding Leibniz algebra $\mathcal{L}$ is one-dimensional i.e. they are algebras $\mathcal{L}_1,\dots\mathcal{L}_7$. Combining an algebra from this list (taking into account the Leibniz dialgebra axioms) with the second part $\mathcal{A}_2=(\mathcal{DL},\vdash)$, we obtain constraints for the structural constants. Then we distinguish non-isomorphic algebras. The following theorem is one of the main results of this paper.

Theorem 2.1. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_1$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathcal{DL}_1(\alpha,m,n,q):\begin{cases} e_2\dashv e_2=\alpha e_1,\ e_3\dashv e_2=e_1,\ e_3\dashv e_3=e_1,\\ e_2\vdash e_2=me_1,\ e_2\vdash e_3=ne_1,\ e_3\vdash e_2=qe_1,\end{cases}$$

where $m\in\mathbb{C}\backslash\{0\},\alpha,n,q\in\mathbb{C}$;

$\mathcal{DL}_2(\alpha,n,q):\ e_2\dashv e_2=\alpha e_1,\ e_3\dashv e_2=e_1,\ e_3\dashv e_3=e_1,\ e_2\vdash e_3=ne_1,\ e_3\vdash e_2=qe_1,$
$\alpha\in\mathbb{C}\backslash\{0\},n,q\in\mathbb{C}$;

Proof. Consider the algebra $\mathcal{A}_1=(\mathcal{DL},\dashv)$ with multiplication table:

$$e_2\dashv e_2=\alpha e_1,e_3\dashv e_2=e_1,e_3\dashv e_3=e_1.$$

The second part $\mathcal{A}_2=(\mathcal{DL},\vdash)$ is defined by the following multiplication table:

$$\begin{array}{lll} e_1\vdash e_1=\alpha_1e_1+\beta_1e_2+\gamma_1e_3, & e_2\vdash e_1=\alpha_4e_1+\beta_4e_2+\gamma_4e_3, & e_3\vdash e_1=\alpha_7e_1+\beta_7e_2+\gamma_7e_3,\\ e_1\vdash e_2=\alpha_2e_1+\beta_2e_2+\gamma_2e_3, & e_2\vdash e_2=\alpha_5e_1+\beta_5e_2+\gamma_5e_3, & e_3\vdash e_2=\alpha_8e_1+\beta_8e_2+\gamma_8e_3,\\ e_1\vdash e_3=\alpha_3e_1+\beta_3e_2+\gamma_3e_3, & e_2\vdash e_3=\alpha_6e_1+\beta_6e_2+\gamma_6e_3, & e_3\vdash e_3=\alpha_9e_1+\beta_9e_2+\gamma_9e_3.\end{array}\tag{2.1}$$

Imposing the Leibniz dialgebra axioms, we obtain

$$e_2\vdash e_2=\alpha_5e_1,\ e_2\vdash e_3=\alpha_6e_1,\ e_3\vdash e_2=\alpha_8e_1,\ e_3\vdash e_3=\alpha_9e_1.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1)=(a_{32}^2\alpha+a_{32}a_{33}+a_{33}^2)\,e_1,\ \varphi(e_2)=a_{21}e_1+a_{22}e_2+a_{23}e_3,\ \varphi(e_3)=a_{31}e_1+a_{32}e_2+a_{33}e_3,$$

with $\begin{cases} a_{22}^2\alpha + a_{22}a_{23} + a_{23}^2 = \alpha\left(a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2\right), \ a_{22}a_{32}\alpha + a_{23}a_{32} + a_{23}a_{33} = 0, \\ a_{22}a_{32}\alpha + a_{22}a_{33} + a_{23}a_{33} = a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2, \ a_{22}a_{33} -_{23} a_{32} \neq 0. \end{cases}$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha_5', \alpha_6', \alpha_8', \alpha_9'\}$ and $\{\alpha_5, \alpha_6, \alpha_8, \alpha_9\}$:

$$\begin{array}{lll} \varphi(e_2) \vdash \varphi(e_2) = \alpha_5'\varphi(e_1), & \Rightarrow & \alpha_5' = \frac{a_{22}^2\alpha_5 + a_{22}a_{23}(\alpha_6+\alpha_8) + a_{23}^2\alpha_9}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, \\ \varphi(e_2) \vdash \varphi(e_3) = \alpha_6'\varphi(e_1), & \Rightarrow & \alpha_6' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_6 + a_{23}a_{32}\alpha_8 + a_{23}a_{33}\alpha_9}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, \\ \varphi(e_3) \vdash \varphi(e_2) = \alpha_8'\varphi(e_1), & \Rightarrow & \alpha_8' = \frac{a_{22}a_{32}\alpha_5 + a_{23}a_{32}\alpha_6 + a_{22}a_{33}\alpha_8 + a_{23}a_{33}\alpha_9}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, \\ \varphi(e_3) \vdash \varphi(e_3) = \alpha_9'\varphi(e_1), & \Rightarrow & \alpha_9' = \frac{a_{32}^2\alpha_5 + a_{32}a_{33}(\alpha_6+\alpha_8) + a_{33}^2\alpha_9}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}. \end{array}$$

Then we have the following cases.

1. Let $(\alpha_5, \alpha_9) \neq (0,0)$. Without loss of generality, we assume that $\alpha_5 \neq 0$. Next, choosing $a_{32} = \frac{-(\alpha_6+\alpha_8) \pm \sqrt{(\alpha_6+\alpha_8)^2 - 4\alpha_5\alpha_9}}{2\alpha_5} a_{33}$, we can put $\alpha_9' = 0$. Thus we obtained the following algebra:

$$e_2 \vdash e_2 = \alpha_5 e_1, \ e_2 \vdash e_3 = \alpha_6 e_1, \ e_3 \vdash e_2 = \alpha_8 e_1.$$

Again by using a change of basis we obtain the following relations:

$$\begin{array}{ll} \alpha_5' = \frac{a_{22}^2\alpha_5 + a_{22}a_{23}(\alpha_6+\alpha_8)}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, & \alpha_6' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_6 + a_{23}a_{32}\alpha_8}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, \\ \alpha_8' = \frac{a_{22}a_{32}\alpha_5 + a_{23}a_{32}\alpha_6 + a_{22}a_{33}\alpha_8}{a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2}, & a_{32}^2\alpha_5 + a_{32}a_{33}\left(\alpha_6 + \alpha_8\right) = 0, \end{array}$$

with $\begin{cases} a_{22}^2\alpha + a_{22}a_{23} + a_{23}^2 = \alpha\left(a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2\right), \ a_{22}a_{32}\alpha + a_{23}a_{32} + a_{23}a_{33} = 0, \\ a_{22}a_{32}\alpha + a_{22}a_{33} + a_{23}a_{33} = a_{32}^2\alpha + a_{32}a_{33} + a_{33}^2, \ a_{22}a_{33} - a_{23}a_{32} \neq 0. \end{cases}$

(1) If $a_{32} \neq 0$ and $\alpha \neq 0$. Then we have $a_{32} = -\frac{a_{33}(\alpha_6+\alpha_8)}{\alpha_5} \neq 0$, $a_{22} = \frac{a_{23}(\alpha_5-\alpha_6-\alpha_8)}{\alpha(\alpha_6+\alpha_8)}$, $a_{33} \neq 0$, $\alpha_5(\alpha_5 - \alpha_6 - \alpha_8) + \alpha\left(\alpha_6 + \alpha_8\right)^2 \neq 0$, $a_{23} = \frac{a_{33}\alpha(\alpha_6+\alpha_8)}{\alpha_5} \neq 0$, and

$$\alpha_5' = \alpha_5 - \alpha_6 - \alpha_8, \ \alpha_6' = -\alpha_8, \ \alpha_8' = -\alpha_6.$$

In this case we derive $\mathcal{DL}_1^1\left(\alpha, m, n, q\right)$, where $\alpha, m \neq 0, n + q \neq 0$.

(2) If $a_{32} \neq 0$ and $\alpha = 0$. Then we have $a_{32} = -\frac{a_{33}(\alpha_6+\alpha_8)}{\alpha_5} \neq 0$, $a_{22} = \frac{a_{33}(\alpha_5-\alpha_6-\alpha_8)}{\alpha_5} \neq 0$, $a_{33} \neq 0$, $a_{23} = 0$, and

$$\alpha_5' = \alpha_5 - \alpha_6 - \alpha_8, \ \alpha_6' = \frac{\alpha_5\alpha_6 - \alpha_6 - \alpha_8}{\alpha_5}, \ \alpha_8' = \frac{\alpha_5\alpha_8 - \alpha_6 - \alpha_8}{\alpha_5}.$$

In this case we derive $\mathcal{DL}_1^2\left(0, m, n, q\right)$, where $m \neq 0, n + q \neq 0$.

(3) If $a_{32} = 0$. Then we have $a_{23} = 0$, $a_{22} \neq 0$, $a_{33} \neq 0$, $a_{22} = a_{33}$, and

$$\alpha_5' = \alpha_5, \ \alpha_6' = \alpha_6, \ \alpha_8' = \alpha_8,$$

we derive $\mathcal{DL}_1^3\left(\alpha, m, n, q\right)$, where $m \neq 0$.

In this case we obtain $\mathcal{DL}_1\left(\alpha, m, n, q\right)$, where $m \in \mathbb{C}\backslash\{0\}, \alpha, n, q \in \mathbb{C}$;

2. Let $\alpha_5 = \alpha_9 = 0$ and $\alpha_6 + \alpha_8 \neq 0$. Then we get $a_{23} = a_{32} = 0$. Thus we obtain $\mathcal{DL}_1^4\left(\alpha, 0, n, q\right)$, where $n + q \neq 0$.

3. Let $\alpha_5 = \alpha_9 = 0$ and $\alpha_6 + \alpha_8 = 0$. Then we get

$$\alpha_5' = 0, \ \alpha_6' = \alpha_6, \ \alpha_8' = -\alpha_6, \ \alpha_9' = 0.$$

Thus we obtain $\mathcal{DL}_1^5\left(\alpha, 0, n, -n\right)$.

In this case we obtain $\mathcal{DL}_2\left(\alpha, n, q\right)$, where $\alpha \in \mathbb{C}\backslash\{0\}, n, q \in \mathbb{C}$;

□

Theorem 2.2. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_2$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$\mathcal{DL}_2^1(m,n):$ $e_3 \dashv e_3 = e_1, e_2 \vdash e_2 = e_1, e_2 \vdash e_3 = me_1, e_3 \vdash e_2 = ne_1,$ *where* $m,n \in \mathbb{C}$;

$\mathcal{DL}_2^2(q):$ $e_3 \dashv e_3 = e_1, e_3 \vdash e_3 = qe_1,$ *where* $q \in \mathbb{C}$;

$\mathcal{DL}_2^3:$ $e_3 \dashv e_3 = e_1, e_3 \vdash e_2 = e_1$;

$\mathcal{DL}_2^4(n):$ $e_3 \dashv e_3 = e_1, e_2 \vdash e_3 = e_1, e_3 \vdash e_2 = ne_1,$ *where* $n \in \mathbb{C}$;

$\mathcal{DL}_2^5:$ $e_3 \dashv e_3 = e_1, e_3 \vdash e_3 = e_2$;

Proof. Consider the algebra $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ with multiplication table:

$$e_3 \dashv e_3 = e_1.$$

The second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$ is defined by unknowns $\alpha_i, \beta_i, \gamma_i$ where $1 \leq i \leq 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$e_2 \vdash e_2 = \alpha_5 e_1,\ e_2 \vdash e_3 = \alpha_6 e_1,\ e_3 \vdash e_2 = \alpha_8 e_1,\ e_3 \vdash e_3 = \alpha_9 e_1 + \beta_9 e_2,$$

with the following constraints

$$\alpha_5\beta_9 = 0,\ \alpha_6\beta_9 = 0,\ \alpha_8\beta_9 = 0.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = a_{33}^2 e_1,\ \varphi(e_2) = a_{21}e_1 + a_{22}e_2,\ \varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3.$$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha_5', \alpha_6', \alpha_8', \alpha_9', \beta_9'\}$ and $\{\alpha_5, \alpha_6, \alpha_8, \alpha_9, \beta_9\}$:

$$\begin{array}{lll}
\varphi(e_2) \vdash \varphi(e_2) = \alpha_5'\varphi(e_1), & \Rightarrow & \alpha_5' = \frac{a_{22}^2}{a_{33}^2}\alpha_5, \\
\varphi(e_2) \vdash \varphi(e_3) = \alpha_6'\varphi(e_1), & \Rightarrow & \alpha_6' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_6}{a_{33}^2}, \\
\varphi(e_3) \vdash \varphi(e_2) = \alpha_8'\varphi(e_1), & \Rightarrow & \alpha_8' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_8}{a_{33}^2}, \\
\varphi(e_3) \vdash \varphi(e_3) = \alpha_9'\varphi(e_1) + \beta_9'\varphi(e_2), & \Rightarrow & \begin{cases} \alpha_9' a_{33}^2 + \frac{a_{21}a_{33}^2}{a_{22}}\beta_9 = a_{32}^2\alpha_5 + a_{32}a_{33}(\alpha_6 + \alpha_8) + a_{33}^2\alpha_9, \\ \beta_9' = \frac{a_{33}^2}{a_{22}}\beta_9. \end{cases}
\end{array}$$

Then we have the following cases.

1. Let $\beta_9 = 0$. Then we have $\beta_9' = 0$ and

$$\begin{array}{ll}
\alpha_5' = \frac{a_{22}^2}{a_{33}^2}\alpha_5, & \alpha_6' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_6}{a_{33}^2}, \\
\alpha_8' = \frac{a_{22}a_{32}\alpha_5 + a_{22}a_{33}\alpha_8}{a_{33}^2}, & \alpha_9' = \frac{a_{32}^2\alpha_5 + a_{32}a_{33}(\alpha_6 + \alpha_8) + a_{33}^2\alpha_9}{a_{33}^2}.
\end{array}$$

   (1) If $\alpha_5 \neq 0$. Next, choosing $a_{33} = a_{22}\sqrt{\alpha_5}$ and $a_{32} = \frac{-(\alpha_6+\alpha_8) \pm \sqrt{(\alpha_6+\alpha_8)^2 - 4\alpha_5\alpha_9}}{2\alpha_5}a_{33}$, we can put $\alpha_5' = 1$ and $\alpha_9' = 0$. Thus we obtained $\mathcal{DL}_2^1(m,n)$, where $m, n \in \mathbb{C}$.

   (2) If $\alpha_5 = 0$. Then we obtain the following relations:

$$\begin{array}{ll}
\alpha_5' = 0, & \alpha_6' = \frac{a_{22}}{a_{33}}\alpha_6, \\
\alpha_8' = \frac{a_{22}}{a_{33}}\alpha_8, & \alpha_9' = \frac{a_{32}(\alpha_6+\alpha_8) + a_{33}\alpha_9}{a_{33}}.
\end{array}$$

   (2.1) If $\alpha_6 = 0$ and $\alpha_8 = 0$. Thus we obtain $\mathcal{DL}_2^2(q)$, where $q \in \mathbb{C}$.

   (2.2) If $\alpha_6 = 0$ and $\alpha_8 \neq 0$. Then choosing $a_{33} = a_{22}\alpha_8$, we get $\mathcal{DL}_2^3$.

   (2.3) If $\alpha_6 \neq 0$. Then choosing $a_{33} = a_{22}\alpha_8$, we get $\mathcal{DL}_2^4(n)$, where $n \in \mathbb{C}$.

2. Let $\beta_9 \neq 0$. Then we get $a_5 = a_6 = a_8 = 0$ and $\alpha_5' = \alpha_6' = \alpha_8' = 0$. Next, choosing $a_{22} = a_{33}^2\beta_9$, we can put $\beta_9' = 1$. In this case we obtain the algebra $\mathcal{DL}_2^5$.

□

Theorem 2.3. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_3$ is isomorphic to one of the following pairwise non-isomorphic algebras:*
$\mathcal{DL}_3\,(m,n,q):\ e_2 \dashv e_2 = e_1,\ e_3 \dashv e_3 = e_1,\ e_2 \vdash e_3 = me_1,\ e_3 \vdash e_2 = ne_1,\ e_3 \vdash e_3 = qe_1,$
where $m,n,q \in \mathbb{C}$;

Proof. Consider the algebra $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ with multiplication table:

$$e_2 \dashv e_2 = e_1, e_3 \dashv e_3 = e_1.$$

The second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$ is defined by unknowns $\alpha_i, \beta_i, \gamma_i$ where $1 \le i \le 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$e_2 \vdash e_2 = \alpha_5 e_1,\ e_2 \vdash e_3 = \alpha_6 e_1,\ e_3 \vdash e_2 = \alpha_8 e_1,\ e_3 \vdash e_3 = \alpha_9 e_1.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = (a_{32}^2 + a_{33}^2)\, e_1,\ \varphi(e_2) = a_{21}e_1 - a_{33}e_2 + a_{32}e_3,\ \varphi(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3,$$

with $a_{22}a_{33} - a_{23}a_{32} \neq 0$.

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_5, \alpha'_6, \alpha'_8, \alpha'_9\}$ and $\{\alpha_5, \alpha_6, \alpha_8, \alpha_9\}$:

$$\begin{aligned}
\varphi(e_2) \vdash \varphi(e_2) &= \alpha'_5\varphi(e_1), &&\Rightarrow\ \alpha'_5 = \tfrac{a_{33}^2\alpha_5 - a_{32}a_{33}(\alpha_6+\alpha_8) + a_{32}^2\alpha_9}{a_{32}^2+a_{33}^2},\\
\varphi(e_2) \vdash \varphi(e_3) &= \alpha'_6\varphi(e_1), &&\Rightarrow\ \alpha'_6 = \tfrac{(\alpha_9-\alpha_5)a_{32}a_{33} - a_{33}^2\alpha_6 + a_{32}^2\alpha_8}{a_{32}^2+a_{33}^2},\\
\varphi(e_3) \vdash \varphi(e_2) &= \alpha'_8\varphi(e_1), &&\Rightarrow\ \alpha'_8 = \tfrac{(\alpha_9-\alpha_5)a_{32}a_{33} + a_{32}^2\alpha_6 - a_{33}^2\alpha_8}{a_{32}^2+a_{33}^2},\\
\varphi(e_3) \vdash \varphi(e_3) &= \alpha'_9\varphi(e_1), &&\Rightarrow\ \alpha'_9 = \tfrac{a_{32}^2\alpha_5 + a_{32}a_{33}(\alpha_6+\alpha_8) + a_{33}^2\alpha_9}{a_{32}^2+a_{33}^2}.
\end{aligned}$$

Then we have the following cases.

1. Let $(\alpha_5 - \alpha_9)^2 + (\alpha_6 + \alpha_8)^2 = 0$. Then we have $\alpha_9 = \alpha_5 - i(\alpha_6 + \alpha_8)$ and

$$\begin{aligned}
\alpha'_5 &= \tfrac{(a_{33}-ia_{32})\alpha_5 - a_{32}(\alpha_6+\alpha_8)}{a_{33}-ia_{32}}, & \alpha'_6 &= \tfrac{a_{32}\alpha_8 + ia_{33}\alpha_6}{a_{32}-ia_{33}},\\
\alpha'_8 &= \tfrac{a_{32}\alpha_6 + ia_{33}\alpha_8}{a_{32}-ia_{33}}, & \alpha'_9 &= \tfrac{(a_{32}+ia_{33})\alpha_5 + a_{33}(\alpha_6+\alpha_8)}{a_{32}+ia_{33}}.
\end{aligned}$$

(1) Let $\alpha_5 + \alpha_9 = 0$ and $\alpha_6 - \alpha_8 = 0$. Then we have $\alpha_5 = -i\alpha_6$ and

$$\alpha'_5 = \tfrac{3a_{32}+ia_{33}}{ia_{32}-a_{33}}\alpha_6,\quad \alpha'_6 = \tfrac{a_{32}+ia_{33}}{a_{32}-ia_{33}}\alpha_6,\quad \alpha'_8 = \tfrac{a_{32}+ia_{33}}{a_{32}-ia_{33}}\alpha_6,\quad \alpha'_9 = \tfrac{3a_{33}-ia_{32}}{a_{32}+ia_{33}}\alpha_6.$$

Furthermore, by choosing $a_{33} = 3ia_{32}$, we obtain the following relations:

$$\alpha'_5 = 0,\quad \alpha'_6 = -\tfrac{1}{2}\alpha_6,\quad \alpha'_8 = -\tfrac{1}{2}\alpha_6,\quad \alpha'_9 = -4i\alpha_6.$$

In this case we obtain $\mathcal{DL}_3^1\,(m)$, where $m \in \mathbb{C}$.

(2) Let $\alpha_5 + \alpha_9 = 0$ and $\alpha_6 - \alpha_8 \neq 0$. Then we have $\alpha_5 = -\frac{i}{2}\,(\alpha_6 + \alpha_8)$ and

$$\begin{aligned}
\alpha'_5 &= \tfrac{3a_{32}+ia_{33}}{2(ia_{32}-a_{33})}(\alpha_6+\alpha_8), & \alpha'_6 &= \tfrac{a_{32}\alpha_8 + ia_{33}\alpha_6}{a_{32}-ia_{33}},\\
\alpha'_8 &= \tfrac{a_{32}\alpha_6 + ia_{33}\alpha_8}{a_{32}-ia_{33}}, & \alpha'_9 &= \tfrac{3a_{33}-ia_{32}}{2(a_{32}+ia_{33})}\,(\alpha_6+\alpha_8)\,.
\end{aligned}$$

Then, by choosing $a_{33} = 3ia_{32}$, we obtain the following relations:

$$\alpha'_5 = 0,\quad \alpha'_6 = \tfrac{1}{4}(-3\alpha_6 + \alpha_8),\quad \alpha'_8 = \tfrac{1}{4}(-3\alpha_8 + \alpha_6),\quad \alpha'_9 = -2i(\alpha_6 + \alpha_8).$$

In this case we obatin $\mathcal{DL}_3^2\,(m,n)$, where $m,n \in \mathbb{C}$ and $m - n \neq 0$.

(3) Let $\alpha_5+\alpha_9\neq 0$ and $\alpha_6-\alpha_8\neq 0$. Then we have $2\alpha_5-i\left(\alpha_6+\alpha_8\right)\neq 0$. Therefore, by choosing $a_{32}=-\frac{i\alpha_5}{\alpha_5-i(\alpha_6+\alpha_8)}a_{33}$, we obtain the following relations:

$$\alpha_5'=0,\quad \alpha_6'=-i\alpha_5-\alpha_6,\quad \alpha_8'=-i\alpha_5-\alpha_8,\quad \alpha_9'=2\alpha_5-i\left(\alpha_6+\alpha_8\right).$$

In this case we obatin $\mathcal{DL}_3^3\left(m,n,q\right)$, where $m,n,q\in\mathbb{C}$, $n+q\neq 0$ and $2m-i\left(n+q\right)\neq 0$.

In this case we obatin $\mathcal{DL}_3^3\left(m,n,q\right)$, where $m,n,q\in\mathbb{C}$, $n+q\neq 0$, $2m-i\left(n+q\right)\neq 0$.

2. Let $(\alpha_5-\alpha_9)^2+(\alpha_6+\alpha_8)^2\neq 0$ and 1) let $(\alpha_5,\alpha_9)\neq(0,0)$. Without loss of generality, we assume that $\alpha_9\neq 0$. Next, choosing $a_{32}=\frac{(\alpha_6+\alpha_8)\pm\sqrt{(\alpha_6+\alpha_8)^2-4\alpha_5\alpha_9}}{2\alpha_9}a_{33}$, we can put $\alpha_5'=0$. Thus we obtained the following relations:

$$\begin{aligned}&a_{32}^2\alpha_9-a_{32}a_{33}\left(\alpha_6+\alpha_8\right)=0, && \alpha_6'=\frac{a_{32}a_{33}\alpha_9-a_{33}^2\alpha_6+a_{32}^2\alpha_8}{a_{32}^2+a_{33}^2},\\ &\alpha_8'=\frac{a_{32}a_{33}\alpha_9-a_{33}^2\alpha_8+a_{32}^2\alpha_6}{a_{32}^2+a_{33}^2}, && \alpha_9'=\frac{a_{32}a_{33}(\alpha_6+\alpha_8)+a_{33}^2\alpha_9}{a_{32}^2+a_{33}^2}.\end{aligned}$$

(1) If $a_{32}\neq 0$. Then we have $a_{32}=\frac{a_{33}(\alpha_6+\alpha_8)}{\alpha_9}$, and

$$\alpha_6'=\alpha_8,\ \alpha_8'=\alpha_6,\ \alpha_9'=\alpha_9.$$

In this case we obtain $\mathcal{DL}_3^4\left(m,n,q\right)$, where $q\neq 0$.

(2) If $a_{32}=0$. Then we have the following relations:

$$\alpha_6'=-\alpha_6,\ \alpha_8'=-\alpha_8,\ \alpha_9'=\alpha_9.$$

In this case we obtain $\mathcal{DL}_3^4\left(-m,-n,q\right)$, where $q\neq 0$.

2) Let $\alpha_5=\alpha_9=0$. Then we get $a_{32}=0$, and

$$\alpha_5'=0,\ \alpha_6'=-\alpha_6,\ \alpha_8'=-\alpha_8,\ \alpha_9'=0.$$

In this case we obtain $\mathcal{DL}_3^4\left(-m,-n,0\right)$.

In general, we obatin $\mathcal{DL}_3\left(m,n,q\right)$, where $m,n,q\in\mathbb{C}$.

□

Theorem 2.4. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_4$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$\mathcal{DL}_4^1:\ e_1\dashv e_3=e_1,\ e_3\vdash e_3=e_1;$
$\mathcal{DL}_4^2:\ e_1\dashv e_3=e_1,\ e_3\vdash e_1=-e_1;$
$\mathcal{DL}_4^3:\ e_1\dashv e_3=e_1,\ e_3\vdash e_1=-e_1,\ e_3\vdash e_3=e_1;$
$\mathcal{DL}_4^4:\ e_1\dashv e_3=e_1,\ e_1\vdash e_3=e_1;$
$\mathcal{DL}_4^5:\ e_1\dashv e_3=e_1,\ e_1\vdash e_3=e_1,\ e_3\vdash e_3=e_1;$

Proof. Consider the algebra $\mathcal{A}_1=(\mathcal{DL},\dashv)$ with multiplication table:

$$e_1\dashv e_3=e_1.$$

The second part $\mathcal{A}_2=(\mathcal{DL},\vdash)$ is defined by unknowns $\alpha_i,\beta_i,\gamma_i$ where $1\leq i\leq 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$e_1\vdash e_3=\alpha_3e_1,\ e_3\vdash e_1=\alpha_7e_1,\ e_3\vdash e_3=\beta_9e_1,$$

with the following constraints

$$\alpha_3\alpha_7=0,\ \alpha_3\left(\alpha_3-1\right)=0,\ \alpha_7\left(\alpha_7+1\right)=0.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1)=a_{11}e_1,\ \varphi(e_2)=a_{22}e_2,\ \varphi(e_3)=a_{32}e_2+e_3.$$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_3, \alpha'_7, \beta'_9\}$ and $\{\alpha_3, \alpha_7, \beta_9\}$:

$$\begin{array}{ll}
\varphi(e_1) \vdash \varphi(e_3) = \alpha'_3\varphi(e_1), & \Rightarrow \quad \alpha'_3 = \alpha_3, \\
\varphi(e_3) \vdash \varphi(e_1) = \alpha'_7\varphi(e_1), & \Rightarrow \quad \alpha'_7 = \alpha_7, \\
\varphi(e_3) \vdash \varphi(e_3) = \beta'_9\varphi(e_1), & \Rightarrow \quad \beta'_9 = \frac{1}{a_{22}}\beta_9.
\end{array}$$

Hence, we see that if $\beta_9 = 0$, then $\beta'_9 = 0$. If $\beta_9 \neq 0$, then by choosing $a_{22} = \beta_9$, we obtain $\beta'_9 = 1$. Therefore, without loss of generality, β'_9 can be reduced to either 0 or 1. Thus we obtain the algebras $\mathcal{DL}_4^1$, $\mathcal{DL}_4^2$, $\mathcal{DL}_4^3$, $\mathcal{DL}_4^4$ and $\mathcal{DL}_4^5$.

□

Theorem 2.5. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_5$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$\mathcal{DL}_5^1$: $e_1 \dashv e_3 = \alpha e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_2 = -e_2,$ *where* $\alpha \in \mathbb{C}\backslash\{0\}$;

$\mathcal{DL}_5^2$: $e_1 \dashv e_3 = e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_2 \vdash e_3 = e_1 + e_2,\ e_3 \vdash e_2 = -e_1 - e_2$;

$\mathcal{DL}_5^3$: $e_1 \dashv e_3 = \alpha e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_1 = -\alpha e_1,\ e_3 \vdash e_2 = -e_2,$ *where* $\alpha \in \mathbb{C}\backslash\{0\}$;

$\mathcal{DL}_5^4$: $e_1 \dashv e_3 = \alpha e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_1 \vdash e_3 = \alpha e_1,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_2 = -e_2,$ *where* $\alpha \in \mathbb{C}\backslash\{0\}$;

Proof. Consider the algebra $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ with multiplication table:

$$e_1 \dashv e_3 = \alpha e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2, \text{ where } \alpha \in \mathbb{C}\backslash\{0\}.$$

The second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$ is defined by unknowns $\alpha_i, \beta_i, \gamma_i$ where $1 \leq i \leq 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$e_1 \vdash e_3 = \alpha_3 e_1,\ e_2 \vdash e_3 = \alpha_6 e_1 + e_2,\ e_3 \vdash e_1 = \alpha_7 e_1,\ e_3 \vdash e_2 = -\alpha_6 e_1 - e_2.$$

with the following constraints

$$\alpha_3(\alpha_3 - \alpha) = 0,\ \alpha_3\alpha_6 = 0,\ \alpha_6(\alpha - 1) = 0,\ \alpha_3\alpha_7 = 0,\ \alpha_3\alpha_6 = 0,\ \alpha_6\alpha_7 = 0,\ \alpha_7(\alpha_7 + \alpha) = 0.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = a_{11}e_1,\ \varphi(e_2) = a_{22}e_2,\ \varphi(e_3) = a_{32}e_2 + e_3.$$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_3, \alpha'_6, \alpha'_7\}$ and $\{\alpha_3, \alpha_6, \alpha_7\}$:

$$\begin{array}{ll}
\varphi(e_1) \vdash \varphi(e_3) = \alpha'_3\varphi(e_1) & \Rightarrow \quad \alpha'_3 = \alpha_3, \\
\varphi(e_2) \vdash \varphi(e_3) = \alpha'_6\varphi(e_1) + \varphi(e_2) & \Rightarrow \quad \alpha'_6 = \frac{a_{22}}{a_{11}}\alpha_6, \\
\varphi(e_3) \vdash \varphi(e_1) = \alpha'_7\varphi(e_1) & \Rightarrow \quad \alpha'_7 = \alpha_7, \\
\varphi(e_3) \vdash \varphi(e_2) = -\alpha'_6\varphi(e_1) - \varphi(e_2) & \Rightarrow \quad \alpha'_6 = \frac{a_{22}}{a_{11}}\alpha_6.
\end{array}$$

Hence, we see that if $\alpha_6 = 0$, then $\alpha'_6 = 0$. If $\alpha_6 \neq 0$, then by choosing $a_{11} = a_{22}\alpha_6$, we obtain $\alpha'_6 = 1$. Therefore, if $\alpha'_6 = 1$, then $\alpha = 1$. Thus, without loss of generality, α'_6 can be reduced to either 0 or 1. Then we obtain the algebras $\mathcal{DL}_5^1$, $\mathcal{DL}_5^2$, $\mathcal{DL}_5^3$ and $\mathcal{DL}_5^4$.

□

Theorem 2.6. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_6$ is isomorphic to the following algebra:*

$\mathcal{DL}_6^1$: $e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_3 \dashv e_3 = e_1,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_2 = -e_2,\ e_3 \vdash e_3 = qe_1,$ *where* $q \in \mathbb{C}$;

Proof. Consider the algebra $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ with multiplication table:

$$e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_3 \dashv e_3 = e_1.$$

The second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$ is defined by unknowns $\alpha_i, \beta_i, \gamma_i$ where $1 \leq i \leq 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$e_2 \vdash e_3 = e_2,\ e_3 \vdash e_2 = -e_2,\ e_3 \vdash e_3 = \alpha_9 e_1.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = e_1,\ \varphi(e_2) = a_{22}e_2,\ \varphi(e_3) = a_{31}e_1 + a_{32}e_2 + e_3.$$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters α'_9 and α_9 :

$$\varphi(e_3) \vdash \varphi(e_3) = \alpha'_9 \varphi(e_1) \quad \Rightarrow \quad \alpha'_9 = \alpha_9.$$

Hence, in this case we have $\mathcal{DL}_6^1$.

□

Theorem 2.7. *Any three-dimensional complex Leibniz dialgebra constructed from the algebra $\mathcal{L}_7$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\mathcal{DL}_7^1 : \begin{cases} e_1 \dashv e_3 = 2e_1,\ e_2 \dashv e_2 = e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_3 \dashv e_3 = e_1, \\ e_1 \vdash e_3 = 2e_1,\ e_2 \vdash e_2 = e_1,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_2 = -e_2; \end{cases}$$

$$\mathcal{DL}_7^2 : \begin{cases} e_1 \dashv e_3 = 2e_1,\ e_2 \dashv e_2 = e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_3 \dashv e_3 = e_1, \\ e_2 \vdash e_2 = -e_1,\ e_2 \vdash e_3 = e_2,\ e_3 \vdash e_1 = -2e_1,\ e_3 \vdash e_2 = -e_2,\ e_3 \vdash e_3 = -e_1; \end{cases}$$

Proof. Consider the algebra $\mathcal{A}_1 = (\mathcal{DL}, \dashv)$ with multiplication table:

$$e_1 \dashv e_3 = 2e_1,\ e_2 \dashv e_2 = e_1,\ e_2 \dashv e_3 = e_2,\ e_3 \dashv e_2 = -e_2,\ e_3 \dashv e_3 = e_1.$$

The second part $\mathcal{A}_2 = (\mathcal{DL}, \vdash)$ is defined by unknowns $\alpha_i, \beta_i, \gamma_i$ where $1 \leq i \leq 9$ as in (2.1). Imposing the Leibniz dialgebra axioms, we obtain

$$\begin{cases} e_1 \vdash e_3 = \alpha_3 e_1,\ e_2 \vdash e_2 = (\alpha_3 - 1)e_1,\ e_2 \vdash e_3 = e_2, \\ e_3 \vdash e_1 = (\alpha_3 - 2)e_1,\ e_3 \vdash e_2 = -e_2,\ e_3 \vdash e_3 = \alpha_9 e_1. \end{cases}$$

with the following constraints

$$\alpha_3 (\alpha_3 - 2) = 0,\ (\alpha_3 - 2)(\alpha_9 + 1) = 0.$$

Let us consider the general change of the generators of basis:

$$\varphi(e_1) = a_{22}^2 e_1,\ \varphi(e_2) = -a_{22}a_{32}e_1 + a_{22}e_2,\ \varphi(e_3) = \tfrac{1}{2}\left(a_{22}^2 - a_{32}^2 - 1\right)e_1 + a_{32}e_2 + e_3.$$

We write the new basis elements $\{\varphi(e_1), \varphi(e_2), \varphi(e_3)\}$ via the basis elements $\{e_1, e_2, e_3\}$. By checking all the multiplications of the algebra in the new basis we obtain the relations between the parameters $\{\alpha'_3, \alpha'_9\}$ and $\{\alpha_3, \alpha_9\}$:

$$\varphi(e_1) \vdash \varphi(e_3) = \alpha'_3 \varphi(e_1) \quad \Rightarrow \quad \alpha'_3 = \alpha_3.$$

$$\varphi(e_2) \vdash \varphi(e_2) = (\alpha'_3 - 1)\varphi(e_1) \quad \Rightarrow \quad \alpha'_3 = \alpha_3.$$

$$\varphi(e_3) \vdash \varphi(e_1) = (\alpha'_3 - 2)\varphi(e_1) \quad \Rightarrow \quad \alpha'_3 = \alpha_3.$$

$$\varphi(e_3) \vdash \varphi(e_3) = \alpha'_9 \varphi(e_1) \quad \Rightarrow \quad \alpha'_9 = \frac{a_{32}^2\alpha_3 - a_{22}^2 + 1 + \alpha_9}{a_{22}^2}.$$

Then we have the following cases.

Case 1. Let $\alpha_3 \neq 0$. Hence, by setting $a_{32} = \sqrt{\frac{a_{22}^2 - \alpha_9 - 1}{2}}$, we obtain $\alpha'_9 = 0$. In this case we have $\mathcal{DL}_7^1$.

Case 2. If $\alpha_3 = 0$, then $\alpha_9 = -1$. In this case we obtain $\mathcal{DL}_7^2$.

□

References

[1] Loday J.-L., Frabetti A., Chapaton F., Goichot F., Dialgebras and Related Operads. Springer, Berlin, (2001).

[2] Loday J.-L., Une version non commutative des algebras de Lie; les algebras de Leibniz. Enseign. Math. (1993) Vol. 39, P. 269–293.

[3] Loday J.-L., Dialgebras. Dialgebras and Related Operads. Springer Verl., (2001), vol. 1763, P. 7–66.

[4] Felipe R., Dendriform algebras and Rota-Baxter operators revisited in several directions. Communicaciotns in CIMAT, (2013),

[5] Gao D., Liu G., Bai C., Anti-dendriform algebras, new splitting of operations and Novikov type algebras. Journal of Algebraic Combinatorics, (2024) Vol. 59, P. 661–696.

[6] Gonzalez C., Associative dialgebras from structural viewpoint. Communications in Algebra, (2013) Vol. 41, P. 1903–1912.

[7] Ayupov S., Omirov B., On Leibniz Algebras. Kluwer Academic Publisher, Notherlands, (1998).

[8] Ayupov S., Omirov B., Rakhimov I., Leibniz Algebras, Structure and Classification. CRC Press, Taylor and Francis Group, (2019).

[9] Casas J., Insua M., Ladra M., Ladra S., An algorithm for the classification of 3-dimensional complex Leibniz algebras. Linear Algebra and Appl. (2012) Vol. 436, P. 3747–3756.

[10] Abdurasulov K., Adashev J., Normatov Z., Solijonova S., Classification of three dimensional anti-dendriform algebras. Communications in Algebra, (2025) Vol. 53, Iss. 5, P. 1887–1903.

[11] Albeverio S., Omirov B., Rakhimov I., Varieties of nilpotent complex Leibniz algebras dimensions less than five. Communications in Algebra, (2005) Vol. 33, P. 1575–1585.

[12] Azizov M., Description of 4-dimensional Leibniz dialgebras which are constructed by null-filiform Leibniz algebra. Bull. Inst. Math. (2023) Vol. 6, Iss. 4, P. 1–8.

[13] Basri W., Rikhsiboev I., On low dimensional diassociative algebras. Proceedings of Third Conference on Research and Education in Mathematics (ICREM-3), Malaysia, (2007).

[14] Rikhsiboev I., Rakhimov I., Basri W., Classification of 3-dimensional complex diassociative algebras. Malaysian Journal of Mathematical Sciences, (2010) Vol. 4, P. 241–254.

[15] Rikhsiboev I., Rakhimov I., Basri W., Four-dimensional nilpotent diassociative algebras. Journal of Generalized Lie Theory and Applications, (2015) Vol. 9,

[16] Kolesnikov P., Varieties of dialgebras and conformal algebras. Siberian Mathematical Journal, (2008) Vol. 49, P. 257–272.

Kurbanbaev T. K.,
Karakalpak State University, Nukus, Uzbekistan,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
e-mail: tuuelbay@mail.ru

Uzakbaev N. E.,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
e-mail: nn.uzakbaev@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 126–131
DOI: 10.29229/uzmj.2026-1-13

Distribution of prime divisors of non-homogeneous Beatty sequences

Low C.W., Sapar S.H., Deraman F., Johari M.A.M., Yunos F.

Abstract. Non-homogeneous Beatty sequence is a sequence of positive integer that taking the floor value of irrational numbers. This paper using the prime counting function, $\pi(x)$ to estimate the cardinality of total distinct prime divisors of a non-homogeneous Beatty sequence. When the parameters in the non-homogeneous Beatty sequence are sufficiently large, a better estimation can be obtained. From this study, we found that for a fixed irrational $\theta > 1$ and a real $\lambda > 0$, the the cardinality of total distinct prime divisors is less than or equals to the prime counting function of the last term. That is $|A| \leq \pi(\lfloor N\theta + \lambda \rfloor)$ where A is the set of total distinct prime divisors of a non-homogeneous Beatty sequence $(\lfloor n\theta + \lambda \rfloor)$ up to N^{th} term. Also, when parameters are sufficiently large, the following estimation is sharper, $|A| \leq \left(\sqrt{\theta^2 + \lambda}\right)(\log\ N)$.

Keywords: Beatty sequence, prime divisor, prime counting function
MSC (2020): 11B83, 11N05

1. Introduction

Numerous studies have been done on Beatty sequence. There are two types of Beatty sequences, namely homogeneous and non-homogeneous. The Beatty sequence that this paper considered is non-homogeneous case and it is defined as follows:

$$(\lfloor n\theta + \lambda \rfloor)_{n \leq N} = \lfloor 1\theta + \lambda \rfloor, \lfloor 2\theta + \lambda \rfloor, \lfloor 3\theta + \lambda \rfloor, ..., \lfloor N\theta + \lambda \rfloor \tag{1.1}$$

where $\theta > 1$ is an irrational number and $\lambda > 0$ is a real number.

In [1], the researchers used Maynard's methods to show the existence of gaps bounded by primes in a homogeneous Beatty sequence. They found that $\#\{x \leq n < 2x :$ there exist m distinct primes of the form $\lfloor \theta r \rfloor, r \in [n, n + \Delta_{\theta,m}]\} \gg \frac{x}{(logx)^B}$ where $\#$ is the number of elements in the set. [2] studied the non-homogeneous Beatty sequence. They used Diophantine properties of θ to established a better estimate bound for its character sums. Below are some important theorems:

Theorem 1.1. *Let θ be a fixed irrational number. For all real numbers λ, integers a, g, m with $gcd(ag, m) = 1$, and positive integers $N \leq t$, where t is the multiplicative order of g modulo m. The character sums of Beatty sequence $(\lfloor n\theta + \lambda \rfloor)_{n \leq N}$ modulo m, $S_m(\theta, \lambda, \chi; N)$ is given by:*

$$S_m(\theta, \lambda, \chi; N) \ll m^{\frac{1}{4}} N^{\frac{1}{2}} + ND_{\theta,\lambda}(N)$$

where $D_{\theta,\lambda}(N)$ is the discrepancy of the Beatty sequence $(\lfloor n\theta + \lambda \rfloor)_{n \leq N}$.

Theorem 1.2. *Let θ be a fixed irrational number. For any fixed $\delta > 0$, there exists a constant $\eta > 0$ such that for all real numbers λ, integers a, g and a prime p with $gcd(ag, p) = 1$, and positive integers $p^\delta < N \leq t$, where t is the multiplicative order of g modulo p, the following bound holds:*

$$S_p(\theta, \lambda, \chi; N) \ll Np^{-\eta} + ND_{\theta,\lambda}(N).$$

Then, [3] continue to estimate the sum of the cardinality of distinct prime divisors after the discovery of [2]. They found that the cardinality of distinct prime divisors $\omega(n)$ for a non-zero integer n, is approximate to $N \log\log N$. [4] studied the Beatty sequence in invariant games. They introduced an infinite binary sequence, called Sturmian word, in order to distinguish any two pairs of complementary Beatty sequences in non-homogeneous case. [5] studied the number on primes within the intersection of any non-homogeneous Beatty sequences and extended a few theorems under various compatibility conditions. [6] solved the trigonometric functions by using the complementary Beatty sequences. They use $\theta = \sqrt{3}$ for tangent function and $\theta = \sqrt{6}$ for sine function while solving the trigonometric inequalities. [7] worked on the disjoint complementary Beatty sequence which covering the rational non-homogeneous Beatty sequence.

[8] studied the new approach found in [2]. They obtained an estimation of the bound associated with composite moduli by using the method in [2] stated below.

Theorem 1.3. *Let θ be a fixed irrational number, λ be any real number and m is any composite number with primitive elements. For positive prime $P \leq m$, the set of prime $\mathcal{P}$ and $\#\mathcal{P}$ is the number of elements in set $\mathcal{P}$, the non-trivial multiplicative characters χ (mod m), the following bound holds:*

$$S_m(\theta, \lambda, \chi; P) \ll \phi(m)^{\frac{1}{4}} \#\mathcal{P}^{\frac{1}{2}} + \#\mathcal{P} D_{\theta,\lambda}(P).$$

[9] investigated the cardinality of character sums with Beatty sequences associated with composite modulo. The Beatty sequence considered is $\lfloor \alpha(n+k) + \beta \rfloor$. Character sums can be used to find the number of solutions of equations over a given finite field. This sums can be obtained by one character or more. Here, we provided the propositions stated in the book [10] on the character sums associated with prime modulo. Suppose that p is a odd prime and $\mathbb{F}_p^*$ is a multiplicative group. Then,

Proposition 1.4. *Let g be a primitive element of $\mathbb{F}_p$ with order $p-1$. For each fixed integer j where $0 \leq j \leq \phi(m) - 1$, the multiplicative character of $\mathbb{F}_p$, denoted by $\chi_j(g^k)$ is given by*

$$\chi_j(g^k) = e^{\frac{2\pi i j k}{p-1}} \text{ where } k = 0, 1, ..., p-1.$$

Proposition 1.5. *For additive character χ_a and χ_b where $a, b \in \mathbb{F}_p$. Then,*

$$\sum_{c \in \mathbb{F}_p} \chi_a(c)\overline{\chi_b(c)} = \begin{cases} p+1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

For multiplicative character, if $a, b \in \mathbb{F}_p^*$. Then,

$$\sum_{\chi} \chi_c(a)\overline{\chi_c(b)} = \begin{cases} p & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

where the sum is extended over all multiplicative character χ of $\mathbb{F}_p$.

In this paper, we are going to improve the result of [3] and introduce a better estimation when the parameters in Beatty sequence are sufficiently large. In our discussion, we let $\theta = \sqrt{k} > 1$ where k is a square-free integer and $\lambda > 0$ is a real number. Next, we expand the expressions in order to obtain the patterns. We will explain more in the next section.

2. Notation

For any real number x, $\lfloor x \rfloor$ denotes the greatest integer less or equals to x and $\{x\}$ denotes the fractional part of x, that is $x - \lfloor x \rfloor$.

In this paper, A denotes as the set of total distinct prime divisors of a non-homogeneous Beatty sequence $(\lfloor n\theta + \lambda \rfloor)$ up to N^{th} term. Our objective is to obtain the approximation for cardinality of A, $|A|$ where the result will be shown in the next section. Before we proceed, we provide an example so that the readers have a better understanding.

Example 2.1. Let $\theta = \sqrt{10}$, $\lambda = 5$ and $N = 100$. Then, (1.1) will be as follow

$$\begin{aligned} \left(\lfloor n\sqrt{10} + 5 \rfloor\right)_{n \leq 100} &= \lfloor 1\sqrt{10} + 5 \rfloor, \lfloor 2\sqrt{10} + 5 \rfloor, \lfloor 3\sqrt{10} + 5 \rfloor, ..., \lfloor 100\sqrt{10} + 5 \rfloor \\ &= 8, 11, 14, ..., 321. \end{aligned}$$

From above, we will find the prime divisors of each terms. Let A denotes as the set of prime divisors of whole sequence $8, 11, 14, ..., 321$. Then the elements of set A is as follow:

$$A = \{2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, 43, 47, 53, 59, 61, 67, 71, 73, 83, 89, 97, 103, 107, \\ 109, 113, 127, 131, 137, 151, 157, 163, 191, 197, 223, 229, 239, 251, 257, 283, 311\}.$$

Thus, the cardinality or the number of elements in set A, $|A| = 41$.

Next, the following is the definition of prime counting function and an example to illustrate it.

Definition 2.2. Prime counting function $\pi(x)$ is the number of primes less than or equal to the positive integer x.

Example 2.3. Let $x = 10$, then we list out the prime less than or equal to 10, which is $\{2, 3, 5, 7\}$. Thus $\pi(10) = 4$.

For a larger number x, we can use online prime counting function calculator to evaluate it.

3. Result and Discussion

The following is the main result of our research.

Theorem 3.1. *Let $\theta > 1$ be a fixed irrational and $\lambda > 0$ be a real number. The cardinality of distinct prime divisors, $|A|$ is given by*

$$|A| \leq \pi(\lfloor N\theta + \lambda \rfloor) \tag{3.1}$$

where $\pi(x)$ is the prime counting function for integer x and N is a natural number.

Proof. The Beatty sequence, $B_{\theta,\lambda}$ with $m_n = \lfloor n\theta + \lambda \rfloor$ for $n = 1, 2, 3, ..., N$ is as follows:

$$B_{\theta,\lambda} = m_1, m_2, m_3, ..., m_{N-2}, m_{N-1}, m_N.$$

Since $\theta > 1$, $B_{\theta,\lambda}$ is an increasing sequence, that is

$$m_1 < m_2 < m_3 < ... < m_N.$$

For each term of m_n, it can be written as a product of prime factors, if m_n is a composite. Then, arrange all primes by ordering number,

$$p_1 < p_2 < p_3 < ... < p_k < m_n.$$

Now, for the last term m_N, it can be either a prime or a composite. We consider two cases as follows:

Case 1: If m_N is a prime.
Let $m_N = p_{\max}$. For any term, m_n with $1 \leq n \leq N - 1$, we have

$$p_1 < p_2 < ... < p_j < m_n < p_{j+1} < ... < p_k < m_N = p_{\max}.$$

Obviously, $p_{\max}$ is the largest prime that $B_{\theta,\lambda}$ consists. Thus,

$$A = \{p_1, p_2, p_3, ..., p_{\max}\}.$$

Note that A may not consists of all the primes that less than $p_{\max}$. Let B be a set consists of all distinct primes that less than or equals to $p_{\max}$,

$$B = \{2, 3, 5, ..., p_{\max}\}.$$

We have $A \subseteq B$. Then,

$$|A| \leq |B|.$$

Since $|B| = \pi(p_{\max})$, therefore

$$|A| \leq \pi(m_N) = \pi(\lfloor N\theta + \lambda \rfloor).$$

We proved for Case 1.

Case 2: If m_N is a composite.
Let $m_N = p_{i_1}^{a_{i_1}} p_{i_2}^{a_{i_2}} p_{i_3}^{a_{i_3}} ... p_{i_k}^{a_{i_k}}$. For any term, m_n with $1 \leq n \leq N-1$, we have

$$p_1 < p_2 < ... < p_j < m_n < p_{j+1} < ... < p_k < m_N = p_{i_1}^{a_{i_1}} p_{i_2}^{a_{i_2}} p_{i_3}^{a_{i_3}} ... p_{i_k}^{a_{i_k}}$$

and

$$p_{i_1} < p_{i_2} < ... < p_{i_k} < m_N.$$

Then,

$$A = C \cup D$$

where

$$C = \{p_1, p_2, p_3, ..., p_k\} \text{ and } D = \{p_{i_1}, p_{i_2}, p_{i_3}, ..., p_{i_k}\}.$$

The largest prime in A is $\max\{p_k, p_{i_k}\}$, but m_N is the largest integer in $B_{\theta,\lambda}$. Thus,

$$\begin{aligned} \max\{p_k, p_{i_k}\} &< m_N \\ \pi(\max\{p_k, p_{i_k}\}) &\leq \pi(m_N) \\ |C \cup D| &\leq \pi(\lfloor N\theta + \lambda \rfloor). \end{aligned}$$

Therefore,

$$|A| \leq \pi(\lfloor N\theta + \lambda \rfloor).$$

We proved for Case 2. As a result, we showed that (3.1) is true,

$$|A| \leq \pi(\lfloor N\theta + \lambda \rfloor).$$

□

The problem of Theorem 3.1 is when θ, λ and N become bigger, the right hand side of (3.1) is much more bigger than the left hand side. Thus, if θ, λ and N are sufficiently large, the below theorem has a better estimation.

Theorem 3.2. *For sufficiently large fixed irrational θ, real λ and N with*

$$\theta \geq \sqrt{\frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2}}, \quad \lambda \geq \frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2} \quad \text{and} \quad N \geq 10^{\sqrt{\pi(\lfloor N\theta + \lambda \rfloor)}}.$$

Then,

$$|A| \leq \left(\sqrt{\theta^2 + \lambda}\right)(\log N).$$

Proof. Firstly, we set

$$\theta \geq \sqrt{\frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2}}, \tag{3.2}$$

$$\lambda \geq \frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2} \tag{3.3}$$

and

$$N \geq 10^{\sqrt{\pi(\lfloor N\theta + \lambda \rfloor)}}. \tag{3.4}$$

By using (3.1) in Theorem 3.1, then (3.2) to (3.4) will become

$$\theta \geq \sqrt{\frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2}} \geq \sqrt{\frac{|A|}{2}}, \tag{3.5}$$

$$\lambda \geq \frac{\pi(\lfloor N\theta + \lambda \rfloor)}{2} \geq \frac{|A|}{2} \tag{3.6}$$

and

$$N \geq 10^{\sqrt{\pi(\lfloor N\theta+\lambda \rfloor)}} \geq 10^{\sqrt{|A|}}. \tag{3.7}$$

From (3.5), we have

$$|A| \leq 2\theta^2. \tag{3.8}$$

From (3.6), we have

$$|A| \leq 2\lambda. \tag{3.9}$$

Then, we sum of (3.8) and (3.9)

$$|A| \leq \theta^2 + \lambda. \tag{3.10}$$

On the other hand, from (3.7) we have

$$|A| \leq (\log N)^2. \tag{3.11}$$

Now, we do the product of (3.10) and (3.11)

$$\begin{aligned}(|A|)^2 &\leq (\theta^2+\lambda)(\log N)^2 \\ &= \left(\left(\sqrt{\theta^2+\lambda}\right)(\log N)\right)^2.\end{aligned}$$

Since $|A|$ is a positive integer, we can remove square from both sides. Thus,

$$|A| \leq \left(\sqrt{\theta^2+\lambda}\right)(\log N).$$

□

4. Conclusion and Recommendation

We found $|A|$ by estimating the prime counting function which stated in Theorem 3.1. For a fixed irrational $\theta > 1$ and a real $\lambda > 0$. Then,

$$|A| \leq \pi(\lfloor N\theta+\lambda \rfloor)$$

where $\pi(x)$ is the prime counting function for integer x. On the other hand, when parameters are sufficiently large, the following estimation (Theorem 3.2) is sharper. For sufficiently large fixed irrational θ, real λ and N with

$$\theta \geq \sqrt{\frac{\pi(\lfloor N\theta+\lambda \rfloor)}{2}},\ \lambda \geq \frac{\pi(\lfloor N\theta+\lambda \rfloor)}{2} \ \text{ and } \ N \geq 10^{\sqrt{\pi(\lfloor N\theta+\lambda \rfloor)}}.$$

Then,

$$|A| \leq \left(\sqrt{\theta^2+\lambda}\right)(\log N).$$

In the future, the estimation of $|A|$ can be sharpened more. Next, the application of prime divisors in Beatty sequence can be an interesting topic to be studied.

5. Acknowledgments

We would like to thank Universiti Putra Malaysia, Serdang Selangor, Malaysia and Institute for Mathematical Research, (INSPEM), UPM Serdang Selangor, Malaysia for providing the necessary resources. The successful completion of this research would not be feasible without their contributions.

References

[1] Chua L., Park S., Smith G. D., Bounded gaps between primes in special sequences. Proc. Amer. Math. Soc. (2015) Vol. 143, P. 4597–4611.

[2] Bank W. D., Shparlinski I. E., Non-residues and primitive roots in Beatty sequences. Bull. Austral. Math. Soc. (2006) Vol. 73, P. 433–443.

[3] Bank W. D., Shparlinski I. E., Prime divisors in Beatty sequences. Journal of Number Theory, (2007) Vol. 123, P. 413–425.

[4] Cassaigne J., Duchêne E., Rigo M., Nonhomogeneous Beatty sequences leading to invariant games. SIAM Journal on Discrete Mathematics, (2016) Vol. 30, P. 1798–1829.

[5] Harman G., Primes in intersections of Beatty sequences. Journal of Integer Sequences, (2015) Vol. 18,

[6] Kimberling C., Beatty sequences and trigonometric functions. Integers, (2016) Vol. 16,

[7] Simpson J., Disjoint Beatty sequences. Integers: Electronic Journal of Combinatorial Number Theory, (2004) Vol. 4,

[8] Deraman F., Sapar S. H., Johari M. A. M., Mohd Atan K. A., Rasedee A. F. N., Extended bounds of Beatty sequence associated with primes. International Journal of Engineering and Advanced Technology (IJEAT), (2019) Vol. 8, P. 115–118.

[9] Allakov I., Deraman F., Sapar S. H., Ismail S., On cardinality of character sums with Beatty sequences associated with composite modules. Chebyshevskii Sbornik, (2023) Vol. 24, Iss. 5, P. 5–15.

[10] Lidl R., Niederreiter H., Introduction to finite fields and their applications. Cambridge University Press, Cambridge, (1986).

Low C. W.,
Institute for Mathematical Research, Universiti Putra Malaysia, Selangor Darul Ehsan, Malaysia.
e-mail: alexlowcheewai@gmail.com

Sapar S. H.,
Department of Mathematics and Statistics, Universiti Putra Malaysia, Selangor Darul Ehsan, Malaysia.
e-mail: sitihas@upm.edu.my

Deraman F.,
Faculty of Applied and Human Sciences, Universiti Malaysia Perlis, Perlis, Malaysia.
e-mail: fatanah@unimap.edu.my

Johari M. A. M.,
Department of Mathematics and Statistics, Universiti Putra Malaysia, Selangor Darul Ehsan, Malaysia.
e-mail: mamj@upm.edu.my

Yunos F.,
Department of Mathematics and Statistics, Universiti Putra Malaysia, Selangor Darul Ehsan, Malaysia.
e-mail: faridahy@upm.edu.my

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 132–140
DOI: 10.29229/uzmj.2026-1-14

On the impact of the exponential functor on some types of continuous mappings

Mamadaliyev N., Toshbuvayev B.

Abstract. This paper investigates the influence of the symmetric product functor SP^n and the exponential functor exp on certain types of continuous mappings, focusing specifically on almost-open and pseudo-open mappings. The primary objective is to analyze how functors SP^n and exp interact with these mappings and alter their topological properties. Through the study, several key lemmas have been proven, providing insights into the behavior of the SP^n-induced mappings. Notably, it is demonstrated that for open sets $U_1, U_2, \ldots, U_n \subset X$, the set $[U_1, U_2, \ldots, U_n]$ retains its openness in $SP^n X$. These findings contribute to a deeper understanding of the topological implications of applying the SP^n functor on continuous mappings, offering new perspectives on its effect on almost-open and pseudo-open transformations.

Keywords: exponential functor, functor of permutation degree, almost-open map, pseudo-open map, sequence-covering map
MSC (2020): 18B05, 18A05, 18F60, 54A05

1. Introduction

The study of continuous mappings and their interactions with topological structures is a fundamental aspect of topology. Continuous mappings encode essential relationships between spaces, and understanding how their properties change under various transformations is a central problem. Functors such as the symmetric product functor SP^n and the exponential functor $\exp_n$ play a significant role in this context. These functors construct new spaces from given ones and induce corresponding mappings, providing a framework to analyze the preservation or alteration of topological properties.

In classical topology, concepts like openness, continuity, and compactness serve as cornerstones for understanding the behavior of mappings and spaces. Among these, almost-open and pseudo-open mappings are essential generalizations of open mappings that retain specific weaker properties. For instance, Michael's work on pseudo-open mappings [1] provides a foundational basis for analyzing mappings that are not strictly open but still preserve significant topological information.

The symmetric product functor SP^n, introduced and studied in depth by Dold [2], constructs the space of unordered n-tuples of points from a topological space X, equipped with a natural topology derived from X. On the other hand, the exponential functor $\exp_n$, closely associated with the Vietoris topology [3, 4], represents the space of non-empty closed subsets of X with at most n points. These constructions preserve certain topological structures of X while introducing new geometric and combinatorial features.

The purpose of this paper is to investigate how the functors SP^n and $\exp_n$ influence almost-open and pseudo-open mappings [5]. Key results presented here extend previous work by Arhangel'skii [6], Fedorchuk and Filippov [7], and Nagata [8] on general mappings, as well as Lin's studies on point-countable covers [9]. For example, we show that:

- For the symmetric product functor SP^n, the mapping $SP^n f\colon SP^n X \to SP^n Y$ is almost-open (Theorem 3.4) and pseudo-open (Theorem 3.5) whenever $f\colon X \to Y$ is almost-open or pseudo-open, respectively.
- For the exponential functor $\exp_n\colon Comp \to Comp$, the mapping retains these properties (Theorems 3.6 and 3.7).

By establishing these results, we contribute to a broader understanding of functorial transformations in topology and their implications for continuous mappings. This research builds upon earlier works in the field [10, 11, 12, 13, 14, 15, 16, 17] and opens avenues for further exploration of functorial interactions in generalized topological settings.

2. Preliminaries

Let X be a topological T_1-space. The set of all non-empty closed subsets of a topological space X is denoted by $\exp X$. The family of all sets of the form

$$O\langle U_1, ..., U_n\rangle = \left\{ F: \ F \in \exp X, \ F \subset \bigcup_{i=1}^{n} U_i, \ F \bigcap U_i \neq \emptyset, \ i = 1, 2, ..., n \right\}.$$

where $U_1, ..., U_n$ are open subsets of X, generates a base of the topology on the set $\exp X$. This topology is called the Vietoris topology. The set $\exp X$ with the Vietoris topology is called exponential space or the hyperspace of a space X [18]. Note that the space $\exp X$ is compact for any compact space X.

Define the following subspaces of $\exp X$:

$$\exp_n X = \{F \in \exp X : |F| \ \leq n\},$$

and

$$\exp_\omega X = \bigcup \{\exp_n X : n = 1, 2, ...\} .$$

Let $f : X \to Y$ be a continuous mapping between topological spaces. For a nonempty closed subset $C \subset X$, define

$$(\exp f)(C) = f(C). \tag{1}$$

Then $\exp f : \exp X \to \exp Y$ is well-defined and continuous.

Equality (1) defines a functor $\exp_n : Comp \to Comp$, which assigns to each topological space X the hyperspace $\exp X$, and to each continuous map $f : X \to Y$ the continuous map $\exp f : \exp X \to \exp Y$.

Let X be a compact Hausdorff space. Consider the mapping

$$\pi_n : X^n \to \exp_n X$$

that assigns to each point $x = (x_1, x_2, \dots, x_n) \in X^n$ the set of its coordinates $\{x_1, x_2, \dots, x_n\}$.

Then π_n is a continuous mapping of the compact space X^n onto the compact space $\exp_n X$. Thus, the hypersymmetric n power of the compact space X is the quotient space of its n power with respect to the partition generated by the following equivalence relation: points $x, y \in X^n$ are equivalent if they have the same set of coordinates.

On the n^{th} power X^n of the compact X, the permutation group S^n acts as the group of coordinate permutations. The set of orbits of this action with the quotient topology is denoted by SP^nX. Consider the quotient mapping

$$\pi_n^s : X^n \to SP^n X$$

that associates to each point $x = (x_1, x_2, \dots, x_n) \in X^n$ the orbit of this point. Thus, the points of the space SP^nX are finite subsets (equivalence classes) of the product X^n.

In this setting, two points $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ are considered equivalent if there exists a permutation $\sigma \in S^n$ such that $y_i = x_{\sigma(i)}$ for all $i = 1, 2, \dots, n$ [13].

The space SP^nX is called the n symmetric power of the space X. Equivalence relations by which the spaces SP^nX and $\exp_n X$ are obtained from X^n are called symmetric and hypersymmetric equivalence relations, respectively. Any two points that are symmetrically equivalent in X^n will also be hypersymmetrically equivalent. However, in general, the converse does not hold. For example, for distinct elements $x, y \in X$ such that $x \neq y$, the points (x, x, y) and $(x, y, y) \in X^3$ are hypersymmetrically equivalent but not symmetrically equivalent [7].

Let $f : X \to Y$ be a continuous mapping between compact Hausdorf spaces X and Y. For the equivalence class $[(x_1, x_2, \dots, x_n)] \in SP^nX$, put

$$(SP^nf)([(x_1, x_2, \dots, x_n)]) = [(f(x_1), f(x_2), \dots, f(x_n))].$$

This defines a mapping

$$SP^nf : SP^nX \to SP^nY.$$

It is easy to verify that the operation SP^n constructed in this way is a covariant functor in the category *Comp* of compact spaces and their continuous mappings [7].

Definition 2.1. A mapping $f\colon X \to Y$ is called almost-open if, for each $y \in Y$, there exists an element $x \in f^{-1}(y)$ such that for every open neighborhood U of x, the image $f(U)$ is a neighborhood of y (i.e. $y \in \mathrm{Int}(f(U))$) [19, 9].

Example 2.2. The notions of almost-open and open maps differ essentially. Let X be the disjoint union of the unit circle S^1 and the interval $(0,1)$; for brevity write $X = S^1 \cup (0,1)$ with the disjoint-union topology. Fix a point $r \in S^1$ and define a map

$$f : X \to S^1, \qquad f(x) = \begin{cases} x, & x \in S^1, \\ r, & x \in (0,1). \end{cases}$$

The map f is continuous and almost-open. Indeed, let $y \in S^1$. Choose the point $x = y \in X$ (here x is regarded as an element of the domain X, while y is the corresponding element of the range S^1). For every open neighborhood $U \subset X$ of x, we have $U \subset S^1$, so $f(U) = U$, which is an open neighborhood of y in S^1. Thus the condition of almost-openness is satisfied.

However, f is not an open map: the set $(0,1) \subset X$ is open, while $f((0,1)) = \{r\}$, which is not open in S^1. Moreover, there exist open neighborhoods in X whose images are neighborhoods in S^1 but not open. For instance, take $y_0 \in S^1$ with $y_0 \neq r$, and let $x_0 = y_0 \in X$. Let $V \subset S^1$ be an open arc containing y_0 but not r, and define $U = V \cup (0,1) \subset X$. Then U is an open neighborhood of x_0 in X, while

$$f(U) = V \cup \{r\}.$$

This set is a neighborhood of y_0 in S^1 (since it contains the open arc V), but it is not open because of the isolated point r. Hence an almost-open map may send an open neighborhood of a preimage point to a neighborhood of the image point without that image being open.

Definition 2.3. A mapping f is called pseudo-open if for each $y \in Y$ and each open neighborhood U of $f^{-1}(y)$ in X , $f(U)$ is a neighborhood of y in Y [6, 19].

Definition 2.4. A mapping f is called a 1-sequence-covering mapping if for each $y \in Y$ there exists $x \in f^{-1}(y)$, such that whenever $\{y_n\}$ is a sequence converging to y in Y, there exists a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$ [6, 19].

Definition 2.5. A mapping f is called a sequence-covering mapping if whenever $\{y_n\}$ is a convergent sequence in Y, there exists a convergent sequence $\{x_n\}$ in X with each $x_n \in f^{-1}(y_n)$ [6, 19].

Definition 2.6. A covariant functor $F\colon Comp \to Comp$ acting in the category of compact Hausdorff spaces and their continuous mappings, is called normal [18], if it
1) preserves the weight;
2) preserves singletons and empty set;
3) monomorphic (preserves embeddings);
4) epimorphic (preserves surjections);
5) preserves intersections of closed subsets;
6) preserves inverse images;
7) is continuous with respect to inverse limits.

Note that the functors exp and SP^n are normal.

3. Main results

For subsets $M_1, M_2, \ldots, M_n$ of a space X define:

$$[(M_1, M_2, \ldots, M_n)] = \big\{[(x_1, x_2, \ldots, x_n)] \in SP^n X : \exists \sigma \in S^n, x_{\sigma(i)} \in M_i, i = 1, \ldots, n\big\} \subset SP^n X.$$

Lemma 3.1. *For open sets $U_1, U_2, \ldots, U_n \subset X$ the set $[(U_1, U_2, \ldots, U_n)]$ is also open in $SP^n X$.*

Proof. By the construction of the set $SP^n X$, for the quotient map $\pi_n^s\colon X^n \to SP^n X$ we have the equality

$$(\pi_n^s)^{-1}([(U_1, U_2, \ldots, U_n)]) = \bigcup_{g \in S^n} U_{g(1)} \times U_{g(2)} \times \cdots \times U_{g(n)}.$$

Therefore, the set $(\pi_n^s)^{-1}([(U_1, U_2, \dots, U_n)])$ is open in X^n as the union of $n!$ number of open sets in the form $U_{g(1)} \times U_{g(2)} \times \dots \times U_{g(n)}$. Consequently, the set $[(U_1, U_2, \dots, U_n)]$ is open in SP^nX. Lemma 3.1 is proved. □

Lemma 3.2. *For every sequence of subsets $A_1, A_2, \dots A_n \subset X$ we have*

$$[(IntA_1, IntA_2, \dots IntA_n)] \subset Int\,[(A_1, A_2, \dots A_n)]$$

Proof. Get an arbitrary element $[(x_1, x_2, \dots, x_n)] \in [(IntA_1, IntA_2, \dots IntA_n)]$. Then there exists $g \in S^n$ such that $x_i \in IntA_{g(i)}$ for every $i = 1, \dots, n$. This means that there are open sets $U_1, U_2, \dots, U_n$ such that $x_i \in U_i \subset IntA_{g(i)}$ for every $i = 1, \dots, n$. Therefore, we obtain $[(x_1, x_2, \dots, x_n)] \in [(U_1, U_2, \dots, U_n)]$. Note that by Lemma 3.1 the set $[(U_1, U_2, \dots, U_n)]$ is an open neighborhood of the point $[(x_1, x_2, \dots, x_n)]$ in the space SP^nX. Now for an arbitrary element $[(y_1, y_2, \dots, y_n)] \in [(U_1, U_2, \dots, U_n)]$ there exists $g' \in S_n$ with $y_i \in U_{g'(i)}$ for every $i = 1, \dots, n$. Since $U_i \subset IntA_{g(i)}$, we have $y_i \in A_{g(g'(i))} = A_{(gg')(i)}$. Consequently, $[(y_1, y_2, \dots, y_n)] \in [(A_1, A_2, \dots, A_n)]$. By the arbitrariness of the choice of the element $[(y_1, y_2, \dots, y_n)]$, we obtain the relation $[(U_1, U_2, \dots, U_n)] \subset [(A_1, A_2, \dots, A_n)]$. This implies $[(IntA_1, IntA_2, \dots IntA_n)] \subset Int\,[(A_1, A_2, \dots A_n)]$. Lemma 3.2 is proved.

□

Proposition 3.3. *The collection of all sets in the form $[(U_1, U_2, \dots, U_n)]$, where $U_1, U_2, \dots, U_n$ are open subsets in X, generates a base in SP^nX.*

Proof. It is clear that $SP^nX = [X]$. For two elements $[(U_1, U_2, \dots, U_n)], [(V_1, V_2, \dots, V_n)] \in SP^nX$ with

$$[(U_1, U_2, \dots, U_n)] \cap [(V_1, V_2, \dots, V_n)] \neq \emptyset,$$

consider any element

$$[(x_1, x_2, \dots, x_n)] \in [(U_1, U_2, \dots, U_n)] \cap [(V_1, V_2, \dots, V_n)].$$

Then there exist $g_1, g_2 \in S^n$ such that $x_i \in U_{g_1(i)} \cap V_{g_2(i)}$ for each $i = 1, \dots, n$. Put $W_i = U_i \cap V_{(g_1^{-1}g_2)(i)}$ (Note that for each $i = 1, \dots, n$ we have $U_i \cap V_{(g_1^{-1}g_2)(i)} \neq \emptyset$, since $x_i \in U_{g_1(i)} \cap V_{g_2(i)}$). Let us take an arbitrary element $[(y_1, y_2, \dots, y_n)] \in [(W_1, W_2, \dots, W_n)]$. Then there exists $g \in S^n$ such that $y_i \in W_{g(i)}$ for each $i = 1, \dots, n$. Taking into account that $y_i \in U_{g(i)} \cap V_{(g_1^{-1}g_2)(i)}$ we obtain

$$[(y_1, y_2, \dots, y_n)] \in [(U_1, U_2, \dots, U_n)] \cap [(V_1, V_2, \dots, V_n)].$$

As a result, we have

$$[(W_1, W_2, \dots, W_n)] \subset [(U_1, U_2, \dots, U_n)] \cap [(V_1, V_2, \dots, V_n)].$$

Proposition 3.3 is proved □

Theorem 3.4. *Let $f\colon X \to Y$ be an almost-open mapping. Then the induced map*

$$SP^nf\colon SP^nX \to SP^nY$$

is also almost-open for every $n \in \mathbb{N}$.

Proof. Get an arbitrary element $[(y_1, y_2, \dots, y_n)] \in SP^nY$. Since $f\colon X \to Y$ is an almost-open mapping, for each y_i there is x_i such that $f(U)$ is an neighborhood of y_i for every open neighborhood U of x_i (i.e. $y_i \in Int(f(U))$). In this case, we have

$$[(x_1, x_2, \dots, x_n)] \in (SP^nf)^{-1}([(y_1, y_2, \dots, y_n)]). \tag{3.1}$$

By Proposition 3.3, without loss of generality, we can choose an arbitrary neighborhood of the point $[(x_1, x_2, \ldots, x_n)] \in SP^n X$ in the form $[(U_1, U_2, \ldots, U_n)]$, where $U_1, U_2, \ldots, U_n$ are open neighborhoods of points $x_1, x_2, \ldots, x_n$, respectively. Clearly, we have

$$[(y_1, y_2, \ldots, y_n)] \in (SP^n f)([(U_1, U_2, \ldots, U_n)]).$$

Further, using Lemma 3.2 we obtain the following relations:

$$[(\operatorname{Int} f(U_1), \operatorname{Int} f(U_2), \ldots, \operatorname{Int} f(U_n))] \subset \operatorname{Int}[(f(U_1), f(U_2), \ldots, f(U_n))] \subset$$

$$\subset [(f(U_1), f(U_2), \ldots, f(U_n))] = (SP^n(f))([(U_1, U_2, \ldots, U_n)]).$$

On the other hand, for each $i = 1, 2, \ldots, n$ we have $y_i \in Int(f(U_i))$. Consequently,

$$[(y_1, y_2, \ldots, y_n)] \in [(Int(f(U_1)), Int(f(U_2)), \ldots, Int(f(U_n)))].$$

Therefore, we obtain $[(y_1, y_2, \ldots, y_n)] \in Int(SP^n f([(U_1, U_2, \ldots, U_n)]))$. The last relation means that the mapping $SP^n f \colon SP^n X \to SP^n Y$ is almost-open. Theorem 3.4 is proved. □

Theorem 3.5. *Let $f \colon X \to Y$ be a pseudo-open mapping. Then the induced map*

$$SP^n f \colon SP^n X \to SP^n Y$$

is also pseudo-open for every $n \in \mathbb{N}$.

Proof. Let $[(y_1, \ldots, y_n)] \in SP^n Y$ be arbitrary and put

$$A := (SP^n f)^{-1}\big([(y_1, \ldots, y_n)]\big) \subset SP^n X.$$

By Proposition 3.3, we can choose open sets

$$U_1, \ldots, U_n \subset X$$

such that each U_i is an open neighborhood of the fiber $f^{-1}(y_i)$ in X, and

$$[(U_1, \ldots, U_n)]$$

is a neighborhood of some point of A in $SP^n X$. Since f is pseudo-open, for each $i = 1, \ldots, n$ we have

$$y_i \in \operatorname{Int}(f(U_i)).$$

Now consider the image of $[(U_1, \ldots, U_n)]$ under $SP^n f$. As in the proof of Theorem 3.4,

$$(SP^n f)([(U_1, \ldots, U_n)]) = [(f(U_1), \ldots, f(U_n))].$$

By Lemma 3.2,

$$[(\operatorname{Int} f(U_1), \ldots, \operatorname{Int} f(U_n))] \subseteq \operatorname{Int}[(f(U_1), \ldots, f(U_n))].$$

Since $y_i \in \operatorname{Int}(f(U_i))$ for each i, it follows that

$$[(y_1, \ldots, y_n)] \in [(\operatorname{Int} f(U_1), \ldots, \operatorname{Int} f(U_n))].$$

Therefore,

$$[(y_1, \ldots, y_n)] \in \operatorname{Int}\big((SP^n f)([(U_1, \ldots, U_n)])\big).$$

Thus every neighborhood in $SP^n X$ of a point of $(SP^n f)^{-1}([(y_1, \ldots, y_n)])$ is mapped by $SP^n f$ to a neighborhood of $[(y_1, \ldots, y_n)]$. Hence $SP^n f$ is pseudo-open. □

Theorem 3.6. *Let $f \colon X \to Y$ be an almost-open and surjective mapping. Then the induced map*

$$\exp_n f \colon \exp_n X \to \exp_n Y$$

is also almost-open for every $n \in \mathbb{N}$.

Proof. Assume that $f\colon X \to Y$ is almost-open and surjective. Let $F = \{y_1, y_2, \dots, y_n\} \in \exp_n Y$ be arbitrary.

Since $\exp_n$ is a normal functor in the category of compact spaces, the induced map $\exp_n f$ is also surjective. We will show that $\exp_n f$ is almost-open.

By the definition of almost-openness, for each $y_i \in F$, there exists a point $x_i \in f^{-1}(y_i)$ such that for every open neighborhood U_i of x_i, the image $f(U_i)$ is a neighborhood of y_i. Let us define the finite set

$$C = \{x_1, x_2, \dots, x_n\} \in \exp_n X.$$

Clearly, $(\exp_n f)(C) = f(C) = \{f(x_1), \dots, f(x_n)\} = F$, so $C \in (\exp_n f)^{-1}(F)$.

Now let $\langle U_1, U_2, \dots, U_n\rangle$ be a basic open neighborhood of C in $\exp_n X$, where each U_i is an open neighborhood of x_i in X.

We want to show that

$$F \in \operatorname{Int}\left((\exp_n f)\left(\langle U_1, U_2, \dots, U_n\rangle\right)\right).$$

First, observe that by the properties of the functor $\exp_n$,

$$(\exp_n f)\left(\langle U_1, U_2, \dots, U_n\rangle\right) = \langle f(U_1), f(U_2), \dots, f(U_n)\rangle.$$

According to Lemma 2.3 in [16], we have:

$$\langle \operatorname{Int}(f(U_1)), \operatorname{Int}(f(U_2)), \dots, \operatorname{Int}(f(U_n))\rangle \subset \langle f(U_1), f(U_2), \dots, f(U_n)\rangle,$$

and this inclusion holds within $\exp_n Y$, where the left-hand side is an open set.

Therefore,

$$\langle \operatorname{Int}(f(U_1)), \dots, \operatorname{Int}(f(U_n))\rangle \subset \operatorname{Int}\left(\langle f(U_1), \dots, f(U_n)\rangle\right).$$

Since $y_i \in \operatorname{Int}(f(U_i))$ for each $i = 1, \dots, n$, we conclude that

$$F = \{y_1, \dots, y_n\} \in \langle \operatorname{Int}(f(U_1)), \dots, \operatorname{Int}(f(U_n))\rangle \subset \operatorname{Int}\left((\exp_n f)(\langle U_1, \dots, U_n\rangle)\right).$$

Hence, $\exp_n f$ is almost-open. □

Theorem 3.7. *Let $f\colon X \to Y$ be a pseudo-open and surjective mapping. Then the induced mapping*

$$\exp_n f\colon \exp_n X \to \exp_n Y$$

is pseudo-open for every $n \in \mathbb{N}$.

Proof. Let $f\colon X \to Y$ be pseudo-open and surjective. Take an arbitrary element $F = \{y_1, y_2, \dots, y_n\} \in \exp_n Y$.

Since $\exp_n$ is a normal functor in the category of compact spaces, the induced map $\exp_n f$ is surjective as well. We will show that $\exp_n f$ is pseudo-open.

By the definition of pseudo-openness, for each $y_i \in F$, there exists a point $x_i^j \in f^{-1}(y_i)$, indexed by $j \in A$, such that for every open neighborhood U_i of x_i^j, the image $f(U_i)$ is a neighborhood of y_i. Here the index $j \in A$ is used to distinguish different possible choices of preimages of y_i, since in general the fiber $f^{-1}(y_i)$ may contain more than one element.

Fix such a collection of points $\{x_1^j, x_2^j, \dots, x_n^j\} \subset X$ and define

$$C^j = \{x_1^j, x_2^j, \dots, x_n^j\} \in \exp_n X.$$

Clearly, $\exp_n f(C^j) = f(C^j) = \{f(x_1^j), \dots, f(x_n^j)\} = F$, so $C^j \in (\exp_n f)^{-1}(F)$.

Let $\langle U_1, U_2, \dots, U_n\rangle$ be a basic open neighborhood of C^j in $\exp_n X$, where each U_i is an open neighborhood of x_i^j. We want to prove that

$$F \in \operatorname{Int}\left((\exp_n f)(\langle U_1, U_2, \dots, U_n\rangle)\right).$$

Using the properties of the $\exp_n$ functor, we have

$$(\exp_n f)(\langle U_1, U_2, \dots, U_n\rangle) = \langle f(U_1), f(U_2), \dots, f(U_n)\rangle. \tag{3.2}$$

By Lemma 2.3 in [16], the following inclusion holds:

$$\langle \mathrm{Int}(f(U_1)), \mathrm{Int}(f(U_2)), \ldots, \mathrm{Int}(f(U_n)) \rangle \subset \langle f(U_1), f(U_2), \ldots, f(U_n) \rangle,$$

and this set is open in $\exp_n Y$. Moreover, since $y_i \in \mathrm{Int}(f(U_i))$ for each $i = 1, \ldots, n$, it follows that

$$F = \{y_1, \ldots, y_n\} \in \langle \mathrm{Int}(f(U_1)), \ldots, \mathrm{Int}(f(U_n)) \rangle.$$

Therefore, by (3.2), we conclude that

$$F \in \mathrm{Int}\left(\langle f(U_1), f(U_2), \ldots, f(U_n) \rangle\right) = \mathrm{Int}\left((\exp_n f)(\langle U_1, \ldots, U_n \rangle)\right).$$

Hence, $\exp_n f$ is pseudo-open. □

Theorem 3.8. *Let $f \colon X \to Y$ be an almost-open surjective mapping between compact spaces X and Y. Then the induced mapping $\exp f \colon \exp X \to \exp Y$ is also almost-open.*

Proof. Take an arbitrary point $E \in \exp Y$. Since f is surjective, the preimage $F = f^{-1}(E) \in \exp X$ we have $F \in (\exp f)^{-1}(E)$.

Consider an arbitrary open neighborhood $O = \langle U_1, \ldots, U_n \rangle$ of F in $\exp X$, where each U_i is an open subset of X. Then:

$$F \cap U_i \neq \emptyset \quad \text{for each } i = 1, 2, \ldots, n,$$

and

$$F \subset \bigcup_{i=1}^{n} U_i.$$

This implies:

$$E \cap f(U_i) \neq \emptyset \quad \text{for each } i, \quad \text{and} \quad E \subset \bigcup_{i=1}^{n} f(U_i).$$

Thus,

$$E \in \langle f(U_1), \ldots, f(U_n) \rangle.$$

We now show that $E \in \langle \mathrm{Int}\, f(U_1), \ldots, \mathrm{Int}\, f(U_n) \rangle \subset \langle f(U_1), \ldots, f(U_n) \rangle$. Let us define $F_i = F \cap U_i$. Then:

$$\bigcup_{i=1}^{n} F_i = F \cap \left(\bigcup_{i=1}^{n} U_i \right) = F.$$

Set $E_i = f(F_i)$. Then:

$$\bigcup_{i=1}^{n} E_i = f\left(\bigcup_{i=1}^{n} F_i \right) = f(F) = E.$$

Now take an arbitrary $y \in E$. Then $y \in E_i$ for some i, so there exists $x_y \in F_i \subset U_i$ with $f(x_y) = y$. Since f is almost-open we clearly have $y \in \mathrm{Int}\, f(U_i)$.

Thus:

$$E \subset \bigcup_{i=1}^{n} \mathrm{Int}\, f(U_i). \tag{3.3}$$

Moreover, for each $j = 1, 2, \ldots, n$, there exists $x \in F_j = F \cap U_j$ such that $f(x) \in \mathrm{Int}\, f(U_j)$. Hence:

$$E \cap \mathrm{Int}\, f(U_j) \neq \emptyset. \tag{3.4}$$

From (3.3), (3.4) and Lemma 2.3.1 in [16], we conclude that

$$E \in \langle \mathrm{Int}\, f(U_1), \ldots, \mathrm{Int}\, f(U_n) \rangle \subset \mathrm{Int} \langle f(U_1), \ldots, f(U_n) \rangle.$$

Therefore, $E \in \mathrm{Int} \langle f(U_1), \ldots, f(U_n) \rangle$, and so $\exp f$ is almost-open. □

Theorem 3.9. *Let $f\colon X\to Y$ be a pseudo-open and surjective mapping between compact spaces. Then the induced mapping $\exp f\colon \exp X\to\exp Y$ is also pseudo-open.*

Proof. Take an arbitrary $E\in\exp Y$. Clearly, $F=f^{-1}(E)\in\exp X$. For any F, consider an arbitrary neighborhood $O\langle U_1,...,U_n\rangle$. We have $F\cap U_i\neq\emptyset$ for each $i=1,2,...,n$ and $F\subset\bigcup\limits_{i=1}^{n}U_i$. This implies $E\cap f(U_i)\neq\emptyset$ for each i , and $E\subset\bigcup\limits_{i=1}^{n}f(U_i)$. Hence, $E\in O\langle f(U_1),...,f(U_n)\rangle$.

Now we need to show pseudo-openness. For any $y\in E$, since f is pseudo-open, there exists $x_y\in f^{-1}(y)$ such that $y\in\operatorname{Int}(f(U_i))$ because $x_y\in F\cap U_i$. Therefore,

$$E\subset\bigcup_{i=1}^{n}\operatorname{Int}(f(U_i)).$$

Moreover, for each $j=1,2,...,n$, we have $E\cap\operatorname{Int}(f(U_j))\neq\emptyset$. Consequently, E lies inside a pseudo-open neighborhood.

Thus, $\exp f:\exp X\to\exp Y$ is pseudo-open. □

Theorem 3.10. *Let $f\colon X\to Y$ be an 1-sequence-covering mapping . Then the induced map $\exp_n f$: $\exp_n X\to\exp_n Y$ is also 1-sequence-covering.*

Proof. Let $F\in\exp_n Y$ and let (F_k) be a sequence in $\exp_n Y$ converging to F. We aim to find a sequence (C_k) in $\exp_n X$ converging to some $C\in\exp_n X$ such that

$$(\exp_n f)(C_k)=F_k\quad\text{for all } k,\quad\text{and}\quad(\exp_n f)(C)=F.$$

Write

$$F_k=\{y_1^{(k)},y_2^{(k)},\dots,y_n^{(k)}\},\quad F=\{y_1,y_2,\dots,y_n\}.$$

Since convergence in $\exp_n Y$ implies that for each $i=1,\dots,n$, the sequences $(y_i^{(k)})$ converge to y_i in Y (possibly after reindexing or relabeling elements if necessary), we can apply the 1-sequence-covering property of f individually to each sequence $(y_i^{(k)})$.

Thus, for each i, there exists a sequence $(x_i^{(k)})$ in X and a point $x_i\in f^{-1}(y_i)$ such that:

$$x_i^{(k)}\to x_i\quad\text{in } X,\quad\text{and}\quad f(x_i^{(k)})=y_i^{(k)}\quad\text{for all } k.$$

Now define

$$C_k=\{x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\},\quad C=\{x_1,x_2,\dots,x_n\}.$$

Then $C_k\to C$ in $\exp_n X$ because each coordinate sequence converges, $(\exp_n f)(C_k)=F_k$ for all k, and $(\exp_n f)(C)=F$.

Therefore, $\exp_n f$ is 1-sequence-covering. □

Theorem 3.11. *Let $f\colon X\to Y$ be a sequence-covering mapping . Then the induced map $\exp_n f\colon \exp_n X\to\exp_n Y$ is also sequence-covering.*

Proof. Let (F_k) be a sequence in $\exp_n Y$ converging to F, where

$$F_k=\{y_1^{(k)},y_2^{(k)},\dots,y_n^{(k)}\},\quad F=\{y_1,y_2,\dots,y_n\}.$$

By the definition of convergence in $\exp_n Y$, for each $i=1,\dots,n$, the sequences $(y_i^{(k)})$ converge to y_i in Y.

Since f is sequence-covering, for each i there exists a sequence $(x_i^{(k)})$ in X and a point $x_i\in X$ such that:

$$x_i^{(k)}\to x_i\quad\text{in } X,\quad f(x_i^{(k)})=y_i^{(k)}\quad\text{for all } k,\quad\text{and}\quad f(x_i)=y_i.$$

Define

$$C_k=\{x_1^{(k)},x_2^{(k)},\dots,x_n^{(k)}\},\quad C=\{x_1,x_2,\dots,x_n\}.$$

Then $C_k\to C$ in $\exp_n X$ since each $x_i^{(k)}\to x_i$, $(\exp_n f)(C_k)=F_k$ for all k, and $(\exp_n f)(C)=F$.

Thus, $\exp_n f$ is a sequence-covering mapping. □

Acknowledgment. The authors would like to thank the referee for their careful reading of the manuscript and valuable suggestions which helped to improve the presentation of the results.

References

[1] Michael E., Quotient and pseudo-open images of separable metric spaces. Journal of Topology, (1968) Vol. 11, Iss. 2, P. 305–320.

[2] Dold A., The fixed point index of fibre-preserving maps. Inventiones Mathematicae, (1958) Vol. 6, Iss. 4, P. 344–356.

[3] Kelley J., General Topology. Van Nostrand Reinhold Company, (1955).

[4] Vietoris L., Bereiche zweiter Ordnung. Mathematische Annalen, (1922) Vol. 85, P. 116–171.

[5] Toshbuvayev B., Acting of covariant functors on mappings. International Journal of Applied Mathematics, (2024) Vol. 37, Iss. 1, P. 47–51.

[6] Arhangelaskii A., Mappings and spaces. Russian Math. Surveys, (1966) Vol. 21, P. 115–162.

[7] Fedorchuk V., Covariant functors in a category of compacta. absolute retracts and μ-manifolds. Uspekhi Mat. Nauk, (1981) Vol. 36, Iss. 3, P. 177–195.

[8] Nagata J., On pseudo-open mappings. Mathematica Scandinavica, (1974) Vol. 34, Iss. 1, P. 55–68.

[9] Lin S., Point-Countable Covers and Sequence-Covering Mappings. Chinese Science Press, Beijing, (2002).

[10] Beshimov R., Georgiou D., Mamadaliev N., Sereti F., On local density and local weak density of the hyperspace of sets with finitely many components. Filomat, (2023) Vol. 37, Iss. 14, P. 4659–4669.

[11] Beshimov R., Georgiou D., Sereti F., Zhuraev R., Metric, stratifiable and uniform spaces of G-permutation degree. Mathematica Slovaca, (2024) Vol. 72, Iss. 1, P. 235–248.

[12] Beshimov R., Georgiou D., Juraev R., Manasipova R., Sereti F., Some topological properties of e-space and description of τ-closed sets. Filomat, (2025) Vol. 39, Iss. 8, P. 2625–2637.

[13] Fedorchuk V., Filippov V., Topology of hyperspaces and its applications. Mathematica, Cybernetica, (1989) Vol. 4, P. 48.

[14] Georgiou D., Mamadaliev N., Zhuraev R., A note on functional tightness and minitightness of space of the G-permutation degree. Comment. Math. Univ. Carolin. (2023) Vol. 64, Iss. 1, P. 97–108.

[15] Hatcher A., Algebraic Topology. Cambridge University Press, (2002).

[16] Michael E., Topologies on spaces of subsets. Transactions of the American Mathematical Society, (1951) Vol. 71, Iss. 2, P. 152–182.

[17] Whitehead J., Combinatorial homotopy. I. Bulletin of the American Mathematical Society, (1949) Vol. 55, Iss. 3, P. 213–226.

[18] Fedorchuk V., Filippov V., General Topology. The Basic Foundation. Fizmatlit, Moscow, (2006).

[19] Ge X., Notes on almost open mappings. Matematicki Vesnik, (2008) Vol. 60, P. 181–186.

Mamadaliyev N.K.,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan,
Kimyo International University in Tashkent,
Tashkent, Uzbekistan
e-mail: nodirbekm.topol25@gmail.com

Toshbuvayev B.M.,
Fergana state university, Fergana , Uzbekistan
e-mail: toshbuvayevboburmirzo@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 141–148
DOI: 10.29229/uzmj.2026-1-15

Inverse source problem for a Hopf-type system

Mukimov A., Mamasoliyev B., Imomnazarov Kh., Iskandarov I.

Abstract. A one-dimensional inverse problem for a quasilinear hyperbolic system with an unknown excitation source is considered. The Cauchy problem for a nonlinear Hopf-type system is studied. A Fourier transform is used to reduce the inverse problem to a direct problem, and an existence and uniqueness theorem for the solution is proved. The approach used can form the basis for constructing an efficient numerical algorithm for the inverse problem.

Keywords: Two-velocity hydrodynamics, viscous fluid, relative velocity, direct problem, inverse problem, Darcy coefficient
MSC (2020): 35F25, 76T06

1. Introduction

The theory of two-phase filtration finds important application in solving problems of petroleum engineering, soil science, biomechanics and others practical areas. Recently, increasing attention has been paid to modeling multiphase flows in connection with the disposal of radioactive waste. Simulation and numerical analysis of two-phase filtration in elastically deformable porous media are an important element in the development of cost-effective and safe treatment devices, allowing for a reduction in the number of laboratory and field experiments, identification of the main mechanisms, optimization of existing strategies, and assessment of potential risks. In recent years, interest in multiphase filtration processes in fractured porous reservoirs with low permeability has increased significantly. One important reason for this is that fractured hydrocarbon deposits contain more than 20 percent of the world's oil reserves [1].

In this paper, we study the inverse problem for a system of Hopf-type equations with an unknown source under the condition of overdetermination of solutions given on a fixed line. The original problem is reduced to the study of the Cauchy problem for a system of ordinary nonlinear integro-differential equations containing a convolution for which a unique solvability has been proven. The unique solvability of the inverse problem is proved, and a representation of its solution is obtained by solving the above-mentioned Cauchy problem [2]. Similar problems for linear and semilinear equations are considered in [3, 4, 5]. Inverse problems with final overdetermination are studied for parabolic equations and equations of viscous incompressible fluids in [6, 7, 8]. For the study of direct problems for Burgers-type equations and systems, see, for example, [9, 10, 11]. The issues of correctness of the linear inverse problem for a three-dimensional, second-order, mixed-type equation of the second kind in an unbounded parallelepiped are considered in [12].

In recent decades, a significant number of publications have been devoted to the study of inverse problems for partial differential equations due to their applied significance, but for the proposed quasilinear equations, this aspect remains poorly understood.

In [13] an inverse problem is considered, which consists of determining a solution-dependent coefficient of a quasilinear system of equations based on additional information about one of the components of the solution of the system, defined at a fixed point in space and being a function of time. The uniqueness of the solution to the inverse problem has been proven. In [14] a theorem on the existence of an inverse problem of determining an unknown coefficient of a quasilinear hyperbolic equation that depends on its solution is proved. Similar problems for a system of Hopf-type equations in the class of analytic functions are considered in [15, 16, 17, 18].

In this paper, we investigate a new inverse source problem for a Hopf-type system, assuming that its solutions are once continuously differentiable with respect to spatial variables and that a Fourier transform exists. The original problem reduces to studying the Cauchy problem for a system of ordinary nonlinear integro-differential equations containing a convolution for which a unique solvability has been proven. An example of a class of such functions is the space of rapidly decreasing functions [19].

2. Hopf type system of equations

The Cauchy problem in a strip $\Pi_{[0,T]} = \{(t,x) : 0 \le t \le T, x \in R\}$ for a system of Hopf-type equations is considered [20, 21, 22]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -b(u-v) + f(x)g_1(t), \tag{2.1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \varepsilon b(u-v) + f(x)g_2(t), \tag{2.2}$$

$$u|_{t=0} = u^0(x), \quad v|_{t=0} = v^0(x). \tag{2.3}$$

where the function $f(x)$ is given, $\varepsilon = \frac{\rho_1}{\rho_2}$ is a dimensionless positive constant, b is a positive constant. The unknown functions $g_k = g_k(t)$ $(k = 1,2)$, $t \in [0,T]$, and solutions u, v of the system of equations (2.1), (2.2) must be determined. The system (2.1), (2.2) differs from the system of two-velocity hydrodynamics in the dissipative case due to the coefficient of friction, the absence of pressure and the condition of incompressibility. For this reason, problems arise associated with the Hopf-type system, which gives the simplest quasi-linear system of equations [23].

Let us define the direct and inverse problems for a Hopf-type system.

Definition 2.1. The problem of determining functions u, v from system (2.1), (2.2) for given parameters ε, b and functions $f(x)$, $u^0(x)$, $v^0(x)$, $g_k = g_k(t)$ $(k = 1,2)$ will be called the direct problem for a Hopf-type system.

3. Inverse source problem for a Hopf-type system

Let us assume we have additional override conditions

$$u|_{x=0} = \varphi(t), \quad v|_{x=0} = \psi(t), \quad t \in [0,T], \tag{3.1}$$

and the functions $\varphi(t)$, $\psi(t)$ satisfy the matching conditions

$$\varphi(0) = u^0(0), \quad \psi(0) = v^0(0). \tag{3.2}$$

Definition 3.1. The problem of determining functions u, v, $g_k = g_k(t)$ $(k = 1,2)$ from system (2.1), (2.2) for given parameters ε, b and functions $f(x)$, $u^0(x)$, $v^0(x)$, $\varphi(t)$, $\psi(t)$ will be called the inverse problem for a Hopf-type system.

The functions $u^0(x)$, $v^0(x)$, $f(x)$ and $\varphi(t)$, $\psi(t)$ are assumed to be real. Next, we study the real solution to the classical inverse problem.

Suppose that there exist the Fourier transforms $U(t,y)$, $V(t,y)$ (with respect to x) of the solution $u(t,x)$, $v(t,x)$ for (2.1)–(2.3)

$$(U(t,y), V(t,y)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(u(t,x), v(t,x))e^{ixy}dx = F(u,v)(t,y), \tag{3.3}$$

$$(u(t,x), v(t,x)) = \int_{-\infty}^{\infty}(U(t,y), V(t,y))e^{-ixy}dx = F^{-1}(U,V)(t,x).$$

Applying the Fourier transform in variables x to (2.1), (2.2) we have

$$\frac{\partial U(t,y)}{\partial t} + \frac{1}{2}F\left(\frac{\partial u^2}{\partial x}\right)(t,y) = -b(U(t,y) - V(t,y)) + \tilde{F}(y)g_1(t), \tag{3.4}$$

$$\frac{\partial V(t,y)}{\partial t} + \frac{1}{2}F\left(\frac{\partial v^2}{\partial x}\right)(t,y) = \varepsilon b(U(t,y) - V(t,y)) + \tilde{F}(y)g_2(t), \tag{3.5}$$

where $\tilde{F}(y) = F(f)(y)$. Let in (2.1) and (2.2) $x = 0$. Using (2.3) and (3.3), taking into account the properties of the Fourier transform, we obtain

$$\varphi_t(t) + i\varphi(t)\int_{-\infty}^{\infty} yU(t,y)dy = -b(\varphi(t) - \psi(t)) + f(0)g_1(t), \tag{3.6}$$

$$\psi_t(t) + i\psi(t)\int_{-\infty}^{\infty} yV(t,y)dy = \varepsilon b(\varphi(t) - \psi(t)) + f(0)g_2(t), \tag{3.7}$$

what does it mean

$$g_1(t) = \frac{1}{f(0)}\left\{\tilde{\varphi}(t) + i\varphi(t)\int_{-\infty}^{\infty} yU(t,y)dy\right\}, \tag{3.8}$$

$$g_2(t) = \frac{1}{f(0)}\left\{\tilde{\psi}(t) + i\psi(t)\int_{-\infty}^{\infty} yV(t,y)dy\right\}. \tag{3.9}$$

In formulas (3.8) and (3.9)

$$\tilde{\varphi}(t) = \varphi_t(t) + b(\varphi(t) - \psi(t)), \quad \tilde{\psi}(t) = \psi_t(t) - \varepsilon b(\varphi(t) - \psi(t)).$$

Further, without loss of generality, we can assume that $f(0) = 1$.

Since we are looking for a real solution $u(t,x)$, $v(t,x)$, $g_1(t)$, $g_2(t)$, it is worth considering the real parts of the functions $g_1(t)$, $g_2(t)$ in (3.8) and (3.9) (see remark 3.1 [24])

$$Re\left\{\tilde{\varphi}(t) + i\varphi(t)\int_{-\infty}^{\infty} yU(t,y)dy\right\}, \quad Re\left\{\tilde{\psi}(t) + i\psi(t)\int_{-\infty}^{\infty} yV(t,y)dy\right\}.$$

Suppose that the functions $U^0(y) = F(u^0)(y)$, $V^0(y) = F(v^0)(y)$ are continuously differentiable on $(-\infty,\infty)$, $\tilde{F}(y)$ and $\tilde{F}_y(y)$ continuous on $(-\infty,\infty)$, the functions $\varphi(t)$, $\psi(t)$ are continuously differentiable in $[0,T]$ and

$$\left(1 + |y|^{k+\lambda}\right)\left|U^0(y)\right| + \left(1 + |y|^{k+\lambda}\right)\left|\tilde{F}(y)\right| + \left|\frac{\partial}{\partial y}U^0(y)\right| + \left|\frac{\partial}{\partial y}\tilde{F}(y)\right| \le d_1(k), y \in (-\infty,\infty), \tag{3.10}$$

$$\left(1 + |y|^{k+\lambda}\right)\left|V^0(y)\right| + \left(1 + |y|^{k+\lambda}\right)\left|\tilde{F}(y)\right| + \left|\frac{\partial}{\partial y}V^0(y)\right| + \left|\frac{\partial}{\partial y}\tilde{F}(y)\right| \le d_2(k), y \in (-\infty,\infty), \tag{3.11}$$

where $\lambda = const > 0$, $d_1(k)$, $d_2(k)$ are positive constants and $k > 0$ is an integer.

Since

$$F(u^2)(t,y) = \int_{-\infty}^{\infty} U(t,z)U(t,y-z)dz$$

we represent

$$F\left(\frac{\partial u^2}{\partial x}\right)(t,y) = iyF(u^2)(t,y) = iy\int_{-\infty}^{\infty} U(t,z)U(t,y-z)dz$$

and substitute the real parts $g_1(t)$, $g_2(t)$ from (3.8), (3.9) into (3.4), (3.5) to obtain the integro-differential equation

$$\frac{\partial U(t,y)}{\partial t} + iy\int_{-\infty}^{\infty} U(t,z)U(t,y-z)dz = -b(U(t,y) - V(t,y)) + Re\left\{\tilde{\varphi}(t) + i\varphi(t)\int_{-\infty}^{\infty} yU(t,y)dy\right\}\tilde{F}(y), \tag{3.12}$$

$$\frac{\partial V(t,y)}{\partial t} + iy\int_{-\infty}^{\infty} V(t,z)V(t,y-z)dz = \varepsilon b(U(t,y) - V(t,y)) + Re\left\{\tilde{\psi}(t) + i\psi(t)\int_{-\infty}^{\infty} yV(t,y)dy\right\}\tilde{F}(y), \tag{3.13}$$

with parameter and initial Cauchy data

$$U(0,y) = U^0(y), \quad V(0,y) = V^0(y). \tag{3.14}$$

Note that system (3.12) and (3.13) are not the result of applying the Fourier transform to system (2.1) and (2.2), since instead of , in (3.8), (3.9) we take only their real parts.

We will prove the existence and uniqueness of solution (3.12)–(3.14) using the method of cutting functions [24]. The essence of the method of cutting functions is as follows. We introduce a sequence of cutting functions in the class such that

$$S_N(y) = \begin{cases} 1, & |y| \le N-2, \\ 0, & |y| > N, \end{cases} \tag{3.15}$$

or we approximate (3.12)–(3.14) by the problem

$$U_t^N(t,y) + iy\int_{-\infty}^{\infty} S_N(z)U^N(t,z)S_N(y-z)U^N(t,y-z)dz =$$

$$= -b(U^N(t,y) - V^N(t,y)) + Re\left\{\tilde{\varphi}(t) + i\varphi(t)\int_{-\infty}^{\infty} yS_N(y)U^N(t,y)dy\right\}\tilde{F}(y), \tag{3.16}$$

$$V_t^N(t,y) + iy\int_{-\infty}^{\infty} S_N(z)V^N(t,z)S_N(y-z)V^N(t,y-z)dz =$$

$$= \varepsilon b(U^N(t,y) - V^N(t,y)) + Re\left\{\tilde{\psi}(t) + i\psi(t)\int_{-\infty}^{\infty} yS_N(y)V^N(t,y)dy\right\}\tilde{F}(y), \tag{3.17}$$

$$U^N(0,y) = S_N(y)U^0(y), \quad V^N(0,y) = S_N(y)V^0(y), \quad N \ge 3. \tag{3.18}$$

By virtue of (3.15), we can replace the integrals in (3.16), (3.17) with integrals over a segment $[-N, N]$ and obtain

$$U_t^N(t,y) + iy\int_{-N}^{N} S_N(z)U^N(t,z)S_N(y-z)U^N(t,y-z)dz =$$

$$= -b(U^N(t,y) - V^N(t,y)) + Re\left\{\tilde{\varphi}(t) + i\varphi(t)\int_{-N}^{N} yS_N(y)U^N(t,y)dy\right\}\tilde{F}(y), \tag{3.19}$$

$$V_t^N(t,y) + iy\int_{-N}^{N} S_N(z)V^N(t,z)S_N(y-z)V^N(t,y-z)dz =$$

$$= \varepsilon b(U^N(t,y) - V^N(t,y)) + Re\left\{\tilde{\psi}(t) + i\psi(t)\int_{-N}^{N} yS_N(y)V^N(t,y)dy\right\}\tilde{F}(y). \tag{3.20}$$

Solving the Cauchy problem for system (3.18)-(3.20) we obtain a system of nonlinear integral Volterra equations of the second kind

$$U^N(t,y) = \frac{\varepsilon + e^{-b(1+\varepsilon)t}}{1+\varepsilon}S_N(y)U^0(y) + \frac{1 - e^{-b(1+\varepsilon)t}}{1+\varepsilon}S_N(y)V^0(y) +$$

$$+\frac{\varepsilon + e^{-b(1+\varepsilon)t}}{(1+\varepsilon)^2}\int_0^t \left[\left(\varepsilon + e^{b(1+\varepsilon)\tau}\right)\left[Re\left\{\tilde{\varphi}(\tau) + i\varphi(\tau)\int_{-N}^{N} yS_N(y)U^N(\tau,y)dy\right\}\tilde{F}(y) -\right.\right.$$

$$\left. - iy\int_{-N}^{N} S_N(z)U^N(\tau,z)S_N(y-z)U^N(\tau,y-z)dz\right] +$$

$$+\left(1 - e^{b(1+\varepsilon)\tau}\right)\left[Re\left\{\tilde{\psi}(\tau) + i\psi(\tau)\int_{-N}^{N} yS_N(y)V^N(\tau,y)dy\right\}\tilde{F}(y) -\right.$$

$$\left.\left. - iy\int_{-N}^{N} S_N(z)V^N(\tau,z)S_N(y-z)V^N(\tau,y-z)dz\right]\right] d\tau +$$

$$+\frac{1 - e^{-b(1+\varepsilon)t}}{(1+\varepsilon)^2}\int_0^t \left[\varepsilon\left(1 - e^{b(1+\varepsilon)\tau}\right)\left[Re\left\{\tilde{\varphi}(\tau) + i\varphi(\tau)\int_{-N}^{N} yS_N(y)U^N(\tau,y)dy\right\}\tilde{F}(y) -\right.\right.$$

$$- iy \int_{-N}^{N} S_N(z) U^N(\tau, z) S_N(y - z) U^N(\tau, y - z) dz\Bigg] +$$

$$+ \left(1 + \varepsilon e^{b(1+\varepsilon)\tau}\right) \left[Re \left\{ \tilde{\psi}(\tau) + i\psi(\tau) \int_{-N}^{N} y S_N(y) V^N(\tau, y) dy \right\} \tilde{F}(y) - \right.$$

$$\left. - iy \int_{-N}^{N} S_N(z) V^N(\tau, z) S_N(y - z) V^N(\tau, y - z) dz \Bigg] \right] d\tau, \tag{3.21}$$

$$V^N(t, y) = \frac{\varepsilon \left(1 - e^{-b(1+\varepsilon)t}\right)}{1 + \varepsilon} S_N(y) U^0(y) + \frac{1 + \varepsilon e^{-b(1+\varepsilon)t}}{1 + \varepsilon} S_N(y) V^0(y) +$$

$$+ \frac{\varepsilon \left(1 - e^{-b(1+\varepsilon)t}\right)}{(1+\varepsilon)^2} \int_0^t \left[\left(\varepsilon + e^{b(1+\varepsilon)\tau}\right) \left[Re \left\{ \tilde{\varphi}(\tau) + i\varphi(\tau) \int_{-N}^{N} y S_N(y) U^N(\tau, y) dy \right\} \tilde{F}(y) - \right.\right.$$

$$- iy \int_{-N}^{N} S_N(z) U^N(\tau, z) S_N(y - z) U^N(\tau, y - z) dz \Bigg] +$$

$$+ \left(1 - e^{b(1+\varepsilon)\tau}\right) \left[Re \left\{ \tilde{\psi}(\tau) + i\psi(\tau) \int_{-N}^{N} y S_N(y) V^N(\tau, y) dy \right\} \tilde{F}(y) - \right.$$

$$\left. - iy \int_{-N}^{N} S_N(z) V^N(\tau, z) S_N(y - z) V^N(\tau, y - z) dz \Bigg] \right] d\tau +$$

$$+ \frac{1 + \varepsilon e^{-b(1+\varepsilon)t}}{(1+\varepsilon)^2} \int_0^t \left[\varepsilon \left(1 - e^{b(1+\varepsilon)\tau}\right) \left[Re \left\{ \tilde{\varphi}(\tau) + i\varphi(\tau) \int_{-N}^{N} y S_N(y) U^N(\tau, y) dy \right\} \tilde{F}(y) - \right.\right.$$

$$- iy \int_{-N}^{N} S_N(z) U^N(\tau, z) S_N(y - z) U^N(\tau, y - z) dz \Bigg] +$$

$$+ \left(1 + \varepsilon e^{b(1+\varepsilon)\tau}\right) \left[Re \left\{ \tilde{\psi}(\tau) + i\psi(\tau) \int_{-N}^{N} y S_N(y) V^N(\tau, y) dy \right\} \tilde{F}(y) - \right.$$

$$\left. - iy \int_{-N}^{N} S_N(z) V^N(\tau, z) S_N(y - z) V^N(\tau, y - z) dz \Bigg] \right] d\tau. \tag{3.22}$$

Using the method of contraction mappings, it can be shown that for fixed $N \geq 3$, there exist classical solutions $U^N(t, y)$, $V^N(t, y) \in C^{1,0}_{t,y}\left(\Pi_{[0,t_N]}\right)$ of problem (3.18)–(3.20) in $\Pi_{[0,t_N]}$. Here the constant t_N is positive and, generally speaking, depends on N.

Following [5], taking into account Lemma 3.1 [24], a priori estimates of solutions $U^N(t, y)$, $V^N(t, y)$ are established:

$$|y|^{3+\lambda} \left|U^N(t, y)\right| \leq c_1, \quad |y|^{3+\lambda} \left|V^N(t, y)\right| \leq c_2, \quad (t, y) \in \Pi_{[0,t_*]}. \tag{3.23}$$

Here and below, the constants c_1, c_2 do not depend on N, while t_* depends on the constants $d_1(4)$, $d_2(4)$, $\|\varphi\|_{C^1[0,T]}$, $\|\psi\|_{C^1[0,T]}$ and does not depend on N, for all $N \geq 3$. From equations (3.19), (3.20) we obtain

$$\left|U^N_t(t, y)\right| \leq c_3, \quad \left|V^N_t(t, y)\right| \leq c_4, \quad (t, y) \in \Pi_{[0,t_*]}. \tag{3.24}$$

Differentiating both parts of system (3.22), (3.23) with respect to y, we can show that the estimates are valid

$$\left|U^N_y(t, y)\right| \leq c_5, \quad \left|V^N_y(t, y)\right| \leq c_4, \quad (t, y) \in \Pi_{[0,t_*]}. \tag{3.25}$$

Using (3.23)–(3.25) and Arzela's compactness theorem in C, we can choose subsequences $\{U^{N_k}\}$, $\{V^{N_k}\}$ such that

$$U^{N_k} \to U, \quad V^{N_k} \to V, \quad N_k \to \infty, \tag{3.26}$$

uniformly on each compact K in $\Pi_{[0,t_*]}$.

The uniqueness of the solution is proved in the usual way. Thus, we arrive at the following theorem.

Theorem 3.1. *Let conditions* (3.10), (3.11) $f(0) = 1$, *be satisfied and* $\varphi, \psi \in C^1[0,T]$. *Then there exists a unique solution* $U(t,y)$, $V(t,y)$ *to system* (3.12)–(3.14) *in the strip* $\Pi_{[0,t_*]}$. *The value* $0 < t_* \leq T$ *depends only on the constants* $d_1(4)$, $d_2(4)$ *and* $\|\varphi\|_{C^1[0,T]}$, $\|\psi\|_{C^1[0,T]}$.

Let us prove that the solution $u(t,x)$, $v(t,x)$, $g_1(t)$, $g_2(t)$ to the original problem (2.1)-(3.1) is

$$(u(t,x), v(t,x)) = \int_{-\infty}^{\infty} (U(t,y), V(t,y)) e^{-ixy} dx, \tag{3.27}$$

$$g_1(t) = Re\left\{\tilde{\varphi}(t) + i\varphi(t)\int_{-\infty}^{\infty} yU(t,y)dy\right\}, \tag{3.28}$$

$$g_2(t) = Re\left\{\tilde{\psi}(t) + i\psi(t)\int_{-\infty}^{\infty} yV(t,y)dy\right\}. \tag{3.29}$$

It is easy to see that $g_1(t)$ and $g_2(t)$ are real functions. We will show that $u(t,x)$, $v(t,x)$ are also real functions and satisfy (2.1)–(2.3), (3.1) (where $g_1(t)$ and $g_2(t)$ are defined in (3.28) and (3.29), respectively). We apply the inverse Fourier transform to (3.12)–(3.14) by y and see that $u(t,x)$, $v(t,x)$ are a solution to the problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -b(u-v) + f(x)g_1(t), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.30}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \varepsilon b(u-v) + f(x)g_2(t), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.31}$$

$$u|_{t=0} = u^0(x), \quad v|_{t=0} = v^0(x). \tag{3.32}$$

or

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} - u_2\frac{\partial u_2}{\partial x} = -b(u_1 - v_1) + f(x)g_1(t), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.33}$$

$$\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x} - v_2\frac{\partial v_2}{\partial x} = \varepsilon b(u_1 - v_1) + f(x)g_2(t), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.34}$$

$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_1}{\partial x} = -b(u_2 - v_2), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.35}$$

$$\frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial x} + v_2\frac{\partial v_1}{\partial x} = \varepsilon b(u_2 - v_2), \quad (t,x) \in \Pi_{[0,t_*]}, \tag{3.36}$$

$$u_1|_{t=0} = u^0(x), \quad v_1|_{t=0} = v^0(x), \quad u_2|_{t=0} = 0, \quad v_2|_{t=0} = 0. \tag{3.37}$$

where u_1, v_1 and u_2, v_2 are the real and imaginary parts of the function u, v ($u = u_1 + iu_2$, $v = v_1 + iv_2$), and $g_1(t)$, $g_2(t)$ are the functions in (3.28), (3.29). Since u_1, v_1 and u_2, v_2 is a classical bounded solution of (3.33)–(3.37) (see (26)), we can consider system (3.35), (3.36) as a linear system with respect to u_2, v_2 and apply the method of characteristics (see, for example, [**25**, 10]) to obtain $u_2 = 0$, $v_2 = 0$. Consequently, $u = u_1$, $v = v_1$ is a real solution of (3.30)–(3.32) or (which is the same) (2.1)–(2.3), and $g_1(t)$, $g_2(t)$ are given by the second equations of (3.28), (3.29).

Let us show that the redefinition conditions (3.1) are satisfied: $u(t,0) = \varphi(t)$, $v(t,0) = \psi(t)$. Let $x = 0$ in (3.30), (3.31), then

$$u_t(t,0) + iu(t,0)\int_{-\infty}^{\infty} yU(t,y)dy = -b(u(t,0) - v(t,0)) + \varphi_t(t) + b(\varphi(t) - \psi(t)) + \varphi(t)i\int_{-\infty}^{\infty} yU(t,y)dy,$$

$$v_t(t,0) + iv(t,0)\int_{-\infty}^{\infty} yV(t,y)dy = \varepsilon b(u(t,0) - v(t,0)) + \psi_t(t) - \varepsilon b(\varphi(t) - \psi(t)) + \psi(t)i\int_{-\infty}^{\infty} yV(t,y)dy,$$

or

$$\Phi_t(t) + \mathrm{K}(t)\Phi(t) = -b(\Phi(t) - \Psi(t)), \tag{3.38}$$

$$\Psi_t(t) + \Lambda(t)\Psi(t) = \varepsilon b(\Phi(t) - \Psi(t)), \tag{3.39}$$

where

$$\Phi(t) = u(t,0) - \varphi(t), \quad \Psi(t) = v(t,0) - \psi(t),$$

$$\mathrm{K}(t) = i\int_{-\infty}^{\infty} yU(t,y)dy, \quad \Lambda(t) = i\int_{-\infty}^{\infty} yV(t,y)dy.$$

Note that the functions $\mathrm{K}(t)$, $\Lambda(t)$ are real functions [24].

By virtue of (3.1) and (3.32)

$$\Phi(t)|_{t=0} = 0, \quad \Psi(t)|_{t=0} = 0. \tag{3.40}$$

The only solutions to the Cauchy problem (3.38)-(3.40) are $\Phi(t) \equiv 0$, $\Psi(t) \equiv 0$ [25] and, therefore, $u(t,0) \equiv \varphi(t)$, $v(t,0) \equiv \psi(t)$. Therefore, the functions $u(t,x)$, $v(t,x)$, $g_1(t)$, $g_2(t)$ are a solution to (2.1)-(2.3), (3.1).

Thus, we have proved the following theorem.

Theorem 3.2. *Let* (3.10) *and* (3.11) *be satisfied and* $\varphi, \psi \in C^1[0,T]$, $f(0) = 1$. *Then problem* (2.1)-(2.3), (3.1) *has a solution* $u(t,x)$, $v(t,x) \in C^1(\Pi_{[0,t_*]})$, $g_1(t)$, $g_2(t) \in C[0,t_*]$, *which is determined by formulas* (3.27)-(3.29). *The value of* t_*, $0 < t_* \le T$, *depends only on the constants* $d_1(4)$, $d_2(4)$ *and* $\|\varphi\|_{C^1[0,T]}$, $\|\psi\|_{C^1[0,T]}$.

4. Acknowledgement

The work of Imomnazarov Kh.Kh. was carried out state contract with of the Institute of Mathematics and Mathematical Geophysics of the Siberian Branch of the Russian Academy of Sciences FWNM-2025-0004

References

[1] Pankratov L., Konyukhov A., Voloshin A., General non-equilibrium matrix imbibition equation for Kondaurov's double porosity model. Proceedings of the 6th BIOT Conference on Poromechanics, ASCE, (2017), P. 531–538.

[2] Anikonov Y., Pseudodifferential operators and inverse problems. Computational Center of Siberian Branch, USSR Academy of Sciences, Novosibirsk, (1986).

[3] Anikonov Y., Belov Y., Single-valued solvability of an inverse problem for a parabolic equation. Doklady Mathematics, (1989) Vol. 39, Iss. 3, P. 601–605.

[4] Akhtamova S., Belov Y., Some inverse problems for parabolic equations. Doklady Mathematics, (1991) Vol. 43, Iss. 1, P. 166–170.

[5] Belov Y., An inverse problem for a semilinear parabolic equation. Doklady Mathematics, (1991) Vol. 43, Iss. 1, P. 216–220.

[6] Prilepko A., Vasin I., Some inverse initial-boundary value problems for nonstationary linearized Navier-Stokes equations. Differential Equations, (1989) Vol. 25, Iss. 1, P. 85–92.

[7] Prilepko A., Vasin I., An inverse initial-boundary value problem for the nonlinear Navier-Stokes system in the case of final overdetermination. Differential Equations, (1989) Vol. 25, Iss. 12, P. 1547–1557.

[8] Prilepko A., Solovyev V., Solvability theorems and the Rothe method in inverse problems for an equation of parabolic type. II. Differential Equations, (1987) Vol. 23, Iss. 11, P. 1971–1980.

[9] Tani N., On the first initial-boundary value problem of the generalized Burgers equation. Publications of the Research Institute for Mathematical Sciences, Kyoto University, (1974), Iss. 10, P. 209–233.

[10] Rozhdestvenskii B., Yanenko N., Systems of quasilinear equations and their applications to gas dynamics. American Mathematical Society, Providence, (1983).

[11] Belov Y., An inverse problem for the Burgers equation. Doklady Mathematics, (1992) Vol. 45, Iss. 2, P. 334–337.

[12] Djamalov S., Sipatdinova B., A linear inverse problem for a three-dimensional mixed-type equation of the second kind, second order with semi-nonlocal boundary condition in an unbounded parallelepiped. Chelyabinsk Physical and Mathematical Journal, (2024) Vol. 9, Iss. 3, P. 471–482.

[13] Denisov A., Inverse problem for a quasilinear system of partial differential equations with a nonlocal boundary condition. Comput. Math. Math. Phys. (2014) Vol. 54, Iss. 10, P. 1513–1521.

[14] Denisov A., Existence of a solution of the inverse coefficient problem for a quasilinear hyperbolic equation. Comput. Math. Math. Phys. (2019) Vol. 59, Iss. 4, P. 550–558.

[15] Salaev D., On solvability of the Cauchy problem for one-dimensional system of the Hopf type equations. Bulletin of the Novosibirsk Computing Center, Series: Mathematical Modeling in Geophysics, (2021), Iss. 23, P. 57–62.

[16] Salaev D., Solvability of inverse dynamic problems for a one-dimensional system of Hopf-type equations. Bulletin of the Novosibirsk Computing Center, Series: Mathematical Modeling in Geophysics, (2022), Iss. 24, P. 51–57.

[17] Salaev D., One-dimensional dynamic inverse problem for a system of Hopf-type equations. Bulletin of the Novosibirsk Computing Center, Series: Mathematical Modeling in Geophysics, (2023), Iss. 25, P. 49–57.

[18] Imomnazarov B., Salaev D., Iskandarov I., Solvability of a one-dimensional dynamic direct and inverse problem for a system of Hopf-type. Journal of Mathematical Sciences, (2024) Vol. 284, Iss. 2, P. 848–856.

[19] Vladimirov V., Zharinov V., Equations of Mathematical Physics. Fizmatlit, Moscow, (2004).

[20] Vasiliev G., Imomnazarov K., Mamasoliyev B., On one system of the Burgers equations arising in the two-velocity hydrodynamics. Journal of Physics: Conference Series, (2016) Vol. 697,

[21] Vasiliev G., Imomnazarov K., Kalimoldayev M., Mamasoliyev B., Cauchy Problem for System of the Burgers Equations Arising in the Two-velocity Hydrodynamics. Mathematical Modelling of Natural Phenomena, (2017) Vol. 12, Iss. 3, P. 134–138.

[22] Vasiliev S., Imomnazarov K., Mamasoliev B., Studying a non-dissipative system of the two-velocity hydrodynamics. AIP Conference Proceedings, (2024).

[23] Vasiliev G., Tan Z.-G., Mamasoliev B., Invariant submodels of the system of equations of two-velocity hydrodynamics with phase pressure equilibrium. Siberian Electronic Mathematical Reports, (2018) Vol. 15, P. 585–602.

[24] Belov Y., Inverse problems for Partial Differential Equations. VSP, Utrecht, (2002).

[25] Imomnazarov B., Imomnazarov K., Mukimov A., On an inverse problem arising in the theory of propagation of nonlinear waves. Bulletin of the Novosibirsk Computing Center, Series: Mathematical Modeling in Geophysics, (2024), Iss. 26, P. 1–7.

Mukimov A.X.,
Karshi State University, Karshi, Uzbekistan
e-mail: asqarmuqumov@gmail.com

Mamasoliyev B.J.,
Branch of the Federal State Budget Educational institution
of Higher Education Natinal Research University ”MPEI”
in Tashkent, Uzbekistan
e-mail: bjmamasoliyev@mail.ru

Imomnazarov Kh.Kh.,
The Institute of Computational Mathematics and Mathematical Geophysics SB RAS,
Novosibirsk, Russia
e-mail: imom@omzg.sscc.ru

Iskandarov I.K.,
Pacific National University, Khabarovsk, Russia
e-mail: imom@omzg.sscc.ru

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 149–156
DOI: 10.29229/uzmj.2026-1-16

On embedding theorems in generalized grand Sobolev spaces

Najafov A., Mammadov R., Gasimov S.

Abstract. In this paper, we introduce generalized grand Sobolev spaces and using the integral representation method, study some properties of functions from these spaces from the point of view of embedding theory.

Keywords: generalized grand Sobolev spaces, integral representation, λ-horn condition, embedding theorems

1. Introduction

Note that the grand Lebesgue $L_{p)}(G)$, ($|G| < \infty$, $1 < p < \infty$) introduced in [1], and after spaces of these types, more precisely, small Lebesgue space $L_{(p}(G)$, grand-grand Lebesgue-Morrey $L_{p),\lambda)}(G)$, grand Sobolev-Morrey $W^l_{p),\varkappa,a}(G)$, grand-grand Sobolev-Morrey $W^l_{p),\varkappa),a,\alpha}(G)$, small small Sobolev-Morrey $W^l_{(p,(\varkappa,a}(G)$, and generalized grand Sobolev-Morrey $W^l_{p),\Phi}(G)$ spaces has been introduced and studied by many mathematicians [2, 3, 4, 5], [6, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

In this paper we introduce a generalized grand Sobolev spaces $W^l_{p),A}(G)$, and using integral representation method we proof Sobolev type integral inequalities for functions from this introduced spaces. It should be noted that the norm introduced in this paper is more general than in previous papers.

Definition 1.1. A generalized grand Sobolev space we denote by $W^l_{p),A}(G)$ a space of locally summable functions f on G having the $D_i^{l_i} f$ ($l_i > 0$ are integers $i = 1, 2, \dots, n$) with the finite norm

$$\|f\|_{W^l_{p),A}(G)} = \|A(\varepsilon, p, |G|, f(\cdot))\|_{p),G} + \sum_{i=1}^{n} \|D_i^{l_i} A(\varepsilon, p, |G|, f(\cdot))\|_{p),G}, \tag{1.1}$$

where

$$\|A(\varepsilon, p, |G|, f(\cdot))\|_{p),G} = \|A(\varepsilon, p, |G|, f(\cdot))\|_{L_{p)}(G)} =$$

$$= \sup_{0<\varepsilon<p-1} \left(\int_G |A(\varepsilon, p, |G|, f(x))|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < \infty, \tag{1.2}$$

$G \subset R^n$ is bounded domain, $1 < p < \infty$, $A : (0, p-1) \times (0, h_0) \times L_1(G) \to R$ is a measurable function on $D = (0, p-1) \times (0, h_0) \times L_1(G)$. Also $A(x, y, z)$ is the differentiable function with respect to argument z, and $\lim\limits_{\varepsilon \to 0+} A(\varepsilon, p, |G|, f) = 0$, for all $f \in L_{p-\varepsilon}(G)$, $|G| < \infty$.

Note that, if

$$A(\varepsilon, p, |G|, f(x)) = \left(\frac{\varepsilon}{|G|} \right)^{\frac{1}{p-\varepsilon}} \cdot f(x),$$

then the space $L_{p),A}(G)$ coincides with the space $L_{p)}(G)$ in [1].

Now we give the definition of domains $G \subset R^n$ satisfying the horn condition (see [20]).

Definition 1.2. Let $l = (l_1, \dots, l_n)$ be a vector with positive components, $0 < h \le \infty$, $\varepsilon > 0$, $\delta > 0$ and $a_i \ne 0$ ($i = 1, \dots, n$). The set

$$V(l) = V(l, h) = \bigcup_{0<v<h} \left\{ x : \frac{x_i}{a_i} > 0,\ v < \left(\frac{x_i}{a_i} \right)^{l_i} < (1+\varepsilon)v \ \ (i = 1, \dots, n) \right\}$$

is called the l-horn of radius h and angle ε. We say that an open set G satisfies the l-horn condition if there exist open sets G_k and l-horns $V_k(l) = V_k(l,h)$ $(k = 1,\dots,N)$ such that

$$G = \bigcup_{k=1}^{N} G_k = \bigcup_{k=1}^{N} (G_k + V_k(l,h))$$

and

$$G = \bigcup_{k=1}^{N} G_k^{(\delta)},$$

where $G_k^{(\delta)} = \{x : \ x \in G_k, \ \rho(x, \partial G_k \setminus \partial G) > \delta\}$.

We now construct an integral representation for studying the properties of functions in $W_p^l(G)$ defined in n-dimensional domain and satisfying the λ-horn condition. In addition, we will assume that $f \in L^{loc}(G)$ has all those generalized derivatives with respect to x that will be included in the consideration.

Let us consider the averaging of functions A, i.e. consider the function (see, [20])

$$A_{v^\lambda}(\varepsilon, p, |G|, f(x)) = v^{-|\lambda|} \int_{R^n} A(\varepsilon, p, |G|, f(x+y))\, \Omega\left(\frac{y}{v^\lambda}\right) dy, \tag{1.3}$$

where $v > 0$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, $\lambda_j > 0$ $(j = 1,2,\dots,n)$, $v^\lambda = (v^{\lambda_1}, \dots, v^{\lambda_n})$, $\Omega \in C_0^\infty(R^n)$, and

$$\Omega(x) = D_x^k \left[\frac{x^{k-1}}{(k-1)!} \int_{R^n} K(z) \ominus (x-z)\, dz\right],$$

$$\int_{R^n} \Omega_{v^\lambda}(x)\, dx = \int_{R^n} v^{-|\lambda|} \Omega_{v^\lambda}\left(\frac{x}{v^\lambda}\right) dx = \int_{R^n} \Omega(x)\, dx = 1,$$

and $k = (k_1, k_2, \dots, k_n), k_i$ $(i = 1,2,\dots,n)$ are sufficiently large natural numbers, $1 = (1,1,\dots,1)$, $K \in C_0^\infty(R^n)$ and

$$\int_{R^n} K(x)\, dx = 1,$$

$\ominus(x) = \prod_{j=1}^n \ominus(x_j)$ is the Heaviside function.

Let us find the derivatives of $\Omega_{v^\lambda}(x)$ with respect to parameters v and obtain

$$\frac{\partial}{\partial v} \Omega_{v^\lambda}(x) = -\sum_{i=1}^{n} \lambda_i v^{-1-|\lambda|} D_i^{k_i} L_i\left(\frac{x}{v^\lambda}\right), \tag{1.4}$$

where

$$L_i(x) = D^{k-k_i e_i}\left[\frac{x^{k-1} x_i}{(k-1)!} \int_{R^{n-1}} K(z_1,\dots,z_{i-1},x_i,z_{i+1},\dots,z_n) \left(\prod_{j}^{(i)} \ominus(x_j - z_j)\, dz^{(i)}\right)\right],$$

$|\lambda| = \sum\limits_{j=1}^{n} \lambda_j$ and $\prod\limits_{j}^{(i)} \ominus(x_j - z_j) = \prod\limits_{j \neq i} \ominus(x_j - z_j)$.

Let the function $A(\varepsilon, p, |G|, f)$ be defined in a domain G containing the support of function $L_i(x)$ and has generalized derivatives on G $D_i^{l_i} f$. Then the following equality is true

$$\int_{G} A(\varepsilon, p, |G|, f(x))\, D_{x_i}^{k_i} L_i(x)\, dx =$$

$$= (-1)^{l_i} \int\limits_G D_{x_i}^{l_i} A\left(\varepsilon, p, |G|, f(x)\right) D_{x_i}^{k_i - l_i} L_i(x)\, dx, \ \ l_i \le k_i, \tag{1.5}$$

which follows from the definition (see [20]) of the generalized derivatives, if $D_{x_i}^{k_i - l_i} L_i(x)$ is taken as the function $\varphi \in C_0^\infty(R^n)$.

By virtue of (1.4), from (1.3) we have

$$\frac{\partial}{\partial v} A_{v^\lambda}\left(\varepsilon, p, |G|, f(x)\right) = \int\limits_{R^n} A\left(\varepsilon, p, |G|, f(x+y)\right) \frac{\partial}{\partial v}\left[v^{-|\lambda|}\Omega\left(\frac{y}{v^\lambda}\right) dy\right] =$$

$$= -\sum_{i=1}^n \lambda_i v^{-1-|\lambda|} \int\limits_{R^n} A\left(\varepsilon, p, |G|, f(x+y)\right) D_i^{k_i} L_i\left(\frac{y}{v^\lambda}\right) dy. \tag{1.6}$$

And using the Newton-Leibniz formula we obtain the following equality

$$A_{\eta^\lambda}\left(\varepsilon, p, |G|, f\right) = A_{h^\lambda}\left(\varepsilon, p, |G|, f\right) +$$

$$+ \int\limits_\eta^h \sum_{i=1}^n \lambda_i v^{-1-|\lambda|} dv \int\limits_{R^n} A\left(\varepsilon, p, |G|, f(x+y)\right) D_i^{k_i} L_i\left(\frac{y}{v^\lambda}\right) dy. \tag{1.7}$$

From here we get

$$A_{\eta^\lambda}\left(\varepsilon, p, |G|, f\right) = A_{h^\lambda}\left(\varepsilon, p, |G|, f\right) +$$

$$+ \int\limits_\varepsilon^h \sum_{i=1}^n v^{-1-|\lambda|+\lambda_i l_i} dv \int\limits_{R^n} D_i^{l_i} A\left(\varepsilon, p, |G|, f(x+y)\right) \tilde{L}_i\left(\frac{y}{v^\lambda}\right) dy, \tag{1.8}$$

where $\tilde{L}_i(x) = (-1)^{l_i} \lambda_i D_i^{k_i - l_i} L_i(x)$.

Equality (1.8) can be considered as representation of the difference in the values of the average functions with parameters η^λ and h^λ at point x through the integrals of the generalized derivatives of the functions A along the coordinate directions.

Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\nu_j \ge 0$ $(j = 1, 2, \ldots, n)$ are integers, and $l_j \le \nu_j$ $(j = 1, 2, \ldots, n;\ j \ne i)$, $l_i < k_i + \nu_i\,(i = 1, 2, \ldots, n)$. Let us apply differentation to both sides of (1.8) and transfer the differentitation operation to the kernel, we have $(x \in G)$

$$A_{\eta^\lambda}^{(\nu)}\left(\varepsilon, p, |G|, f(x)\right) = A_{h^\lambda}^{(\nu)}\left(\varepsilon, p, |G|, f(x)\right) +$$

$$+ \int\limits_\eta^h \sum_{i=1}^n v^{-1-|\lambda|+\lambda_i l_i - (\nu,\lambda)} \int\limits_{R^n} D_i^{l_i} A\left(\varepsilon, p, |G|, f(x+y)\right) \tilde{L}_i^{(\nu)}\left(\frac{y}{v^\lambda}\right) dy dv, \tag{1.9}$$

where

$$A_{\eta^\lambda}^{(\nu)}\left(\varepsilon, p, |G|, f(x)\right) = h^{-|\lambda|-(\nu,\lambda)} \int\limits_{R^n} A\left(\varepsilon, p, |G|, f(x+y)\right) \Omega^{(\nu)}\left(\frac{y}{h^\lambda}\right) dy, \tag{1.10}$$

and $(\nu, \lambda) = \sum\limits_{i=1}^n \nu_j \lambda_j$. Note that the λ-norm

$$x + V(\lambda, \delta) = x + \bigcup_{0<\delta<h} \left(a v^\lambda + v^\lambda \delta^\lambda I\right) \subset G$$

is the support of this representations for $x \in U$, where $U = \{x : x \in G,\ x + V \subset G\}$.

Let us now show that if the inequalities

$$m_i = \lambda_i l_i - (\nu, \lambda) > 0 \ \ (i = 1, 2, \ldots, n) \tag{1.11}$$

are satisfied, then there is a generalized derivatives $D^{\nu}A\left(\varepsilon,p,|G|,\ f\ \right)\in L^{loc}\left(G\right)$ and we will obtain an integral representations for them. Let us first establish that

$$A^{(\nu)}_{\eta^{\lambda}}-A^{(\nu)}_{h^{\lambda}}\to 0, \tag{1.12}$$

with $0<\eta<h\to 0$ in $L^{loc}\left(U\right)$. Let the compact $F\subset U$, then at some $\rho>0$, $F+\rho I\subset U$. By virtue of Minkowski‘s inequality we have

$$\|A^{(\nu)}_{\eta^{\lambda}}\left(\varepsilon,p,|G|,f\left(\cdot\right)\right)-A^{(\nu)}_{h^{\lambda}}\left(\varepsilon,p,|G|,f\left(\cdot\right)\right)\|_{1,U}\le$$

$$\le\int\limits_0^h\sum_{i=1}^n v^{-1-|\lambda|+\lambda_i l_i-(\nu,\lambda)}\|D_i^{l_i}A\|_{1,F+\rho I}\left\|\tilde{L}_i\left(\frac{\cdot}{v^{\lambda}}\right)\right\|_1 dv\le\sum_{i=1}^n\left\|D_i^{l_i}A\right\|_{1,U}\|\tilde{L}_i^{(\nu)}\|_1\frac{h^{m_i}}{m_i}$$

From here, by virtue of (1.11) it follows (1.12). Let us assume that generalized derivatives $D^{\nu}A$ exist on G, and passing to the limit in (1.9) as $\eta\to 0$, for almost all $x\in U$ we obtain with the same kernels the following equality

$$D^{\nu}A\left(\varepsilon,p,|G|,f\left(x\right)\right)=A^{(\nu)}_{h^{\lambda}}\left(\varepsilon,p,|G|,f\left(x\right)\right)+$$

$$+\sum_{i=1}^n\int\limits_0^h v^{-1-|\lambda|+\lambda_i l_i-(\nu,\lambda)}\int\limits_{R^n}D_i^{l_i}A\left(\varepsilon,p,|G|,f\left(x+y\right)\right)\tilde{L}_i^{(\nu)}\left(\frac{y}{v^{\lambda}}\right)dvdy. \tag{1.13}$$

2. Main results

Now let‘s prove the main theorems on the properties of functions from the introduced spaces.

Theorem 2.1. *Let $G\subset R^n$ be a bounded domain satisfying the λ-horn condition, $1<p<q\le\infty$, $\nu=(\nu_1,\ldots,\nu_n)$, $\nu_j\ge 0$ $(j=1,2,\ldots,n)$ are integers, $f\in W^l_{p),A}\left(G\right)$, and*

$$\beta_i=\lambda_i l_i-(\nu,\lambda)-\frac{|\lambda|}{p-\varepsilon}+\frac{|\lambda|}{q-\varepsilon}>0,\ \ i=1,2,\ldots,n.$$

Then $D^{\nu}:W^l_{p),A}\left(G\right)\hookrightarrow L_{q-\varepsilon}\left(G\right)$, $0<\varepsilon<p-1$, i.e. for all $f\in W^l_{p),A}\left(G\right)$ on the domain G there exist generalized mixed derivatives $D^{\nu}f\in L_{q-\varepsilon}\left(G\right)$ and there are positive numbers h_0,C^1 and C^2 such that

$$\|D^{\nu}A\|_{q-\varepsilon;G}\le C^1h^{\beta_0}\|A\left(\varepsilon,p,|G|,f\left(\cdot\right)\right)\|_{p);G}+C^2\sum_{i=1}^n h^{\beta_i}\|D_i^{l_i}A\left(\varepsilon,p,|G|,f\left(\cdot\right)\right)\|_{p);G} \tag{2.1}$$

where $\beta_0=\beta_i-\lambda_i l_i$.

In particular, if $\beta_i^0=\lambda_i l_i-(\nu,\lambda)-\frac{|\lambda|}{p-\varepsilon}>0$ $(i=1,2,\ldots,n)$, then $D^{\nu}f\left(x\right)$ is continuous on G and

$$\underset{x\in G}{esssup}|D^{\nu}A\left(\varepsilon,p,|G|,f\left(x\right)\right)|\le C^1h^{\beta_0^0}\|A\left(\varepsilon,p,|G|,f\left(\cdot\right)\right)\|_{p);G}+C^2\sum_{i=1}^n h^{\beta_i^0}\left\|D_i^{l_i}A\left(\varepsilon,p,|G|,f\left(\cdot\right)\right)\right\|_{p),G}, \tag{2.2}$$

C^1 and C^2 are constants do not depend on h and f.

Proof. Initially note that under the conditions, of our theorem, there exist generalized derivatives $D^{\nu}A$ on G. Indeed $p<q$, $\beta_i>0$ $(i=1,2,\ldots,n)$, then $\lambda_i l_i-(\nu,\lambda)>0$, $(i=1,2,\ldots,n)$, if follows that exist $D^{\nu}A$, and the following integral representation (1.13).

We assume that $U+V\subset G$ and based on the Minkowski from equality (1.13) we have

$$\|D^{\nu}A\|_{q-\varepsilon;G}\le\|A^{(\nu)}_{h^{\lambda}}\|_{q-\varepsilon;G}+\sum_{i=1}^n\|B_i\|_{q-\varepsilon,G}, \tag{2.3}$$

where

$$B_i(x)=\int\limits_0^h v^{-1-|\lambda|+\lambda_i l_i-(\nu,\lambda)}\int\limits_{R^n} D_i^{l_i}A(\varepsilon,p,|G|,f(x+y))\,L_i^{(\nu)}\left(\frac{y}{v^\lambda}\right)dydv. \quad (2.4)$$

For $|B_i|$ presented in the form (2.4) we apply the generalized Minkovskii inequality and obtain that

$$\|B_i\|_{q-\varepsilon,G}\leq\int\limits_0^h v^{-1-|\lambda|+\lambda_i l_i-(\nu,\lambda)}\|F_i(\cdot,v)\|_{q-\varepsilon;G}dv, \quad (2.5)$$

where

$$F_i(x,v)=\int\limits_{R^n} D_i^{l_i}A(\varepsilon,p,|G|,f(x+y))\,L_i^{(\nu)}\left(\frac{y}{v^\lambda}\right)dy. \quad (2.6)$$

Let us represent the integrand of the expression presented in formula (2.6), in the form

$$\left|D_i^{l_i}AL_i^{(\nu)}\right|=\left(\left|D_i^{l_i}A\right|^{p-\varepsilon}\left|L_i^{(\nu)}\right|^s\right)^{\frac{1}{q-\varepsilon}}\left(\left|D_i^{l_i}A\right|^{p-\varepsilon}\varkappa\right)^{\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}}\left(\left|L_i^{(\nu)}\right|^s\right)^{\frac{1}{s}-\frac{1}{q-\varepsilon}},\ \ \frac{1}{s}=1-\frac{1}{p-\varepsilon}+\frac{1}{q-\varepsilon},$$

and apply Holders inequality for $|F_i|$ in this case

$$\frac{1}{q-\varepsilon}+\left(\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}\right)+\left(\frac{1}{s}-\frac{1}{q-\varepsilon}\right)=1,$$

we get

$$\|F_i(\cdot,v)\|_{q-\varepsilon,G}\leq\sup_{x\in U}\left(\int\limits_{R^n}\left|D_i^{l_i}A(\varepsilon,p,|G|,f(x+y))\right|^{p-\varepsilon}\varkappa\left(\frac{y}{v^\lambda}\right)dy\right)^{\left(\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}\right)}\times$$

$$\times\sup_{y\in V}\left(\int\limits_U\left|D_i^{l_i}A(\varepsilon,p,|G|,f(x+y))\right|^{p-\varepsilon}dx\right)^{\frac{1}{q-\varepsilon}}\left(\int\limits_{R^n}\left|L_i^{(\nu)}\left(\frac{y}{v^\lambda}\right)\right|^s dy\right)^{\frac{1}{s}}, \quad (2.7)$$

$\varkappa$ be the characteristic function of set $S\left(L_i^{(\nu)}\right)$.

For $x\in U$, we have

$$\int\limits_{R^n}\left|D_i^{l_i}A(\varepsilon,p,|G|,f(x+y))\right|^{p-\varepsilon}\varkappa\left(\frac{y}{v^\lambda}\right)dy\leq$$

$$\leq\int\limits_G\left|D_i^{l_i}A(\varepsilon,p,|G|,f(x+y))\right|^{p-\varepsilon}dy\leq\|D_i^{l_i}A\|_{p-\varepsilon;G}^{p-\varepsilon}, \quad (2.8)$$

for all $y\in V$, we have

$$\int\limits_U\left|D_i^{l_i}A(\varepsilon,p,|G|,f(x+y))\right|^{p-\varepsilon}dx\leq\|D_i^{l_i}A\|_{p-\varepsilon;G}^{p-\varepsilon}, \quad (2.9)$$

and

$$\int\limits_{R^n}\left|L_i^{(\nu)}\left(\frac{y}{v^\lambda}\right)\right|^s dy=v^{|\lambda|}\|L_i^{(\nu)}\|_s^s. \quad (2.10)$$

From inequalities (2.7)-(2.10) follows, that

$$\|F_i(\cdot,v)\|_{q-\varepsilon,G}\leq C_1\|D_i^{l_i}A(\varepsilon,p,|G|,f(\cdot))\|_{p-\varepsilon;G}\,v^{|\lambda|-\frac{|\lambda|}{p-\varepsilon}+\frac{|\lambda|}{q-\varepsilon}}, \quad (2.11)$$

$$\|A_{h^\lambda}^{(\nu)}\|_{q-\varepsilon,G}\leq C_2\|A\|_{p-\varepsilon;G}\cdot h^{-|\lambda|-(\nu,\lambda)}\,h^{|\lambda|-\frac{|\lambda|}{p-\varepsilon}+\frac{|\lambda|}{q-\varepsilon}}=$$

$$= C_2 h^{-(\nu,\lambda)-\frac{|\lambda|}{p-\varepsilon}+\frac{|\lambda|}{q-\varepsilon}} \| A(\varepsilon, p, |G|, f(\cdot)) \|_{p-\varepsilon;G}. \tag{2.12}$$

From inequalities (2.3), (2.5), (2.11) and (2.12) follows that inequality (2.1).

Show that $D^\nu f$ is continuous on G. By (2.1) and (2.3), for $q = \infty$ we obtain

$$\|D^\nu A - A_{h^\lambda}^{(\nu)}\|_{q-\varepsilon,G} \leq \overline{C} \sum_{i=1}^{n} h^{\beta_i} \left\| A\left(\varepsilon, p, |G|, D_i^{l_i} f\right) \right\|_{p-\varepsilon;G}.$$

It follows that the left-hand side of the last inequality tends to zero as $h \to 0$. Since $A_{h^\lambda}^{(\nu)}$ is continuous on G, in our case the convergence in $L_\infty(G)$ coincides with uniform convergence; concequently $D^\nu A$ is continuous on G.

This completes the proof.

Let γ be an n-dimensional vector.

Theorem 2.2. *Suppose that the domain G, the parameters p, q and vector v satisfy the condition of Theorem 2.1.*

Let $l_j^1 \in N$, $j = 1, 2, \ldots, n$, and also let

$$\beta_{i,j} = \lambda_i l_i - (\nu, \lambda) - \frac{|\lambda|}{p-\varepsilon} + \frac{|\lambda|}{q-\varepsilon} - \lambda_j l_j^1 > 0, \quad i, j = 1, 2, \ldots, n.$$

Then $D^\nu : W_{p),A}^{l}(G) \hookrightarrow W_{q-\varepsilon,A}^{l^1}(G)$, i.e., the inequality holds for $f \in W_{p),A}^{l}(G)$

$$\|D^\nu A\|_{W_{q-\varepsilon,A}^{l^1}(G)} \leq C^1 h^{\beta_0} \|A(\varepsilon, p, |G|, f(\cdot))\|_{p);G} +$$

$$+ C^2 \sum_{i=1}^{n} h^{\beta_i} \left\| D_i^{l_i} A(\varepsilon, p, |G|, f(\cdot)) \right\|_{p);G}, \tag{2.13}$$

where h is an arbitrary number from $(0, h_0)$, C^1 and C^2 are constants and do not depend on f.

Proof. *Note that*

$$\|f\|_{W_{q-\varepsilon,A}^{l^1}(G)} = \|A(\varepsilon, p, |G|, f(\cdot))\|_{q-\varepsilon,G} + \sum_{i=1}^{n} \left\| D_i^{l_i} A(\varepsilon, p, |G|, f(\cdot)) \right\|_{q-\varepsilon;G},$$

where

$$\|A(\varepsilon, p, |G|, f(\cdot))\|_{q-\varepsilon,G} = \left(\int_G |A(\varepsilon, p, |G|, f(x))|^{q-\varepsilon} dx \right)^{\frac{1}{q-\varepsilon}}.$$

To obtain inequality (2.13) on the identity (1.13), in the second term on the right side instead of ν we will take $\nu + l_j^1$, $j = 1, 2, \ldots, n$, i.e.,

$$D^{\nu + l_j^1} A(\varepsilon, p, |G|, f(x)) = A_{h^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(x)) +$$

$$+ \sum_{j=1}^{n} \int_{i=1}^{n} \int_{R^n} v^{-1-|\lambda|+\lambda_i l_i - (\nu,\lambda) - \lambda_j l_j^1} D_i^{l_i + l_j^1} A(\varepsilon, p, |G|, f(x+y)) \tilde{L}_i^{(\nu)} \left(\frac{y}{v^\lambda} \right) dv dy, \tag{2.14}$$

where

$$A_{h^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(x)) = h^{-|\lambda| - (\nu,\lambda)} \int_{R^n} A(\varepsilon, p, |G|, f(x+y)) \Omega^{(\nu)} \left(\frac{y}{v^\lambda} \right) dy. \tag{2.15}$$

As in Theorem 2.1 here too

$$\left\| D^{\nu + l_j^1} A(\varepsilon, p, |G|, f(\cdot)) \right\|_{q-\varepsilon,G} \leq \left\| A_{h^\lambda}^{(\nu)}(\varepsilon, p, |G|, f(\cdot)) \right\|_{q-\varepsilon,G} + \sum_{i=1}^{n} \|B_{i,j}\|_{q-\varepsilon,G}, \tag{2.16}$$

where

$$B_{i,j}(x)=\int_0^h\int_{R^n} v^{-1-|\lambda|+\lambda_i l_i-(\nu,\lambda)-\lambda_j l_j^1} D_i^{l_i+l_j^1} A(\varepsilon,p,|G|,f(x+y))\,\tilde{L}_i^{(\nu)}\left(\frac{y}{v^\lambda}\right) dvdy.$$

Similarly, using inequalities (2.5) and (2.12) here we also obtain the following inequalities

$$\left\| A_{h^\lambda}^{(\nu)}(\varepsilon,p,|G|,f(\cdot))\right\|_{q-\varepsilon,G} \le C_1 h^{\beta_0}\left\| A(\varepsilon,p,|G|,f(\cdot))\right\|_{p-\varepsilon,G}, \tag{2.17}$$

and

$$\|B_{i,j}\|_{q-\varepsilon,G} \le C_2 h^{\beta_{i,j}}\left\| D_i^{l_i+l_j^1} A(\varepsilon,p,|G|,f(\cdot))\right\|_{p-\varepsilon,G}. \tag{2.18}$$

It is known that

$$\|D^\nu A\|_{W^{l^1}_{q-\varepsilon,A}(G)} = \left\| A_{h^\lambda}^{(\nu)}(\varepsilon,p,|G|,f)\right\|_{q-\varepsilon,G} + =$$

$$= \|A^{(\nu)}\|_{L_{q-\varepsilon,A}(G)} + \sum_{i=1}^{n}\left\| D^{\nu+l_j^1} A\right\|_{L_{q-\varepsilon,A}(G)}. \tag{2.19}$$

Then, taking into account inequalities (2.17)-(2.19) and (2.16), we obtain the required inequality (2.13).

Theorem 2.2 is proved.

References

[1] Iwaniec T., Sbordone C., On the integrability of the Jacobian under minimal hypotheses. Archive for Rational Mechanics and Analalysis, (1992) Vol. 119, P. 129–149.

[2] Capone C., Fiorenza A., On small Lebesgue spaces. Jour. of funct. spaces and appl. (2005) Vol. 3, Iss. 1, P. 75–79.

[3] Fiorenza A., Karadzhov C., Grand and small Lebesgue spaces and their analogues. Jour. of Analytical and Applied, (2004) Vol. 23, Iss. 4, P. 657–681.

[4] Fiorenza A., Formica M., Gogatishvili A., On grand and small Lebesgue spaces and some applications to PDE's. Diff. Equation and Appications, (2018) Vol. 10, Iss. 1, P. 21–46.

[5] He S., Tao S., Boundedness of some operators on grand generalized Morrey spaces over nonhomogeneous spaces. AIMS Mathematics, (2022) Vol. 7, Iss. 1, P. 1000–1014.

[6] Kokilashvili V. M., Meskhi A., Trace inequalities for fractional integrals in grand Lebesgue spaces. Studia Math. (2012) Vol. 210, Iss. 2, P. 159–176.

[7] Makharadze D., Meskhi A., Ragusa M., Commutators of calderon-zygmind operators in grand variable exponent Morrey spaces, and applications to PDE's. Trends in Mathematics, (2024) Vol. 7, P. 131–141.

[8] Meskhi A., Maximal functions, potentials and singular integrals in grand Morrey spaces. Complex variables and elliptic equations, (2011) Vol. 56, Iss. 10-11, P. 1003–1019.

[9] Mizuta Y., Ohno T., Trudingers exponential integrability for Riesz potentials of function in generalized grand Morrey spaces. Jour. of Mathematical Analysis and Applications, (2014) Vol. 420, Iss. 1, P. 268–278.

[10] Najafov A. M., Rustamova N. R., Some properties of anisotropic grand sobolev-Morrey type spaces. Proceedings of A. Razmadze Mathematical Institute, (2018) Vol. 172, Iss. 1, P. 82–89.

[11] Najafov A. M., Rustamova N. R., On some properties of functions from a Besov-Morrey type spaces. Afrika Matematika, (2018) Vol. 29, Iss. 7-8, P. 1007–1017.

[12] Najafov A. M., Gasimova A. M., On embedding theorems in grand grand Nikolski - Morrey spaces. European Journal of Pure Applied Mathematics, (2019) Vol. 12, Iss. 4, P. 1602–1611.

[13] Najafov A. M., Babayev R. F., On embedding of grand grand Sobolev-Morrey spaces with dominant mixed derivatives. Tbilisi Math. J. (2020) Vol. 13, Iss. 1, P. 1–10.

[14] Najafov A. M., Babayev R. F., On some differential properties of grand fractional Sobolev-Morrey spaces. Uzbek Math. J. (2021) Vol. 65, Iss. 2, P. 128–139.

[15] Najafov A. M., On some differential properties of small small Sobolev-Morrey spaces. Eurasian Matematical Journal, (2021) Vol. 12, Iss. 1, P. 57–67.

[16] Najafov A. M., Eroglu A., Mustafayeva F. F., On some embedding theorems in grand Nikolskii-Morrey spaces with dominant mixed derivatives. Trans. of A. Razm. Math. Inst. (2023) Vol. 77, Iss. 1, P. 85–92.

[17] Najafov A. M., Eroglu A., Mustafayeva F. F., On some differential properties functions from generalized grand Sobolev-Morrey spaces. Commun. Faculty of sciences of Ankara ser. A1 Math. and Statist. (2023) Vol. 72, Iss. 2, P. 429–437.

[18] Najafov A. M., Eroglu A., Mustafayeva F. F., On compactness of embedding operators in grand grand and small small Besov spaces. mathematical Methods in the Applied Sciences, (2025), P. 1–7.

[19] Rafeiro H., A note on boundedness of operators in grand grand Morrey spaces. Advances in harmonic analysis and operator theory. Birkhauser/Springer Basel AG, (2013), vol. 229, P. 349–356.

[20] Besov O., Ilyin V., Nikolskii S., Integral representations of functions and embedding theorems. Nauka, Moscow, (1996).

Najafov A.M.,
Higher Mathematics Department,Azerbaijan University of Architecture and Construction,, Baku, Azerbaijan, Ministry of Science and Education Republic of Azerbaijan Institute of Mathematics and Mechanics,, Baku, Azerbaijan
e-mail: aliknajafov@gmail.com

Mammadov R.S.,
Azerbaijan State University of Oil and Industry,
Baku, Azerbaijan
e-mail: rasadmammedov@gmail.com

Gasimov S.Yu.,
Azerbaijan State University of Oil and Industry,
Baku, Azerbaijan
e-mail: sardarkasumov1955@mail.ru

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 157–164
DOI: 10.29229/uzmj.2026-1-17

On the existence and uniqueness of a strong solution to the antiperiodic problem for a 2-parabolic equation with a deviating argument

Otarova J., Uzaqbaeva D.

Abstract. This paper investigates the antiperiodic boundary value problem for a 2-parabolic equation with a time-deviating argument. A corresponding spectral problem is constructed, the symmetry of the differential operator is proven, and the properties of eigenvalues and eigenfunctions are established. It is shown that the eigenvalues have multiplicity two, and the corresponding eigenfunctions form an orthonormal basis in a Hilbert space. A strong solution to the problem is obtained in the form of a series expansion using the orthonormal basis of eigenfunctions corresponding to the spectrum of the operator generated by the boundary value problem. Conditions for the existence and uniqueness of a strong solution are established, and an explicit form of the inverse operator is constructed. Furthermore, it is proven that the problem operator is essentially self-adjoint. The proven statements complement the theoretical foundation for problems with deviating arguments in classes with antiperiodic boundary conditions, which is significant in modeling processes with memory and delay.

Keywords: deviating argument, 2-parabolic equation, spectral problem, eigenfunctions, eigenvalues
MSC (2020): 35D35, 35K35, 35P10, 47B25

1. Introduction and problem statement

The study of boundary value problems for partial differential equations plays a main role in theoretical and applied mathematics, especially in modeling complex physical processes described by parabolic-type equations. In recent years, there has been growing interest in problems with time-deviated arguments, which reflect memory effects or delays in the evolution of processes. Such problems naturally arise in thermal physics, biology, economics, and other applied fields.

Of particular interest is the formulation of antiperiodic boundary value problems, where the function and its derivatives at opposite ends of the interval differ in sign. Such conditions often model processes with symmetrical oscillatory regimes, where periodic influences reverse direction in each cycle.

Theoretical methods for analyzing differential equations with deviating arguments largely rely on the classical approaches presented in the monograph [1]. Fundamental works [2, 3, 4] on the theory of parabolic equations have formed a methodological basis for investigating a wide range of evolutionary problems in mathematical physics. The issues of well-posedness and construction of solutions to problems with time-deviating arguments have been addressed in numerous studies. Works [5, 6, 7, 8] yielded important results on the spectral properties of boundary value problems with a deviating argument. These studies laid the foundation for the further development of the theory of problems with non-traditional boundary conditions and time-deviating arguments. Significant contributions to the advancement of this field were made by works [9, 10, 11, 12, 13, 14, 15], which examined the existence, uniqueness, and regular and strong solutions of various classes of differential equations with deviating arguments.

A significant contribution to the development of the spectral theory of differential operators in functional spaces was made in the monograph [16], which consistently presents the modern concept of spectral geometry. The authors consider a wide range of problems related to studying the spectrum of differential and pseudo-differential operators on Riemannian manifolds and Lie groups, allowing for a deep analysis of the behavior of solutions to boundary value problems from the perspectives of geometry and functional analysis. Special attention is paid to the conditions for basis property of systems of eigenfunctions and associated functions, which are widely used in solving non-trivial boundary value problems of mathematical physics, including problems with antiperiodic conditions.

A substantial contribution to the study of spectral formulations of boundary value problems with deviating argument was made in the work [17], which conducted a systematic analysis of all possible boundary conditions for a first-order differential equation with involution. It has been shown that the structure of the spectrum and the properties of solutions significantly depend on the type of boundary conditions and the nature of the involutive operator.

In the work [18], the conditions for the basis property of the system of eigenfunctions and associated functions of a differential operator with involution are examined. Establishing the basis property plays a key role in constructing generalized expansions of solutions, which is especially relevant for problems with antiperiodic conditions, where the symmetric structure of boundary conditions is related to the involutive action of the operator. The results of [18] provide important analytical tools for proving the uniqueness and stability of solutions, as well as for developing spectral-analytical methods for studying boundary value problems with involution.

This work investigates the issues of unique strong solvability of the boundary value problem for a 2-parabolic equation with a deviating argument and homogeneous antiperiodic boundary conditions in the Hilbert space of square-integrable functions.

Let $\Omega = \{(x,t) : 0 < x < l,\ \ 0 < t < T\}$ and consider the following problem in the domain Ω:

Problem AP**.** To find the solution of the equation

$$Lu \equiv u_t\left(x, T-t\right) - u_{xxxx}\left(x,t\right) = f\left(x,t\right), \tag{1.1}$$

satisfying the boundary conditions

$$\left.\frac{\partial^k u}{\partial x^k}\right|_{x=0} + \left.\frac{\partial^k u}{\partial x^k}\right|_{x=l} = 0, \quad 0 \le t \le T, \quad k = \overline{0,3}, \tag{1.2}$$

$$u(x,0) = 0, \quad 0 \le x \le l, \tag{1.3}$$

where $f(x,t)$ is a given function.

According to the classification proposed in the work [19], the equations of the form

$$\frac{\partial u}{\partial t} = (-1)^{p-1}\frac{\partial^{2p} u}{\partial x^{2p}} + f\left(x,t\right),$$

are called $p-$parabolic equations. At $p=2$, the classical 2-parabolic equation is obtained

$$u_t\left(x,t\right) + u_{xxxx}\left(x,t\right) = f\left(x,t\right).$$

The deviating argument $T-t$ in equation (1.1) represents an involution of the variable t, since the double application of this transformation returns the original value $(T-(T-t))$. This involution significantly affects the structure of the equation and leads to a change in the sign before the fourth-order derivative compared to the classical Mihailov formula. Thus, equation (1.1) represents a 2-parabolic equation with involute deviation, which can be considered as an inverse problem for the classical 2-parabolic equation in the Mikhailov sense.

Let us introduce the following notations: $V(\Omega) = \{u(x,t) : u \in C^{3,0}_{x,t}(\bar{\Omega}) \cap C^{4,1}_{x,t}(\Omega)$, satisfies conditions (1.2),(1.3)$\}$.

Definition 1.1. The function $u(x,t) \in V(\Omega)$ is called a regular solution of the problem AP for $f(x,t) \in C(\Omega)$, if it satisfies equation (1.1) and conditions (1.2), (1.3) in the domain Ω.

Definition 1.2. The function $u(x,t) \in L_2(\Omega)$ is called a strong solution of the problem AP for $f(x,t) \in L_2(\Omega)$, if there exists a sequence $\{u_n(x,t)\}_{n=1}^{\infty}$ of regular solutions such that $\{u_n(x,t)\}_{n=1}^{\infty}$ and $\{Lu_n(x,t)\}_{n=1}^{\infty}$ converge in $L_2(\Omega)$ to $u(x,t)$ and $f(x,t)$ respectively.

Definition 1.3. A boundary value problem AP is called strongly solvable if a strong solution of the problem exists for any right-hand side $f(x,t) \in L_2(\Omega)$ and unique.

2. On the spectrum of the anti-periodic problem AP.

2.1. On symmetry. On the set $V(\Omega)$, we define the operator L_0, which acts from $V(\Omega)$ to $C(\Omega)$ from $\forall u \in V(\Omega)$ according to rule (1.1). Due to the relationship $C_0^{\infty}(\Omega) \in V(\Omega) \in L_2(\Omega)$, the domain $D(L_0) = V(\Omega)$ of the operator L_0 is densely packed in $L_2(\Omega)$. Let L be the closure of the operator L_0, in Hilbert space $L_2(\Omega)$, which is the minimum closed extension of the operator L_0.

Definition 2.1. [20]. An operator A acting in a Hilbert space H is called symmetric if $\overline{D(A)} = H$ and if for any $u, v \in D(A)$, the identity $(Au, v)_H = (u, Av)_H$ holds, where $(u, v)_H$ the inner product in the space H. In our case, $H = L_2(\Omega)$, and the inner product is defined by the formula

$$(u,v)_0 = (u,v)_{L_2(\Omega)} = \iint\limits_{\Omega} u(x,t)v(x,t)dxdt.$$

Lemma 2.2. *The operator L corresponding to the boundary value problem AP is symmetric.*

Proof. Note that $\overline{D(L)} = L_2(\Omega)$ is constructed. To prove the symmetry of the operator L, it is necessary to prove that for any $u, v \in D(L)$, the equality $(Lu, v)_0 = (u, Lv)_0$ holds.

$$(Lu,v) = \int\limits_0^T \int\limits_0^l [u_t(x,T-t) - u_{xxxx}(x,t)]v(x,t)dxdt,$$

using the substitution of the $s = T - t$ variable and integration by parts by t, we obtain

$$\int\limits_0^T \int\limits_0^l u_t(x,T-t)v(x,t)dxdt = \int\limits_0^T \int\limits_0^l u(x,T-t)v_t(x,t)dxdt.$$

The boundary conditions become zero due to the initial condition $u(x,0) = 0$. Applying quadruple integration by parts by x, we have:

$$\int\limits_0^T \int\limits_0^l u_{xxxx}(x,t)v(x,t)dxdt = \int\limits_0^T \int\limits_0^l u(x,t)v_{xxxx}(x,t)dxdt,$$

all boundary conditions become zero due to the antiperiodic conditions (1.2). We get $(Lu,v)_0 = (u,Lv)_0$, which proves the symmetry of the operator.

□

2.2 On the basis property of eigenvectors. Let us consider the spectral problem for the operator L corresponding to the boundary value problem (1.1) - (1.3):

$$u_t(x,T-t) - u_{xxxx}(x,t) = \lambda u(x,t), \tag{2.1}$$

$$\left.\frac{\partial^k u}{\partial x^k}\right|_{x=0} + \left.\frac{\partial^k u}{\partial x^k}\right|_{x=l} = 0, \quad 0 \le t \le T, \quad k = \overline{0,3}, \tag{2.2}$$

$$u(x,0) = 0. \tag{2.3}$$

To solve the problem, we use the method of separation of variables. Assuming

$$u(x,t) = w(x) \cdot v(t), \tag{2.4}$$

and substituting (2.4) into (2.1), we have

$$\frac{v'(T-t)}{v(t)} = \frac{w^{IV}(x) + \lambda w(x)}{w(x)} = \gamma,$$

where is $\gamma-$the spectral parameter. Thus, if the solutions of problem (2.1) - (2.3) have the form (2.4), then the functions $w(x)$ and $v(t)$ are respectively solutions of the following spectral problems

$$\begin{cases} v'(T-t) - \gamma v(t) = 0 \\ v(0) = 0 \end{cases}, \tag{2.5}$$

$$\begin{cases} w^{IV}(x) - \beta w(x) = 0 \\ \left.\dfrac{d^k w}{dx^k}\right|_{x=0} + \left.\dfrac{d^k w(x)}{dx^k}\right|_{x=l} = 0, k = \overline{0,3}, \end{cases} \tag{2.6}$$

where $\beta = \gamma - \lambda$.

Lemma 2.3. *The spectral problem (2.6) has an infinite set of eigenvalues*

$$\beta_k = \frac{\pi^4 (2k+1)^4}{l^4}, \quad k = 0, 1, 2..., \tag{2.7}$$

and corresponding eigenfunctions

$$w_k^{(1)}(x) = \sqrt{\frac{2}{l}} \cos \frac{\pi\,(2k+1)}{l} x, \qquad w_k^{(2)}(x) = \sqrt{\frac{2}{l}} \sin \frac{\pi\,(2k+1)}{l} x, \quad k = 0, 1, 2..., \tag{2.8}$$

which form an orthonormal basis in $L_2\,(0, l)$.

Proof. Let us find the eigenvalues of the antiperiodic problem. The characteristic equation is written in the form, $m^4 = \beta$, let us consider the cases $\beta > 0$, $\beta = 0$, $\beta < 0$.

Let $\beta = 0$, then the characteristic equation $m^4 = 0$ has four roots $m_{1,2,3,4} = 0$. The general solution is written as $w\,(x) = C_1 x^3 + C_2 x^2 + C_3 x + C_4$ using the antiperiodic conditions of problem (2.2) we have $C_1 = C_2 = C_3 = C_4 = 0$. From this $X\,(x) = 0$.

Let $\beta < 0$, be $\beta = -4\mu^4$, $(\mu > 0)$, then the characteristic equation $m^4 = -4\mu^4$ has complex conjugate roots $m_{1,2} = \mu\,(1 \pm i)\,;\ \ m_{3,4} = \mu\,(-1 \pm i)\,;$ the solution is written as:

$$w\,(x) = C_1 ch\mu x\, \cos \mu x + C_2 ch\mu x\, \sin \mu x + C_3 sh\mu x\, \cos \mu x + C_4 sh\mu x\, \sin \mu x,$$

using the conditions of problem (2.6), we have $C_1 = C_2 = C_3 = C_4 = 0$. From this $w\,(x) = 0$.

Thus, the problem (2.6) at $\beta \le 0$ has only a trivial solution.

Let $\beta = \mu^4$, where $\mu > 0$. Then the characteristic equation $m^4 = \mu^4$ has roots $m_{1,2} = \pm\mu$; $m_{3,4} = \pm\mu i$, and the general solution can be written as:

$$w\,(x) = C_1 e^{\mu x} + C_2 e^{-\mu x} + C_3 \cos \mu x + C_4 \sin \mu x,$$

where $C_i, i = \overline{1,4}$ are arbitrary real numbers. Further, considering the boundary conditions of problem (2.6), to find these constants, we obtain the system

$$\begin{cases} \left(1 - e^{\mu l}\right) C_1 + \left(1 + e^{-\mu l}\right) C_2 + \left(1 + \cos \mu l\right) C_3 + \sin \mu l C_4 = 0, \\ \left(1 + e^{\mu l}\right) C_1 - \left(1 + e^{-\mu l}\right) C_2 - \sin \mu l C_3 + \left(1 + \cos \mu l\right) C_4 = 0, \\ \left(1 + e^{\mu l}\right) C_1 + \left(1 - e^{-\mu l}\right) C_2 - \left(1 + \cos \mu l\right) C_3 - \sin \mu l C_4 = 0, \\ \left(1 - e^{\mu l}\right) C_1 - \left(1 - e^{-\mu l}\right) C_2 + \sin \mu l C_3 - \left(1 + \cos \mu l\right) C_4 = 0. \end{cases} \tag{2.9}$$

The resulting system has a non-trivial solution only for the values μ, at which its determinant becomes zero. Let us denote the determinant of this system by $\Delta(\mu)$. Then it is not difficult to see, that $\Delta(\mu) = -16(1 + \cos \mu l)$. From this we find eigenvalues, that have the form (2.7).

Now let us examine the multiplicity of eigenvalues. Since the rank of the matrix corresponding to system (2.9) is equal to 2 when $\mu_k l = \pi + 2\pi k$, it follows that the geometric multiplicity of the eigenvalues is equal to 2. Therefore, each eigenvalue corresponds to a pair of eigenfunctions. The algebraic multiplicity is the order of multiplicity of the root in μ_k the equation $\Delta\,(\mu) = 0$. Since

$$\Delta'\,(\mu) = 16 l \sin \mu l, \quad \Delta'\,(\mu_k) = 16 l \sin\left[(2k+1)\,\pi\right] = 0,$$

$$\Delta''\,(\mu) = 16 l^2 \cos \mu l, \ \ \Delta''\,(\mu_k) = 16 l^2 \cos\left[(2k+1)\,\pi\right] = -16 l^2 \ne 0,$$

therefore, the algebraic multiplicity of eigenvalues is also equal to 2. Consequently, all eigenvalues of problem (2.6) are of multiplicity two, and the eigenfunctions are the functions (2.8). The orthonormality of the resulting system in $L_2(0, 1)$ is verified directly. Then, by the Riesz-Fischer theorem, the system of eigenfunctions (2.8) of problem (2.6) forms an orthonormal basis in $L_2\,(0, 1)$. □

Lemma 2.4. *[15]. The spectral problem (2.5) has an infinite set of eigenvalues*

$$\gamma_n = (-1)^k \left(\frac{1}{2} + 2k\right) \frac{\pi}{T}, \quad k = 0, 1, 2.... \tag{2.10}$$

and corresponding eigenfunctions

$$v_k(t) = \sqrt{\frac{2}{T}} \sin \frac{\pi(2k+1)}{2T} t, \quad k = 0, 1, 2, ..., \tag{2.11}$$

which form an orthonormal basis of the space $L_2(0;T)$.

The following statements hold. [[20], p.65]:

Lemma 2.5. *If the system of functions* $\{\varphi_m(x)\}, m = 1, 2, ...$*forms an orthonormal basis of the space* $L_2(0, l)$*, and the system of functions* $\{\psi_n(x)\}, n = 1, 2, ...$*forms an orthonormal basis of the space* $L_2(0, T)$*, then the system of functions* $\{\varphi_m(x)\psi_n(x)\}$, $m, n = 1, 2, ...$ *forms an orthonormal basis of the space* $L_2[(0, l) \times (0, T)]$.

From this lemma and from formulas (2.4), (2.7), (2.11), it follows that:

Theorem 2.6. *The spectral problem (2.1) - (2.3) has an infinite set of eigenvalues*

$$\lambda_{kn} = (-1)^n \frac{\pi}{T}\left(\frac{1}{2} + 2n\right) + \frac{\pi^4(2k+1)^4}{l^4}; \quad k, n = 0, 1, 2, ..., \tag{2.12}$$

and corresponding eigenfunctions

$$u_{kn}^{(1)}(x,t) = \frac{2}{\sqrt{Tl}} \cos \frac{\pi(2n+1)}{l} x \cdot \sin \frac{\pi(2k+1)}{2T} t, \quad k, n = 0, 1, 2, ..., \tag{2.13}$$

$$u_{kn}^{(2)}(x,t) = \frac{2}{\sqrt{Tl}} \sin \frac{\pi(2n+1)}{l} x \cdot \sin \frac{\pi(2k+1)}{2T} t, \quad k, n = 0, 1, 2, ..., \tag{2.14}$$

which form an orthonormal basis of the space $L_2(\Omega)$.

3. On the existence and uniqueness of a strong solution to the antiperiodic problem

Consider the linear operator L corresponding to the boundary value problem (1.1)-(1.3). Suppose that for some $u \in D(L)$, $u \neq 0$ the equality $Lu = 0$ holds. Then, due to the symmetry of the operator L we have the equality

$$0 = \left(Lu, u_{kn}^{(i)}\right) = \left(u, Lu_{kn}^{(i)}\right) = \lambda_{kn}\left(u, u_{kn}^{(i)}\right), \quad i = 1, 2.$$

If $\lambda_{kn} \neq 0$, then due to the completeness of the system (2.13), (2.14), we obtain $u = 0$, which contradicts our assumption. Therefore, for some values of the indices, the equality $\lambda_{kn} = 0$ holds. Conversely, if there is a zero eigenvalue among the eigenvalues, then for some $u \neq 0$ the equality $Lu = 0$ holds.

For the existence of the inverse operator L^{-1} it is necessary and sufficient that the kernel of the operator L consists only of the zero element, i.e.

$$\ker L = \{u \in D(L),\ Lu = 0\} = \{0\},$$

and for this, it is necessary and sufficient that the condition is satisfied$\lambda_{kn} \neq 0,\ \forall\, k, n = 0, 1, 2, ...$

If $\lambda_{kn} \neq 0,\ \forall\ k, n \in N$ then according to the theory, there exists a unique inverse operator L^{-1}, i.e. the solution to the problem AP exists and unique. Obviously, for $n = 2m$ we have

$$\lambda_{k,2m} = \frac{\pi}{T}\left(\frac{1}{2} + 4m\right) + \frac{\pi^4(2k+1)^4}{l^4} \neq 0, \quad k,\ m = 0, 1, 2, ...$$

Therefore, from (2.12), when $n = 2m + 1$ we obtain the necessary and sufficient condition for the invertibility of the operator

$$\lambda_{k,2n+1} = \frac{\pi^4(2k+1)^4}{l^4} - \frac{\pi}{T}\left(\frac{5}{2} + 4m\right) \neq 0, \quad k,\ m = 0, 1, 2, ...,$$

which holds when the following conditions are met

$$\frac{l^4}{2\pi^3 T} \neq \frac{(2k+1)^4}{4m+1}. \tag{3.1}$$

Thus, the necessary and sufficient condition for the invertibility of the operator L will be condition (3.1), which excludes the coincidence of eigenvalues (2.12) with the zero eigenvalue.

Since the right side of the inequality (3.1) is a rational number for any $k, m \in N$, and π is an irrational number, then, for example, this inequality holds for any fixed $l, T \in Q$.

Let us now construct the inverse operator L^{-1}. Let $u \in D(L)$, $f \in R(L)$ and the equality $Lu = f$ is hold. Expanding the left and right sides of this equality into a Fourier series with respect to the system $\{u_{kn}^{(i)}\}$, $k, n = 0, 1, 2, ...$, we have

$$Lu = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2} (Lu, u_{kn}^{(i)}) u_{kn}^{(i)}(x,t) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2} \lambda_{kn}(u, u_{kn}^{(i)}) u_{kn}^{(i)}(x,t),$$

$$f = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2} (f, u_{kn}^{(i)}) u_{kn}^{(i)}(x,t).$$

Substituting all of this into the equation , and comparing the coefficients, we get

$$(u, u_{kn}^{(i)}) = \frac{(f, u_{kn}^{(i)})}{\lambda_{kn}}.$$

Then the sought solution $u(x,t)$ of the equation $Lu = f$ can be written as

$$u(x,t) = L^{-1}f = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2} \frac{f_{kn}^{(i)}}{\lambda_{kn}} u_{kn}^{(i)}(x,t), \tag{3.2}$$

provided that $\lambda_{kn} \neq 0$ for $\forall\, k, n$. This is valid, since

$$\lambda_{kn} = (-1)^n \frac{\pi}{T}\left(\frac{1}{2} + 2n\right) + \frac{\pi^4 (2k+1)^4}{l^4} \neq 0, \quad \forall\ k,\ n\ \in\ N_0.$$

Thus, the inverse operator $L^{-1} : R(L) \to D(L)$ is formally defined by expression (3.2). The resulting solution (3.2) is a strong solution to the problem (1.1)-(1.3) [9]. Let us denote by the closure of the operator $\overline{L}$, originally defined L on the set of regular functions $D(L)$. Then, if $R\left(\overline{L}\right) = L_2(\Omega)$, any function f can be the right-hand side of the equation, and there exists a strong solution; this condition is equivalent to the condition $\lambda_{kn} \neq 0,\ \forall\ k,\ n\ \in\ N_0$,i.e., there must be no zeros among the eigenfunctions. Thus, the following theorem is proven.

Theorem 3.1. *For the uniqueness of the strong solution of the boundary value problem (1.1) - (1.3), it is necessary and sufficient that condition (3.1) be satisfied. When this condition is satisfied, the strong solution of the problem exists and has the form*

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2} \frac{\left(f,\ u_{kn}^{(i)}\right)}{\lambda_{kn}} u_{kn}^{(i)}(x,t),$$

for all $f(x,t) \in L_2(\Omega)$, *satisfying the condition*

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=1}^{2} \left| \frac{\left(f,\ u_{kn}^{(i)}\right)}{\lambda_{kn}} \right|^2 < \infty,$$

where $u_{kn}^{(i)}(x,t)$, $i = 1, 2$ *and* λ_{kn} *are defined by (2.12) - (2.14).*

4. On self-adjointness in the essential sense of an operator L.

The following statements hold [9]:

Lemma 4.1. *Let A be a symmetric linear operator in a Hilbert space H. If the operator A has a complete system of eigenvectors, then its closure $\overline{A}$ is a self-adjoint operator in H.*

Theorem 4.2. *The operator $Lu \equiv u_t(x, T-t) - u_{xxxx}(x,t)$, acting in the Hilbert space $H = L_2(\Omega)$, where $\Omega = (0,l) \times (0,T)$, with the domain*

$$D(L) := \left\{ u \in C^{4,1}(\Omega) \cap C(\overline{\Omega}) \;\middle|\; \begin{array}{ll} u(x,0) = 0, & 0 \le x \le l \\ \left.\frac{\partial^k u}{\partial x^k}\right|_{x=0} + \left.\frac{\partial^k u}{\partial x^k}\right|_{x=l} = 0, \quad k = \overline{0,3}, & 0 \le t \le T \end{array} \right\}$$

which is symmetric and allows for a self-adjoint closure in $L_2(\Omega)$. That is, its closure $\overline{L}$coincides with the adjoint operator L^*, $\overline{L} = L^*$, and therefore the operator is essentially self-adjoint.

From Theorems 3.1 and 4.2, it follows

Theorem 4.3. *If*

$$\frac{l^4}{2\pi^3 T} \neq \frac{(2k+1)^4}{4n+1}, \quad \forall\, k, n \in N_0,$$

then the inverse operator L^{-1} exists and self-adjoint.

Proof. According to Lemma 4.1, a symmetric operator with a complete system of eigenfunctions is essentially self-adjoint, i.e. $\overline{L} = L^*$ is its closure. From this, it follows that the L^{-1}−inverse operator is also self-adjoint, $(L^{-1})^* = (L^*)^{-1} = (\overline{L})^{-1} = L^{-1}$. □

References

[1] Elsgols L., Norkin S., Introduction to the Theory of Differential Equations with Deviating Arguments. Nauka, Moscow, (1971).

[2] Ladyzhenskaya O., Solonnikov V., Uralseva N., Linear and Quasi-Linear Equations of a Parabolic Type. Nauka, Moscow, (1967).

[3] Friedman A., Equations with Partial Derivatives of the Parabolic Type. Mir, Moscow, (1968).

[4] Lions J., Some methods of solving problems with nonlinear limits. Dunod, Paris, (1969).

[5] Kalmenov T., Spectral properties of boundary value problems for differential equations with deviating arguments. Automated systems and related problems of analysis: Collection of scientific works, Nalchik, (1989), P. 146–149.

[6] Kalmenov T., Shomanbayeva M., On one criterion for the existence of a strong solution to the Cauchy-Neumann problem for the heat conduction equation with a deviating argument. Differential Equations, Theory of Functions and Applications: Thesis. doc. internar.conf. Novosibirsk, (2007), P. 183–184.

[7] Kalmenov T., Shaldanbayev A., Shomanbayeva M., On the existence and uniqueness of a strong solution to the periodic problem for the heat conduction equation with a deviating argument. Mathematical Journal, (2007) Vol. 7, Iss. 4, P. 44–50.

[8] Kalmenov T., Shaldanbayev A., Shomanbayeva M., On the nature of the spectrum of the operator of the periodic problem for the heat conduction equation with a deviating argument. Mathematical Journal, (2008) Vol. 8, Iss. 1, P. 40–49.

[9] Shaldanbayev A., Shomanbayeva M., On the Volterra behavior of the heat conductivity operator. Search. Series of Natural and Technical Sciences, (2006), P. 166–169.

[10] Shaldanbayev A., Shomanbayeva M., On the strong solvability of the Dirichlet condition mixed problem for the heat conduction equation with a deviating argument. Science and Education of Southern Kazakhstan. Ser. fiz., mat., inf. (2006), Iss. 10, P. 133–136.

[11] Shomanbayeva M., On the Neumann problem for the heat conduction equation with a deviating argument. Search. Series of Natural and Technical Sciences, (2006), P. 179–183.

[12] Shomanbayeva M., On the nature of the spectrum of the Cauchy-Neumann operator for the heat conduction equation with a deviating argument. Science and Education of Southern Kazakhstan. Ser.phys., mat., inf, (2006) Vol. 10, Iss. 59,

[13] Orazov I., Shaldanbayev A., Shomanbayeva M., On the nature of the spectrum of the periodic problem for the heat equation with a deviating argument. Abstract and Applied Analysis, (2013) Vol. 2013, P. 16.

[14] Shomanbayeva M., Criterion of strong solvability of a semi-fixed problem for the heat conduction equation with a deviating argument. Search. Series of Natural and Technical Sciences, (2010), Iss. 1, P. 141–144.

[15] Shomanbayeva M., On the existence and uniqueness of a strong solution to the antiperiodic problem for the heat conduction equation with a deviating argument. Bulletin of Karaganda University. Mathematics Series, (2016) Vol. 81, Iss. 3, P. 83–91.

[16] Ruzhansky M., Sadybekov M. A., Suragan D., Spectral Geometry of Partial Differential Operators. Taylor Francis Group, New York, (2020).

[17] Sadybekov M. A., Sarsenbi A. M., Solving the basic spectral questions of all boundary value problems for one first-order differential equation with a divergent argument. Uzbek Mathematical Journal, (2020), P. 88–94.

[18] Granilchikova Y. A., Shkalikov A. A., Spectral properties of the differential operator with involution. Bulletin of Moscow University. 1st episode. Mathematics. Mechanics, (2022), P. 67–71.

[19] Mikhailov V., On the Potentials of Parabolic Equations. Doc. USSR Academy of Sciences, (1959) Vol. 129, Iss. 6, P. 1226–1229.

[20] Rid M., Simon B., Methods of Modern Mathematical Physics. Vol. 1. Functional Analysis. Mir, Moscow, (1977).

Otarova J. A,
Karakalpak State University, Nukus, Uzbekistan
e-mail: j.otarova@mail.ru

Uzaqbaeva D. E,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Science,
Tashkent, Uzbekistan.
e-mail: uzaqbaevadilfuza1606@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 165–173
DOI: 10.29229/uzmj.2026-1-18

Thermodynamic analysis of the three-state SOS model on the binary tree

Rahmatullaev M., Karshiboev O.

Abstract. In this work, we investigate the thermodynamic properties of the three-state solid-on-solid (SOS) model on a binary Cayley tree. Employing recurrence relations, we analyze the partition function and derive explicit expressions for the local magnetization and the quadrupolar moment. One of the main results of the paper is the demonstration of the existence of a second-order phase transition in the model. To explore the system's dynamical behavior, we compute the Lyapunov exponent, which reveals transitions between distinct dynamical regimes. Our findings demonstrate the model's rich phase structure, characterized by the emergence of periodic regimes and the absence of chaos, as confirmed by Lyapunov analysis.

Keywords: SOS model, Second-order phase transition, Magnetization, Quadrupolar moment, Lyapunov exponent
MSC (2020): Primary 82B05 · 82B20; Secondary 60K35

1. Introduction

The solid-on-solid (SOS) model is a fundamental framework in statistical mechanics, extensively employed to investigate surface growth, interface dynamics, and phase transitions in lattice systems [1, 2, 3, 4, 5, 6]. By focusing on the height differences between neighboring sites, the SOS model simplifies complex interactions while preserving essential physical phenomena, rendering it a powerful tool for both theoretical and computational studies.

In this work, we examine the three-state SOS model on a Cayley tree of order two (also referred to as a binary tree), which introduces a hierarchical lattice structure that accentuates the role of recursive interactions [3]. The three-state SOS model, in which each site assumes one of three discrete height values, serves as a rich platform for studying phase transitions driven by temperature and coupling strength. Prior investigations of related models have revealed the existence of multiple phases which confirms the existence of the first-order phase transition and critical phenomena in tree-like structures [7, 8, 9, 10, 11, 12].

Here, we employ recursive methods to derive key thermodynamic quantities such as the partition function and magnetization, leveraging the symmetry of the Cayley tree to facilitate analytical progress. We show that the model exhibits the second-order phase transition using stability analysis of fixed points. In addition, we investigate the model's dynamical behavior via the Lyapunov exponent, providing insight into the transition between periodic and chaotic regimes in the recursive dynamics. Our analysis integrates classical methods from statistical physics while offering novel insights into the interplay between thermal and dynamical properties in hierarchical systems. These results contribute to the broader understanding of disordered systems and phase transitions in non-standard geometries.

The structure of the paper is as follows. In Section 2, we define the three-state SOS model on a Cayley tree of order two and derive a system of recurrence relations for the partition function. Section 3 focuses on the thermodynamic behavior of the model, including explicit expressions for the magnetization and quadrupolar moment associated with each fixed point, showing the presence of the second-order phase transition. In Section 4, we examine the dynamical stability of the system through cobweb diagrams, iterative maps, and Lyapunov exponents to characterize periodic and chaotic behavior. Finally, in Section 5, we summarize the main results.

2. Model definition and recursive formulation

We consider the three-state SOS model on a Cayley tree of order two, also known as a binary tree. A Cayley tree of order two is a connected, acyclic graph in which each vertex is connected to exactly three neighbors, except for the vertices on the boundary (leaves). The tree can be constructed recursively by starting from a root vertex (referred to as the central site) and attaching two new branches to each non-terminal vertex at each successive level. The number of layers, denoted by n, determines the depth of the tree.

The Hamiltonian of the model is given by

$$H = -J \sum_{\langle i,j \rangle} |s_i - s_j|, \tag{2.1}$$

where each spin variable s_i takes one of three possible values: -1, 0, or 1; J is the coupling constant; and the summation is carried out over all nearest-neighbor pairs $\langle i, j \rangle$.

The partition function of the model is

$$Z \equiv \sum_{s} \exp\left(-\beta H(s)\right) = \sum_{s} \exp\left(K \sum_{\langle i,j \rangle} |s_i - s_j| \right), \tag{2.2}$$

and here $\beta = 1/T$ (with T being the temperature) and $K = \beta J$. The summation over s in Eq. (2.2) denotes the sum over all possible spin configurations on the tree.

Phase transitions are fundamental phenomena in statistical physics, often identified through the emergence of multiple Gibbs measures. Following [9], the existence of at least two distinct Gibbs states at a temperature T indicates a first-order phase transition. A second-order phase transition is characterized by the continuous emergence of nonzero spontaneous magnetization from zero as the parameter crosses a critical value, accompanied by a change in stability of the fixed points in the recursive dynamical system.

The model defined by the Hamiltonian (2.1) on a Cayley tree of order two can be studied analytically and numerically using the method of recursion relations. The approach is based on the hierarchical structure of the tree.

Since the analysis focuses on the bulk behavior of the system (i.e., deep inside the Cayley tree), it is important to note that all interior sites are statistically equivalent due to the tree's symmetry [8]. When the Cayley tree is cut at the central site (denoted as site 0; see Fig. 1), it splits into three identical subtrees.

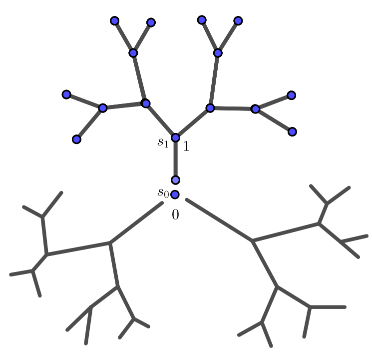

Figure 1. Structure of the Cayley tree of order two rooted at site 0.

This symmetry allows the partition function (2.2) to be rewritten in the simplified form:

$$Z = \sum_{s_0} \left[g_n(s_0) \right]^3, \tag{2.3}$$

where $s_0 \in \{-1, 0, 1\}$, and the functions $g_n(s_0)$ satisfy the recursion relation

$$g_n(s_0) = \sum_{s_1} \exp\left(K |s_0 - s_1|\right) \left[g_{n-1}(s_1) \right]^2. \tag{2.4}$$

Here, $g_n(s_0)$ represents the contribution to the partition function from a subtree of depth n rooted at a site with spin value s_0, and s_1 denotes the spin values at the next layer of the tree. It is assumed that the tree has n layers in total.

It is often more convenient to work with recursion relations defined as ratios of the functions in Eq. (2.4), namely,

$$x_n = \frac{g_n(-1)}{g_n(1)}, \qquad y_n = \frac{g_n(0)}{g_n(1)}. \tag{2.5}$$

The explicit form of the recursion relations for x_n and y_n in Eq. (2.5) is given by

$$\begin{cases} x_n = \dfrac{x_{n-1}^2 + \theta y_{n-1}^2 + \theta^2}{\theta^2 x_{n-1}^2 + \theta y_{n-1}^2 + 1}, \\ \\ y_n = \dfrac{\theta x_{n-1}^2 + y_{n-1}^2 + \theta}{\theta^2 x_{n-1}^2 + \theta y_{n-1}^2 + 1}, \end{cases} \tag{2.6}$$

where $\theta = \exp(K)$. The fixed point values of x_n and y_n in the limit $n \to \infty$ are obtained by solving the steady-state version of Eq. (2.6), which leads to the system:

$$\begin{cases} x = \dfrac{x^2 + \theta y^2 + \theta^2}{\theta^2 x^2 + \theta y^2 + 1}, \\ \\ y = \dfrac{\theta x^2 + y^2 + \theta}{\theta^2 x^2 + \theta y^2 + 1}. \end{cases} \tag{2.7}$$

In [3], the complete solution of the system (2.7) is presented, and the occurrence of a first-order phase transition for the model is established. In the present paper, we further demonstrate that the model also exhibits a second-order phase transition. Moreover, [3] shows that there exist two critical values of θ:

- $\theta_c \approx 0.1414$, which is the solution of the algebraic equation

$$4\theta^7 + 12\theta^5 + 71\theta^4 + 12\theta^3 - 38\theta^2 + 12\theta - 1 = 0,$$

- and $\theta_c' = \frac{1}{3}\left(\sqrt[3]{26 + 6\sqrt{33}} - \frac{8}{\sqrt[3]{26+6\sqrt{33}}} - 1\right) \approx 0.2956.$

The set of solutions to the system (2.7) is summarized in the following result:

Lemma 2.1 ([3])**.** *The number of fixed points of the system* (2.7) *depends on the value of the parameter* θ *as follows:*

- *If $\theta > \theta_c'$, the system has a unique solution $(x^{(1)}, y^{(1)})$;*
- *If $\theta = \theta_c'$, the system has three solutions: $(x^{(1)}, y^{(1)})$, $(x^{(4)}, y^{(4)})$, and $(x^{(6)}, y^{(6)})$;*
- *If $\theta_c < \theta < \theta_c'$, the system has five solutions: $(x^{(1)}, y^{(1)})$ and $(x^{(i)}, y^{(i)})$ for $i = 4, 5, 6, 7$;*
- *If $\theta = \theta_c$, the system has six solutions: $(x^{(1)}, y^{(1)})$ and $(x^{(i)}, y^{(i)})$ for $i = 3, 4, 5, 6, 7$;*
- *If $\theta < \theta_c$, the system has seven solutions: $(x^{(i)}, y^{(i)})$ for $i = 1, 2, 3, 4, 5, 6, 7$.*

Here, the values $y^{(i)}$ for $i = 1, 2, 3$ (ordered as $y^{(3)} < y^{(2)} < y^{(1)}$, if they exist) are the roots of the cubic equation

$$\theta y^3 - y^2 + (\theta^2 + 1)y - 2\theta = 0,$$

which can be solved explicitly using Cardano's formula. For these roots, $x^{(1)} = x^{(2)} = x^{(3)} = 1$. The remaining $x^{(i)}$ and $y^{(i)}$ values for $i = 4, 5, 6, 7$ are given by:

$$x^{(4)} = \frac{1}{2}\left(\xi_2 - \sqrt{\xi_2^2 - 4}\right), \quad x^{(5)} = \frac{1}{2}\left(\xi_1 - \sqrt{\xi_1^2 - 4}\right),$$
$$x^{(6)} = \frac{1}{2}\left(\xi_1 + \sqrt{\xi_1^2 - 4}\right), \quad x^{(7)} = \frac{1}{2}\left(\xi_2 + \sqrt{\xi_2^2 - 4}\right),$$

where

$$\xi_1 = \frac{1-3\theta^2-\sqrt{(\theta-1)(\theta^3+\theta^2+3\theta-1)}}{2\theta^2}, \quad \xi_2 = \frac{1-3\theta^2+\sqrt{(\theta-1)(\theta^3+\theta^2+3\theta-1)}}{2\theta^2},$$

and the corresponding $y^{(i)}$ *values are given by*

$$y^{(i)} = \frac{1}{\sqrt{\theta}}\sqrt{(1-\theta^2)x^{(i)}-\theta^2\left((x^{(i)})^2+1\right)}, \quad i=4,5,6,7.$$

3. The presence of the second-order phase transition

In this section, we analyze the thermodynamic behavior of the three-state SOS model on the binary Cayley tree using the recursive framework established in the previous section. By utilizing the explicit expressions derived from the recurrence relations, we compute key physical observables such as the local magnetization and the quadrupolar moment. Using these, we show that there is a second-order phase transition.

3.1. **Local Magnetization.** The local magnetization is defined by

$$M = Z^{-1}\sum_{s_0} s_0 \exp\{-\beta H\}, \tag{3.1}$$

where the sum is taken over the possible spin states at the root of the Cayley tree.

Using the representation of the recursion relations given in Eqs. (2.2)–(2.6), the spontaneous magnetization per site is obtained as

$$M = \frac{1-x^3}{x^3+y^3+1}, \tag{3.2}$$

where (x, y) is any solution of the fixed point equations (2.7).

We denote by M_i the magnetization corresponding to the fixed point $(x^{(i)}, y^{(i)})$, $i = 1, \ldots, 7$, i.e.,

$$M_i = \frac{1-(x^{(i)})^3}{(x^{(i)})^3+(y^{(i)})^3+1}, \qquad i = 1, \ldots, 7. \tag{3.3}$$

Note that for $i = 1, 2, 3$, we have $x^{(i)} = 1$, hence $M_i = 0$ for these indices. In Fig. 2, plots of magnetizations M_i for $i = 4, 5, 6, 7$ are drawn.

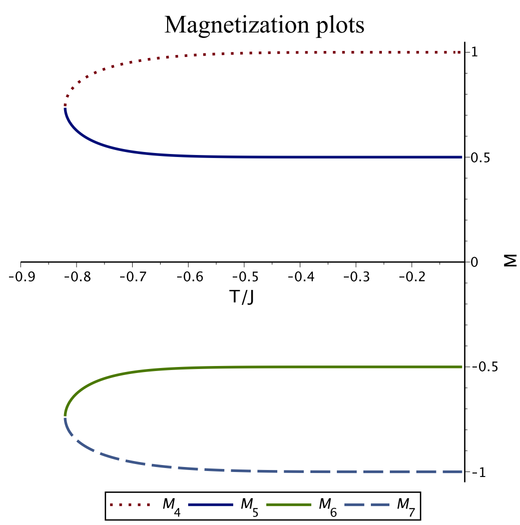

Figure 2. Plots of the magnetizations M_i for $i = 4, \ldots, 7$ are presented. At low temperatures and for $J < 0$, the magnetizations M_4, M_5, M_6, and M_7 take nonzero values with opposite signs. As the temperature increases, their magnitudes gradually decrease and eventually vanish continuously at the critical temperature.

Equation (3.2) shows that the local magnetization has zero value only when $x = 1$. Therefore, the second-order phase transition can be realized at some value of the temperature only for $x = 1$ with a corresponding definite value of y. The model exhibits a second-order phase transition at the critical point θ_c' (≈ 0.2956), where for $\theta < \theta_c'$, there are two stable fixed points with $x \neq 1$, corresponding to nonzero local magnetization.

Now, we show that the system (2.7) has two stable fixed points with $x \neq 1$. Note that the Jacobian at a fixed point (x, y) of (2.7) can be calculated as follows (see [12]):

$$\mathbb{J}(x,y,\theta) = \begin{pmatrix} \frac{-2x(\theta^2-1)(\theta y^2+\theta^2+1)}{(\theta^2x^2+\theta y^2+1)^2} & \frac{2\theta y(\theta^2-1)(x^2-1)}{(\theta^2x^2+\theta y^2+1)^2} \\ \frac{-2x\theta(\theta^2-1)}{(\theta^2x^2+\theta y^2+1)^2} & \frac{-2y\theta(\theta^2-1)}{(\theta^2x^2+\theta y^2+1)^2} \end{pmatrix}. \tag{3.4}$$

We find the eigenvalues of the matrix (3.4):

$$\lambda_\pm(x,y,\theta) = \frac{-B \pm (\theta^2-1)\sqrt{D}}{A},$$

where

$$A = A(x,y,\theta) := \left(\theta^2x^2 + \theta\, y^2 + 1\right)^2,$$

$$B = B(x,y,\theta) := \theta\, xy^2 + \theta^2 x + \theta\, y + x,$$

$$\begin{aligned} D = D(x,y,\theta) :=& \theta^4x^2 + \left(2\,x^2y^2 - 2\,xy\right)\theta^3 + \left(2\,x^2y^2 - 2\,xy\right)\theta + x^2 + \\ &+ \left(x^2y^4 - 2\,xy^3 + y^2 + \left(-4\,x^3 + 4\,x\right)y + 2\,x^2\right)\theta^2 \end{aligned}$$

By Lemma 2.1, it is known that $x^{(i)} \neq 1$, $i = 4, 5, 6, 7$. Note that the functions $\lambda_\pm$ at fixed points depend solely on θ and do not involve any additional parameters. From the graphs (see Figs. 3 and 4), one can observe that

- $|\lambda_\pm(x^{(4)}, y^{(4)}, \theta)| < 1$ for $\theta < \theta_c'$;
- $|\lambda_\pm(x^{(5)}, y^{(5)}, \theta)| < 1$ for $\theta < \theta_c'$;
- $|\lambda_+(x^{(6)}, y^{(6)}, \theta)| < 1$ for $\theta < \theta_c'$ and $|\lambda_-(x^{(6)}, y^{(6)}, \theta)| > 1$ for $\theta < \theta_c'$;
- $|\lambda_+(x^{(7)}, y^{(7)}, \theta)| < 1$ for $\theta < \theta_c'$ and $|\lambda_-(x^{(7)}, y^{(7)}, \theta)| < 1$ for $\theta < \widehat{\theta}$ (≈ 0.2949) and $|\lambda_-(x^{(7)}, y^{(7)}, \theta)| \geq 1$ for $\widehat{\theta} \leq \theta < \theta_c'$.

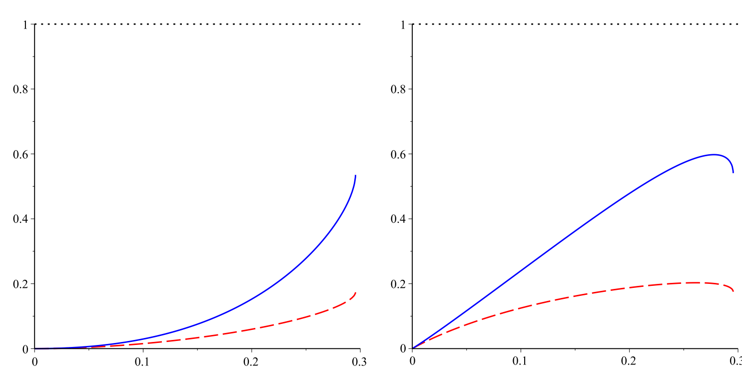

Figure 3. Plots of the functions (left) $\mid \lambda_\pm(x^{(4)}, y^{(4)}, \theta) \mid$ for $\theta < \theta_c'$ and (right)$\mid \lambda_\pm(x^{(5)}, y^{(5)}, \theta) \mid$ for $\theta < \theta_c'$.

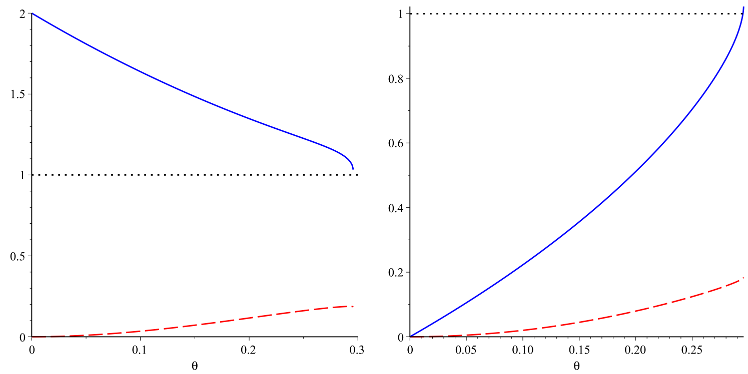

Figure 4. Plots of the functions (left) $\mid \lambda_{\pm}(x^{(6)}, y^{(6)}, \theta) \mid$ for $\theta < \theta_c'$ and (right)$\mid \lambda_{\pm}(x^{(7)}, y^{(7)}, \theta) \mid$ for $\theta < \theta_c'$.

Thus, we conclude that the model exhibits the second-order phase transition.

3.2. **Quadrupolar Moment.** In the subsection, we calculate quadrupolar moment of the model.

The quadrupolar moment is defined by

$$Q = Z^{-1} \sum_{s_0} s_0^2 \exp\{-\beta H\}. \tag{3.5}$$

Using the same recursive framework, the quadrupolar moment is given by

$$Q = \frac{1 + x^3}{x^3 + y^3 + 1}, \tag{3.6}$$

where (x, y) is a fixed point of Eq. (2.7).

Let Q_i denote the quadrupolar moment corresponding to $(x^{(i)}, y^{(i)})$, $i = 1, \ldots, 7$, i.e.,

$$Q_i = \frac{1 + (x^{(i)})^3}{(x^{(i)})^3 + (y^{(i)})^3 + 1}, \qquad i = 1, \ldots, 7. \tag{3.7}$$

One can show that $Q_4 = Q_7$ and $Q_5 = Q_6$ for $\theta \le \theta_c'$. In Fig 5, plots of quadrupolar moments Q_i for $i = 1, \ldots, 5$ are drawn.

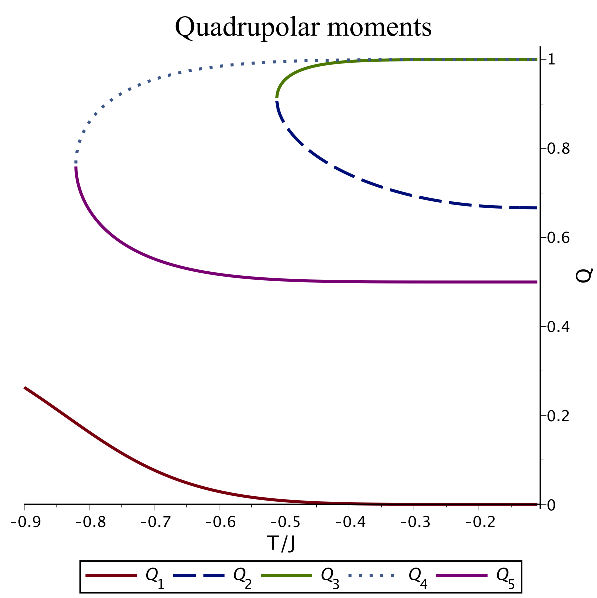

Figure 5. Plots of the quadrupolar moments Q_i for $i = 1, \ldots, 5$ are presented. At low temperatures and for $J < 0$, the quadrupolar moments Q_1, Q_2, Q_3, Q_4, and Q_5 take distinct nonzero values. As the temperature increases, they show a slight decrease.

4. Dynamical analysis

In this section, we investigate the regions where the model exhibits chaotic or periodic behavior. This is achieved through the numerical computation of Lyapunov exponents and visualization of the iteration dynamics via cobweb diagrams.

4.1. Cobweb Diagrams and Map Iterations.

We begin by analyzing the recurrence equation (2.7) under the simplifying assumption $x = 1$. Under this constraint, the equation reduces to the following rational map:

$$y = f(y) = \frac{y^2 + 2\theta}{\theta y^2 + \theta^2 + 1}. \tag{4.1}$$

The stability of fixed points in such dynamical systems is a key factor in understanding the nature of phase transitions. Cobweb diagrams and iterative maps are classical numerical tools used to study the qualitative behavior of these systems.

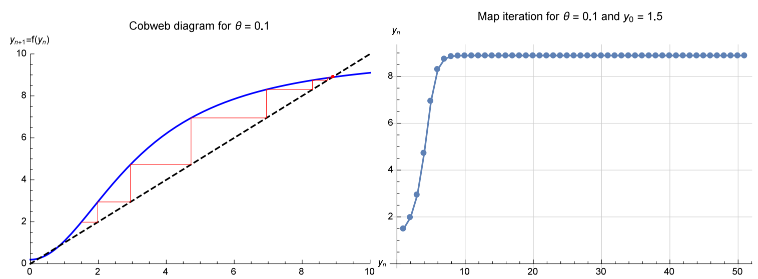

Figure 6. Cobweb diagram and iteration map for the dynamical system (4.1) with parameters $\theta = 0.1$, $y_0 = 1.5$. The system is iterated 50 times. Three distinct fixed points are observed, one of which is repelling.

Figure 6 shows the cobweb and iteration plots generated using *Mathematica* [13] for $\theta = 0.1$ and initial condition $y_0 = 1.5$. The iteration reveals three fixed points, with one repelling and two attracting.

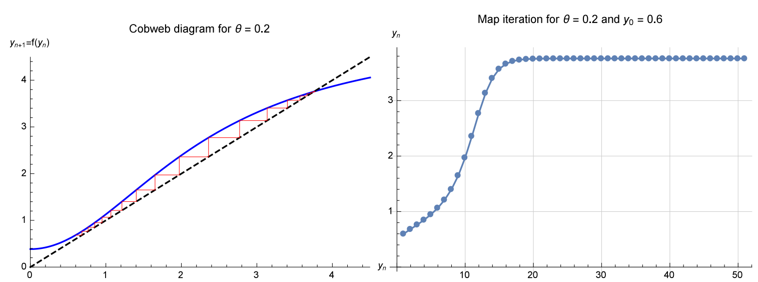

Figure 7. Cobweb diagram and iteration map for the dynamical system (4.1) with parameters $\theta = 0.2$, $y_0 = 0.6$. The system is iterated 50 times. A single attracting fixed point is observed.

In Figure 7, for $\theta = 0.2$ and $y_0 = 0.6$, the system converges to a unique attracting fixed point, highlighting parameter sensitivity in the system's long-term behavior.

4.2. Lyapunov Exponent.

To further assess the stability and possible chaotic behavior of the system, we compute the Lyapunov exponent. This quantity quantifies the average exponential rate of divergence (or convergence) of nearby trajectories in phase space [14, 15]. For the rational map (4.1),

$$y_n = f(y_{n-1}) = \frac{y_{n-1}^2 + 2\theta}{\theta y_{n-1}^2 + \theta^2 + 1}, \tag{4.2}$$

the Lyapunov exponent is defined as

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln |f'(y_n)|, \tag{4.3}$$

where the derivative of $f(y)$ is given by

$$f'(y) = \frac{2y(1-\theta^2)}{(\theta y^2+\theta^2+1)^2}. \tag{4.4}$$

Thus, the Lyapunov exponent becomes

$$\lambda = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \ln\left|\frac{2(1-\theta^2)y_n}{(\theta y_n^2+\theta^2+1)^2}\right|. \tag{4.5}$$

To numerically compute λ, the map is iterated for different values of θ, with transient dynamics discarded, and the long-term average evaluated. This provides insight into the periodic or chaotic nature of the system.

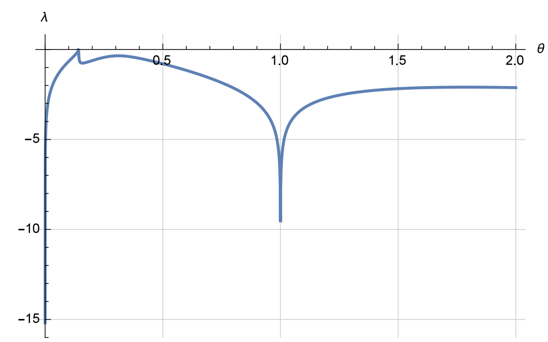

Figure 8. Lyapunov exponent λ of the rational map (4.2) as a function of the parameter θ. Each data point is based on 1000 iterations. The negative values of λ indicate stable periodic behavior.

As seen in Fig. 8, the Lyapunov exponent remains negative across the range of θ, indicating that the system exhibits periodic (non-chaotic) behavior. These findings are consistent with previous studies of lattice models on Cayley trees [16, 17], where the dynamical systems were found to be predominantly regular, with chaotic behavior being rare or absent.

5. Conclusion

In this paper, we analyzed the three-state SOS model on the binary Cayley tree using a recursive approach. Specifically, we studied the temperature dependence of magnetization and quadrupolar moments in the case $J < 0$. Our results show that M_1, M_2, and M_3 remain zero at all temperatures, while nonzero magnetization appears only for M_4, M_5, M_6, and M_7. These exhibit opposite signs at low temperatures, indicating antiferromagnetic order. As temperature increases, the magnetizations decrease and vanish continuously at the critical point, confirming a second-order phase transition.

The quadrupolar moments behave differently: they stay finite at low temperatures, decrease slightly with increasing temperature, but do not vanish at the critical point. This demonstrates that once magnetic order is lost, a nematic (hidden) phase still persists in the system. Hence, for $J < 0$, the three-state SOS model exhibits the following sequence of phases with increasing temperature: antiferromagnetic $\to$ nematic $\to$ paramagnetic.

A dynamical systems analysis further shows that the model's behavior is periodic, with the Lyapunov exponent remaining negative across all values of θ, confirming the absence of chaos.

Acknowledgments. We thank the referees for the careful reading of the manuscript and especially for a number of suggestions that have improved the paper.

References

[1] Jahnel B., Rozikov U., Three-state p-SOS models on binary Cayley trees. J. Stat. Mech. Theory Exp. (2024) Vol. 2024, Iss. 11,

[2] Karshiboev O., Periodic Gibbs measures for the three-state SOS model on a Cayley tree with a translation-invariant external field. Theor. Math. Phys. (2022) Vol. 212, P. 1276–1283.

[3] Kuelske C., Rozikov U., Extremality of translation-invariant phases for a three-state SOS model on the binary tree. J. Stat. Phys. (2015) Vol. 160, P. 659–680.

[4] Rahmatullaev M., Abraev B., Non-translation invariant Gibbs measures of an SOS model on a Cayley tree. Rep. Math. Phys. (2020) Vol. 86, Iss. 3, P. 345–360.

[5] Rahmatullaev M., Karshiboev O., Gibbs measures for the three-state SOS model with external field on a Cayley tree. Positivity, (2022),

[6] Rozikov U., Suhov Y., Gibbs measures for SOS models on a Cayley tree. Inf. Dim. Anal. Quant. Prob. Rel. Fields, (2006),

[7] Rozikov U., Gibbs Measures on Cayley Trees. World Scientific, Singapore, (2013).

[8] Baxter R., Exactly Solved Models in Statistical Mechanics. Academic Press, London/New York, (1982).

[9] Friedli S., Velenik Y., Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction. Cambridge University Press, (2017).

[10] Sinai Y., Theory of Phase Transitions: Rigorous Results. Science, Moscow, (1980).

[11] Mukhamedov F., Extremality of disordered phase of λ-model on Cayley tree. Algorithms, (2022) Vol. 15, P. 18.

[12] Rozikov U., An Introduction to Mathematical Billiards. World Scientific, Hackensack, NJ, (2019).

[13] Wolfram Research I., Mathematica, Version 8.0. Wolfram Research, Inc., Champaign, IL, (2010).

[14] Alligood K., Sauer T., Yorke J., Chaos: An Introduction to Dynamical Systems. Springer, (2000).

[15] Devaney R., An Introduction to Chaotic Dynamical Systems. Benjamin-Cummings, Menlo Park, CA, (1986).

[16] Akin H., Calculation of some thermodynamic quantities for the Ising model on a kth order Cayley tree. Physica B, (2023) Vol. 662,

[17] Akin H., On the periodicity of the rational dynamical system corresponding to the Vannimenus-Ising model. J. Comput. Nonlinear Dynam. (2023) Vol. 18, Iss. 1,

Rahmatullaev M.M.,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
e-mail: mrahmatullaev@rambler.ru

Karshiboev O.Sh.,
Oriental university, Samarkand campus, Uzbekistan
e-mail: okarshiboevsher@mail.ru

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 174–178
DOI: 10.29229/uzmj.2026-1-19

Normality of quasitrace and AW*-completion of the real C*-subalgebras

Rakhmonova N.

Abstract. The paper studies quasitraces on real C*-algebras, and AW*-completion of C*-subalgebras with respect to the d_τ-metric generated by quasitrace τ. It is proved that the d_τ-closure of unital real C*-subalgebra B of real C*-algebra R is the smallest real AW*-subalgebra of R containing B. To prove this, it was necessary to obtain a key result concerning the maximal Abelian self-adjoint subalgebra (masa), in connection with which Abelian algebras are studied separately. It is proved that for a compact Hausdorff space X the algebra $C_r(X)$ of all continuous real functions on X is a real abelian AW*-algebra if and only if X is Stonean. Moreover, it has been proven that a unital real C*-algebra is a real AW*-algebra if and only if every masa has Stonean spectrum, and this is equivalent to the fact that every masa is monotone complete.

Keywords: Real C*-algebras, AW*-algebras, quasitrace, monotone completeness, maximal abelian self-adjoint subalgebra (masa).
MSC (2020): 46L10, 46K10, 46L05

1. Introduction

The theory of AW*-algebras, originally introduced by Kaplansky as a generalization of von Neumann algebras without assuming a distinguished Hilbert space, continues to play an important role in the study of non-commutative measure theory, projection lattices, and completions of C*-algebras. While the complex case has been extensively developed over the past decades, the corresponding theory for real C*-algebras and real AW*-algebras remains less explored, despite growing interest in real structures in operator algebras, Jordan algebras, and applications in mathematical physics.

This paper is devoted to the study of quasitraces on real C*-algebras and to the description of AW*-completions of real C*-subalgebras with respect to the metric d_τ induced by a quasitrace τ. The central result establishes that, under suitable conditions, the d_τ-closure of a unital real C*-subalgebra B inside a real C*-algebra R coincides with the smallest real AW*-subalgebra of R containing B.

A key step in proving this statement requires a detailed understanding of maximal abelian self-adjoint subalgebras (masas) in the real setting. For this reason, the paper first investigates abelian real C*-algebras and their AW*-properties.

It is shown that, for a compact Hausdorff space X, the algebra $C_r(X)$ of all continuous real-valued functions on X is a real abelian AW*-algebra if and only if X is Stonean (extremely disconnected). Furthermore, a unital real C*-algebra is a real AW*-algebra if and only if every masa has Stonean spectrum; this condition is in turn equivalent to every masa being monotone complete.

These characterizations closely parallel well-known results in the complex setting [1], but their proofs in the real case require careful adaptation, often via the method of passing to the enveloping complex C*-algebra $R + iR$ and applying known complex results there.

The notion of quasitrace plays a crucial role throughout the work. We adopt the definition suitable for real C*-algebras and establish its close relationship with quasitraces on the complexification. Building on this, we study normality of quasitraces (in the sense of additivity over orthogonal projections) and prove that if the closed unit ball is complete with respect to the d_τ-metric, then the algebra is a real AW*-algebra and the quasitrace is normal.

Finally, combining these tools, we obtain the main theorem concerning the AW*-completion in the real finite setting (when both R and its complexification are AW*-algebras), which provides a natural real analogue of corresponding completion results in the complex theory [1, 2].

The proofs rely heavily on the enveloping complex algebra technique, together with results from [1, 2] and structural properties of real AW*-algebras developed in [3, 4, 5]. The proofs of basically all the results were obtained by the so-called method of enveloping (complex) algebra. In this case, the results from the papers [1] and [2] were used, in which the above results were obtained in the complex case.

1.1. **Preliminaries.** A Banach *-algebra A over a field $\mathbb{C}$ is called a *C*-algebra* if $\|x^*x\| = \|x\|^2$, for any $x \in A$. By a *real C*-algebra* we mean a real Banach *-algebra A such that the relation $\|x^*x\| = \|x\|^2$ holds and the element $1 + x^*x$ is invertible for any $x \in A$. Let A be a ring and S a non-empty subset of A. Assume that $R(S) = \{x \in A \mid sx = 0 \text{ for all } s \in S\}$ and call $R(S)$ the right annihilator of S. Similarly, $L(S) = \{x \in A \mid xs = 0 \text{ for all } s \in S\}$ denotes the left annihilator of S. A *Baer *-ring* is a ring A such that for every non-empty subset S of A, $R(S) = gA$ for a suitable projection g. The equality $L(S) = ((R(S^*)))^* = ((hA))^* = Ah$ (for some projection h) shows that this definition can also be given through the left annihilator. *AW*-algebra* is a C*-algebra, which is also a Baer *-ring.

Let A be a C*-algebra with identity and let A_h be its Hermitian part. A *quasitrace* τ on A is a function $\tau : A \to \mathbb{C}$ satisfying the conditions:

(*i*) $\overline{\tau}(x^*x) = \overline{\tau}(xx^*) \geq 0$, for all $x \in A$;

(*ii*) $\overline{\tau}(a + ib) = \overline{\tau}(a) + i\overline{\tau}(b)$, for all $a, b \in A_h$;

(*iii*) $\overline{\tau}$ is linear on any abelian C*-subalgebra B of A.

Definition 1.1. Let R be a unital real C*-algebra. A *quasitrace* τ on R is a function $\tau : R \to \mathbb{R}$ that satisfies:

(i') $\tau(x^*x) = \tau(xx^*) \geq 0$, for $x \in R$;

(ii') $\tau(a + b) = \tau(a)$, for $a \in R_h$, $b \in R_k$, where $R_k = \{b = -b^*, b \in R\}$;

(iii') τ is linear on any abelian C*-subalgebra B of R.

Theorem 1.2. *[5] If $\overline{\tau}$ is a quasitrace on the C*-algebra $A = R + iR$, then its restriction to the real C*-algebra R, defined as $\tau(a + b) = \overline{\tau}(a)$, $a \in R_h, b \in R_k$ is a quasitrace on R.*

Conversely, if τ is a quasitrace on R, then its extension $\overline{\tau}$ to $A = R + iR$, defined as $\overline{\tau}(x + iy) = \tau(x) + i\tau(y)$, is a quasitrace on A, where $x, y \in R$.

2. Abelian and monotone complete AW*-algebras.

Proposition 2.1. *Let R be a real AW*-algebra. Let $Q \subseteq R$ be a *-subalgebra, and let Q' be the relative commutant of Q, that is, $Q' = \{x \in R : xy = yx, \forall y \in Q\}$.*

(1) *If $Q = Q''$, then Q is a real AW*-subalgebra.*

(2) *The center $Z(R)$ of R is a real AW*-subalgebra.*

Proof. 1) Let $N = Q + iQ$. Then $N' = Q' + iQ'$ (see [4]), therefore $N'' = Q'' + iQ'' = Q + iQ = N$. By [1, Proposition 1.8] N is AW*-subalgebra of $R + iR$, hence by [3, Proposition 4.3.1] Q is a real AW*-subalgebra of R.

2) By [1, Proposition 1.8] $Z(R + iR)$ is AW*-subalgebra of $R + iR$, therefore $Z(R)$ is a real AW*-subalgebra of R. □

Definition 2.2. A compact Hausdorff space X is called *Stonean* (or extremely disconnected) if the closure of every open set is open again. X is called *Hyperstonean* if it is Stonean (i.e., extremely disconnected) and the support of every positive Radon measure on X is clopen.

Theorem 2.3. *Let X be Stonean, and let $C_r(X)$ be the algebra of all continuous real functions on X. Then $C_r(X)$ is a real AW*-algebra.*

Proof. Let's consider algebra $C(X)$ of all continuous complex function on X. It is easily to see that $C(X) = C_r(X) + iC_r(X)$. By [1, Theorem 1.10] $C(X)$ is AW*-algebra. Then by [3, Proposition 4.3.1] $C_r(X)$ is a real AW*- algebra. □

Theorem 2.4. *Let X be compact Hausdorff space. If $C_r(X)$ is a real abelian AW*-algebra such that $C(X) = C_r(X) + iC_r(X)$ AW*-algebra, then X is Stonean.*

Proof. Since $C(X)$ is AW*-algebra, by [1, Theorem 1.11] the space X is Stonean. □

Remark 2.5. Following the same scheme of proof of [1, Theorem 1.11], one can prove Theorem 2.4 without the assumption that $C(X) = C_r(X) + iC_r(X)$ is AW*-algebra.

Let us recall that a complex or real C*-algebra called *monotone complete* if every upward directed and norm-bounded set of self-adjoint elements has a least upper bound.

Proposition 2.6. *Let R be a real C*-algebra. If $A = R + iR$ is monotone complete, then R is monotone complete.*

Proof. Let $(a_n) \subset R$ be a bounded monotone increasing sequence of self-adjoint elements. Since the algebra A is monotone complete, then the sequence $(a_n) \subset A$ has a least upper bound in A, which we denote by a. Since $a_n \in R$ $(n \in \mathbb{N})$, then $a \in R$, hence R is monotone complete. □

Theorem 2.7. *If X is Stonean space, then $C_r(X)$ is monoton complete.*

Proof. By [1, Theorem 1.13] *-algebra $C(X)$ is monotone complete. Then by Proposition2.6, $C_r(X)$ is also monoton complete. □

In the future, the *maximal abelian self-adjoint subalgebra* is briefly written as *masa*. Let us present one auxiliary result.

Lemma 2.8. *Let R be a unital real C^*-algebra such that every maximal abelian self-adjoint subalgebra (masa) is monotone complete. Let P be a family of commuting projections and L be the set of all projections that are lower bounds for P. Then:*

(1) *L is upward directed.*

(2) *P has the greatest lower bound.*

Proof. The proof of the lemma is similar to the proof of [1, Lemma 1.14]. □

Theorem 2.9. *Let X be a compact Hausdorff space. Then $C_r(X)$ is isomorphic to a real von Neumann algebra if and only if X is a Hyperstonean space.*

Proof. Since $C(X) = C_r(X) + iC_r(X)$ is isomorphic to a (complex) von Neumann algebra, then by [6, Theorem 1.18] X is a Hyperstonean space, and conversely. □

Theorem 2.10. *Let R be a unital real C*-algebra. Then the following are equivalent:*

1) *R is a real AW^*-algebra;*

2) *every masa has Stonean spectrum;*

3) *every masa is monotone complete.*

Proof. 1) $\Rightarrow$ 2). Let $Q \subseteq R$ be a masa. Then $Q = Q' = Q''$, hence by Proposition2.1, Q is a real AW*-subalgebra. From the Theorem 2.4 , we know that the spectrum X is Stonean.

2) $\Rightarrow$ 3). This follows from Theorem 2.7.

Further, the equivalence of these conditions to condition 1) is shown similarly to the complex case. □

3. Normality of τ and AW*-completion of the *-subalgebras with respect to the d_τ-metric.

Lemma 3.1. *Let R be a real C^* algebra and τ a faithful quasitrace on R. Then the closed unit ball of A is closed in d_τ.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the closed unit ball of R, converging to x in d_τ. Consider the sequence $a_n = x_n^* x_n$ and the element $a = x^*x$. Since the product is continuous in d_τ on norm-bounded sets, it is obvious that the sequence a_n converges to a in d_τ, and we can also deduce that for every $p \in \mathbb{N}$, the sequence a_n^p converges to a^p in d_τ. By continuity we obtain $\tau(a_n^p) \to \tau(a^p)$ for every $p \in \mathbb{N}$. Let μ_n be the measure on the spectrum $\sigma(a_n)$ given by the linear functional $\tau|_{C^*(a_n,1)}$, and let μ be the measure on the spectrum $\sigma(a)$ given by the linear functional $\tau|_{C^*(a,1)}$. We can consider all the measures as measures in the interval $J = [0, \max\{1, \|a\|\}]$, because all a_n are in the closed unit ball of A. Since $\tau(a_n^p) \to \tau(a^p)$ for all $p \in \mathbb{N}$, we see that μ_n converges to μ in the w^*-topology on $C(J)^*$. Furthermore, μ_n has support in $[0, 1]$ for all $n \in \mathbb{N}$, hence, μ has also support in $[0, 1]$. From the fact that τ is faithful, we obtain $\mathrm{supp}(\mu) = \sigma(a)$, and then the C^*-equation gives $\|x\|^2 = \|a\| \leq 1$. □

Let us recall that a quasitrace τ is called *normal* if for every orthogonal family of projections $(p_i)_{i\in I}$ the following holds: $\tau\Big(\sup_{i\in I} p_i\Big) = \sum_{i\in I} \tau(p_i)$. Put $\|x\|_{2,\tau} = \tau(x^*x)^{1/2}$ and $d_\tau(x, y) = ||x - y||_2^{2/3}$, $x, y \in A$. Then d is a metric on A (see [1],[2]).

Proposition 3.2. *Let R be a real C*-algebra and τ a faithful normalized trace on R. If the closed unit ball of R is complete in the $\|\cdot\|_{2,\tau}$-norm, then R is a real von Neumann algebra and τ is normal.*

Proof. Let $A = R + iR$. It is not difficult to show that the closed unit ball of A is also complete in the $\|\cdot\|_{2,\overline{\tau}}$-norm. Then by [1, Lemma 2.20] A is a (complex) von Neumann algebra and $\overline{\tau}$ is normal. Hence R is a real von Neumann algebra and τ is normal. □

Theorem 3.3. *Let R be a real C*-algebra and τ a faithful quasitrace on R. If the closed unit ball of R is complete in d_τ, then R is an real AW*-algebra and τ is normal.*

Proof. From the Theorem 2.10, we know that it suffices to show that every masa has Stonean spectrum. So, let Q be a masa in R. By Lemma 3.1 the closed unit ball of B is closed in d_τ, therefore it is also complete in d_τ. Since τ is linear on B, $\|.\|_{2,\tau}$ is a norm, and the closed unit ball is also complete in this norm. By Proposition 3.2 B is a real von Neumann algebra, and $\tau|_B$ is normal. By Theorem 2.9 B has Hyperstonean spectrum, in particular, it is Stonean. Then by Theorem 2.10 R is a real AW*-algebra. The normality of τ on every masa as a trace ensures that τ is also normal as a quasitrace. □

Now we will prove one of the main results of the paper.

Theorem 3.4. *Let R be a real finite AW*-algebra such that $R + iR$ is AW*-algebra. If B is a unital real C*-subalgebra of R, then the d_τ-closure of B is the smallest real AW*-subalgebra of R containing B.*

Proof. Let τ a faithful normal quasitrace on R and let $\overline{\tau}$ be its extension to $A = R + iR$, which is also a faithful normal. Let $B_c = B + iB$. By [1, Theorem 2.26] algebra $\overline{B_c}^{d_{\overline{\tau}}} = \overline{B}^{d_\tau} + i\overline{B}^{d_\tau}$ is the smallest (complex) AW*-subalgebra of A containing B_c. Then by [3, Proposition 4.3.1] $\overline{B}^{d_\tau}$ is a real AW*-subalgebra of R, and in view of the above $\overline{B}^{d_\tau}$ is the smallest real AW*-subalgebra of R containing B. □

References

[1] Fehlker F., Quasitrace and AW*-bundles. University of Muenster, Muenster, (2018).

[2] Haagerup U., Quasitraces on exact C*-algebras are traces. C.R. Math. Acad. Sci. Soc. R. Can. (2014) Vol. 36, Iss. 2, P. 67–92.

[3] Ayupov S., Rakhimov A., Real W*-algebras. Actions of groups and Index theory for real factors. VDM Publishing House Ltd., Beau-Bassin, (2010).

[4] Ayupov S., Rakhimov A., Usmanov S., Jordan, Real and Lie Structures in Operator Algebras. Kluwer Academic Publishers, Dordrecht, (1997).

[5] Kim D., Rakhimov A., Quasitraces on real C*- and AW*-algebras. Uzbek Mathematical Journal, (2025), Iss. 3, P. 121–125.

[6] Takesaki M., Theory of Operator Algebras. Springer-Verlag, New York, (1979).

Nilufarkhon Rakhmonova,
Department of Digital technologies and Mathematics,
Kokand university, Kokand, Uzbekistan
e-mail: rahmonovanilufar406@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 179–182
DOI: 10.29229/uzmj.2026-1-20

Quasitraces on exact real C*-algebras are traces

Ramazonova L., Rakhimov A.

Abstract. In this paper, n-quasitraces on real C*-algebras are studied. It is proved that if R is a real C*-algebra, then the natural extension of an n-quasitrace of R to $R + iR$ is also an n-quasitrace, and conversely, the restriction of an n-quasitrace from $R + iR$ to R is also an n-quasitrace. In 1982, Blackadar and Handelman proved that every quasitrace on an AW*-algebra is a 2-quasitrace. In this paper, a real analogue of that result is obtained. However, in the general case (i.e., for C*-algebras), this result does not hold. Using Kirchberg's example – where a unital C*-algebra and its quasitrace are constructed such that the quasitrace is not a 2-quasitrace (and therefore is not a trace) – a similar example is constructed in the real case. The paper also studies the properties of 2-quasitrace on real C*-algebras. As is known, Kaplansky asked whether every (2-) quasitrace on a C*-algebra linear, i.e., a trace. This question remains open to this day. Haagerup has a positive answer to this question in the case where the C*-algebra is unital and exact. In this paper, a real analogue of Haagerup's result is proved.

Keywords: C*-algebras, AW*-algebras, quasitraces on C*- and AW*-algebras
MSC (2020): 46L10, 47C15

1. Introduction

It is known that AW*-algebras are a generalization of W*-algebras (von Neumann algebras), and naturally, the question arises about the generalization of results obtained for W*-algebras to AW*-algebras, which is quite relevant. It is also pertinent to investigate under which conditions (or which) AW*-algebras are W*-algebras. As is well know, in the study and classification W*-algebras, the concept of a trace–alongside the role of projections–plays a significant part. For example, as shown by Takesaki, a W*-algebra is finite if and only if there exists a separating family of finite normal traces on it. This illustrates why C*-algebras are relatively less studied: some of them do not even have non-trivial projections, let alone traces.

On the other hand, AW*-algebras are relatively better studied because these algebras possess a sufficient number of projections. However, there are also problems concerning traces for these algebras. In some works (for example, those by Wright), the existence of a trace on an AW*-algebra is assumed for convenience.

In 1982, Blackadar and Handelman introduced an analogue of the trace called a quasitrace. Although this concept does not fully replace the trace, it has allowed some researchers to obtain results that are analogous to those available for traces. In partcular, U. Haagerup studied certain properties of quasitraces and proved that, in an exact C*-algebra, every quasitrace is in fact a trace.

In this paper, we study quasitraces on real C*-algebras and obtain a real analogue of Haagerup's result.

2. Preliminaries

A complex Banach *-algebra A is called *C*-algebra* if $\|x^*x\| = \|x\|^2$ for all $x \in A$. By a *real C*-algebra* we mean a real Banach *-algebra R such that the relation $\|a^*a\| = \|a\|^2$ holds and the element $1 + a^*a$ is invertible for any $a \in R$ (see [1], [2]). A bijective linear mapping $\alpha : A \to A$ is called a **-antiautomorphism*, if $\alpha(x^*) = \alpha(x)^*$ and $\alpha(xy) = \alpha(y)\alpha(x)$, for all $x, y \in A$. A mapping α is called *involutive* if $\alpha^2 = id$. It is directly shown that a real C*-algebra R generates a natural involutive *-antiautomorphism of $A = R + iR$, namely $\alpha(x + iy) = x^* + iy^*$, where $x, y \in R$. It is clear that $R = \{z \in A : \ \alpha(z) = z^*\}$. Conversely, given a C*-algebra A and any involutive *-antiautomorphism α on A, the set $\{z \in A : \ \alpha(z) = z^*\}$ is real C*-algebra.

Let A be an *-algebra and let S be a nonempty subset of A. The sets $R(S) = \{x \in A \ : \ sx = 0$ for all $s \in S\}$ and $L(S) = \{x \in A \ : \ xs = 0$ for all $s \in S\}$ are called the right-annihilator and the left-annihilator of S, respectively. An *-algebra A is called a Baer *-algebra if for any non-empty $S \subset A$ we have $R(S) = gA$ for an appropriate projection g. Since $L(S) = (R(S^*))^* = (hA)^* = Ah$ the definition is symmetric and can be given in terms of the left-annihilator and a suitable projection h. Here $S^* = \{s^* | \ s \in S\}$. A real (or complex) C*-algebra R which is a Baer *-algebra is called a real (or complex) AW*-algebra. A linear functional τ on A is called *positive* if $\tau(x^*x) \geq 0$ for all $x \in A$. A positive linear functional with $\|\tau\| = 1$ is called a *state*. A state is called *trace* if it satisfies the condition $\tau(xy) = \tau(yx)$, for all $x, y \in A_+$.

Let everywhere R be a unital real C*-algebra and $A = R + iR$ be an enveloping C*-algebra of R.

Definition 2.1. [3] A quasitrace τ on A is a function $\tau : A \to \mathbb{C}$ that satisfies the following conditions

(*i*) $\tau(x^*x) = \tau(xx^*) \geq 0$, $x \in A$;

(*ii*) $\tau(a + ib) = \tau(a) + i\tau(b)$, for $a, b \in A_h$;

(*iii*) τ is linear on an abelian C*-subalgebra B of A.

Furthermore, τ is called an $n-$*quasitrace* for $n \in \mathbb{N}, n \geq 2$ if there exists a 1-quasitrace $\tau_n : M_n(A) \to \mathbb{C}$ such that $\tau(x) = \tau_n(x \otimes e_{11})$.

Definition 2.2. [4] A quasitrace τ on R is a function $\tau : R \to \mathbb{R}$ that satisfies the following conditions

(*i′*) $\tau(x^*x) = \tau(xx^*) \geq 0$, $x \in R$;

(*ii′*) $\tau(a + b) = \tau(a)$, for $a \in R_h, b \in R_k$;

(*iii′*) τ is linear on an abelian C*-subalgebra B of R.

Furthermore, τ is called an $n-$*quasitrace* for $n \in \mathbb{N}, n \geq 2$ if there exists a 1-quasitrace $\tau_n : M_n(R) \to \mathbb{R}$ such that $\tau(x) = \tau_n(x \otimes e_{11})$.

We can see that definitions of quasitrace in real and complex cases are slightly different. In the next two theorems we naturally consider the *restriction* of a quasitrace from A to R, and conversely, the *extension* of a quasitrace from R to A

Theorem 2.3. *[4] If $\overline{\tau}$ is a quasitrace on the C*-algebra $A = R + iR$, then its restriction to the real C*-algebra R, defined as $\tau(a + b) = \overline{\tau}(a)$, $a \in R_h, b \in R_k$ is a quasitrace on R.*

Conversely, If τ is a quasitrace on R, then its extension $\overline{\tau}$ to $A = R + iR$, defined as $\overline{\tau}(x + iy) = \tau(x) + i\tau(y)$, is a quasitrace on A, where $x, y \in R$.

3. 2-quasitraces on a real C*-algebras.

Using Theorem 2.3 the following results are directly proved.

Proposition 3.1. *Let $\overline{\tau}$ be an n-quasitrace on A. Then $\tau(a + b) = \overline{\tau}(a)$, $a \in R_h$, $b \in R_k$ is an n-quasitrace on R.*

Proof. Let $\overline{\tau}$ be an n-quasitrace on A. Then according to the definition of n-quasitrace, there exists a 1-quasitrace $\overline{\tau}_n : M_n(A) \to \mathbb{C}$, such that

$$\overline{\tau}(x) = \overline{\tau}_n(x \otimes e_{11}), \qquad x \in A.$$

If we define the restriction of $\overline{\tau}_n$ to $M_n(\mathbb{R})$ as follows

$$\tau_n : M_n(R) \to \mathbb{R}, \ \tau_n((a + b) \otimes e_{11}) = \overline{\tau}_n(a \otimes e_{11}).$$

Hence $\tau : R \to \mathbb{R}$ and $\tau(a+b) = \tau_n((a+b) \otimes e_{11}) = \overline{\tau}_n(a \otimes e_{11}) = \overline{\tau}(a)$. Therefore, τ is an n-quasitrace on R. □

Proposition 3.2. *Let τ be an n-quasitrace on R. Then $\overline{\tau}(x + iy) = \tau(x) + i\tau(y)$, $x, y \in R$ is an n-quasitrace on A.*

Proof. Let τ be an n-quasitrace on R. According to the definition of n-quasitrace, there exists a 1-quasitrace $\tau_n : M_n(R) \to \mathbb{R}$, such that

$$\tau(c) = \tau_n(c \otimes e_{11}), \qquad c \in R.$$

So, we can extend τ to quasitrace $\overline{\tau}$ on $M_n(A)$. So $\overline{\tau}_n : M_n(A) \to \mathbb{C}$, $\overline{\tau}_n((x+iy)\otimes e_{11}) = \tau_n(x\otimes e_{11}) + i\tau_n(y\otimes e_{11})$. This $\overline{\tau}_n$ is a 1-quasitrace on $M_n(A)$. Then for arbitrary $z \in A$, we have

$$\begin{aligned}\overline{\tau}(z) &= \overline{\tau}(x+iy) = \tau(x) + i\tau(y) = \tau_n(x\otimes e_{11}) + i\tau_n(y\otimes e_{11}) \\ &= \overline{\tau}_n((x+iy)\otimes e_{11}) = \overline{\tau}(z\otimes e_{11}).\end{aligned}$$

Therefore, $\overline{\tau}$ is an n-quasitrace on A. □

In [3, Corollary II.1.10] Blackadar and Handelman proved that *every quasitrace on an AW*-algebra is a* 2*-quasitrace*. Below, we will prove a real analogue of this result.

Theorem 3.3. *Every quasitrace on a real AW*-algebra is a* 2*-quasitrace.*

Proof. Let τ be a quasitrace on a real AW*-algebra. Then, by Theorem 2.3, its extension $\overline{\tau}$ to $A = R + iR$, such that $\overline{\tau}(x+iy) = \tau(x) + i\tau(y),\ x, y \in R$ is a quasitrace on A. Then by [3, Corollary II.1.10] $\overline{\tau}$ is a 2-quasitrace on A. According to Proposition 3.1, τ is a 2-quasitrace on R, defined as $\tau(a+b) = \overline{\tau}(a),\ a \in R_h,\ b \in R_k$. □

In the general case, i.e. for C*-algebras, this result is not true. Kirchberg gave an example of a quasitrace on a unital C*-algebra which is not a 2-quasitrace (see [5, 23§, 1274p.]). In particular, this quasitrace is not a trace. Using Kirchberg's example we can construct its real analogue as follows.

Example 3.4. Let A be a unital C*-algebra, constructed in [5, 23§, 1274p.] and let $\overline{\tau}$ is a quasitrace on A, which is not a 2-quasitrace (therefore it is not a trace – nonlinear). Let $\alpha : A \to A$ be an arbitrary involutive *-antiautomorphism. Then consider a real C*-algebra $R = \{x \in A : \alpha(x) = x^*\}$. Then we have $A = R + iR$. By Theorem 2.3 $\tau(a+ib) = \overline{\tau}(a)$ $(a \in R_s, b \in R_k)$ is a quasitrace on R, which according to Propositions 3.1 and 3.2 is not a 2-quasitrace, therefore, it is also not a trace.

In [3, Proposition II.4.1] Blackadar and Handelman also proved, that *every 2-quasitrace is an n-quasitrace*, for every $n \in N$. We will prove the real analogue of this result in a more general form. More specifically, we will prove the real analogue of F.Fehlker's result [6, Corollary 2.11].

Theorem 3.5. *Let τ be a 2-quasitrace on a real C*-algebra R. Then*

(i) *τ is an n-quasitrace for every $n \in \mathbb{N}$;*

(ii) *τ is order-preserving on R_{sa};*

(iii) *τ is continuous;*

(iv) *τ is bounded.*

Proof. (i) By Proposition 3.2, $\overline{\tau}(x+iy) = \tau(x) + i\tau(y)$ $(x, y \in R)$ is a 2-quasitrace on A. Then by [6, Corollary 2.11], $\overline{\tau}$ is an n-quasitrace for every $n \in \mathbb{N}$. Therefore, according to Proposition 3.1, a quasitrace τ is an n-quasitrace on R.
(ii) Since $\overline{\tau}$ is a 2-quasitrace, by [6, Corollary 2.11] $\overline{\tau}$ is order-preserving on A_{sa}, therefore τ is order-preserving on R_{sa}.
(iii) By [6, Corollary 2.11] $\overline{\tau}$ is continuous, therefore, τ is also continuous.
(iv) By [6, Corollary 2.11] $\overline{\tau}$ is bounded, hence τ is also bounded. □

In [7, Corollary 6.4] Alex Gow demonstrated that if a unital C*-algebra A admits a 2-quasitrace $\tau : A \to \mathbb{C}$, then A also admits a tracial functional. In particular, this implies that τ is linear. We have proved this result in the real case as a theorem.

Corollary 3.6. *If a unital real C*-algebra R admits a 2-quasitrace $\tau : R \to \mathbb{R}$, then it admits a tracial functional. In particular, τ is linear.*

Proof. By Proposition 3.2, the extension $\overline{\tau}$ to A is a 2-quasitrace on A. By [7, Corollary 6.4], A admits a tracial functional and $\overline{\tau}$ is linear. Then τ is also linear. □

4. Quasitraces on exact real C*-algebras.

Recall that a sequence $0 \to M \to N \to F \to 0$ is called *exact*, if there exists a monomorphism f from the algebra M to N and an epimorphism g from the algebra N to F such that $Im(f) = Ker(g)$. A real or complex C*-algebra A is called *exact* if for all pairs (B, J) of a C*-algebra B and a closed two-sided ideal J in B, the sequence

$$0 \to A \otimes J \to A \otimes B \to A \otimes B/J \to 0$$

is exact.

Whether every (2-) quasitrace on a C*-algebra is linear, i.e. a trace, is a well-known open question (as asked by Kaplansky). Haagerup has a positive answer to this question in the case that C*-algebra is unital and exact [8, Theorem 5.11]. In [9, Lemma 6.2] was proved that a real C*-algebra R is exact if and only if $A = R + iR$ is exact. Using this below we will prove a real analogue of Haagerup's result.

Theorem 4.1. *Quasitraces on exact unital real C*-algebras are traces.*

Proof. Let R be an exact unital real C*-algebra and let τ be a quasitrace on R. Then by [9, Lemma 6.2], A is also exact. By Theorem 2.3, the extension $\overline{\tau}$ to $A = R + iR$ of τ is a quasitrace on A. By [8, Theorem 5.11] $\overline{\tau}$ is a trace on A. Then τ is also trace on R. □

References

[1] Li B., Real operator algebras. World Scientific Publishing Co. Pte. Ltd., (2003).

[2] Ayupov S., Rakhimov A., Usmanov S., Jordan, Real and Lie Structures in Operators Algebras. Kluwer Academic Publisher, (1997).

[3] Blackadar B., Handelman D., Dimension functions and traces on C*-algebras. J. Funct. Anal. (1982) Vol. 45, Iss. 3, P. 297–340.

[4] Kim D., Rakhimov A., Quasitraces on real C*- and AW*-algebras. Uzb. Math. Journal, (2025),

[5] Kirchberg E. The classification of purely infinite C*-algebras using Kasparov's theory. 2006.

[6] Fehlker F. Quasitraces and AW*-bundles. MA thesis. Munster, 2018, p. 27.

[7] Gow A., Every 2-quasitrace is a trace. arXiv:2501.13088v1 [math.OA], (2025),

[8] Haagerup U., Quasitraces on exact C*-algebras are traces. C. R. Math. Acad. Sci. Soc. R. Can. (2014) Vol. 36, Iss. 2-3, P. 67–92.

[9] Jeffrey L., Efren R., Stacey P., The classification of real purely infinite simple C*-algebras. Documenta Mathematica, (2011) Vol. 16, P. 639.

Ramazonova L.D.,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
e-mail: rlaylo2405@gmail.com

Rakhimov.A.A.,
National University of Uzbekistan,
Tashkent, Uzbekistan
e-mail: rakhimov@ktu.edu.tr

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 183–191
DOI: 10.29229/uzmj.2026-1-21

Periodic quasi Gibbs measures for the p-adic Potts model with an external field

Samijonova N.

Abstract. In the present paper, we study G_2-periodic p-adic quasi Gibbs measures for the p-adic Potts model with an external field on a Cayley tree of order two. We find G_2-periodic (non-translation-invariant) p-adic quasi Gibbs measures. Moreover, for the corresponding model, we show that if $|q(q-1)|_p = 1$, $\sqrt{1-q} \in \mathbb{Q}_p$ then a phase transition occurs; If $|q|_p < 1$, a quasi phase transition occurs.

Keywords: p-adic numbers, Potts model, external field, p-adic quasi Gibbs measure, phase transition, quasi phase transition
MSC (2020): 46S10, 12J12, 11S99, 30D05, 54H20

1. Introduction

A central problem in statistical mechanics involves understanding how infinite systems behave based on their energy (Hamiltonian). This includes identifying phase transitions, where the system can exist in multiple distinct stable states. While determining all possible stable states for a given system is often extremely complex, researchers often focus their efforts on studying these states within specific simplified structures known as Cayley trees. The problem of phase transitions for some models on a Cayley tree was studied (for example, see [1, 2, 3, 4, 5, 6]).

Theories of p-adic and non-Archimedean stochastic processes have been established in previous research [7, 8]. Building upon these theories, researchers have constructed a wide range of stochastic processes using finite-dimensional probability distributions [9, 10, 11, 7, 12]. Furthermore, p-adic statistical mechanics has been developed within the framework of p-adic probability and stochastic processes [11, 13, 14, 15, 16, 17, 18]. This research has specifically focused on investigating the p-adic Ising and Potts models on the Cayley tree.

In the present paper, we study p-adic quasi Gibbs measures (including p-adic Gibbs measures) for the Potts model with an external field on the Cayley tree of order two. Note that p-adic quasi Gibbs measures were first introduced by F. Mukhamedov [5]. p-adic quasi Gibbs measures for the Potts model (without external field) studied in [5, 19, 20]. In the real case, Gibbs measures for the Potts model with external field were studied in [21, 22]. In [20] it was found that a phase transition occurs for any p for the three state Potts model. Moreover, in [5] it was proved that if $|q|_p = 1$, a quasi phase transition occurs for the $(q+1)$- state Potts model. By comparing these works, we prove that, if $|q(q-1)|_p = 1$, $\sqrt{1-q} \in \mathbb{Q}_p$ a phase transition occurs, if $|q|_p < 1$, a quasi phase transition occurs for the p-adic Potts model with an external field.

2. Preliminaries

2.1. **p-adic numbers and p-adic measure.** Let $\mathbb{Q}$ be a field of rational numbers. For a fixed prime number p, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$ where, $r, n \in \mathbb{Z}$, m is a positive integer, and n and m are relatively prime with p. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, \ x \neq 0, \\ 0, \quad x = 0. \end{cases}$$

This norm is non-Archimedean, i.e. it satisfies the strong triangle inequality: for all $x, y \in \mathbb{Q}$ $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. From this property, one gets the following facts:
1) if $|x|_p \neq |y|_p$, then $|x \pm y|_p = \max\{|x|_p, |y|_p\}$;
2) if $|x|_p = |y|_p$, then $|x - y|_p \leq |x|_p$.

The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field $\mathbb{Q}_p$. Any p-adic number $y \neq 0$ can be uniquely represented in the canonical form

$$y = p^{\alpha(y)}(y_0 + y_1 p + y_2 p^2 + ...),$$

where $\alpha(y) \in \mathbb{Z}$ and the integers y_j satisfy: $y_0 > 0,\ 0 \leq y_j \leq p-1$. In this case $|y|_p = p^{-\alpha(y)}$. An integer $a \in \mathbb{Z}$ is called *quadratic residue modulo* p if the congruent equation $x^2 \equiv a(\mathrm{mod}\, p)$ has a solution $x \in \mathbb{Z}$.

Lemma 2.1. *[23] The equation $y^2 = a,\ 0 \neq a = p^{\alpha(a)}(a_0 + a_1 p + a_2 p^2 + ...),\ 0 \leq a_j \leq p-1,\ a_0 > 0$ has a solution in $y \in \mathbb{Q}_p$ if and only if the following conditions hold:*
i) $\alpha(a)$ is even;
ii) $y^2 \equiv a_0(\mathrm{mod}\, p)$ is solvable for $p \neq 2$; the equality $a_1 = a_2 = 0$ hold if $p = 2$.

In [24] authors have introduced new symbols "O" and "o" which allowed to simplify certain calculations. Roughly speaking, these symbols replace the notation $\equiv\ (mod\ p^k)$ without noticing about power of k. Let us recall them. A given p-adic number y by $O[y]$ we mean a p-adic number with the norm $p^{-\alpha(y)}$, i.e. $|y|_p = |O(y)|_p$. By $o[y]$, we mean a p-adic number with a norm strictly less than $p^{-\alpha(y)}$, i.e. $|o(y)|_p < |y|_p$. For instance, if $y = 1 - p + p^2$, we can write $O[1] = y$, $o[1] = y - 1$ or $o[p] = y - 1 + p$. Therefore, the symbols $O[\cdot]$ and $o[\cdot]$ make our work easier when we need to calculate the p-adic norm of p-adic numbers. It is easy to see that $y = O[x]$ if and only if $x = O[y]$.

For $c \in \mathbb{Q}_p$ and $r > 0$ we denote

$$B(c, r) = \{x \in \mathbb{Q}_p :\ |x - c|_p < r\},$$

and the set of all p-adic integers $\mathbb{Z}_p := B(0, p)$. The set $\mathbb{Z}_p^* = \mathbb{Z}_p \backslash p\mathbb{Z}_p$ is called a set of p-adic units. p-adic *exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, \frac{1}{2})$ if $p = 2$ and $x \in B(0, 1)$ if $p \neq 2$.

Put

$$\mathcal{E}_p = \left\{x \in \mathbb{Q}_p : |x - 1|_p < p^{-1/(p-1)}\right\}.$$

The set $\mathcal{E}_p$ has following properties.

Lemma 2.2. *Let p be a prime. Then the set $\mathcal{E}_p$ has the following properties:*
(a) $\mathcal{E}_p$ is a group under multiplication;
(b) $|a - b|_p < \begin{cases} \frac{1}{2}, & p = 2; \\ 1, & p \neq 2 \end{cases}$ for all $a, b \in \mathcal{E}_p$;
(c) $|a + b|_p = \begin{cases} \frac{1}{2}, & p = 2; \\ 1, & p \neq 2 \end{cases}$ for all $a, b \in \mathcal{E}_p$;
(d) If $a \in \mathcal{E}_p$, then there is an element $h \in B(0, p^{-1/(p-1)})$ such that $a = \exp_p(h)$.

A more detailed description of p-adic calculus and p-adic mathematical physics can be found in [25], [26].

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a p-adic measure if for any disjoin $U_1, U_2, ..., U_n \in \mathcal{B}$, the following holds:

$$\mu\left(\bigcup_{j=1}^{n} U_j\right) = \sum_{j=1}^{n} \mu(U_j).$$

A p-adic measure is called *probability* if $\mu(X) = 1$. One of the important conditions is boundedness, namely a p-adic measure μ is called *bounded* if $\sup\{|\mu(U)|_p : U \in \mathcal{B}\} < \infty$. For more detail information about p-adic measures, we refer to [27, 25].

2.2. **Cayley Tree.** Let $\Gamma^k_+ = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (whose each vertex has exactly $k+1$ edges, except for the root x^0, which has k edges)[5]. Here V is the set of vertices and L is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = \langle x, y\rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1\rangle, \dots, \langle x_{d-1}, y\rangle$ is called a *path* from the point x to the point y. The distance $d(x, y)$ on the Cayley tree is the length (number of edges) of the shortest path from x to y. Let us set

$$W_n = \{x \in V : \; d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m,$$

$$L_n = \{\langle x, y\rangle \in L : \; x, y \in V_n\}.$$

We define a coordinate structure in Γ^k_+: every vertex x (except for x^0) of Γ^k_+ has coordinates $(i_1, \dots, i_n)$, here $i_m \in \{1, \dots, k\}$, $1 \leq m \leq n$ and for the vertex x^0 we put (0). Namely, the symbol (0) constitutes level 0, and the sites $(i_1, \dots, i_n)$ form level n (i.e. $d(x^0, x) = n$) of the lattice. Let us define on Γ^k_+ binary operation $\circ : \Gamma^k_+ \times \Gamma^k_+ \to \Gamma^k_+$ as follows: for any two elements $x = (i_1, \dots, i_n)$ and $y = (j_1, \dots, j_m)$ put

$$x \circ y = (i_1, \dots, i_n) \circ (j_1, \dots, j_m) = (i_1, \dots, i_n, j_1, \dots, j_m) \tag{2.1}$$

and

$$x \circ x^0 = x^0 \circ x = (i_1, \dots, i_n) \circ (0) = (i_1, \dots, i_n). \tag{2.2}$$

By means of the defined operation Γ^k_+ becomes a noncommutative semigroup with a unit. Let us denote this group $(G^k, \circ)$. Using this semigroup structure one defines translations $\tau_g : G^k \to G^k$, $g \in G_k$ by

$$\tau_g(x) = g \circ x.$$

It is clear that $\tau_{(0)} = id$.

Let $G \subset G^k$ be a sub-semigroup of G^k and $h : G^k \to Y$ be a Y-valued function defined on G^k. We say that h is G- *periodic* if $h(\tau_g(x)) = h(x)$ for all $g \in G$ and $x \in G^k$. Any G^k-periodic function is called *translation invariant*.

Now for each $m \geq 2$ we put

$$G_m = \{x \in G^k : d(x, x^0) \equiv 0 (\mathrm{mod}\, m)\}. \tag{2.3}$$

One can check that G_m is a sub-semigroup of G^k.

3. p-adic quasi Gibbs measure for the Potts model

Let $\mathbb{Q}_p$ be field of p-adic numbers and $\Phi = \{1, 2, ..., q\}$ be a finite set. A configuration σ on $A \subset V$ is defined by the function $x \in A \to \sigma(x) \in \Phi$. The set of all configurations on A is denoted by $\Omega_A = \Phi^A$ and $\Omega = \Omega_V$.

For given configurations $\sigma \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x), \; if \;\; x \in V_{n-1}, \\ \omega(x), \;\; if \; x \in W_n. \end{cases}$$

It is clear that $\sigma \vee \omega \in \Omega_{V_n}$.

The (formal) Hamiltonian of p-adic Potts model with an external field is

$$H(\sigma) = J \sum_{\langle x, y\rangle \in L} \delta_{\sigma(x)\sigma(y)} + \alpha \sum_{x \in V} \delta_{q\sigma(x)} \tag{3.1}$$

where $J, \alpha \in B(0, p^{-1/(p-1)})$ J is a coupling constant, α is an external field and δ_{ij} is the Kronecker symbol, i.e.,

$$\delta_{ij} = \begin{cases} 0, \;\; \text{if} \;\; i \neq j, \\ 1, \;\; \text{if} \;\; i = j. \end{cases}$$

Assume that $h : V \to \mathbb{Q}_p^{\Phi}$ is a mapping, i.e., $\mathbf{h}_x = (h_{1,x}, h_{2,x}, ..., h_{q,x})$, where $h_{i,x} \in \mathbb{Q}_p$ $(i \in \Phi)$ and $x \in V$. Given $n \in \mathbb{N}$, we consider a p-adic probability measure $\mu_{\mathbf{h},\sigma}^{(n)}$ on Ω_{V_n} defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{Z_n^{(\mathbf{h})}} \exp\{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x}. \tag{3.2}$$

Here, $\sigma \in \Omega_{V_n}$, and $Z_n^{(\mathbf{h})}$ is the corresponding normalizing factor

$$Z_n^{(h)} = \sum_{\sigma \in \Omega_{V_n}} \exp\{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x}. \tag{3.3}$$

We say that p-adic probability distributions (3.2) are compatible if all $n \geq 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega \in \Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \omega) = \mu_h^{(n-1)}(\sigma_{n-1}). \tag{3.4}$$

We notice that a non-Archimedean analogue of the Kolmogorov extension theorem was proved in [6, 7]. According to this theorem there exists a unique p-adic measure μ_h on $\Omega = \Phi^V$ such that for all $n \geq 1$ and $\sigma \in \Phi^{V_{n-1}}$:

$$\mu(\sigma \in \Omega : \sigma|_{V_n} \equiv \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n).$$

Such measure is called a *p-adic quasi Gibbs measure* corresponding to the Hamiltonian (3.1) and vector-valued function $\mathbf{h}_x$, $x \in V$. By $QG(H)$ we denote the set of all p-adic quasi Gibbs measure associated with function $\mathbf{h} = \{\mathbf{h}_x, x \in V\}$ If all values of h_x belong to the set $\mathcal{E}_p$ then it is called p-adic Gibbs measure.

Definition 3.1. [16] If there are at least two distinct $\mu, \nu \in QG(H)$ such that μ is bounded and ν is unbounded, then we say that *a phase transition occurs.* If there are two different $\mu, \nu \in QG(H)$, either both μ, ν are bounded or unbounded, then we say *a quasi phase transition occurs.*

The following statement describe conditions $\mathbf{h}_x$ guaranteing compatibility of $\mu_{\mathbf{h}}^{(n)}(\sigma)$.

Theorem 3.2. *[28] The measure $\mu_h^{(n)}(\sigma), n = 1, 2, ...$ (see (3.2)) associated with the q-state Potts model (3.1) satisfy the compatibility condition (3.4) if and only if for any $n \in \mathbb{N}$ the following equation holds:*

$$\widehat{\boldsymbol{h}}_x = \prod_{y \in S(x)} F(\widehat{\boldsymbol{h}}_y, \theta, \eta), \tag{3.5}$$

here and below a vector $\widehat{\boldsymbol{h}} = \left(\widehat{h}_1, \widehat{h}_2, ..., \widehat{h}_{q-1}\right) \in \mathbb{Q}_p^{q-1}$ is defined by a vector $\boldsymbol{h} = (h_1, h_2, ..., h_q) \in \mathbb{Q}_p^q$ as follows

$$\widehat{h}_i = \frac{h_i}{h_q}, \ i = 1, 2, ..., q-1 \tag{3.6}$$

and mapping
$F : \mathbb{Q}_p^{q-1} \times \mathbb{Q}_p \to \mathbb{Q}_p^{q-1}$ is defined by $F(x; \theta, \eta) = (F_1(x; \theta, \eta), ..., F_{q-1}(x; \theta, \eta))$ with

$$F_i(x; \theta, \eta) = \frac{(\theta - 1)x_i + \sum_{j=1}^{q-1} x_j + \eta}{\sum_{j=1}^{q-1} x_j + \theta\eta}, \ \ x = \{x_i\} \in \mathbb{Q}_p^{q-1}, \ i = 1, 2, ..., q-1 \tag{3.7}$$

4. Non translation-invariant two periodic quasi Gibbs measure for p-adic Potts model with an external field

In this section, we are going to construct G_2-periodic $QGMs$ for the considered model. Let G_2 be a sub semigroup of G^k (see (2.3)). We denote that

$$\mathbf{h}_x = \begin{cases} \mathbf{h}_1, \ x \in G_2, \\ \mathbf{h}_2, \ x \in G^k \setminus G_2. \end{cases}$$

From equation (3.5), we get the following system

$$\begin{cases}\widehat{\mathbf{h}}_1 = (F(\widehat{\mathbf{h}}_2, \theta, \eta))^k, \\ \widehat{\mathbf{h}}_2 = (F(\widehat{\mathbf{h}}_1, \theta, \eta))^k.\end{cases} \tag{4.1}$$

Lemma 4.1. *If the pair of numbers (x, y) satisfies the system of equations (4.1), then the pair (y, x) also satisfies the system of equations (4.1).*

Proof. The proof follows from the fact that equation (4.1) is symmetric with respect to x and y. □

We assume $\widehat{\mathbf{h}}_i = \left(\widehat{h}_i^{(1)}, \widehat{h}_i^{(2)}, ..., \widehat{h}_i^{(q-1)}\right)$. Let $\widehat{h}_i^{(j)} = h_i, j = \overline{1, q-1}$. For the sake of simplicity, we consider $k = 2$.

In this case, from (4.1) we can obtain following system of equations

$$\begin{cases}\widehat{h}_1 = \left(\dfrac{(\theta + q - 2)\widehat{h}_2 + \eta}{(q-1)\widehat{h}_2 + \theta\eta}\right)^2, \\ \widehat{h}_2 = \left(\dfrac{(\theta + q - 2)\widehat{h}_1 + \eta}{(q-1)\widehat{h}_1 + \theta\eta}\right)^2.\end{cases} \tag{4.2}$$

Let us denote

$$f(\widehat{h}) = \left(\frac{(\theta + q - 2)\widehat{h} + \eta}{(q-1)\widehat{h} + \theta\eta}\right)^2.$$

For equation (4.2), if $\widehat{h}_1 = \widehat{h}_2$ then we get translation-invariant Gibbs measures. Our aim is to find G_2-periodic (non translation-invariant) p-adic quasi Gibbs measures. It demands to solve the following equation

$$\frac{f(f(\widehat{h})) - \widehat{h}}{f(\widehat{h}) - \widehat{h}} = 0. \tag{4.3}$$

Simplifying the last equation we get

$$A\widehat{h}^2 + B\widehat{h} + C = 0, \tag{4.4}$$

where

$$\begin{aligned}
&A = (\theta\eta q + \theta^2 - \theta\eta + 2\theta q + q^2 - 4\theta - 4q + 4)^2, \\
&B = \eta^3 q\theta^3 - 2\eta^3\theta^3 + \eta^2 q^2\theta^2 + 2\eta^2 q\theta^3 + \eta^2\theta^4 + 4\eta^2 q^2\theta - 4\eta^2\theta^3 - \\
&\eta^2 q^2 - 12\eta^2 q\theta + 2\eta q^3 + 6\eta q^2\theta + 6\eta q\theta^2 + 2\eta\theta^3 + 2\eta^2 q + 8\eta^2\theta - \\
&12\eta q^2 - 24\eta q\theta - 12\eta\theta^2 - \eta^2 + 24\eta q + 24\eta\theta - 16\eta, \\
&C = \eta^2(\theta^2\eta + \theta + q - 2)^2.
\end{aligned}$$

Note that, the case $q \in \mathcal{E}_p$ requires more calculus. Therefore, we investigate this case for future work.

Lemma 4.2. *Let $q \notin \mathcal{E}_p$, $p \geq 3$. Equation (4.4) has two distinct solutions if $|q|_p = 1$ and $\sqrt{(1-q)} \in \mathbb{Q}_p$ or $|q|_p < 1$. Otherwise, there is not any solution.*

Proof. We set

$$D(\theta, \eta, q) = B^2 - 4AC.$$

We note that, equation (4.4) has a solution in $\mathbb{Q}_p$ if and only if $\sqrt{D(\theta, \eta, q)} \in \mathbb{Q}_p$.

We can rewrite $D(\theta, q)$ by the following

$$D(\theta, q) = \eta^2(\theta - 1)^2(\theta + q - 1)^2 D^*.$$

where,

$D^* = -4m^3qs^2 - 3m^4s - 14m^3qs + 4m^3s^2 - 3m^2q^2s - 12m^2qs^2 - 3m^4 - 10m^3q + 8m^3s -$

$3m^2q^2 - 36m^2qs + 12m^2s^2 - 12mq^2s - 12mqs^2 - 36m^2q + 36m^2s - 24mq^2 - 12mqs +$

$$12ms^2 - 8q^2s - 4qs^2 + 36m^2 + 24mq + 24ms + 8qs + 4s^2 - 4q^3 + 4q^2, \tag{4.5}$$

$m = \theta - 1,\ s = \eta - 1$.

It can be seen that $\sqrt{D(\theta, q)} \in \mathbb{Q}_p$ if and only if $\sqrt{D^*} \in \mathbb{Q}_p$. At first, we consider the following case.

Let $|q|_p = 1,\ q \notin \mathcal{E}_p$. Since $|m|_p < 1,\ |s|_p < 1$, we write

$$D^* = 4q^2(1-q) + o[1].$$

According to Lemma 2.1, $\sqrt{D^*} \in \mathbb{Q}_p$ is equivalent to $\sqrt{1-q} \in \mathbb{Q}_p$. So, we get that equation (4.2) has two solutions if $\sqrt{1-q} \in \mathbb{Q}_p$.

Next, we consider following case: $p \geq 3,\ |q|_p < 1$. Using (4.5), we have $D^* = 4(3m+s+q)^2 + o[(3m+s+q)^2]$. We find the condition that $\sqrt{D^*} \in \mathbb{Q}_p$ holds. So, if $p \mid q$, then equation (4.4) has two distinct solutions. It is known that the solutions of (4.4) has following forms

$$h_{1,2} = \frac{-B \pm \sqrt{D}}{2A}.$$

Lemma is proved.

According to Lemma 4.1, the pair of (h_1, h_2) is a solution of (4.2), then (h_2, h_1) also satisfies (4.2). Finding the first coefficient of these solutions of (4.4) in the canonical form, is necessary to ascertain the solution's norm. Let $|q|_p = 1, q \notin \mathcal{E}_p$.

$A = q^2(q-1)^2 + o[1]$ $B = 2q^2(q-1) + o[1]$, $D = \eta^2(\theta-1)^2(\theta+q-1)^2 D^*$. From these equalities, we conclude that

$$h_{1,2} = \frac{1}{1-q} + o[1].$$

We study the case $|q|_p < 1$. Then we get

$B = -2(3(\theta-1) + (\eta-1) + q)^2 + o[p^2]$, $D = \eta^2(\theta-1)^2(\theta+q-1)^2 D^* = o[p^4]$, $A = (3(\theta-1) + (\eta-1) + q)^2 + o[p^2]$. It yields that

$$h_{1,2} = \frac{2(3(\theta-1) + (\eta-1) + q)^2 + o[p^2]}{2(3(\theta-1) + (\eta-1) + q)^2 + o[p^2]} = 1 + o[1].$$

In [28], translation-invariant p-adic quasi Gibbs measures ($TIQGM$) associated with $\mathbf{h} = \{h, h, ..., h\} \in \mathbb{Q}_p^{q-1}$ for the p-adic Potts model with an external field was examined. Here the following result was given.

Theorem 4.3. *[28] Let $q \notin \mathcal{E}_p$. Then the following assertions holds:*

a) *if $|q|_p = 1,\ p > 3,\ \sqrt{1-q} \in \mathbb{Q}_p$, then there exist three $TIQGMs$ such that, one of them is bounded, the others are unbounded;*

b) *if $p = 3,\ |q|_3 = 1$ or $p > 3,\ |q|_p = 1,\ \sqrt{1-q} \notin \mathbb{Q}_p$, then there exist a unique bounded $TIQGM$;*

c) *if $p \geq 3,\ |q|_p < 1$, then there does not exist any $TIQGM$.*

Using Theorem 4.3 and Lemma 4.2, we obtain the following result.

Theorem 4.4. *Let $q \notin \mathcal{E}_p$ and $p \geq 3$. The following statements hold for the Potts model on the Cayley tree of order two:*

1) *If $p \neq 3,\ |q|_p = 1$ and $\sqrt{1-q} \in \mathbb{Q}_p$, then there exist three $TIQGMs$ and two G_2- periodic $QGMs$;*

2) *If $p = 3,\ |q|_3 = 1$ and $\sqrt{1-q} \in \mathbb{Q}_3$, then there exist a unique $TIQGM$ and two G_2- periodic $QGMs$;*

3) *If $|q|_p = 1$ and $\sqrt{1-q} \notin \mathbb{Q}_p$, then there exist a unique $TIQGM$;*

4) *If $|q|_p < 1$, then there exist two G_2-periodic $QGMs$.*

Remark 4.5. From Theorem 4.3, it can be seen that when $|q|_p = 1,\ p > 3,\ \sqrt{1-q} \in \mathbb{Q}_p$, there are three $TIQGMs$. Based on this condition and $p = 3,\ |q|_3 = 1,\ \sqrt{(1-q)} \in \mathbb{Q}_3$, we identified two different G_2- periodic $QGMs$. Furthermore, it was shown that when $p \geq 3,\ |q|_p < 1$, $TIQGMs$ do not exist, but in this work, two distinct G_2- periodic $QGMs$ were found when $p \geq 3,\ |q|_p < 1$.

5. Boundedness of two periodic p-adic quasi Gibbs measures and phase transitions

Lemma 5.1. Let $\mathbf{h}$ be a solution of (3.5), and $\mu_{\mathbf{h}}$ be an associated p-adic quasi Gibbs measure. Then for the corresponding partition function $Z_n^{(\mathbf{h})}$ the following equality holds:

$$Z_n^{(\mathbf{h})} = A_{\mathbf{h},n-1} Z_{n-1}^{(\mathbf{h})}, \tag{5.1}$$

where $A_{\mathbf{h},n} = \prod\limits_{x\in W_n} a_{\mathbf{h}}(x)$, $\prod\limits_{y\in S(x)} \sum\limits_{j=1}^{q} \exp_p\{J\delta_{i,j}\}h_{j,y} = a_{\mathbf{h}}(x)h_{i,x}$, $a_{\mathbf{h}}(x) \in \mathbb{Q}_p$, $i = 1, 2, ..., q$.

Proof. The proof of this lemma follows a similar argument to that of Lemma 3.2 in [16]. Using Lemma 5.1, we get the following statement.

Lemma 5.2. Let $k = 2$. If $\mathbf{h}^{(1,2)}$ is G_2-periodic (non translation-invariant) solution of (3.5) then for the corresponding partition function $Z_n^{(\mathbf{h})}$ the following assertions true:
If n is odd, then

$$Z_n^{(\mathbf{h})} = Z_n^{(h)} = ((q-1)h_1 + \theta\eta)^{\frac{2^{n+1}-4}{3}} ((q-1)h_2 + \theta\eta)^{\frac{2^{n+1}-4}{3}} Z_1^{(\mathbf{h})}; \tag{5.2}$$

If n is even, then

$$Z_n^{(\mathbf{h})} = ((q-1)h_1 + \theta\eta)^{\frac{2^{n+2}-4}{3}} ((q-1)h_2 + \theta\eta)^{\frac{2^{n}-4}{3}} Z_1^{(\mathbf{h})}. \tag{5.3}$$

Proof. Let

$$h_{\sigma(x),x} = \begin{cases} h_x, \text{ if } \ \sigma(x) \in \overline{1, q-1}; \\ 1, \text{ if } \ \sigma(x) = q. \end{cases}$$

and

$$h_x = \begin{cases} h_1, \text{ if } \ |x| \ \text{ is } \ \text{even}; \\ h_2, \text{ if } \ |x| \ \text{ is } \ \text{odd}. \end{cases}$$

Consider the following cases
Case 1. Let n be odd. By Lemma 5.1, we get

$$a_h(x) = \frac{(\theta + (q-2)h_1 + \eta)^2}{h_2} = ((q-1)h_1 + \theta\eta)^2,$$

$$A_{h,n} = ((q-1)h_1 + \theta\eta)^{2^{n+1}}, \quad A_{h,n-1} = ((q-1)h_2 + \theta\eta)^{2^n}.$$

$$Z_n^{(h)} = ((q-1)h_1 + \theta\eta)^{\frac{2^{n+1}-4}{3}} ((q-1)h_2 + \theta\eta)^{\frac{2^{n+1}-4}{3}} Z_1^{(\mathbf{h})}.$$

Case 2. Let n be even. By Lemma 5.1, we get

$$a_h(x) = \frac{(\theta + (q-2)h_2 + \eta)^2}{h_1} = ((q-1)h_2 + \theta\eta)^2,$$

$$A_{h,n} = ((q-1)h_2 + \theta\eta)^{2^{n+1}}, \quad A_{h,n-1} = ((q-1)h_1 + \theta\eta)^{2^n}.$$

$$Z_n^{(h)} = ((q-1)h_1 + \theta\eta)^{\frac{2^{n+2}-4}{3}} ((q-1)h_2 + \theta\eta)^{\frac{2^{n}-4}{3}} Z_1^{(\mathbf{h})}.$$

Lemma is proved.

Theorem 5.3. *Let $q \notin \mathcal{E}_p$ and $p \geq 3$. If $|q|_p = 1$ and $\sqrt{1-q} \in \mathbb{Q}_p$ or $|q|_p < 1$, then the measures $\mu_{\boldsymbol{h}^{(1,2)}}$ are unbounded.*

Proof. Let $|q|_p = 1$, $\sqrt{1-q} \in \mathbb{Q}_p$. Then the measures $\mu_{\mathbf{h}^{(1,2)}}$ exist.
By Lemma 5.2 and (3.2), we get

$$|\mu^{(n)}_{\mathbf{h}^{(1,2)}}|_p = \left| \frac{\exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}}{((q-1)h_{1,2}+\theta\eta)^{\frac{2^{n+1}-4}{3}}((q-1)h_{2,1}+\theta\eta)^{\frac{2^{n+1}-4}{3}} Z_1^{\mathbf{h}^{(1,2)}}} \right|_p = \left| \frac{1}{(\theta\eta-1)^{\frac{2^{n+1}-4}{3}}(\theta\eta-1)^{2^n} Z_1^{\mathbf{h}^{(1,2)}}} \right|_p.$$

Since $|\theta\eta - 1|_p < 1$, we get following result

$$\lim_{n\to\infty} |\mu_{\mathbf{h}^{(1,2)(n)}}|_p = \infty.$$

Case 2. If $|q|_p < 1$, then there exist measures $\mu_{\mathbf{h}^{(1,2)}}$. Note that, for $|q|_p < 1$, we get $h_1^{(1,2)} = 1 + o[1]$, By Lemma 5.2, we have

$$|\mu^{(n)}_{\mathbf{h}^{(1,2)}}|_p = \left| \frac{\exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}}{((q-1)h_{1,2}+\theta\eta)^{\frac{2^{n+1}-4}{3}}((q-1)h_{2,1}+\theta\eta)^{\frac{2^{n+1}-4}{3}} Z_1^{\mathbf{h}^{(1,2)}}} \right|_p = \frac{1}{|(q+\theta\eta-1)^{\frac{2^{n+2}-8}{3}} Z_1^{\mathbf{h}^{(1,2)}}|_p}.$$

It yields that

$$\lim_{n\to\infty} |\mu^{(n)}_{\mathbf{h}^{(1,2)}}|_p = \infty.$$

Theorem 5.3 is proved.

Due to Remark 4.3 and Theorem 5.3, we have the following assertions belong to a phase transition.

Theorem 5.4. *Let $q \notin \mathcal{E}_p$ and $p \geq 3$. The following statements hold for the p-adic q-state Potts model on the Cayley tree of order two:*

- *If $|q|_p = 1$ and $\sqrt{1-q} \in \mathbb{Q}_p$, then there exists a phase transition;*
- *If $|q|_p < 1$ then there exist a quasi phase transition.*

Remark 5.5. Note that the first part of Theorem 5.4 coincides with the result of [28]. However, under the case where $|q|_p = 1$ and $\sqrt{1-q} \in \mathbb{Q}_p$, we found two distinct $QG(H)s$. Moreover, in the case $|q|_p < 1$, we found that a quasi phase transition occurs for the p-adic Potts model with an external field.

References

[1] Mukhamedov F. M., Dogan M., On p-adic λ-model on the Cayley tree II: phase transitions. Rep. Math. Phys. (2015) Vol. 75, P. 25–46.

[2] Mukhamedov F., Akin H., Phase transitions for p-adic Potts model on the Cayley tree of order three. J. Stat. Mech. (2013), Iss. 7, P. P07014.

[3] Mukhamedov F., A dynamical system approach to phase transitions p-adic Potts model on the Cayley tree of order two. Rep. Math. Phys. (2012) Vol. 70, P. 385–406.

[4] Mukhamedov F., Khakimov O., Phase transition and chaos: p-adic Potts model on a Cayley tree. Chaos, Solitons and Fractals, (2016) Vol. 87, P. 190–196.

[5] Mukhamedov F., On dynamical systems and phase transitions for $q+1$-state p-adic Potts model on Cayley tree. Math. Phys. Anal. Geom. (2013) Vol. 53, P. 49–87.

[6] Ganikhodjaev N. N., Mukhamedov F. M., Rozikov U. A., Phase transitions of the Ising model on $\mathbb{Z}$ in the p-adic number field. Uzbek. Math. Jour. (1998) Vol. 4, P. 23–29.

[7] Khrennikov A. Y., Ludkovsky S., Stochastic process on non-Archimedean space with values in non-Archimedean fields. Markov Proc.Rel.Fields, (2003) Vol. 9, P. 131–162.

[8] Ludkovsky S. V., Non-Archimedean valued quasi-invariant descending at infinity measure. Int.J.Math.Math.Sci. (2005), Iss. 23, P. 3799–3817.

[9] Khakimov O., On a generalized p-adic Gibbs measure for Ising model on trees. p-Adic Numbers, Ultrametric Analysis and Applications, (2014) Vol. 6, Iss. 3, P. 207–217.

[10] Khakimov O., p-Adic Solid-on-Solid model on a Cayley tree. Theor. Math. Phys, (2017) Vol. 193, Iss. 3, P. 1881–1894.

[11] Ganikhodjaev N. N., Mukhamedov F. M., Rozikov U. A., Existence of a phase transition for the Potts p-adic model on the set Z. Theor. Math. Phys. (2002) Vol. 130, P. 425–431.

[12] Mukhamedov F., Khakimov O., On Julia set and chaos in p-adic Ising model on the Cayley tree. Math. Phys. Anal. Geom. (2017) Vol. 20, Iss. 23,

[13] Gandolfo D., Rozikov U., Ruiz J., On p-adic Gibbs measures for hard core model on a Cayley Tree. Markov Proc. Rel. Fields, (2012) Vol. 18, Iss. 4, P. 701–720.

[14] Khrennikov A. Y., Mukhamedov F., Mendes J. F. F., On p-adic Gibbs measures of countable state Potts model on the Cayley tree. Nonlinearity, (2007) Vol. 20, P. 2923–2937.

[15] Mukhamedov F., On existence of generalized Gibbs measures for one dimensional p-adic countable state Potts model. Proc. Steklov Inst. Math. (2009) Vol. 265, P. 165–176.

[16] Mukhamedov F., On p-adic quasi Gibbs measures for $q+1$-state Potts model on the Cayley tree. p-Adic Numbers Ultrametric Analysis and Applications, (2010) Vol. 2, P. 241–251.

[17] Rozikov U. A., Khakimov O. N., Description of all translation-invariant p-adic Gibbs measures for the Potts model on the Cayley tree. Markov Processes Rel. Fields, (2015) Vol. 21, P. 177–204.

[18] Rahmatullaev M. M., Abdukahorova Z. T., On H_A-weakly periodic p-adic generalized Gibbs measures of p-adic Ising model with an external field on a Cayley tree. Rep. Math. Phys. (2025) Vol. 95, Iss. 1, P. 93–107.

[19] Tukhtabaev A. M., On G_2-Periodic Quasi Gibbs Measures of p-Adic Potts Model on a Cayley Tree. p-Adic Numbers, Ultrametric Analysis and Applications, (2021) Vol. 13, Iss. 4, P. 291–307.

[20] Rahmatullaev M. M., Tukhtabaev A. M., Samijonova N. D., Weakly periodic p-adic quasi Gibbs measures for the Potts model on a Cayley tree. Letters in Mathematical Physics,

[21] Rasulova M., Ground states for the Potts model with an external field. Rep. Math. Phys. (2024) Vol. 94, Iss. 3, P. 325–334.

[22] Rozikov U. A., Rahmatulaev M. M., Khakimov R. M., Periodic Gibbs measures for the Potts model in translation-invariant and periodic external fields on the Cayley tree. Theor. Math. Phys, (2022) Vol. 210, Iss. 1, P. 156–176.

[23] Vladimirov V. S., Volovich I. V., Zelenov E. I., p-Adic analysis and Mathematical physics. World Sci., Singapore, (1994).

[24] Mukhamedov F., Khakimov O., On equation $x^k = a$ over Q_p and its applications. Izvestiya Math. (2020) Vol. 84, P. 348–360.

[25] Khrennikov A. Y., Nilsson M., p-Adic Deterministic and Random Dynamical Systems. Kluwer, Dordrecht, (2004).

[26] Khrennikov A. Y., Yamada S., Rooij A., Measure-theoretical approach to p-adic probability theory. Annals Math.Blaise Pascal, (1999) Vol. 6, P. 21–32.

[27] Anashin V., Khrennikov A., Applied Algebraic Dynamics. Walter de Gruyter, Berlin, New York, (2009).

[28] Rahmatullaev M. M., Samijonova N. D., Translation-invariant p-adic quasi gibbs measures for the Potts model with an external field on the Cayley tree. Nanosystems: Phys. Chem. Math. (2025) Vol. 16, Iss. 2, P. 164–175.

Samijonova N.D.,
Namangan State University, Namangan, Uzbekistan.
e-mail: nurxonsamijonova@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 192–201
DOI: 10.29229/uzmj.2026-1-22

On an optimal interpolation formula with derivatives in the Sobolev space

Shadimetov Kh.M., Hayotov A.R., Olimov N.N.

Abstract. In this work, the problem of constructing an optimal interpolation formula involving derivatives is studied. The values of the unknown function are required not only at the nodal points but also the values of its first three derivatives at these nodes. An upper bound for the interpolation error is obtained using an extremal function, whose explicit form is determined. Furthermore, the squared norm of the corresponding error functional is derived. Since this norm depends on the coefficients, the Lagrange function is introduced, and its partial derivatives with respect to the coefficients are computed and set equal to zero, leading to a system of equations. The resulting system is solved by the method proposed by Sobolev. To this end, the discrete analogue of the differential operator $\frac{d^2}{dx^2}$ is employed to solve the system and to determine the coefficients of the interpolation formula.

Keywords: Interpolation, splines, extremal function, error functional, norm, Sobolev space
MSC (2020): 65D05, 41A15

1. Introduction

Interpolation splines play a crucial role in modern technological advancements across various fields. They are essential in numerical analysis, computer graphics, data fitting, and numerical solving partial differential equations.

In numerical analysis, spline interpolation involves fitting low-degree polynomials to subsets of data points, ensuring a smooth and accurate approximation of complicated functions. This method is often preferred over high-degree polynomial interpolation due to its stability and precision. Splines are particularly effective in scattered data fitting, where they provide smooth curves and surfaces that pass through or near given data points. This capability is invaluable in fields like geospatial analysis and computer-aided design [1].

Additionally, in numerical solutions of partial differential equations (PDEs), especially in fluid dynamics and aerodynamics, splines are widely used to approximate solutions to the Navier–Stokes equations. For instance, cubic splines can be employed in weather prediction models to interpolate sparse atmospheric data accurately and smoothly, enhancing the efficiency and stability of numerical simulations. [2].

Moreover, in the Galerkin method, the selection of basis functions is crucial for the accuracy and efficiency of the solution. A notable approach involves using interpolation-based basis functions, which are constructed to satisfy the governing partial differential equations (PDEs) locally. This technique is particularly advantageous in problems with variable coefficients, where traditional polynomial basis functions may not capture the solution's behavior effectively.

For instance, the study [3] explored the construction of generalized plane waves (GPWs) as basis functions within a discontinuous Galerkin framework. These GPWs are designed to approximately satisfy the PDEs locally, thereby enhancing the method's efficiency and accuracy in solving wave-related boundary value problems with variable coefficients. The paper provides a detailed algorithm for constructing these functions and discusses their interpolation properties, offering valuable insights into their implementation in numerical methods.

This approach exemplifies how interpolation-based basis functions can be effectively utilized in the Galerkin method to address complex PDEs, leading to improved solution strategies in computational mathematics.

In the work [4], an optimal interpolation formula with derivatives in $L_2^{(m)}$ was also constructed, but the difference from our work is that only the values of the derivatives of the unknown function on the boundaries of the interval [0,1] were required. In addition to the above, the construction of interpolation formulas in other spaces has been studied in [5], [6], and [7]. Moreover, interpolation formulas with derivatives and their applications have been investigated in the following works: [8], [9], [10].

The next sections of the paper are organized as follows:
In Section 2, the problem of constructing an interpolation formula in the $L_2^{(4)}$ space is stated, derives an upper bound for the error of the formula, presents the norm of the error functional, and obtains the system of equations for the unknown coefficients of the interpolation formula.
Section 3 describes the algorithm for determining the unknown coefficients using the Sobolev method.
Section 4 presents numerical results and their analysis based on the analytically obtained coefficients.

2. Statement of the problem

Let the functions φ belong to the Sobolev space $L_2^{(4)}(0,1)$. Here $L_2^{(4)}(0,1)$ is the Hilbert space of functions that are square integrable with first four order derivative in the interval [0,1].
The space is equipped with the norm

$$\|\varphi\|_{L_2^{(4)}} = \sqrt{\int_0^1 (f^{IV}(x))^2 dx}.$$

Let a grid

$$\Delta : \ 0 = x_0 < x_1 < \ldots < x_N = 1$$

be given on the interval [0,1]. Assume that on this grid, the values of the itself and first three derivatives of the function are given, i.e. we have

$$\varphi(x_\beta), \ \varphi'(x_\beta), \ \varphi''(x_\beta), \ \varphi'''(x_\beta), \quad \beta = 0, 1, \ldots, N. \tag{2.1}$$

Here, $x_\beta = h\beta, h = \frac{1}{N}$.
In this work, we consider the problem of optimal interpolation of functions $\varphi(x)$ given by values (2.1) at points $x_\beta, \ \beta = 0, 1, \ldots, N$ in the space $L_2^{(4)}(0,1)$. For this, we consider the following

$$P_\varphi(x) = \sum_{\beta=0}^{N} \sum_{\alpha=0}^{3} C_{\beta,\alpha} \varphi^{(\alpha)}(x_\beta) \tag{2.2}$$

In this case, we have the following approximation

$$\varphi(x) \cong P_\varphi(x). \tag{2.3}$$

Here the coeffitcients $C_{\beta,\alpha}(x), \beta = 0, 1, ..., N$ and $\alpha = 0, 1, 2$ are known and $\alpha = 0$ have the form [11] :

$$C_{0,0}(x) = \begin{cases} \frac{h-x}{h}, & 0 \le x \le h, \\ 0, & h < x \le 1, \end{cases}$$

$$C_{\beta,0}(x) = \begin{cases} \frac{x+h-h\beta}{h}, & h(\beta-1) \le x \le h\beta, \\ \frac{h-x+h\beta}{h}, & h\beta < x \le h(\beta+1), \\ 0, & \text{otherwise}, \end{cases} \quad \beta = 1, 2, \ldots, N-1,$$

$$C_{N,0}(x) = \begin{cases} 0, & 0 \le x \le h(N-1), \\ \frac{h-1+x}{h}, & h(N-1) < x \le 1. \end{cases}$$

The following coefficients were found in [12]:

$$C_{0,1}(x) = \begin{cases} \frac{x(h-x)}{2h}, & 0 \le x \le h, \\ \\ 0, & h < x \le 1, \end{cases}$$

$$C_{\beta,1}(x) = \begin{cases} \frac{(x-h\beta)^2+h(x-h\beta)}{2h}, & h(\beta-1) \le x < h\beta, \\ \frac{-(x-h\beta)^2+h(x-h\beta)}{2h}, & h\beta \le x \le h(\beta+1), \\ 0, \ \ otherwise, \end{cases} \quad \beta = 1, 2, \ldots, N-1,$$

$$C_{N,1}(x) = \begin{cases} 0, & 0 \le x \le h(N-1), \\ \frac{(x-1)(x-1+h)}{2h}, & h(N-1) < x \le 1. \end{cases}$$

$$C_{0,2}(x)=\begin{cases}\frac{x(h-x)(2x-h)}{12h}, & 0\le x\le h,\\ 0, & h<x\le 1,\end{cases}$$

$$C_{\beta,2}(x)=\begin{cases}\frac{(x-h(\beta-1))(x-h\beta)(2x-2h\beta+h)}{12h}, & h(\beta-1)\le x\le h\beta,\\ \frac{(h\beta-x)(2h\beta+h-2x)(h\beta+h-x)}{12h}, & h\beta<x\le h(\beta+1),\\ 0, & \text{otherwise},\end{cases}\qquad \beta=1,2,\dots,N-1,$$

$$C_{N,2}(x)=\begin{cases}0, & 0\le x\le (N-1)h,\\ \frac{(x-1)(2x+h-2)(x+h-1)}{12h}, & h(N-1)<x\le 1.\end{cases}$$

and the coefficients $C_{\beta,3}(x)$ are unknown.

The error associated with the approximate equality (2.2) takes the form of the difference

$$E_\varphi(x)=\varphi(x)-P_\varphi(x). \tag{2.4}$$

It should be noted that in this work, when we consider the approximation of the form (2.3), we impose the condition that the class of functions that transforms this approximate equality into an exact equality in $L_2^{(4)}(0,1)$ space should be the class of all polynomials up to degree three. If we take $\varphi_0(x)=1$, $\varphi_1(x)=x$, $\varphi_2(x)=x^2$ and $\varphi_3(x)=x^3$ as the basis functions, the imposition is

$$E_{\varphi_i}(x)=\varphi_i(x)-P_{\varphi_i}(x)=0 \text{ or } (R,x^i)=0, i=0,1,2,3. \tag{2.5}$$

conditions on the error functional $R(x)$ is enough for the approximation formula (2.3) to be exact for polynomials up to degree three.

Then in the space $L_2^{(4)}(0,1)$ at every fixed point $x=z$ of the interval $[0,1]$ the error (2.4) defines a linear continuous functional

$$R(x,z)=\delta(x-z)-\sum_{\beta=0}^{N}C_{\beta,0}(z)\cdot\delta(x-x_\beta)+\sum_{\beta=0}^{N}C_{\beta,1}(z)\cdot\delta'(x-x_\beta)$$
$$-\sum_{\beta=0}^{N}C_{\beta,2}(z)\cdot\delta''(x-x_\beta)+\sum_{\beta=0}^{N}C_{\beta,3}(z)\cdot\delta'''(x-x_\beta). \tag{2.6}$$

In order to construct an optimal interpolation formula of the form (2.3), it is necessary to calculate the norm $\|R\|$, then we find the smallest value of this quantity in the given $C_{\beta,0}$, $C_{\beta,1}$and $C_{\beta,2}$ by the coefficients $C_{\beta,3}$. This necessity arises from the fact that, according to the Cauchy-Schwarz inequality, the estimation of the error (2.3) is expressed by the norm as follows:

$$|(R,\varphi)|\le \|R\|_{L_2^{(4)*}}\cdot\|\varphi\|_{L_2^{(4)}}.$$

It is easy to see that the norm $\|R\|_{L_2^{(4)*}}$ depends on the coefficients $C_{\beta,3}$. Then it should be found the smallest value of the norm $\|R\|_{L_2^{(4)*}}$ by the coefficient $C_{\beta,3}$. That is, it should be calculated the quantity

$$\inf_{C_{\beta,3}}\|R\|_{L_2^{(4)*}}. \tag{2.7}$$

The coefficients $\mathring{C}_{\beta,3}$ reaching the value (2.7) we call the optimal coefficients.

Thus, consequently in order to get optimal formula

- we calculate the norm $\|R\|_{L_2^{(4)*}}$,
- we find $\mathring{C}_{\beta,3}$ which gives (2.7).

To calculate $\|R\|_{L_2^{(4)*}}$, we use the definition of the extremal function [13]. The function $U_R(x)$ satisfying the following equality is called an extremal for the interpolation formula (2.3):

$$(R,U_R)=\|R\|_{L_2^{(4)*}}\cdot\|U_R\|_{L_2^{(4)}}$$

here, the extremal function corresponding to a linear continuous functional defined in the $L_2^{(m)}$ space was found by S. L. Sobolev [13], from which, as a particular case, we obtain the following:

$$U_R(x) = R(x) * G_4(x) + p_3x^3 + p_2x^2 + p_1x + p_0, \tag{2.8}$$

where

$$G_4(x) = \frac{|x|^7}{2\cdot 7!}.$$

Now, we first calculate the convolution in equation (2.8).

$$R(x) * G_4(x) = \int_{-\infty}^{\infty} R(y)\cdot G_4(x-y)dy = \frac{|x-z|^7}{2\cdot 7!} - \sum_{\beta=0}^{N} C_{\beta,0}\frac{|x-z|^7}{2\cdot 7!}$$
$$+\sum_{\beta=0}^{N} C_{\beta,1}\frac{\operatorname{sgn}(x-x_\beta)(x-x_\beta)^6}{2\cdot 6!} - \sum_{\beta=0}^{N} C_{\beta,2}\frac{|x-z|^5}{2\cdot 5!} + \sum_{\beta=0}^{N} C_{\beta,3}\frac{\operatorname{sgn}(x-x_\beta)(x-x_\beta)^4}{48}. \tag{2.9}$$

Then the extremal function has the following form:

$$U_R(x) = \frac{|x-z|^7}{2\cdot 7!} - \sum_{\beta=0}^{N} C_{\beta,0}\frac{|x-z|^7}{2\cdot 7!} + \sum_{\beta=0}^{N} C_{\beta,1}\frac{\operatorname{sgn}(x-x_\beta)(x-x_\beta)^6}{2\cdot 6!} - \sum_{\beta=0}^{N} C_{\beta,2}\frac{|x-z|^5}{2\cdot 5!}$$
$$+\sum_{\beta=0}^{N} C_{\beta,3}\frac{\operatorname{sgn}(x-x_\beta)(x-x_\beta)^4}{48} + p_3x^3 + p_2x^2 + p_1x + p_0.$$

Taking into account the last expression, we get

$$(R, U_R) = \int_{-\infty}^{\infty} R(x)\cdot U_R(x)dx = \int_{-\infty}^{\infty} R(x)\cdot\left(R(x)*G_4(x) + p_3x^3 + p_2x^2 + p_1x + p_0\right)dx$$
$$= \int_{-\infty}^{\infty} R(x)\cdot(R(x)*G_4(x))\,dx + p_3(R,x^3) + p_2(R,x^2) + p_1(R,x) + p_0(R,1)$$
$$= \int_{-\infty}^{\infty} R(x)\cdot(R(x)*G_4(x))\,dx. \tag{2.10}$$

Using expression (2.9), from expression (2.10), we obtain

$$(R, U_R) = \int_{-\infty}^{\infty}\Bigg(\delta(x-z) - \sum_{\beta=0}^{N} C_{\beta,0}(z)\cdot\delta(x-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,1}(z)\cdot\delta'(x-x_\beta)$$
$$-\sum_{\beta=0}^{N} C_{\beta,2}(z)\cdot\delta''(x-x_\beta) + \sum_{\beta=0}^{N} C_{\beta,3}(z)\cdot\delta'''(x-x_\beta)\Bigg)\cdot\Bigg(\frac{|x-z|^7}{2\cdot 7!} - \sum_{\gamma=0}^{N} C_\gamma\frac{|x-x_\gamma|^7}{2\cdot 7!}$$
$$+\sum_{\gamma=0}^{N} C_{\gamma,1}\frac{\operatorname{sgn}(x-x_\gamma)(x-x_\gamma)^6}{2\cdot 6!} - \sum_{\gamma=0}^{N} C_{\gamma,2}\frac{|x-x_\gamma|^5}{2\cdot 5!} + \sum_{\gamma=0}^{N} C_{\gamma,3}\frac{\operatorname{sgn}(x-x_\gamma)(x-x_\gamma)^4}{48}\Bigg)$$
$$= -\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,3}C_{\gamma,3}\frac{|x_\beta-x_\gamma|}{2} + \sum_{\beta=0}^{N} C_{\beta,3}\Bigg[\sum_{\gamma=0}^{N} C_{\gamma,2}\frac{\operatorname{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^2}{2}$$
$$-\sum_{\gamma=0}^{N} C_{\gamma,1}\frac{|x_\beta-z|^3}{6} + \sum_{\gamma=0}^{N} C_{\gamma,0}\frac{\operatorname{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^4}{24} - \frac{|x_\beta-z|^3}{12}\Bigg]$$

$$-\sum_{\beta=0}^{N} C_{\beta,2}\cdot\left[\sum_{\gamma=0}^{N} C_{\gamma,1}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^4}{4}-\sum_{\gamma=0}^{N} C_{\gamma,0}\frac{|x_\beta-x_\gamma|^5}{120}-\frac{|x_\beta-z|^5}{120}\right]$$

$$+\sum_{\beta=0}^{N} C_{\beta,1}\cdot\left[\sum_{\gamma=0}^{N} C_{\gamma,0}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^6}{720}-\frac{\mathrm{sgn}(x_\beta-z)(x_\beta-z)^6}{720}+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,2}C_{\gamma,2}\frac{|x_\beta-x_\gamma|^3}{12}\right]$$

$$-\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,1}C_{\gamma,1}\frac{|x_\beta-x_\gamma|^5}{240}+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,0}C_{\gamma,0}\frac{|x_\beta-x_\gamma|^7}{2\cdot 7!}-\sum_{\beta=0}^{N} C_{\beta,0}\frac{|x_\beta-z|^7}{2\cdot 7!}. \tag{2.11}$$

We also have the following equalities [12]:

$$(R,1)=0, \text{then } \sum_{\beta=0}^{N} C_{\beta,0}=1,$$

$$(R,x)=0, \text{then } \sum_{\beta=0}^{N} C_{\beta,1}=0,$$

$$(R,x^2)=0, \text{then } \sum_{\beta=0}^{N} C_{\beta,2}=0.$$

Similarly, we come

$$(R,x^3)=\int_{-\infty}^{\infty} R(x)\cdot x^3 dx$$

$$=\int_{-\infty}^{\infty}\left(\delta(x-z)-\sum_{\beta=0}^{N} C_{\beta,0}\delta(x-x_\beta)+\sum_{\beta=0}^{N} C_{\beta,1}\delta'(x-x_\beta)-\sum_{\beta=0}^{N} C_{\beta,2}\delta''(x-x_\beta)\right.$$

$$\left.+\sum_{\beta=0}^{N} C_{\beta,3}\delta'''(x-x_\beta)\right)\cdot x^3 dx=\frac{z^3}{6}-\frac{1}{6}\sum_{\beta=0}^{N} C_{\beta,0}\cdot x_\beta^3-\frac{1}{2}\sum_{\beta=0}^{N} C_{\beta,1}x_\beta^2-\sum_{\beta=0}^{N} C_{\beta,2}x_\beta-\sum_{\beta=0}^{N} C_{\beta,3}=0.$$

Then,

$$\sum_{\beta=0}^{N} C_{\beta,3}=\frac{z^3}{6}-\frac{1}{6}\sum_{\beta=0}^{N} C_{\beta}\cdot x_\beta^3-\frac{1}{2}\sum_{\beta=0}^{N} C_{\beta,1}x_\beta^2-\sum_{\beta=0}^{N} C_{\beta,2}x_\beta. \tag{2.12}$$

So, we get the following expression for the norm of the error functional of the interpolation formula: .

$$\|R\|^2_{L_2^{(4)*}}=-\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,3}C_{\gamma,3}\frac{|x_\beta-x_\gamma|}{2}+\sum_{\beta=0}^{N} C_{\beta,3}\left[\sum_{\gamma=0}^{N} C_{\gamma,2}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^2}{2}\right.$$

$$\left.-\sum_{\gamma=0}^{N} C_{\gamma,1}\frac{|x_\beta-z|^3}{6}+\sum_{\gamma=0}^{N} C_{\gamma,0}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^4}{24}-\frac{|x_\beta-z|^3}{12}\right]$$

$$-\sum_{\beta=0}^{N} C_{\beta,2}\cdot\left[\sum_{\gamma=0}^{N} C_{\gamma,1}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^4}{4}-\sum_{\gamma=0}^{N} C_{\gamma,0}\frac{|x_\beta-x_\gamma|^5}{120}-\frac{|x_\beta-z|^5}{120}\right]$$

$$+\sum_{\beta=0}^{N} C_{\beta,1}\cdot\left[\sum_{\gamma=0}^{N} C_{\gamma,0}\frac{\mathrm{sgn}(x_\beta-x_\gamma)(x_\beta-x_\gamma)^6}{720}-\frac{\mathrm{sgn}(x_\beta-z)(x_\beta-z)^6}{720}+\right]$$

$$+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,2}C_{\gamma,2}\frac{|x_\beta-x_\gamma|^3}{12}-\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,1}C_{\gamma,1}\frac{|x_\beta-x_\gamma|^5}{240}$$

$$+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_{\beta,0}C_{\gamma,0}\frac{|x_\beta - x_\gamma|^7}{2\cdot 7!} - \sum_{\beta=0}^{N} C_{\beta,0}\frac{|x_\beta - z|^7}{2\cdot 7!}. \tag{2.13}$$

We find the minimum of expression (2.13) under condition (2.12). For this, we come to the problem of finding the conditional extremum of a multivariable Lagrange function.
We construct the Lagrange function:

$$\Lambda = \|R\|^2 + 2\lambda(R, x^3) = \|R\|^2 + 2\lambda\left(\frac{z^3}{6} - \frac{1}{6}\sum_{\beta=0}^{N} C_{\beta,0}\cdot x_\beta^3 - \frac{1}{2}\sum_{\beta=0}^{N} C_{\beta,1}x_\beta^2 - \sum_{\beta=0}^{N} C_{\beta,2}x_\beta - \sum_{\beta=0}^{N} C_{\beta,3}\right),$$

where, λ is a Lagrange multiplier.
In that case, equating to zero the partial derivatives of the function Λ by $C_{\beta,3}$ and λ, we get the following system of the linear equations

$$\sum_{\gamma=0}^{N} C_{\gamma,3}\frac{|x_\beta - x_\gamma|}{2} + \lambda = f(x_\beta, z), \quad \beta = 0, 1, ..., N, \tag{2.14}$$

$$\sum_{\gamma=0}^{N} C_{\gamma,3} = \frac{z^3}{6} - \frac{1}{6}\sum_{\beta=0}^{N} C_{\beta,0}\cdot x_\beta^3 - \frac{1}{2}\sum_{\beta=0}^{N} C_{\beta,1}x_\beta^2 - \sum_{\beta=0}^{N} C_{\beta,2}x_\beta. \tag{2.15}$$

We simplify expression (2.14) , then we have

$$\sum_{\gamma=0}^{N} C_{\gamma,3} = 0, \tag{2.16}$$

$$\sum_{\gamma=0}^{N} C_{\gamma,3}\cdot\frac{|x_\beta - x_\gamma|}{2} + \lambda = f(x_\beta, z), \quad \beta = 0, 1, ..., N, \tag{2.17}$$

where

$$f(x_\beta, z) = -\sum_{\gamma=0}^{N} C_{\gamma,0}\frac{\mathrm{sgn}(x_\beta - x_\gamma)(x_\beta - x_\gamma)^4}{48} + \sum_{\gamma=0}^{N} C_{\gamma,1}\frac{|x_\beta - x_\gamma|^3}{12}$$

$$-\sum_{\gamma=0}^{N} C_{\gamma,2}\frac{\mathrm{sgn}(x_\beta - x_\gamma)(x_\beta - x_\gamma)^2}{4} + \frac{\mathrm{sgn}(x_\beta - z)(x_\beta - z)^4}{24}. \tag{2.18}$$

3. An algorithm of finding the coefficients of the interpolation formula

In order to find an analytical solution to the system (2.14), we need the discrete analogue

$$D_1(h\beta) = \begin{cases} 0, & |\beta| \geq 2, \\ \frac{1}{h^2}, & |\beta| = 1, \\ -\frac{2}{h^2}, & \beta = 0 \end{cases} \tag{3.1}$$

of the differential operator $\frac{d^2}{dx^2}$. The discrete operator (3.1) has the following properties [14]

$$\begin{aligned} &D_1(h\beta) * 1 = 0, \\ &D_1(h\beta) * (h\beta) = 0, \\ &hD_1(h\beta) * \frac{|h\beta|}{2} = \delta_d(h\beta), \end{aligned} \tag{3.2}$$

where

$$\delta_d(h\beta) = \begin{cases} 1, & \beta = 0, \\ 0, & \beta \neq 0. \end{cases}$$

We consider the left-hand side of the expression (2.17) as a new function

$$U_1(h\beta) = \sum_{\gamma=0}^{N} C_{\gamma,3} \cdot \frac{|h\beta - h\gamma|}{2} + \lambda. \tag{3.3}$$

Here, $C_{\gamma,3}$ is considered as a discrete function of the integer-valued argument. For $\gamma = -1, -2, \dots$ and $\gamma = N+1, N+2, \dots$ we assume $C_{\gamma,3} = 0$.

As a result, based on the definition of the convolution operation of functions with discrete arguments, from expression (2.18) we arrive at the following

$$U_1(h\beta) = C_{\beta,3} * \frac{|h\beta|}{2} + \lambda.$$

In that case, according to properties (3.1), we get the following

$$C_{\beta,3} = hD_1(h\beta) * U_1(h\beta). \tag{3.4}$$

In order to find the coefficients $C_{\beta,3}$ from relation (3.4), we must first determine the function $U_1(h\beta)$ at all integer values of β.

Depending on (2.14), the following equality

$$U_1(h\beta) = f(h\beta, z), \tag{3.5}$$

is valid for $\beta = 0, 1, \dots, N$. Now we find representation of $U_1(h\beta)$ at $\beta < 0$ and $\beta > N$.

Let $\beta = -1, -2, \dots$. Then, from (3.3), we get the following:

$$U_1(h\beta) = L + \lambda,$$

where, $L = \sum_{\gamma=0}^{N} C_{\gamma,3} \cdot \frac{h\gamma}{2}$
Similarly, for $\beta = N+1, N+2, \dots$, we have

$$U_1(h\beta) = -L + \lambda. \tag{3.6}$$

From (3.4)-(3.6), we get the following:

$$U_1(h\beta) = \begin{cases} L + \lambda, & \beta = -1, -2, \dots, \\ f(h\beta, z), & \beta = 0, 1, \dots, N, \\ -L + \lambda, & \beta = N+1, N+2, \dots. \end{cases}$$

It is easy to show that

$$\lambda - L = f(1, z), \quad \lambda + L = f(0, z).$$

So,

$$U_1(h\beta) = \begin{cases} f(0), & \beta < 0, \\ f(h\beta, z), & 0 \le \beta \le 1, \\ f(1), & \beta > N. \end{cases} \tag{3.7}$$

Using equation (3.7), we find coefficients $C_{\beta,3}$ based on equation (3.3). Then

$$C_{\beta,3} = hD_1(h\beta) * U_1(h\beta) = h \sum_{\gamma=-\infty}^{\infty} D_1(h\beta - h\gamma) \cdot U_1(h\gamma)$$

$$= h \left[\sum_{\gamma=0}^{N} D_1(h\beta - h\gamma) \cdot U_1(h\gamma) + \sum_{\gamma=1}^{\infty} D_1(h(\gamma + N - \beta) \cdot U_1(h(N+\gamma)) + \sum_{\gamma=1}^{\infty} D_1(h\beta + h\gamma) U_1(-h\gamma) \right] \tag{3.8}$$

From the above expression, for $\beta = 0$, we have the following:

$$\begin{aligned} C_{0,3} &= h\left[\sum_{\gamma=0}^{N} D_1(h\gamma)\cdot U_1(h\gamma) + \sum_{\gamma=1}^{\infty} D_1(h\gamma)\cdot U_1(-h\gamma) + \sum_{\gamma=1}^{\infty} D_1(h(N+\gamma))\cdot U_1(h(N+\gamma))\right] \\ &= \frac{1}{h}[f(h,z) - f(0,z)]. \end{aligned}$$

Now, from (3.8) for $\beta = 1, 2, \ldots, N-1$, we have the following:

$$\begin{aligned} C_{\beta,3} &= h\sum_{\gamma=0}^{N} D_1(h\beta - h\gamma)\cdot U_1(h\gamma) \\ &= h\left[D_1(h)\cdot U_1(h(\beta-1)) + D_1(0)\cdot U_1(h\beta) + D_1(h)\cdot U_1(h(\beta+1))\right] \\ &= \frac{1}{h}\left[f(h(\beta-1),z) - 2f(h\beta,z) + f(h(\beta+1),z)\right]. \end{aligned}$$

Finally, from (3.8) for $\beta = N$, we get the following:

$$\begin{aligned} C_{N,3} &= h\left[\sum_{\gamma=0}^{N} D_1(hN - h\gamma)\cdot U_1(h\gamma, z) + \sum_{\gamma=1}^{\infty} D_1(h\gamma)\cdot U_1(h(N+\gamma)) + \sum_{\gamma=1}^{\infty} D_1(1+h\gamma)\cdot U_1(-h\gamma)\right] \\ &= \frac{1}{h}\left[f(1-h,z) - f(1,z)\right]. \end{aligned}$$

Then, based on (2.18) we get the following result:

Theorem 3.1. Coefficients of the optimal interpolation formula of the form (2.2) in the space $L_2^{(4)}(0,1)$ have the form:

$$\begin{aligned} C_{0,3}(z) &= \frac{1}{24h}\begin{cases} z^2\left(h(2z-h) - z^2\right), & 0 \le z \le h, \\ 0, & h < z \le 1, \end{cases}, \\ C_{\beta,3}(z) &= \frac{1}{24h}\begin{cases} (h\beta - z)^2(h(\beta-1) - z)^2, & h(\beta-1) \le z \le h\beta, \\ -(h\beta - z)^2(h(\beta+1) - z)^2, & h\beta < z \le h(\beta+1), \\ 0, & \textit{otherwise}, \end{cases} \\ C_{N,3}(z) &= \frac{1}{24h}\begin{cases} 0, & 0 \le z \le h(N-1), \\ (z-1)^2(h+z-1)^2, & h(N-1) \le z \le 1. \end{cases} \end{aligned}$$

4. Numerical results and discussion

Using the theoretical results obtained, we approximate several functions. Additionally, we compare the numerical results with those obtained in similar studies.

We analyze the approximation of the function $\varphi(x) = x^4$ using the optimal interpolation formula (2.2) in the interval $[0,1]$ with a step size $h = \frac{1}{N}$ for both $N = 10$ and $N = 100$.

We analyze the approximation of the function $\varphi(x) = \sin(x)$ using the optimal interpolation formula (2.2) in the interval $[0,1]$ with a step size $h = \frac{1}{N}$ for both $N = 10$ and $N = 100$.

We consider the approximation of the function $\varphi(x) = e^x$ using the optimal interpolation formula (2.2) in the interval $[0,1]$ with a step size $h = \frac{1}{N}$ for both $N = 10$ and $N = 100$.

5. Conclusion

In this work, we addressed the problem of optimal interpolation for functions in the Sobolev space $L_2^{(4)}(0,1)$. Our goal was to construct an optimal interpolation formula for a function based on its values and the values of its first three derivatives at a given set of points. To achieve this, we solved the

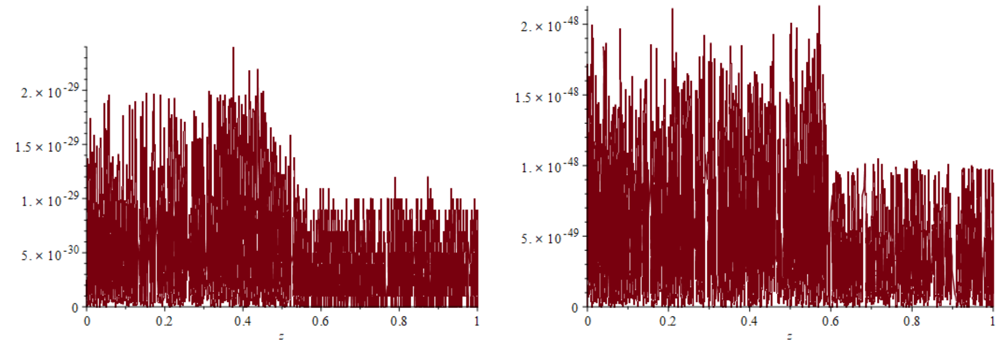

FIGURE 2. The absolute error $|z^4 - P_{z^4}(z)|$ for $N = 10$ and $N = 100$.

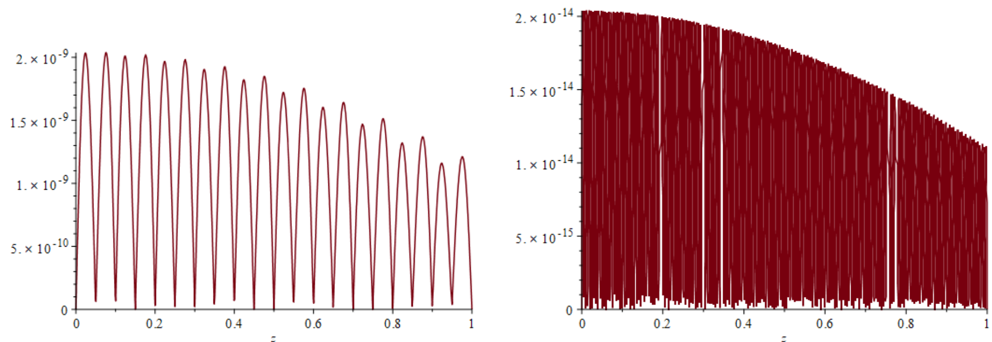

FIGURE 3. The absolute error $|\sin(z) - P_{\sin(z)}(z)|$ for $N = 10$ and $N = 100$.

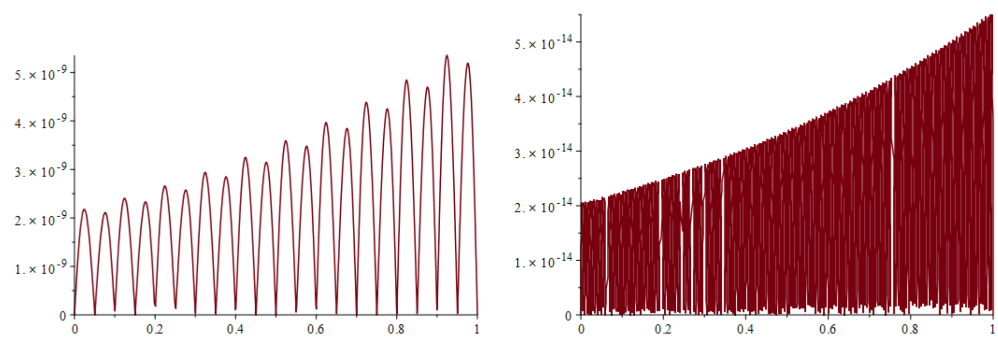

FIGURE 4. The absolute error $|\exp(z) - P_{\exp(z)}(z)|$ for $N = 10$ and $N = 100$.

problem of minimizing the norm of the error functional. As a result, we formulated this problem as a conditional extremum problem using a Lagrange function, which led to a system of linear equations for the optimal coefficients. By solving this system using the discrete operator method, we found the exact analytical expressions for the optimal coefficients and constructed the desired optimal interpolation formula. This approach allows for increased accuracy and efficiency in the interpolation process.

To verify and confirm our theoretical findings, we performed numerical calculations for several functions. Specifically, we approximated the functions $\varphi(x) = x^4$, $\varphi(x) = \sin(x)$, and $\varphi(x) = e^x$ using different values of N (e.g., $N = 10$ and $N = 100$). Our calculations showed that for any function with a fourth derivative, the interpolation error approaches zero proportionally to a higher power of the step size h. These results demonstrate the practical significance of our optimal interpolation method and its potential application in various engineering and scientific computations.

References

[1] Lai M. J., Multivariate Splines and Their Applications. Encyclopedia of Complexity and Systems Science. Springer, (2019).

[2] De Boor C., A Practical Guide to Splines. Springer, (2001).

[3] Imbert-Garard L. M., Sylvand G., A roadmap for Generalized Plane Waves and their interpolation properties. Numer. Math. (2021) Vol. 149, P. 87–137.

[4] Shadimetov K. M., Hayotov A. R., Nuraliev F. A., Optimal interpolation formulas with derivative in the space $L_2^{(m)}(0,1)$. Filomat, (2019) Vol. 33, P. 5661–5675.

[5] Shadimetov K. M., Hayotov A. R., Construction of interpolation splines minimizing semi-norm in $W_2^{(m,m-1)}(0,1)$ space. Bit Numer. Math. (2013) Vol. 53, Iss. 2, P. 545–563.

[6] Babaev S. S., Olimov N., Imomova S., Kuvvatov B., Construction of a natural L spline in $W_{2,\sigma}^{(2,1)}$ space. AIP Conf. Proc. (2024), P. 060021.

[7] Ogniewski J., Cubic Spline Interpolation in Real-Time Applications using Three Control Points. International Conference in Central Europe on Computer Graphics, Visualization and Computer Vision, Plzen, Czech Republic, (2019).

[8] Novak E., Zhang S., Optimal algorithms for numerical integration with Hermite information. J. Complexity, (2022) Vol. 68, P. 101603.

[9] Kumari A., Kumar M., Sharma S., A Hermite interpolation based numerical method for fractional differential equations. Chaos, Solitons and Fractals, (2021) Vol. 152, P. 111427.

[10] Schneider C., Quadrature and Generalized Hermite Interpolation. Numerical Integration. Birkhauser, (1982), P. 212–221.

[11] Babaev S. S., Mamatova N. M., Hayotov A. R., Optimal interpolation formula in space $L_2^{(m)}(0,1)$. Uzb. Math. Journ. (2017), P. 23–31.

[12] Hayotov A. R., Babaev S. S., Olimov N. N., An optimal interpolation formula of Hermite type in the Sobolev space. Filomat, (2024) Vol. 23, P. 8305–8322.

[13] Sobolev S. L., Introduction to the theory of cubature formulas. Nauka, Moscow, (1974).

[14] Shadimetov K. M., Hayotov A. R., Construction of a Discrete Analogue of Differential Operator. Uzb. Math. Journ. (2003), Iss. 2, P. 59–69.

Shadimetov Kh.M.,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences, Tashkent, Uzbekistan;
Tashkent State Transport University, Tashkent, Uzbekistan
e-mail: kholmatshadimetov@mail.ru

Hayotov A.R,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences, Tashkent, Uzbekistan;
Bukhara State University, Bukhara, Uzbekistan
Central Asian University, Tashkent, Uzbekistan
e-mail: hayotov@mail.ru

Olimov N.N.,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences Tashkent, Uzbekistan.
Tashkent International Unversity, Tashkent, Uzbekistan.
Bukhara State University, Bukhara, Uzbekistan
e-mail: olimovnurali8@gmail.com

Uzbek Mathematical Journal
2026, Volume 70, Issue 1, pp. 202–211
DOI: 10.29229/uzmj.2026-1-23

Pauli Gaussian Leonardo quaternions

Yağmur T.

Abstract. In this paper, a new family of Pauli quaternions whose components are the Gaussian Leonardo numbers is defined. These new Pauli quaternions are called Pauli Gaussian Leonardo quaternions. Furthermore, several properties of Pauli Gaussian Leonardo quaternions, including relations with the Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions are investigated. In addition, Binet-like formula, (ordinary) generating function, exponential generating function, and some summation formulas for these Pauli quaternions are given. Moreover, some results are illustrated with examples.

Keywords: Leonardo number, Gaussian Leonardo number, Pauli matrix, Pauli quaternion, Pauli Gaussian Leonardo quaternion
MSC (2020): 11B37, 11B83, 11R52

1. Introduction

The sequences of Fibonacci and Lucas numbers [1] are defined by the relations

$$F_0 = 0, F_1 = 1; F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \tag{1.1}$$

and

$$L_0 = 2, L_1 = 1; L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \tag{1.2}$$

respectively. The Binet formulas for Fibonacci and Lucas numbers are given as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$L_n = \alpha^n + \beta^n,$$

respectively, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Here, α and β are the roots of the characteristic equation $x^2 - x - 1 = 0$ of (1.1) and (1.2). For more information, we refer to [1].

Over the past few years, the sequence of Leonardo numbers, which is listed as $A001595$ in the OEIS [2], has garnered enormous interest. In [3], Catarino and Borges studied some properties of the Leonardo number sequence.

The Leonardo sequence is defined by the relations

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2$$

or

$$Le_n = 2Le_{n-1} - Le_{n-3}, \quad n \geq 3$$

with initial conditions $Le_0 = Le_1 = 1$ and $Le_2 = 3$. Here, Le_n represents the n-th Leonardo number [3].

The Leonardo numbers have a strong relationship with the well-known Fibonacci numbers. Let Le_n be the n-th Leonardo number and F_{n+1} be the $(n+1)$-th Fibonacci number. Then, the Leonardo and Fibonacci numbers are related in the way that follows [3]:

$$Le_n = 2F_{n+1} - 1.$$

Furthermore, the n-th Leonardo number is given as

$$Le_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ (see [3]).

A Gaussian number is a complex number with integer coefficients that was investigated by Gauss [4]. The concept of complex Fibonacci numbers was first introduced and studied by Horadam [5]. Then, the Gaussian Fibonacci and Gaussian Lucas numbers were studied by Jordan [6].

The n-th Gaussian Fibonacci number is defined recursively by the relation

$$GF_n = GF_{n-1} + GF_{n-2}, \quad n \geq 2$$

with $GF_0 = i$ and $GF_1 = 1$. Similarly, the n-th Gaussian Lucas number is defined recursively by

$$GL_n = GL_{n-1} + GL_{n-2}, \quad n \geq 2$$

with $GL_0 = 2 - i$ and $GL_1 = 1 + 2i$ (see [6]).

Exactly like the Fibonacci numbers, the complex Leonardo numbers [7] and the Gaussian Leonardo numbers [8, 9, 10] are introduced and studied. Then, in [11], some new results involving the Gaussian Leonardo numbers are given.

The n-th Gaussian Leonardo number is defined by

$$GLe_n = GLe_{n-1} + GLe_{n-2} + (1+i), \quad n \geq 2 \tag{1.3}$$

or

$$GLe_n = 2GLe_{n-1} - GLe_{n-3}, \quad n \geq 3 \tag{1.4}$$

with $GLe_0 = 1 - i$, $GLe_1 = 1 + i$ and $GLe_2 = 3 + i$. Moreover, for non-negative integer n, the n-th Gaussian Leonardo number is given as

$$GLe_n = \frac{2(\alpha^{n+1} - \beta^{n+1}) + 2i(\alpha^n - \beta^n)}{\alpha - \beta} - (1+i), \tag{1.5}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ (see [8, 9, 10]).

Besides, for Gaussian Leonardo numbers, the followings hold [8, 9, 10, 11]:

$$GLe_n = Le_n + Le_{n-1}i, \tag{1.6}$$

$$GLe_n = 2GF_{n+1} - (1+i), \tag{1.7}$$

$$GLe_{n-1} + GLe_{n+1} = 2GL_{n+1} - 2(1+i), \tag{1.8}$$

$$GLe_{n+1} - GLe_n = 2GF_n, \tag{1.9}$$

$$GLe_{n+2} - GLe_{n-2} = 2GL_{n+1}, \tag{1.10}$$

$$GLe_n + GF_n + GL_n = 2GLe_n + (1+i), \tag{1.11}$$

$$\sum_{k=0}^{n} GLe_k = GLe_{n+2} - (n+2)(1+i), \tag{1.12}$$

$$\sum_{k=0}^{n} GLe_{2k} = GLe_{2n+1} - n - (n+2)i, \tag{1.13}$$

$$\sum_{k=0}^{n} GLe_{2k+1} = GLe_{2n+2} - (n+2) - ni, \tag{1.14}$$

where Le_n is the n-th Leonardo number, GLe_n is the n-th Gaussian Leonardo number, GF_n is the n-th Gaussian Fibonacci number, and GL_n is the n-th Gaussian Lucas number.

A (real) quaternion, introduced by W. R. Hamilton in 1843, is a hyper-complex number. A quaternion q is represented as follows:

$$q = q_0 + q_1 i + q_2 j + q_3 k,$$

where q_0, q_1, q_2, and q_3 are real numbers, and i, j, and k are quaternionic units such that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

For further information, we refer to [12, 13].

The Pauli quaternions are quaternions formed by using the Pauli matrices. The Pauli matrices comprise a collection of 2×2 complex matrices as follows [14, 15]:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma_2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with multiplication rules given by

$$\boldsymbol{\sigma_1}^2 = \boldsymbol{\sigma_2}^2 = \boldsymbol{\sigma_3}^2 = \mathbf{1},$$

$$\boldsymbol{\sigma_1}\boldsymbol{\sigma_2} = -\boldsymbol{\sigma_2}\boldsymbol{\sigma_1} = i\boldsymbol{\sigma_3}, \quad \boldsymbol{\sigma_2}\boldsymbol{\sigma_3} = -\boldsymbol{\sigma_3}\boldsymbol{\sigma_2} = i\boldsymbol{\sigma_1}, \quad \boldsymbol{\sigma_3}\boldsymbol{\sigma_1} = -\boldsymbol{\sigma_1}\boldsymbol{\sigma_3} = i\boldsymbol{\sigma_2}. \tag{1.15}$$

Moreover, the Pauli matrices, named after the German physicist Wolfgang E. Pauli, are Hermitian and unitary (see [14, 15]). The Pauli matrices have found wide applications in various areas, including mathematics, physics, and mathematical physics (see, e.g., [16, 17, 18, 14, 15, 19, 20, 21]).

The set of $\{\mathbf{1}, i\boldsymbol{\sigma_1}, i\boldsymbol{\sigma_2}, i\boldsymbol{\sigma_3}\}$ is the basis of Pauli quaternions. This set is isomorphic to the set of (real) quaternions [15].

A Pauli quaternion is defined by Kim [15] as

$$p = p_0\mathbf{1} + p_1\boldsymbol{\sigma_1} + p_2\boldsymbol{\sigma_2} + p_3\boldsymbol{\sigma_3},$$

where $\boldsymbol{\sigma_1}$, $\boldsymbol{\sigma_2}$, and $\boldsymbol{\sigma_3}$ satisfy the rules (1.15).

The conjugate of a Pauli quaternion p, denoted by $\overline{p}$, is

$$\overline{p} = p_0\mathbf{1} - p_1\boldsymbol{\sigma_1} - p_2\boldsymbol{\sigma_2} - p_3\boldsymbol{\sigma_3}.$$

Furthermore, in [15], Kim investigated algebraic and analytic properties of Pauli quaternions.

In [22], Aydın introduced the Pauli Fibonacci quaternions and obtained some properties involving these Pauli quaternions. Then, in [23], İşbilir et al. defined and studied the Pauli Leonardo quaternions.

The n-th Pauli Leonardo quaternion is defined as

$$Q_P Le_n = Le_n\mathbf{1} + Le_{n+1}\boldsymbol{\sigma_1} + Le_{n+2}\boldsymbol{\sigma_2} + Le_{n+3}\boldsymbol{\sigma_3}, \tag{1.16}$$

where Le_n is the n-th Leonardo number (see [23]).

More recently, in [24], Azak defined the Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions as follow:

The n-th Pauli Gaussian Fibonacci quaternion is

$$Q_P GF_n = GF_n\mathbf{1} + GF_{n+1}\boldsymbol{\sigma_1} + GF_{n+2}\boldsymbol{\sigma_2} + GF_{n+3}\boldsymbol{\sigma_3}, \tag{1.17}$$

where GF_n is the n-th Gaussian Fibonacci number.

The n-th Pauli Gaussian Lucas quaternion is

$$Q_P GL_n = GL_n \mathbf{1} + GL_{n+1}\sigma_{\mathbf{1}} + GL_{n+2}\sigma_{\mathbf{2}} + GL_{n+3}\sigma_{\mathbf{3}}, \tag{1.18}$$

where GL_n is the n-th Gaussian Lucas number.

Furthermore, the author obtained some identities and formulas involving the Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions (see [24]).

Quaternions are an extension of complex numbers. Both complex numbers and quaternions have a rich representation capacity in many scientific disciplines, especially in applied sciences. Also, Pauli matrices are very useful in quantum mechanics as well as in classical mechanics. On the other hand, due to the extraordinary patterns of Fibonacci numbers in nature, the Fibonacci sequence and other sequences that are closely related to this sequence are the focus of many researchers' studies. The Leonardo sequence, a non-homogeneous extension of the Fibonacci sequence, has recently attracted considerable attention by researchers.

Inspired and motivated by some of the above mentioned papers, by bringing together complex (Gaussian) numbers, quaternions, Pauli matrices, and Leonardo numbers, we aim to introduce a new family of quaternions. These numbers will be referred to as Pauli Gaussian Leonardo quaternions. The Pauli Gaussian Leonardo quaternions are an extended description of the Pauli Leonardo quaternions in [23] to the complex case. For our purpose, we first define Pauli quaternions with Gaussian Leonardo number coefficients. Then, we derivate some identities and formulas involving these Pauli quaternions.

2. Main Results

Definition 2.1. For $n \geq 0$, the n-th Pauli Gaussian Leonardo quaternion, denoted by $Q_P GLe_n$, is defined by

$$Q_P GLe_n = GLe_n \mathbf{1} + GLe_{n+1}\sigma_{\mathbf{1}} + GLe_{n+2}\sigma_{\mathbf{2}} + GLe_{n+3}\sigma_{\mathbf{3}}, \tag{2.1}$$

where GLe_n is the n-th Gaussian Leonardo number, and $\sigma_{\mathbf{1}}$, $\sigma_{\mathbf{2}}$, and $\sigma_{\mathbf{3}}$ satisfy the rules (1.15).

Note that, by virtue of (1.6) and (1.16), it is easy to see that

$$Q_P GLe_n = Q_P Le_n + Q_P Le_{n-1} i.$$

By considering the definition of the conjugate of a Pauli quaternion, the conjugate of $Q_P GLe_n$ is defined as

$$\overline{Q_P GLe_n} = GLe_n \mathbf{1} - GLe_{n+1}\sigma_{\mathbf{1}} - GLe_{n+2}\sigma_{\mathbf{2}} - GLe_{n+3}\sigma_{\mathbf{3}}. \tag{2.2}$$

Throughout the paper, let $P = (1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}})$. From the definition of the Pauli Gaussian Leonardo quaternions, we can get the recurrence relations as

$$Q_P GLe_n = Q_P GLe_{n-1} + Q_P GLe_{n-2} + P, \quad n \geq 2 \tag{2.3}$$

or

$$Q_P GLe_n = 2Q_P GLe_{n-1} - Q_P GLe_{n-3}, \quad n \geq 3 \tag{2.4}$$

with

$$\begin{aligned} Q_P GLe_0 &= (1-i)\mathbf{1} + (1+i)\sigma_{\mathbf{1}} + (3+i)\sigma_{\mathbf{2}} + (5+3i)\sigma_{\mathbf{3}}, \\ Q_P GLe_1 &= (1+i)\mathbf{1} + (3+i)\sigma_{\mathbf{1}} + (5+3i)\sigma_{\mathbf{2}} + (9+5i)\sigma_{\mathbf{3}}, \\ Q_P GLe_2 &= (3+i)\mathbf{1} + (5+3i)\sigma_{\mathbf{1}} + (9+5i)\sigma_{\mathbf{2}} + (15+9i)\sigma_{\mathbf{3}}. \end{aligned}$$

Theorem 2.2. *Let Q_PGLe_n be the n-th Pauli Gaussian Leonardo quaternion. Then, for $n \geq 0$, we have*

$$Q_PGLe_n + \overline{Q_PGLe_n} = 2GLe_n\mathbf{1}, \tag{2.5}$$

$$(Q_PGLe_n)^2 = Q_PGLe_n\left(2GLe_n\mathbf{1} - \overline{Q_PGLe_n}\right), \tag{2.6}$$

$$Q_PGLe_n\mathbf{1} - Q_PGLe_{n+1}\sigma_\mathbf{1} - Q_PGLe_{n+2}\sigma_\mathbf{2} - Q_PGLe_{n+3}\sigma_\mathbf{3} = -\left(GLe_{n+1} + 2GL_{n+6} - (1+i)\right)\mathbf{1}. \tag{2.7}$$

Proof. (2.5): From (2.1) and (2.2), it is straightforward.
(2.6): By considering (2.5), we get

$$(Q_PGLe_n)^2 = Q_PGLe_n.Q_PGLe_n = Q_PGLe_n\left(2GLe_n\mathbf{1} - \overline{Q_PGLe_n}\right).$$

(2.7): By virtue of the multiplication rules (1.15) and Equation (2.1), we obtain

$$\begin{aligned}
&Q_PGLe_n\mathbf{1} - Q_PGLe_{n+1}\sigma_\mathbf{1} - Q_PGLe_{n+2}\sigma_\mathbf{2} - Q_PGLe_{n+3}\sigma_\mathbf{3}\\
&\quad= \left(GLe_n\mathbf{1} + GLe_{n+1}\sigma_\mathbf{1} + GLe_{n+2}\sigma_\mathbf{2} + GLe_{n+3}\sigma_\mathbf{3}\right)\mathbf{1}\\
&\qquad- \left(GLe_{n+1}\mathbf{1} + GLe_{n+2}\sigma_\mathbf{1} + GLe_{n+3}\sigma_\mathbf{2} + GLe_{n+4}\sigma_\mathbf{3}\right)\sigma_\mathbf{1}\\
&\qquad- \left(GLe_{n+2}\mathbf{1} + GLe_{n+3}\sigma_\mathbf{1} + GLe_{n+4}\sigma_\mathbf{2} + GLe_{n+5}\sigma_\mathbf{3}\right)\sigma_\mathbf{2}\\
&\qquad- \left(GLe_{n+3}\mathbf{1} + GLe_{n+4}\sigma_\mathbf{1} + GLe_{n+5}\sigma_\mathbf{2} + GLe_{n+6}\sigma_\mathbf{3}\right)\sigma_\mathbf{3}\\
&\quad= \left((GLe_n - GLe_{n+2}) - (GLe_{n+4} + GLe_{n+6})\right)\mathbf{1}.
\end{aligned}$$

From (1.3) and (1.8), we have

$$Q_PGLe_n\mathbf{1} - Q_PGLe_{n+1}\sigma_\mathbf{1} - Q_PGLe_{n+2}\sigma_\mathbf{2} - Q_PGLe_{n+3}\sigma_\mathbf{3} = -\left(GLe_{n+1} + 2GL_{n+6} - (1+i)\right)\mathbf{1}.$$

This completes the proof. □

We now give some identities involving the Pauli Gaussian Leonardo quaternion Q_PGLe_n, including relations with the Pauli Gaussian Fibonacci quaternion Q_PGF_n and Pauli Gaussian Lucas quaternion Q_PGL_n.

Theorem 2.3. *For $n \geq 0$, let Q_PGLe_n be the n-th Pauli Gaussian Leonardo quaternion. Then, the followings hold true:*

$$Q_PGLe_{n-1} + Q_PGLe_{n+1} = 2Q_PGL_{n+1} - 2P, \tag{2.8}$$

$$Q_PGLe_{n+1} - Q_PGLe_n = 2Q_PGF_n, \tag{2.9}$$

$$Q_PGLe_{n+2} - Q_PGLe_{n-2} = 2Q_PGL_{n+1}, \tag{2.10}$$

$$Q_PGLe_n + Q_PGF_n + Q_PGL_n = 2Q_PGLe_n + P. \tag{2.11}$$

Proof. (2.8): By virtue of (1.8), (1.18), and (2.1), we have

$$\begin{aligned}
Q_PGLe_{n-1} + Q_PGLe_{n+1} &= GLe_{n-1}\mathbf{1} + GLe_n\sigma_\mathbf{1} + GLe_{n+1}\sigma_\mathbf{2} + GLe_{n+2}\sigma_\mathbf{3}\\
&\quad+ GLe_{n+1}\mathbf{1} + GLe_{n+2}\sigma_\mathbf{1} + GLe_{n+3}\sigma_\mathbf{2} + GLe_{n+4}\sigma_\mathbf{3}\\
&= (GLe_{n-1} + GLe_{n+1})\mathbf{1} + (GLe_n + GLe_{n+2})\sigma_\mathbf{1}\\
&\quad+ (GLe_{n+1} + GLe_{n+3})\sigma_\mathbf{2} + (GLe_{n+2} + GLe_{n+4})\sigma_\mathbf{3}\\
&= 2\left(GL_{n+1}\mathbf{1} + GL_{n+2}\sigma_\mathbf{1} + GL_{n+3}\sigma_\mathbf{2} + GL_{n+4}\sigma_\mathbf{3}\right)\\
&\quad- 2(1+i)(\mathbf{1} + \sigma_\mathbf{1} + \sigma_\mathbf{2} + \sigma_\mathbf{3})\\
&= 2Q_PGL_{n+1} - 2P.
\end{aligned}$$

(2.9): By virtue of (1.9), (1.17), and (2.1), the desired result can be obtained in a similar manner as Equation (2.8).

(2.10): By virtue of (1.10), (1.18), and (2.1), the proof is similar as Equation (2.8).
(2.11): By virtue of (1.11), (1.17), (1.18), and (2.1), we have

$$\begin{aligned} Q_PGLe_n + Q_PGF_n + Q_PGL_n &= (GLe_n + GF_n + GL_n)\mathbf{1} + (GLe_{n+1} + GF_{n+1} + GL_{n+1})\sigma_\mathbf{1} \\ &\quad + (GLe_{n+2} + GF_{n+2} + GL_{n+2})\sigma_\mathbf{2} + (GLe_{n+3} + GF_{n+3} + GL_{n+3})\sigma_\mathbf{3} \end{aligned}$$

$$\begin{aligned} &= (2GLe_n + (1+i))\mathbf{1} + (2GLe_{n+1} + (1+i))\sigma_\mathbf{1} + (2GLe_{n+2} + (1+i))\sigma_\mathbf{2} + (2GLe_{n+3} + (1+i))\sigma_\mathbf{3} \\ &= 2\left(GLe_n\mathbf{1} + GLe_{n+1}\sigma_\mathbf{1} + GLe_{n+2}\sigma_\mathbf{2} + GLe_{n+3}\sigma_\mathbf{3}\right) + (1+i)(\mathbf{1} + \sigma_\mathbf{1} + \sigma_\mathbf{2} + \sigma_\mathbf{3}) = 2Q_PGLe_n + P \end{aligned}$$

which completes the proof. □

Example 2.4. If we take $n = 1$ in Equation (2.8), $n = 0$ in Equations (2.9) and (2.11), and $n = 2$ in Equation (2.10) in Theorem 2.3 then, we get

$$\begin{aligned} Q_PGLe_0 + Q_PGLe_2 &= (1-i)\mathbf{1} + (1+i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (5+3i)\sigma_\mathbf{3} \\ &\quad + (3+i)\mathbf{1} + (5+3i)\sigma_\mathbf{1} + (9+5i)\sigma_\mathbf{2} + (15+9i)\sigma_\mathbf{3} \\ &= 4.\mathbf{1} + (6+4i)\sigma_\mathbf{1} + (12+6i)\sigma_\mathbf{2} + (20+12i)\sigma_\mathbf{3} \end{aligned}$$

$$\begin{aligned} 2Q_PGL_2 - 2P &= 2\left((3+i)\mathbf{1} + (4+3i)\sigma_\mathbf{1} + (7+4i)\sigma_\mathbf{2} + (11+7i)\sigma_\mathbf{3}\right) - 2(1+i)(\mathbf{1} + \sigma_\mathbf{1} + \sigma_\mathbf{2} + \sigma_\mathbf{3}) \\ &= 4.\mathbf{1} + (6+4i)\sigma_\mathbf{1} + (12+6i)\sigma_\mathbf{2} + (20+12i)\sigma_\mathbf{3}, \end{aligned}$$

$$\begin{aligned} Q_PGLe_1 - Q_PGLe_0 &= (1+i)\mathbf{1} + (3+i)\sigma_\mathbf{1} + (5+3i)\sigma_\mathbf{2} + (9+5i)\sigma_\mathbf{3} \\ &\quad - ((1-i)\mathbf{1} + (1+i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (5+3i)\sigma_\mathbf{3}) \\ &= 2i\mathbf{1} + 2\sigma_\mathbf{1} + (2+2i)\sigma_\mathbf{2} + (4+2i)\sigma_\mathbf{3} \\ 2Q_PGF_0 &= 2\left(i\mathbf{1} + 1\sigma_\mathbf{1} + (1+i)\sigma_\mathbf{2} + (2+i)\sigma_\mathbf{3}\right) \\ &= 2i\mathbf{1} + 2\sigma_\mathbf{1} + (2+2i)\sigma_\mathbf{2} + (4+2i)\sigma_\mathbf{3}, \end{aligned}$$

$$\begin{aligned} Q_PGLe_4 - Q_PGLe_0 &= (9+5i)\mathbf{1} + (15+9i)\sigma_\mathbf{1} + (25+15i)\sigma_\mathbf{2} + (41+25i)\sigma_\mathbf{3} \\ &\quad - ((1-i)\mathbf{1} + (1+i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (5+3i)\sigma_\mathbf{3}) \\ &= (8+6i)\mathbf{1} + (14+8i)\sigma_\mathbf{1} + (22+14i)\sigma_\mathbf{2} + (36+22i)\sigma_\mathbf{3} \\ 2Q_PGL_3 &= 2\left((4+3i)\mathbf{1} + (7+4i)\sigma_\mathbf{1} + (11+7i)\sigma_\mathbf{2} + (18+11i)\sigma_\mathbf{3}\right) \\ &= (8+6i)\mathbf{1} + (14+8i)\sigma_\mathbf{1} + (22+14i)\sigma_\mathbf{2} + (36+22i)\sigma_\mathbf{3} \end{aligned}$$

and

$$\begin{aligned} Q_PGLe_0 + Q_PGF_0 + Q_PGL_0 &= (1-i)\mathbf{1} + (1+i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (5+3i)\sigma_\mathbf{3} \\ &\quad + i\mathbf{1} + 1\sigma_\mathbf{1} + (1+i)\sigma_\mathbf{2} + (2+i)\sigma_\mathbf{3} \\ &\quad + (2-i)\mathbf{1} + (1+2i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (4+3i)\sigma_\mathbf{3} \\ &= (3-i)\mathbf{1} + (3+3i)\sigma_\mathbf{1} + (7+3i)\sigma_\mathbf{2} + (11+7i)\sigma_\mathbf{3} \end{aligned}$$

$$\begin{aligned} 2Q_PGLe_0 + P &= 2\left((1-i)\mathbf{1} + (1+i)\sigma_\mathbf{1} + (3+i)\sigma_\mathbf{2} + (5+3i)\sigma_\mathbf{3}\right) + (1+i)(\mathbf{1} + \sigma_\mathbf{1} + \sigma_\mathbf{2} + \sigma_\mathbf{3}) \\ &= (3-i)\mathbf{1} + (3+3i)\sigma_\mathbf{1} + (7+3i)\sigma_\mathbf{2} + (11+7i)\sigma_\mathbf{3}, \end{aligned}$$

respectively.

Theorem 2.5. *For $n \geq 0$, the Binet-like formula for the Pauli Gaussian Leonardo quaternions is given by*

$$Q_PGLe_n = \frac{2(\alpha+i)\alpha^*\alpha^n - 2(\beta+i)\beta^*\beta^n}{\alpha-\beta} - P, \tag{2.12}$$

where $\alpha^* = \mathbf{1} + \alpha\sigma_\mathbf{1} + \alpha^2\sigma_\mathbf{2} + \alpha^3\sigma_\mathbf{3}$ and $\beta^* = \mathbf{1} + \beta\sigma_\mathbf{1} + \beta^2\sigma_\mathbf{2} + \beta^3\sigma_\mathbf{3}$.

Proof. From (1.5) and (2.1), we have

$$\begin{aligned} Q_P GLe_n &= GLe_n \mathbf{1} + GLe_{n+1}\sigma_{\mathbf{1}} + GLe_{n+2}\sigma_{\mathbf{2}} + GLe_{n+3}\sigma_{\mathbf{3}} \\ &= \left(\frac{2(\alpha^{n+1}-\beta^{n+1}) + 2i(\alpha^n - \beta^n)}{\alpha-\beta} - (1+i)\right)\mathbf{1} \\ &\quad + \left(\frac{2(\alpha^{n+2}-\beta^{n+2}) + 2i(\alpha^{n+1} - \beta^{n+1})}{\alpha-\beta} - (1+i)\right)\sigma_{\mathbf{1}} \\ &\quad + \left(\frac{2(\alpha^{n+3}-\beta^{n+3}) + 2i(\alpha^{n+2} - \beta^{n+2})}{\alpha-\beta} - (1+i)\right)\sigma_{\mathbf{2}} \\ &\quad + \left(\frac{2(\alpha^{n+4}-\beta^{n+4}) + 2i(\alpha^{n+3} - \beta^{n+3})}{\alpha-\beta} - (1+i)\right)\sigma_{\mathbf{3}} \\ &= \frac{2\alpha^{n+1}(\mathbf{1} + \alpha\sigma_{\mathbf{1}} + \alpha^2\sigma_{\mathbf{2}} + \alpha^3\sigma_{\mathbf{3}}) - 2\beta^{n+1}(\mathbf{1} + \beta\sigma_{\mathbf{1}} + \beta^2\sigma_{\mathbf{2}} + \beta^3\sigma_{\mathbf{3}})}{\alpha-\beta} \\ &\quad + \frac{2i\alpha^{n}(\mathbf{1} + \alpha\sigma_{\mathbf{1}} + \alpha^2\sigma_{\mathbf{2}} + \alpha^3\sigma_{\mathbf{3}}) - 2i\beta^{n}(\mathbf{1} + \beta\sigma_{\mathbf{1}} + \beta^2\sigma_{\mathbf{2}} + \beta^3\sigma_{\mathbf{3}})}{\alpha-\beta} \\ &\quad - (1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) \\ &= \frac{2(\alpha+i)\alpha^*\alpha^n - 2(\beta+i)\beta^*\beta^n}{\alpha-\beta} - P. \end{aligned}$$

Thus, the proof is completed. □

Example 2.6. If we take $n = 2$ in Theorem 2.5 , then $Q_P GLe_2$ can be obtained as

$$\begin{aligned} Q_P GLe_2 &= \frac{2(\alpha+i)\alpha^*\alpha^2 - 2(\beta+i)\beta^*\beta^2}{\alpha-\beta} - P \\ &= \frac{2\left(\frac{1+\sqrt{5}}{2}+i\right)\left(\frac{3+\sqrt{5}}{2}\right)\left(\mathbf{1}+\left(\frac{1+\sqrt{5}}{2}\right)\sigma_{\mathbf{1}}+\left(\frac{3+\sqrt{5}}{2}\right)\sigma_{\mathbf{2}}+\left(2+\sqrt{5}\right)\sigma_{\mathbf{3}}\right)}{\sqrt{5}} \\ &\quad - \frac{2\left(\frac{1-\sqrt{5}}{2}+i\right)\left(\frac{3-\sqrt{5}}{2}\right)\left(\mathbf{1}+\left(\frac{1-\sqrt{5}}{2}\right)\sigma_{\mathbf{1}}+\left(\frac{3-\sqrt{5}}{2}\right)\sigma_{\mathbf{2}}+\left(2-\sqrt{5}\right)\sigma_{\mathbf{3}}\right)}{\sqrt{5}} - (1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) \\ &= \frac{\left(4\sqrt{5}.\mathbf{1}+6\sqrt{5}\sigma_{\mathbf{1}}+10\sqrt{5}\sigma_{\mathbf{2}}+16\sqrt{5}\sigma_{\mathbf{3}}\right)+i\left(2\sqrt{5}.\mathbf{1}+4\sqrt{5}\sigma_{\mathbf{1}}+6\sqrt{5}\sigma_{\mathbf{2}}+10\sqrt{5}\sigma_{\mathbf{3}}\right)}{\sqrt{5}} - (1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) \\ &= (4+2i)\mathbf{1} + (6+4i)\sigma_{\mathbf{1}} + (10+6i)\sigma_{\mathbf{2}} + (16+10i)\sigma_{\mathbf{3}} - (1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) \\ &= (3+i)\mathbf{1} + (5+3i)\sigma_{\mathbf{1}} + (9+5i)\sigma_{\mathbf{2}} + (15+9i)\sigma_{\mathbf{3}}. \end{aligned}$$

Theorem 2.7. *The (ordinary) generating function of the Pauli Gaussian Leonardo quaternions is*

$$g(x) = \frac{Q_P GLe_0 + (Q_P GLe_1 - 2Q_P GLe_0)x + (Q_P GLe_2 - 2Q_P GLe_1)x^2}{1 - 2x + x^3}.$$

Proof. Let $g(x)$ be the generating function of the Pauli Gaussian Leonardo quaternions. From the definition of the generating function of a sequence, we can write

$$g(x) = \sum_{n=0}^{\infty} Q_P GLe_n x^n. \tag{2.13}$$

By virtue of (2.4) and (2.13), we have

$$\begin{aligned} g(x) &= Q_P GLe_0 + Q_P GLe_1 x + Q_P GLe_2 x^2 + \sum_{n=3}^{\infty} Q_P GLe_n x^n \\ &= Q_P GLe_0 + Q_P GLe_1 x + Q_P GLe_2 x^2 + \sum_{n=3}^{\infty} (2Q_P GLe_{n-1} - Q_P GLe_{n-3}) x^n \\ &= Q_P GLe_0 + Q_P GLe_1 x + Q_P GLe_2 x^2 + 2x\sum_{n=3}^{\infty} Q_P GLe_{n-1} x^{n-1} - x^3 \sum_{n=3}^{\infty} Q_P GLe_{n-3} x^{n-3} \\ &= Q_P GLe_0 + (Q_P GLe_1 - 2Q_P GLe_0)x + (Q_P GLe_2 - 2Q_P GLe_1)x^2 \\ &\quad + 2x\sum_{n=0}^{\infty} Q_P GLe_n x^n - x^3 \sum_{n=0}^{\infty} Q_P GLe_n x^n. \end{aligned}$$

Then, it follows that

$$g(x)(1-2x+x^3) = Q_PGLe_0 + (Q_PGLe_1 - 2Q_PGLe_0)x + (Q_PGLe_2 - 2Q_PGLe_1)x^2$$

which completes the proof. □

Theorem 2.8. *The exponential generating function of the Pauli Gaussian Leonardo quaternions is*

$$g(t) = \frac{2(\alpha+i)\alpha^* e^{\alpha t} - 2(\beta+i)\beta^* e^{\beta t}}{\alpha-\beta} - Pe^t.$$

Proof. Let $g(t) = \sum_{n=0}^{\infty} Q_PGLe_n \dfrac{t^n}{n!}$ be the exponential generating function for the Pauli Gaussian Leonardo quaternions. Then, from (2.12), we have

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty}\left(\frac{2(\alpha+i)\alpha^*\alpha^n - 2(\beta+i)\beta^*\beta^n}{\alpha-\beta} - P\right)\frac{t^n}{n!} \\ &= \frac{2(\alpha+i)\alpha^*}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!} - \frac{2(\beta+i)\beta^*}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\beta t)^n}{n!} - P\sum_{n=0}^{\infty}\frac{t^n}{n!} \\ &= \frac{2(\alpha+i)\alpha^*}{\alpha-\beta}e^{\alpha t} - \frac{2(\beta+i)\beta^*}{\alpha-\beta}e^{\beta t} - Pe^t. \end{aligned}$$

Hence, the proof is completed. □

Theorem 2.9. *Let Q_PGLe_n be the n-th Pauli Gaussian Leonardo quaternion. Then, we have*

$$\sum_{k=0}^{n} Q_PGLe_k = Q_PGLe_{n+2} - (n+2)P - 2(\sigma_{\mathbf{1}} + (2+i)\sigma_{\mathbf{2}} + (4+2i)\sigma_{\mathbf{3}}), \tag{2.14}$$

$$\sum_{k=0}^{n} Q_PGLe_{2k} = Q_PGLe_{2n+1} - (n+2)P + 2(\mathbf{1} + i\sigma_{\mathbf{1}} - \sigma_{\mathbf{3}}), \tag{2.15}$$

$$\sum_{k=0}^{n} Q_PGLe_{2k+1} = Q_PGLe_{2n+2} - (n+2)P + 2(i\mathbf{1} - \sigma_{\mathbf{2}} - (2+i)\sigma_{\mathbf{3}}). \tag{2.16}$$

Proof. (2.14): By virtue of (1.12), and (2.1), we get

$$\begin{aligned} \sum_{k=0}^{n} Q_PGLe_k &= \sum_{k=0}^{n}\left(GLe_k\mathbf{1} + GLe_{k+1}\sigma_{\mathbf{1}} + GLe_{k+2}\sigma_{\mathbf{2}} + GLe_{k+3}\sigma_{\mathbf{3}}\right) \\ &= \left(\sum_{k=0}^{n} GLe_k\right)\mathbf{1} + \left(\sum_{k=0}^{n} GLe_{k+1}\right)\sigma_{\mathbf{1}} + \left(\sum_{k=0}^{n} GLe_{k+2}\right)\sigma_{\mathbf{2}} + \left(\sum_{k=0}^{n} GLe_{k+3}\right)\sigma_{\mathbf{3}} \\ &= \left(GLe_{n+2} - (n+2)(1+i)\right)\mathbf{1} + \left(GLe_{n+3} - (n+2)(1+i) - 2\right)\sigma_{\mathbf{1}} \\ &\quad + \left(GLe_{n+4} - (n+2)(1+i) - (4+2i)\right)\sigma_{\mathbf{2}} + \left(GLe_{n+5} - (n+2)(1+i) - (8+4i)\right)\sigma_{\mathbf{3}} \\ &= GLe_{n+2}\mathbf{1} + GLe_{n+3}\sigma_{\mathbf{1}} + GLe_{n+4}\sigma_{\mathbf{2}} + GLe_{n+5}\sigma_{\mathbf{3}} \\ &\quad - (n+2)(1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) - 2(\sigma_{\mathbf{1}} + (2+i)\sigma_{\mathbf{2}} + (4+2i)\sigma_{\mathbf{3}}) \\ &= Q_PGLe_{n+2} - (n+2)P - 2(\sigma_{\mathbf{1}} + (2+i)\sigma_{\mathbf{2}} + (4+2i)\sigma_{\mathbf{3}}). \end{aligned}$$

By considering (1.13) and (1.14), Equations (2.15) and (2.16) can be obtained similarly. □

Example 2.10. For $n = 3$ in Theorem 2.9, we obtain

$$\begin{aligned} \sum_{k=0}^{3} Q_PGLe_k &= Q_PGLe_0 + Q_PGLe_1 + Q_PGLe_2 + Q_PGLe_3 \\ &= (1-i)\mathbf{1} + (1+i)\sigma_{\mathbf{1}} + (3+i)\sigma_{\mathbf{2}} + (5+3i)\sigma_{\mathbf{3}} \\ &\quad + (1+i)\mathbf{1} + (3+i)\sigma_{\mathbf{1}} + (5+3i)\sigma_{\mathbf{2}} + (9+5i)\sigma_{\mathbf{3}} \\ &\quad + (3+i)\mathbf{1} + (5+3i)\sigma_{\mathbf{1}} + (9+5i)\sigma_{\mathbf{2}} + (15+9i)\sigma_{\mathbf{3}} \\ &\quad + (5+3i)\mathbf{1} + (9+5i)\sigma_{\mathbf{1}} + (15+9i)\sigma_{\mathbf{2}} + (25+15i)\sigma_{\mathbf{3}} \\ &= (10+4i)\mathbf{1} + (18+10i)\sigma_{\mathbf{1}} + (32+18i)\sigma_{\mathbf{2}} + (54+32i)\sigma_{\mathbf{3}} \end{aligned}$$

$$
\begin{aligned}
&Q_P GLe_5 - 5P - 2(\sigma_{\mathbf{1}} + (2+i)\sigma_{\mathbf{2}} + (4+2i)\sigma_{\mathbf{3}}) \\
&\quad = (15+9i)\mathbf{1} + (25+15i)\sigma_{\mathbf{1}} + (41+25i)\sigma_{\mathbf{2}} + (67+41i)\sigma_{\mathbf{3}} \\
&\qquad - 5(1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) - 2(\sigma_{\mathbf{1}} + (2+i)\sigma_{\mathbf{2}} + (4+2i)\sigma_{\mathbf{3}}) \\
&\quad = (10+4i)\mathbf{1} + (18+10i)\sigma_{\mathbf{1}} + (32+18i)\sigma_{\mathbf{2}} + (54+32i)\sigma_{\mathbf{3}},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{3} Q_P GLe_{2k} &= Q_P GLe_0 + Q_P GLe_2 + Q_P GLe_4 + Q_P GLe_6 \\
&= (1-i)\mathbf{1} + (1+i)\sigma_{\mathbf{1}} + (3+i)\sigma_{\mathbf{2}} + (5+3i)\sigma_{\mathbf{3}} \\
&\quad + (3+i)\mathbf{1} + (5+3i)\sigma_{\mathbf{1}} + (9+5i)\sigma_{\mathbf{2}} + (15+9i)\sigma_{\mathbf{3}} \\
&\quad + (9+5i)\mathbf{1} + (15+9i)\sigma_{\mathbf{1}} + (25+15i)\sigma_{\mathbf{2}} + (41+25i)\sigma_{\mathbf{3}} \\
&\quad + (25+15i)\mathbf{1} + (41+25i)\sigma_{\mathbf{1}} + (67+41i)\sigma_{\mathbf{2}} + (109+67i)\sigma_{\mathbf{3}} \\
&= (38+20i)\mathbf{1} + (62+38i)\sigma_{\mathbf{1}} + (104+62i)\sigma_{\mathbf{2}} + (170+104i)\sigma_{\mathbf{3}}
\end{aligned}
$$

$$
\begin{aligned}
&Q_P GLe_7 - 5P + 2(\mathbf{1} + i\sigma_{\mathbf{1}} - \sigma_{\mathbf{3}}) = (41+25i)\mathbf{1} + (67+41i)\sigma_{\mathbf{1}} + (109+67i)\sigma_{\mathbf{2}} + (177+109i)\sigma_{\mathbf{3}} \\
&\qquad - 5(1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) + 2(\mathbf{1} + i\sigma_{\mathbf{1}} - \sigma_{\mathbf{3}}) \\
&\quad = (38+20i)\mathbf{1} + (62+38i)\sigma_{\mathbf{1}} + (104+62i)\sigma_{\mathbf{2}} + (170+104i)\sigma_{\mathbf{3}}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{3} Q_P GLe_{2k+1} &= Q_P GLe_1 + Q_P GLe_3 + Q_P GLe_5 + Q_P GLe_7 \\
&= (1+i)\mathbf{1} + (3+i)\sigma_{\mathbf{1}} + (5+3i)\sigma_{\mathbf{2}} + (9+5i)\sigma_{\mathbf{3}} \\
&\quad + (5+3i)\mathbf{1} + (9+5i)\sigma_{\mathbf{1}} + (15+9i)\sigma_{\mathbf{2}} + (25+15i)\sigma_{\mathbf{3}} \\
&\quad + (15+9i)\mathbf{1} + (25+15i)\sigma_{\mathbf{1}} + (41+25i)\sigma_{\mathbf{2}} + (67+41i)\sigma_{\mathbf{3}} \\
&\quad + (41+25i)\mathbf{1} + (67+41i)\sigma_{\mathbf{1}} + (109+67i)\sigma_{\mathbf{2}} + (177+109i)\sigma_{\mathbf{3}} \\
&= (62+38i)\mathbf{1} + (104+62i)\sigma_{\mathbf{1}} + (170+104i)\sigma_{\mathbf{2}} + (278+170i)\sigma_{\mathbf{3}}
\end{aligned}
$$

$$
\begin{aligned}
&Q_P GLe_8 - 5P + 2(i\mathbf{1} - \sigma_{\mathbf{2}} - (2+i)\sigma_{\mathbf{3}}) = (67+41i)\mathbf{1} + (109+67i)\sigma_{\mathbf{1}} + (177+109i)\sigma_{\mathbf{2}} \\
&\qquad + (287+177i)\sigma_{\mathbf{3}} - 5(1+i)(\mathbf{1} + \sigma_{\mathbf{1}} + \sigma_{\mathbf{2}} + \sigma_{\mathbf{3}}) + 2(i\mathbf{1} - \sigma_{\mathbf{2}} - (2+i)\sigma_{\mathbf{3}}) \\
&\quad = (62+38i)\mathbf{1} + (104+62i)\sigma_{\mathbf{1}} + (170+104i)\sigma_{\mathbf{2}} + (278+170i)\sigma_{\mathbf{3}},
\end{aligned}
$$

respectively.

3. Conclusions

In the current study, complex (Gaussian) numbers, quaternions, Pauli matrices, and Leonardo numbers are combined to develop a new class of quaternions. These newly defined quaternions are referred to as Pauli Gaussian Leonardo quaternions. Moreover, many formulas for these quaternions, such as the recurrence relation, Binet-like formula, ordinary generating function, exponential generating function, and some summation formulas, are presented. Also, some relationships between the Pauli Gaussian Leonardo quaternions, Pauli Gaussian Fibonacci quaternions, and Pauli Gaussian Lucas quaternions are established. Additionally, examples of some results obtained in this paper are provided.

Pauli matrices and quaternions are extensively used in mathematics and physics, especially in quantum mechanics. We believe that the new number family we propose may offer a new perspective to researchers working in the related fields.

References

[1] Koshy T., Fibonacci and Lucas Numbers with Applications. John Wiley and Sons, New York, (2001).

[2] Sloane N., The Online Encyclopedia of Integer Sequences. The OEIS Foundation Inc., Highland Park, New York,

[3] Catarino P., Borges A., On Leonardo numbers. Acta Math. Univ. Comenianae, (2020) Vol. 89, Iss. 1, P. 75–86.

[4] Gauss C., Theoria Residuorum Biquadraticorum: Commentatio Prima. Sumtibus Dieterichianis, Gottingae, (1832).

[5] Horadam A., Complex Fibonacci numbers and Fibonacci quaternions. American Math. Monthly, (1963) Vol. 70, P. 289–291.

[6] Jordan J., Gaussian Fibonacci and Lucas numbers. Fibonacci Quart. (1965) Vol. 3, P. 315–318.

[7] Karataş A., On complex Leonardo numbers. Notes on Number Theory and Discrete Mathematics, (2022) Vol. 28, Iss. 3, P. 458–465.

[8] Kara N., Yilmaz F., On hybrid numbers with Gaussian Leonardo coefficients. Mathematics, (2023) Vol. 11, 1551.

[9] Prasad K., Mohanty R., Kumari M., Mahato H., Some new families of generalized k-Leonardo and Gaussian Leonardo numbers. Commun. Combinatorics Optim. (2024) Vol. 9, Iss. 3, P. 539–553.

[10] Taşcı D., On Gaussian Leonardo numbers. Contrib. Math. (2023) Vol. 7, P. 34–40.

[11] Yağmur T., On Gaussian Leonardo hybrid polynomials. Symmetry, (2023) Vol. 15, 1422.

[12] Hamilton W., Lectures on Quaternions. Hodges and Smith, Dublin, (1853).

[13] Ward J., Quaternions and Cayley Numbers: Algebra and Applications. Kluwer, London, (1997).

[14] Condon E., Morse P., Quantum Mechanics. McGraw-Hill, New York, (1931).

[15] Kim J., A representation of De Moivre's formula over Pauli-quaternions. Ann. Acad. Rom. Sci. Ser. Math. Appl. (2017) Vol. 9, Iss. 2, P. 145–151.

[16] Akbıyık M., Akbıyık S., Yılmaz F., The matrices of Pauli quaternions, their De Moivre's and Euler's formulas. Int. J. Geom. Methods Mod. Phys. (2022) Vol. 19, Iss. 11, 2250175.

[17] Arfken G., Mathematical Methods for Physicists. Academic Press, San Diego, California, (1985).

[18] Bhatia R., Jain T., The geometric mean of exponentials of Pauli matrices. J. Ramanujan Math. Soc. (2015) Vol. 30, Iss. 2, P. 199–204.

[19] Longe P., The properties of the Pauli matrices A, B, C and the conjugation of charge. Physica, (1966) Vol. 32, Iss. 3, P. 603–610.

[20] Patera J., Zassenhaus H., The Pauli matrices in n dimensions and finest gradings of simple Lie algebras of type A_{n-1}. J. Math. Phys. (1988) Vol. 29, Iss. 3, P. 665–673.

[21] Soeken M., Miller D., Drechsler R., Quantum circuits employing roots of the Pauli matrices. Physical Review A, (2013) Vol. 88, 042322.

[22] Aydın F., Pauli-Fibonacci quaternions. Notes on Number Theory and Discrete Mathematics, (2021) Vol. 27, Iss. 3, P. 184–193.

[23] İşbilir Z., Akyiğit M., Tosun M., Pauli-Leonardo quaternions. Notes on Number Theory and Discrete Mathematics, (2023) Vol. 29, Iss. 1, P. 1–16.

[24] Azak A., Pauli Gaussian Fibonacci and Pauli Gaussian Lucas quaternions. Mathematics, (2022) Vol. 10, 4655.

Yağmur T.,
Department of Mathematics, Aksaray University,
Aksaray, Türkiye
e-mail: tulayyagmurr@gmail.com

Contents

Computer imposition: *A.F. Aliyev*

The journal was registered by the Press and Information Agency of the Republic of Uzbekistan on December 22, 2006. Register. No 0044.

Handed over to the set on 24/03/2026. Signed for printing on 03/04/2026
Format 60×84 1/16. Literary typeface. Offset printing.

V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences 9 University st. 100174

Printed in a printing house "MERIT-PRINT"
Tashkent city, Yakkasaray district, Sh. Rustaveli street, 91