

UZBEKISTAN ACADEMY OF SCIENCES
V.I. ROMANOVSKIY INSTITUTE OF MATHEMATICS

UZBEK MATHEMATICAL JOURNAL

Journal was founded in 1957. Until 1991 it was named
“Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk”. Since 1991 it is known as
“Uzbek Mathematical Journal”. It has 4 issues annually.

Volume 70 Issue 3 2026

Uzbek Mathematical Journal is abstracted and indexed by

SCOPUS

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MathSciNet

Zentralblatt MATH

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Small-noise asymptotics of exit times for a class of nonlinear autoregressive processes

Aliev A., Dzhililov A., Fontana R.

Abstract. We study exit times from the interval $[-1, 1]$ for the nonlinear autoregressive process given, for $n \geq 0$, by

$$X_{n+1}^{(\varepsilon)}(\varphi_{f,g}) = \varphi_{f,g}(X_n^{(\varepsilon)}) + \varepsilon \xi_{n+1},$$

where $\varepsilon > 0$ is a small noise parameter, $\{\xi_n\}$ is a sequence of independent standard Gaussian random variables, and $\varphi_{f,g}$ is constructed from two continuous nondecreasing maps $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $f(0) = g(0) = 0$. We consider two classes of such maps. In the first class, the expansive behavior persists up to the endpoints of the interval. In the second class, the maps are expansive on inner intervals and linear and contractive on the corresponding outer intervals. For the first class, we prove that the logarithmic growth rate of the expected exit time vanishes as $\varepsilon \rightarrow 0$. For the second class, we determine the exact logarithmic rate of the finite-horizon exit probability and derive an explicit upper bound for the expected exit time. More precisely, our analysis shows that the inner expansive nonlinear part of the drift has zero variational cost, while in the contractive case the outer linear part determines the explicit exponential rate. This extends earlier estimates obtained for particular piecewise-linear autoregressive models to broader classes of monotone nonlinear processes.

Keywords: Exit time, nonlinear autoregressive process, small-noise asymptotics, Markov chain, large deviation principle

MSC (2020): 60F10, 60J05, 37H10

1. INTRODUCTION

The study of exit times plays an important role in understanding the behavior of stochastic processes and has applications in population dynamics (see [1], [2], [3]), financial analysis [4], and surveillance analysis [5], among other areas. In this paper, we investigate exit times for nonlinear autoregressive processes driven by Gaussian noise. Throughout, the drift map $f : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be continuous and nondecreasing. Such processes generalize classical linear autoregressive models and may exhibit regime-dependent dynamics, making their first-passage behavior particularly interesting.

Let $\{\xi_n\}_{n \geq 1}$ be a sequence of i.i.d. random variables with $\xi_1 \sim N(0, 1)$. Consider the autoregressive process

$$X_{n+1}^{(\varepsilon)}(f) = f(X_n^{(\varepsilon)}) + \varepsilon \xi_{n+1}, \quad n \geq 0, \quad X_0^{(\varepsilon)} = x_0 \in (-1, 1). \quad (1.1)$$

We define the exit time from the interval $[-1, 1]$ by (see [6], [7], [8], [9])

$$\tau^{(\varepsilon)}(f) := \min\{k \geq 1 : |X_k^{(\varepsilon)}(f)| \geq 1\}, \quad (1.2)$$

where $X_k^{(\varepsilon)}(f)$ is defined by (1.1). Several techniques have been used to study exit times for autoregressive models, including martingale methods and large deviation principles (LDPs). In particular, martingale methods have been used to obtain analytical approximations for the distribution and expectation of exit times in AR(1) processes (see [10], [11], [12], [13]). These methods have also proved effective in the analysis of Ornstein–Uhlenbeck processes in continuous time. In [6], Klebaner and Liptser established an LDP for a general nonlinear autoregressive process of the form

$$X_{n+1}^{(\varepsilon)}(g) := g(X_n^{(\varepsilon)}, \dots, X_{n-m+1}^{(\varepsilon)}, \varepsilon \xi_{n+1}), \quad (1.3)$$

where $\varepsilon > 0$, $m \geq 1$, and $g(x_1, \dots, x_m, y)$ is continuous. For comparison, consider a linear autoregressive AR(1) sequence $\mathbb{X}(\lambda) := \{X_n(\lambda), n = 1, 2, \dots\}$ defined by

$$X_{n+1}^{(\varepsilon)}(\lambda) = \lambda X_n^{(\varepsilon)}(\lambda) + \varepsilon \xi_{n+1}, \quad n \geq 0, \quad X_0^{(\varepsilon)} = x_0, \quad (1.4)$$

where $\{\xi_n\}$ is a sequence of i.i.d. random variables with $\xi_1 \sim N(0, 1)$ and λ is a constant. Applying the LDP to the linear AR(1) process $\mathbb{X}(\lambda)$, Klebaner and Liptser [6] obtained the following upper bound for the expected exit time.

Theorem 1.1 (see [6]). *Let $\tau^{(\varepsilon)}(\lambda)$ be the exit time from the interval $[-1, 1]$ of the random process $\mathbb{X}(\lambda)$. If $|\lambda| < 1$, then*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\lambda) \leq \frac{1}{2}(1 - \lambda^2).$$

A corresponding lower bound can be obtained by a different method [9]. The multivariate case was studied in [8], and the results were later extended to ARMA processes in [14]. Various aspects of exit-time problems for linear autoregressive processes have been investigated by many authors (see, for example, [15], [16], [17]). A natural problem is to extend these results from linear contractive maps to nonlinear maps. Allen et al. [1] considered examples involving nonlinear contractive maps. Piecewise-linear maps provide a natural intermediate setting between linear and general nonlinear models. Högnäs and Jung [7] studied AR(1) processes driven by continuous increasing contractive maps with a fixed point at $x = 0$. In particular, obtaining an upper bound for $\mathbb{E}\tau^{(\varepsilon)}(f)$ can be reduced to minimizing sums of the form

$$S_N(y_0, y_1, \dots, y_N) := \frac{1}{2} \sum_{i=1}^N (y_i - f(y_{i-1}))^2.$$

For details, see the next section. Högnäs and Jung [7] established several useful properties of these sums. Applying those properties to particular piecewise-linear maps, they obtained upper bounds for $\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{E}\tau^{(\varepsilon)}(f)$ and computed the corresponding minimum values of S_N . In the present work, we extend their analysis beyond particular piecewise-linear examples. We consider two-sided drift maps $\varphi_{f,g}$ constructed from continuous nondecreasing one-sided maps f and g that are expansive on inner intervals. The expansive behavior either persists up to the endpoints or is followed by linear contractive behavior on the outer intervals. The precise classes of maps are introduced below.

The contribution of the present paper lies in the explicit solution of the resulting variational problem for these classes of monotone nonlinear drift maps. Unlike the particular piecewise-linear examples treated previously, the maps considered here may exhibit general continuous nondecreasing nonlinear behavior on the inner interval. We show that this inner expansive part has zero variational cost for reaching the interface point. Consequently, in the contractive case, the logarithmic exit cost is determined entirely by the outer linear part of the drift. This leads to an explicit formula for the finite-horizon exit probability starting from the origin and to an upper bound for the expected exit time.

Let $\mathcal{M}$ denote the class of all continuous nondecreasing functions $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $h(0) = 0$. We introduce the following subclasses of $\mathcal{M}$ (see Fig. 1).

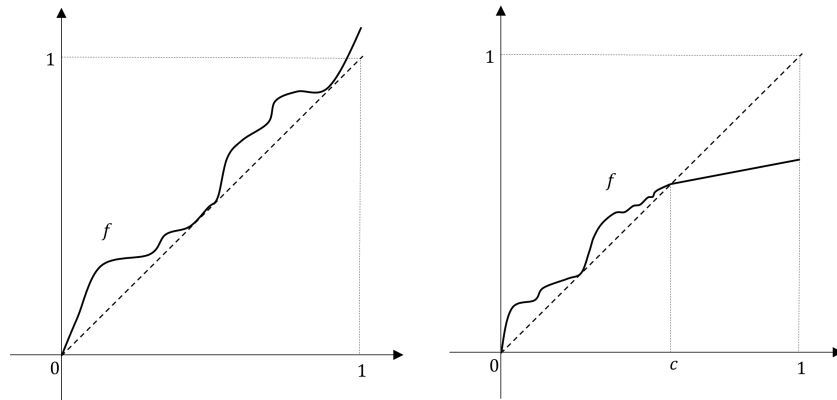


FIGURE 1. Typical graphs of functions from the classes $\mathcal{C}_{\text{up}}$ (left) and $\mathcal{C}_{\text{int}}$ (right).

- The class $\mathcal{C}_{\text{up}}$ consists of all functions $h \in \mathcal{M}$ such that $h(x) \geq x$ for every $x \in [0, 1]$.
- The class $\mathcal{C}_{\text{int}}$ consists of all functions $h \in \mathcal{M}$ for which there exist constants $c_h \in (0, 1)$ and $\alpha_h \in (0, 1)$ such that

$$h(x) \geq x \quad \text{for every } x \in [0, c_h],$$

and

$$h(x) = \alpha_h x + (1 - \alpha_h)c_h \quad \text{for every } x \in [c_h, 1].$$

In particular, $h(c_h) = c_h$ and $h(x) < x$ for every $x \in (c_h, 1]$.

Let $\mathcal{M}^{(c)}$ denote the class of two-sided maps $\varphi_{f,g} : \mathbb{R} \rightarrow \mathbb{R}$ associated with pairs $f, g \in \mathcal{M}$ and defined by

$$\varphi_{f,g}(x) = \begin{cases} -f(-x), & \text{if } x \leq 0, \\ g(x), & \text{if } x > 0. \end{cases}$$

The main results are as follows.

Theorem 1.2. *Let $\varphi_{f,g} \in \mathcal{M}^{(c)}$, and let $\mathbb{X}(\varphi_{f,g})$ be the process defined by (1.1) with drift map $\varphi_{f,g}$. Then the following assertions hold.*

- (i) *If $f, g \in \mathcal{C}_{\text{up}}$, then, for every $x_0 \in [-1, 1]$,*

$$\lim_{M \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log P \left(\max_{1 \leq k \leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = x_0 \right) = 0.$$

- (ii) *If $f, g \in \mathcal{C}_{\text{int}}$, then, for the initial point $x_0 = 0$,*

$$\lim_{M \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log P \left(\max_{1 \leq k \leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = 0 \right) = -\min\{K(f), K(g)\} \quad (1.5)$$

where, for $h \in \{f, g\}$,

$$K(h) := \frac{1 + \alpha_h}{1 - \alpha_h} \cdot \frac{(1 - h(1))^2}{2}.$$

Theorem 1.3. *Let $\varphi_{f,g} \in \mathcal{M}^{(c)}$, and let $\mathbb{X}(\varphi_{f,g})$ be the process defined by (1.1) with drift map $\varphi_{f,g}$. Then the following statements hold.*

- (i) *If $f, g \in \mathcal{C}_{\text{up}}$, then, for every $x_0 \in [-1, 1]$,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{f,g}) = 0.$$

- (ii) *If $f, g \in \mathcal{C}_{\text{int}}$, then*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{f,g}) \leq \max\{K(f), K(g)\}. \quad (1.6)$$

In the symmetric case $K(f) = K(g)$ this bound coincides with the constant in (1.5).

Here $K(h)$ is defined as in Theorem 1.2.

Remark 1.1. The upper bound in the second assertion is attained in a simple symmetric case. Indeed, let $f = g = h$, where, for some $c \in (0, 1)$ and $\alpha \in (0, 1)$,

$$h(x) = \begin{cases} x, & 0 \leq x \leq c, \\ \alpha x + (1 - \alpha)c, & c \leq x \leq 1. \end{cases}$$

For this example, a direct computation gives

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{h,h}) = \frac{1 - \alpha^2}{2} (1 - c)^2 = K(h),$$

so equality holds in the upper bound for this example.

2. THE LARGE DEVIATION PRINCIPLE AND PAST-DEPENDENT PROCESSES

We recall the large deviation framework used by Klebaner and Liptser [6] for past-dependent recursions. Let $\Omega = (\mathbb{R}^\infty, \rho)$ be the complete separable metric space equipped with the metric

$$\rho(x, z) := \sum_{j \geq 1} 2^{-j} \frac{|x_j - z_j|}{1 + |x_j - z_j|}.$$

Klebaner and Liptser proved an LDP for nonlinear recursions with small noise. In the Markov case, they derived finite-horizon asymptotics for probabilities of the form

$$\mathbb{P} \left(\max_{1 \leq k \leq M} |X_k^{(\varepsilon)}| \geq 1 \right).$$

We use this LDP reduction below.

Definition 2.1. Let $q(\varepsilon) > 0$ be a normalizing function (the speed of the principle) such that $q(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We say that the family of Borel probability measures $\{P_\varepsilon, \varepsilon > 0\}$ satisfies a large deviation principle (LDP) in Ω with speed $q(\varepsilon)$ and rate function $I : \Omega \rightarrow [0, \infty]$ if the following conditions hold:

I) $0 \leq I(\underline{x}) \leq \infty$ for all $\underline{x} \in \Omega$, I is lower semicontinuous, and the set $\{\underline{x} : I(\underline{x}) \leq r\}$ is compact in Ω for all $r < \infty$;

II) For any $C \subset \Omega$,

$$-\inf_{\underline{x} \in \text{int}(C)} I(\underline{x}) \leq \liminf_{\varepsilon \rightarrow 0} q(\varepsilon) \log P_\varepsilon(C) \leq \limsup_{\varepsilon \rightarrow 0} q(\varepsilon) \log P_\varepsilon(C) \leq -\inf_{\underline{x} \in \overline{C}} I(\underline{x}), \quad (2.1)$$

where $\text{int}(C)$ and $\overline{C}$ denote the interior and closure of the set C , respectively.

In the present paper we apply this result to the one-dimensional Markov recursion

$$X_{n+1}^{(\varepsilon)} = h(X_n^{(\varepsilon)}) + \varepsilon \xi_{n+1}, \quad X_0^{(\varepsilon)} = x_0,$$

where h is a continuous drift map and $\xi_1 \sim N(0, 1)$. In this case the speed is $q(\varepsilon) = \varepsilon^2$, and the corresponding rate function is

$$I_h(\underline{y}) = \begin{cases} \frac{1}{2} \sum_{n=1}^{\infty} (y_n - h(y_{n-1}))^2, & y_0 = x_0, \\ \infty, & y_0 \neq x_0. \end{cases}$$

For each $M \geq 1$ put

$$\mathbb{B}_M^{\geq}(x_0) := \{(y_0, y_1, \dots, y_M) \in \mathbb{R}^{M+1} : y_0 = x_0, \max_{1 \leq i \leq M} |y_i| \geq 1\},$$

$$\mathbb{B}_M^{>}(x_0) := \{(y_0, y_1, \dots, y_M) \in \mathbb{R}^{M+1} : y_0 = x_0, \max_{1 \leq i \leq M} |y_i| > 1\}.$$

Applying the LDP to the closed set $\mathbb{B}_M^{\geq}(x_0)$ and to the open set $\mathbb{B}_M^{>}(x_0)$ gives

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log P \left(\max_{1 \leq k \leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = x_0 \right) \leq -\inf_{\mathbb{B}_M^{\geq}(x_0)} S_M^h,$$

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log P \left(\max_{1 \leq k \leq M} |X_k^{(\varepsilon)}| > 1 \mid X_0^{(\varepsilon)} = x_0 \right) \geq -\inf_{\mathbb{B}_M^{>}(x_0)} S_M^h,$$

where

$$S_M^h(y_0, \dots, y_M) := \frac{1}{2} \sum_{n=1}^M (y_n - h(y_{n-1}))^2.$$

The two variational infima are equal. Indeed, $\mathbb{B}_M^>(x_0) \subset \mathbb{B}_M^{\geq}(x_0)$, and the reverse inequality follows by a small perturbation of one coordinate for which $|y_i| = 1$, using the continuity of S_M^h . Hence the above upper and lower bounds give the finite-horizon logarithmic asymptotic

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log P \left(\max_{1 \leq k \leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = x_0 \right) = - \inf_{\mathbb{B}_M^{\geq}(x_0)} S_M^h. \quad (2.2)$$

Equivalently, if

$$V_M^h(x_0) := \min_{1 \leq N \leq M} \inf_{\substack{y_0 = x_0, |y_N| \geq 1, \\ |y_k| < 1, 1 \leq k \leq N-1}} \frac{1}{2} \sum_{n=1}^N (y_n - h(y_{n-1}))^2,$$

then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log P \left(\max_{1 \leq k \leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = x_0 \right) = -V_M^h(x_0). \quad (2.3)$$

The reduction from S_M^h to the first-exit sum follows by choosing $y_n = h(y_{n-1})$ after the first hitting time; all subsequent terms then have zero cost.

Thus the finite-horizon exit probability is governed by a deterministic variational problem. This is the same mechanism used by Klebaner and Liptser in the linear Markov case and by Högnäs and Jung for nonlinear autoregressive maps.

For later use, when the drift is clear from the context, we write

$$S_N(y_0, y_1, \dots, y_N) := \frac{1}{2} \sum_{n=1}^N (y_n - h(y_{n-1}))^2. \quad (2.4)$$

Högnäs and Jung [7] studied variational problems of this type for several classes of nonlinear autoregressive maps.

The following form of their sign-reduction lemma is the version needed below.

Lemma 2.1. *Assume that h is continuous and nondecreasing on $[-1, 1]$ and that $h(0) = 0$. Then, for every $N \geq 1$,*

$$\inf_{y_0=0, |y_N| \geq 1} S_N(y_0, y_1, \dots, y_N) = \min \left\{ \inf_{y_0=0, y_N=1, y_1, \dots, y_{N-1} \geq 0} S_N(y_0, y_1, \dots, y_N), \right. \\ \left. \inf_{y_0=0, y_N=-1, y_1, \dots, y_{N-1} \leq 0} S_N(y_0, y_1, \dots, y_N) \right\}.$$

In particular, a minimizing path to the right endpoint may be chosen nonnegative, while a minimizing path to the left endpoint may be chosen nonpositive.

For the two-sided drift $h = \varphi_{f,g}$ and the initial point $x_0 = 0$, Lemma 2.1 allows us to treat the positive and negative exits separately. The positive exit is governed by the one-sided map g on $[0, 1]$, and the negative exit is governed, after the change of variables $x \mapsto -x$, by the one-sided map f on $[0, 1]$.

3. PROOF OF THEOREMS 1.2–1.3

For the proof of Theorem 1.2, we first formulate several lemmas, whose proofs are given in the next section.

For a continuous drift map h and $u \in [-1, 1]$, let $V_M^h(u)$ be the finite-horizon variational quantity defined in (2.3).

Lemma 3.1. *Let $h = \varphi_{f,g}$.*

- *If $f, g \in \mathcal{C}_{\text{up}}$, then, for every $u \in [-1, 1]$,*

$$\lim_{M \rightarrow \infty} V_M^h(u) = 0.$$

- If $f, g \in \mathcal{C}_{\text{int}}$, then

$$\lim_{M \rightarrow \infty} V_M^h(0) = \min\{K(f), K(g)\}.$$

Moreover, for every $u \in [-1, 1]$,

$$\limsup_{M \rightarrow \infty} V_M^h(u) \leq \max\{K(f), K(g)\}.$$

Equation (2.3) and Lemma 3.1 are the main tools in the proof of the theorems.

For $a, b \in \mathbb{R}$ we write $\vec{y}_N(a, b) := (a, y_1, \dots, y_{N-1}, b)$, so that $y_0 = a$ and $y_N = b$, and, consistently with (2.4),

$$S_N(\vec{y}_N(a, b)) := \frac{1}{2} \sum_{n=1}^N (y_n - f(y_{n-1}))^2. \quad (3.1)$$

Before proving Theorem 1.2, we state the following two lemmas; their proofs will be given in the next section.

Lemma 3.2. *Let $[a, b] \subseteq [0, 1]$ and let $f : [a, b] \rightarrow [a, \infty)$ be a continuous, nondecreasing function such that*

$$f(a) = a, \quad f(b) \geq b, \quad f(x) \geq x \text{ for all } x \in [a, b].$$

Then

$$\inf_{N \in \mathbb{N}, \vec{y}_N(a, b)} S_N(\vec{y}_N(a, b)) = 0. \quad (3.2)$$

Lemma 3.3. *Let $f(x) = \alpha x + \beta$ be defined on the interval $[c, 1]$, where $\alpha, c \in (0, 1)$ and $f(c) = c$ (equivalently, $\beta = (1 - \alpha)c$). Then $f(x) < x$ for all $x \in (c, 1]$, and*

$$\inf_{N \in \mathbb{N}, \vec{y}_N(c, 1)} S_N(\vec{y}_N(c, 1)) = \frac{(1 + \alpha)(1 - f(1))^2}{2(1 - \alpha)}.$$

Proof of Theorem 1.2. By the finite-horizon asymptotic (2.3), for every fixed M and every initial point $x_0 \in [-1, 1]$ we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log P \left(\max_{1 \leq k \leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = x_0 \right) = -V_M^{\varphi_{f,g}}(x_0).$$

If $f, g \in \mathcal{C}_{\text{up}}$, Lemma 3.1 gives $\lim_{M \rightarrow \infty} V_M^{\varphi_{f,g}}(x_0) = 0$ for every $x_0 \in [-1, 1]$. This proves the first assertion of Theorem 1.2.

Assume now that $f, g \in \mathcal{C}_{\text{int}}$ and take $x_0 = 0$. By Lemma 2.1, a path from 0 to the right endpoint can be chosen nonnegative and is governed by the one-sided map g , while a path from 0 to the left endpoint is transformed, by $x \mapsto -x$, into a nonnegative path governed by f . Hence Lemmas 3.2 and 3.3 give

$$\lim_{M \rightarrow \infty} V_M^{\varphi_{f,g}}(0) = \min\{K(f), K(g)\}.$$

Combining this with (2.3) proves (1.5). Theorem 1.2 is proved.

Proof of Theorem 1.3. Since $\tau^{(\varepsilon)}(\varphi_{f,g})$ takes natural values, $\log E\tau^{(\varepsilon)}(\varphi_{f,g}) \geq 0$. We use the standard block estimate of Klebaner–Liptser and Jung,

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{f,g}) \leq - \lim_{M \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \inf_{|x_0| \leq 1} \varepsilon^2 \log P \left(\max_{1 \leq k \leq M} |X_k^{(\varepsilon)}| \geq 1 \mid X_0^{(\varepsilon)} = x_0 \right),$$

whenever the right-hand side is estimated by the finite-horizon variational problem.

If $f, g \in \mathcal{C}_{\text{up}}$, Lemma 3.1 gives a zero limiting cost uniformly over initial points in $[-1, 1]$, and therefore

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{f,g}) = 0.$$

If $f, g \in \mathcal{C}_{\text{int}}$, Lemma 3.1 gives the uniform variational upper bound $\max\{K(f), K(g)\}$ for the cost of exiting from $[-1, 1]$. Hence

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log E\tau^{(\varepsilon)}(\varphi_{f,g}) \leq \max\{K(f), K(g)\}.$$

This proves Theorem 1.3.

4. PROOFS OF LEMMAS 3.1–3.3

We first prove the variational Lemma 3.1.

Proof of Lemma 3.1.

We first prove the corresponding one-sided assertions. Let $r : [0, 1] \rightarrow [0, \infty)$ be a continuous and nondecreasing function, with $r(0) = 0$.

Assume first that $r \in \mathcal{C}_{\text{up}}$. Fix $u \in [0, 1]$. If some deterministic iterate of u under r reaches or crosses 1, then the path following these iterates has zero cost up to the exit. Otherwise, the sequence of iterates is nondecreasing and bounded above by 1, hence it converges to a fixed point $a \in [u, 1]$. Following the deterministic orbit from u to a point arbitrarily close to a costs zero, and the remaining cost from a to 1 is zero by Lemma 3.2, applied on $[a, 1]$. Therefore

$$\lim_{M \rightarrow \infty} V_M^r(u) = 0, \quad u \in [0, 1].$$

Assume now that $r \in \mathcal{C}_{\text{int}}$. Let $c = c_r$ and $\alpha = \alpha_r$. For every $u \in [0, c]$, the cost of reaching c can be made arbitrarily small by the same deterministic-orbit argument, followed by Lemma 3.2 on the interval whose left endpoint is the limiting fixed point. On the linear interval $[c, 1]$, Lemma 3.3 gives the limiting cost $K(r)$ from c to 1. Hence the limiting one-sided cost from any $u \in [0, c]$ to 1 is at most $K(r)$.

If $u \in [c, 1]$, the same quadratic minimization as in Lemma 3.3, but with $y_0 = u$ and $y_N = 1$, gives

$$\inf_{\vec{y}_N(u, 1)} S_N(\vec{y}_N(u, 1)) = \frac{(1 - \alpha^2)(1 - c - \alpha^N(u - c))^2}{2(1 - \alpha^{2N})}.$$

Letting $N \rightarrow \infty$ yields the same limiting value $K(r)$. Consequently the limiting cost from any $u \in [0, 1]$ to 1 is at most $K(r)$.

It remains to justify the equality for the starting point $u = 0$. Let $(0 = y_0, y_1, \dots, y_N = 1)$ be any admissible path. Let $k := \max\{n : y_n \leq c\}$; then $y_k \leq c < y_{k+1}$, unless c itself is already a node of the path. Since r is nondecreasing and $r(c) = c$, we have $r(y_k) \leq c$. Inserting the point c between y_k and y_{k+1} does not increase the cost, because

$$(y_{k+1} - r(y_k))^2 \geq (c - r(y_k))^2 + (y_{k+1} - c)^2.$$

Thus every path from 0 to 1 has cost at least the sum of the optimal costs from 0 to c and from c to 1. By Lemmas 3.2 and 3.3, this lower bound is $K(r)$. The opposite inequality was proved above, hence the limiting one-sided cost from 0 to 1 is exactly $K(r)$.

We now return to the two-sided drift $h = \varphi_{f,g}$. If $u \geq 0$, a positive exit is governed by the one-sided map g . If $u \leq 0$, a negative exit is transformed by $x \mapsto -x$ into the corresponding one-sided problem for f . This proves the first assertion when $f, g \in \mathcal{C}_{\text{up}}$.

If $f, g \in \mathcal{C}_{\text{int}}$ and $u = 0$, Lemma 2.1 reduces the centered two-sided problem to the minimum of the two one-sided costs, and therefore

$$\lim_{M \rightarrow \infty} V_M^h(0) = \min\{K(f), K(g)\}.$$

For arbitrary $u \in [-1, 1]$, choosing the exit on the same side as u gives a cost not exceeding $K(g)$ for $u \geq 0$ and not exceeding $K(f)$ for $u \leq 0$. Hence

$$\limsup_{M \rightarrow \infty} V_M^h(u) \leq \max\{K(f), K(g)\}.$$

The lemma is proved. □

To prove Lemma 3.2 we first establish three auxiliary lemmas.

Lemma 4.1. *Let $f : [p, q] \rightarrow [p, q]$, $0 \leq p < q \leq 1$, be a continuous, nondecreasing function with $f(p) = p$, $f(q) = q$ and $f(x) > x$ for all $x \in (p, q)$. Then*

$$\inf_{N \in \mathbb{N}, \vec{y}_N(p, q)} S_N(\vec{y}_N(p, q)) = 0.$$

Proof of Lemma 4.1. Fix $\varepsilon \in (0, q - p)$. Set $y_1 = p + \varepsilon$, $y_n = f(y_{n-1})$ for $2 \leq n \leq N - 1$, and $y_N = q$. Since $y_n = f(y_{n-1})$ for $2 \leq n \leq N - 1$, all the middle terms vanish and

$$S_N(\vec{y}_N(p, q)) = \frac{1}{2}(\varepsilon^2 + (q - f^{N-1}(p + \varepsilon))^2).$$

The iterates $f^n(p + \varepsilon)$ form a nondecreasing sequence (since $f(x) \geq x$) bounded above by q , and hence converge to a fixed point of f in $(p, q]$. As $f(x) > x$ on (p, q) , this fixed point is q , so $f^{N-2}(p + \varepsilon) \rightarrow q$ as $N \rightarrow \infty$. Therefore $\limsup_N S_N(\vec{y}_N(p, q)) \leq \frac{1}{2}\varepsilon^2$, and letting $\varepsilon \rightarrow 0$ gives the claim. $\square$

Lemma 4.2. Let $[p, q] \subseteq [0, 1]$ and suppose $f(x) = x$ for all $x \in [p, q]$. Then

$$\inf_{N \in \mathbb{N}, \vec{y}_N(p, q)} S_N(\vec{y}_N(p, q)) = 0.$$

Proof of Lemma 4.2. For $N \in \mathbb{N}$ take the uniform partition $y_j = p + \frac{j}{N}(q - p)$, $0 \leq j \leq N$. Then $f(y_{j-1}) = y_{j-1}$ and

$$S_N(\vec{y}_N(p, q)) = \frac{1}{2} \sum_{j=1}^N (y_j - y_{j-1})^2 = \frac{(q - p)^2}{2N} \rightarrow 0, \quad N \rightarrow \infty.$$

$\square$

Lemma 4.3. Let $f : [p, b] \rightarrow [p, \infty)$, $0 \leq p < b \leq 1$, be a continuous, nondecreasing function with $f(p) = p$, $f(b) \geq b$ and $f(x) > x$ for all $x \in (p, b)$. Then

$$\inf_{N \in \mathbb{N}, \vec{y}_N(p, b)} S_N(\vec{y}_N(p, b)) = 0.$$

Proof of Lemma 4.3. We construct a finite backward chain of preimages of b . Set $v_0 = b$. Given $v_k \in (p, b]$, we have $f(p) = p < v_k \leq f(v_k)$ (indeed $f(v_k) > v_k$ if $v_k \in (p, b)$, and $f(b) \geq b$ if $v_k = b$); since f is continuous, by the intermediate value theorem the set $\{x \in [p, v_k] : f(x) = v_k\}$ is nonempty, and we put

$$v_{k+1} := \min\{x \in [p, v_k] : f(x) = v_k\}.$$

Then $v_{k+1} > p$ (because $f(p) = p < v_k$) and $v_{k+1} \in (p, b)$, so $v_k = f(v_{k+1}) > v_{k+1}$; thus (v_k) is strictly decreasing and bounded below by p , hence $v_k \downarrow L$ with $f(L) = L$. Since $f(x) > x$ on (p, b) , necessarily $L = p$, i.e. $v_k \downarrow p$.

Given $\varepsilon > 0$, choose K with $v_K - p < \varepsilon$ and set

$$y_0 = p, \quad y_1 = v_K, \quad y_j = v_{K-j+1} = f(y_{j-1}) \quad (2 \leq j \leq K + 1),$$

so that $y_{K+1} = v_0 = b$. Every step except the first costs zero, hence

$$S_{K+1}(\vec{y}_{K+1}(p, b)) = \frac{1}{2}(v_K - f(p))^2 = \frac{1}{2}(v_K - p)^2 < \frac{1}{2}\varepsilon^2.$$

Letting $\varepsilon \rightarrow 0$ proves the claim. $\square$

Proof of Lemma 3.2. For $a \leq u \leq v \leq b$ write $\mathcal{C}(u, v) := \inf_{N \in \mathbb{N}, \vec{y}_N(u, v)} S_N(\vec{y}_N(u, v)) \geq 0$. Concatenating admissible sequences at a common endpoint gives, for every finite partition $a = u_0 < u_1 < \dots < u_M = b$,

$$\mathcal{C}(a, b) \leq \sum_{i=1}^M \mathcal{C}(u_{i-1}, u_i). \quad (4.1)$$

It suffices to prove $\mathcal{C}(a, b) = 0$.

Let $E := \{x \in [a, b] : f(x) = x\}$, a closed set containing a . Each component (α, β) of $[a, b] \setminus E$ satisfies $f(x) > x$, with $\alpha \in E$ and $\beta \in E$ or $\beta = b$; hence $\mathcal{C}(\alpha, \beta) = 0$ by Lemma 4.1 (if $\beta \in E$, since then f maps $[\alpha, \beta]$ into itself and fixes its endpoints) or by Lemma 4.3 (if $\beta = b$). Moreover $f \equiv \text{id}$ on any interval contained in E , so $\mathcal{C} = 0$ there by Lemma 4.2.

Fix $\delta > 0$. Only finitely many components of $[a, b] \setminus E$ have length $\geq \delta$; each has cost 0, and removing them leaves finitely many closed intervals $I_0, \dots, I_P$ (one of which may about a or b) inside each of which every component of $[a, b] \setminus E$ is shorter than δ ; equivalently, E is δ -dense in each I_m . Fix $I_m = [u, v]$; its endpoints lie in E , with the only possible exception $v = b$. Put $w := v$ if $v \in E$, and $w := \max(E \cap [u, v))$ otherwise (then $v - w < \delta$). Setting $e_0 := u$ and, while $e_i < w$,

$$e_{i+1} := \sup\{x \in E \cap [u, w] : x \leq e_i + \delta\} \in E,$$

δ -density yields a strictly increasing sequence with steps $< \delta$ which, being unable to accumulate before w , reaches w after finitely many steps $u = e_0 < \dots < e_P = w$. Since $f(e_i) = e_i$, the vector $(e_0, \dots, e_P)$ is admissible and

$$\mathcal{C}(u, w) \leq \sum_{i=0}^{P-1} \frac{1}{2} (e_{i+1} - e_i)^2 \leq \frac{\delta}{2} \sum_{i=0}^{P-1} (e_{i+1} - e_i) = \frac{\delta}{2} (w - u).$$

If $w = v$ this bounds $\mathcal{C}(I_m)$; if $w < v = b$, the single step $w \rightarrow b$ adds at most $\frac{1}{2}(b - w)^2 < \frac{1}{2}\delta^2$. Summing over m via (4.1) and recalling that the removed components have cost 0,

$$\mathcal{C}(a, b) \leq \sum_m \mathcal{C}(I_m) \leq \frac{\delta}{2} (b - a) + \frac{1}{2} \delta^2 \xrightarrow{\delta \rightarrow 0} 0.$$

Hence $\mathcal{C}(a, b) = 0$, which completes the proof. $\square$

Proof of Lemma 3.3. On $[c, 1]$ we have $f(y_{n-1}) = \alpha y_{n-1} + \beta$, so for a fixed $N \geq 1$

$$S_N(\vec{y}_N(c, 1)) = \frac{1}{2} \sum_{n=1}^N (y_n - \alpha y_{n-1} - \beta)^2, \quad y_0 = c, \quad y_N = 1.$$

We first determine its minimum over the free variables $(y_1, \dots, y_{N-1}) \in \mathbb{R}^{N-1}$. Put

$$d_n := y_n - \alpha y_{n-1} - \beta = y_n - f(y_{n-1}) \quad (1 \leq n \leq N),$$

so that $S_N = \frac{1}{2} \sum_{n=1}^N d_n^2$. Each d_n is an affine function of $(y_1, \dots, y_{N-1})$, and the linear map $(y_1, \dots, y_{N-1}) \mapsto (d_1, \dots, d_N)$ is injective, since y_k enters d_k with coefficient 1. Hence S_N is a strictly convex quadratic function, and its unique global minimum is attained at the (unique) critical point.

Each variable y_k ($1 \leq k \leq N-1$) occurs in exactly two terms, namely in d_k (with coefficient 1) and in $d_{k+1} = y_{k+1} - \alpha y_k - \beta$ (with coefficient $-\alpha$); therefore

$$\frac{\partial S_N}{\partial y_k} = d_k - \alpha d_{k+1} = 0, \quad 1 \leq k \leq N-1.$$

This recurrence $d_k = \alpha d_{k+1}$ gives, with $\lambda := d_N$,

$$d_n = \lambda \alpha^{N-n}, \quad 1 \leq n \leq N.$$

To find λ , multiply $d_n = y_n - \alpha y_{n-1} - \beta$ by α^{N-n} and sum; the y -terms telescope,

$$\sum_{n=1}^N \alpha^{N-n} (y_n - \alpha y_{n-1}) = y_N - \alpha^N y_0 = 1 - \alpha^N c,$$

so, using $\beta = (1 - \alpha)c$ (which is equivalent to $f(c) = c$),

$$\sum_{n=1}^N \alpha^{N-n} d_n = (1 - \alpha^N c) - \beta \frac{1 - \alpha^N}{1 - \alpha} = (1 - \alpha^N c) - c(1 - \alpha^N) = 1 - c.$$

On the other hand, $\sum_{n=1}^N \alpha^{N-n} d_n = \lambda \sum_{n=1}^N \alpha^{2(N-n)} = \lambda \frac{1 - \alpha^{2N}}{1 - \alpha^2}$, whence

$$\lambda = \frac{(1 - \alpha^2)(1 - c)}{1 - \alpha^{2N}}.$$

Consequently the minimum for fixed N equals

$$\min_{\vec{y}_N(c,1)} S_N = \frac{1}{2} \sum_{n=1}^N d_n^2 = \frac{1}{2} \lambda^2 \sum_{n=1}^N \alpha^{2(N-n)} = \frac{1}{2} \lambda^2 \frac{1 - \alpha^{2N}}{1 - \alpha^2} = \frac{(1 - \alpha^2)(1 - c)^2}{2(1 - \alpha^{2N})}.$$

It remains to check that this minimizer is admissible, i.e. lies in $[c, 1]$. Since $\beta = (1 - \alpha)c$, the recurrence $y_n = \alpha y_{n-1} + \beta + d_n$ may be written as

$$y_n - c = \alpha(y_{n-1} - c) + \lambda \alpha^{N-n}, \quad y_0 - c = 0.$$

Solving this recursion gives

$$y_n - c = \lambda \alpha^{N-n} \frac{1 - \alpha^{2n}}{1 - \alpha^2}, \quad 1 \leq n \leq N.$$

This expression is positive. Furthermore,

$$(y_{n+1} - c) - (y_n - c) = \lambda \alpha^{N-n-1} \frac{(1 - \alpha)(1 + \alpha^{2n+1})}{1 - \alpha^2} > 0,$$

so $c = y_0 < y_1 < \dots < y_N = 1$. Thus the minimizing vector belongs to $[c, 1]^{N+1}$.

Finally, since $\alpha \in (0, 1)$, the quantity $1 - \alpha^{2N}$ increases to 1, so the above minima decrease in N , and

$$\inf_{N \in \mathbb{N}, \vec{y}_N(c,1)} S_N(\vec{y}_N(c,1)) = \lim_{N \rightarrow \infty} \frac{(1 - \alpha^2)(1 - c)^2}{2(1 - \alpha^{2N})} = \frac{(1 - \alpha^2)(1 - c)^2}{2} = \frac{(1 + \alpha)(1 - f(1))^2}{2(1 - \alpha)}.$$

□

ACKNOWLEDGEMENTS

This work was partially carried out during a visit of the second author to Great Bay University. The second author is deeply grateful for the warm hospitality extended to him during the visit.

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Approximate Solution for Two-Dimensional Free Surface Flow Using Perturbation Methods

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Abstract. The two-dimensional free-surface flow of an inviscid and incompressible fluid passing over a triangular hollow at the bottom of a channel is studied in this paper. The flow is considered irrotational and steady. The effect of surface tension is taken into account, while the force of gravity is neglected. The region in the potential W plane is mapped onto the auxiliary λ plane using the Schwarz–Christoffel transformation. The primary objective of this research is to provide an approximate solution for this problem using the perturbation approach together with the Hilbert transform; the results are computed at high Weber numbers and small inclination interior angles. The solution is provided and illustrated up to first-order approximation. The shape of the free surface for different values of: the Weber number α , the depth of the hollow l and the inclination interior angles γ and δ are plotted. The findings demonstrate the simplicity of the method used and give an approximate solution for similar problems.

Keywords: Surface tension, conformal mapping, Weber number, Hilbert’s transforms, perturbation technique

MSC (2020): 76B10, 30E25, 30C20, 76M45

1. INTRODUCTION

The phenomenon of fluid flow over an irregular terrain when there is a free surface has significant theoretical and practical interest. This class of phenomena, often referred to as Neumann-Kelvin problems, is distinguished by the nonlinear boundary conditions on the free surface, as dictated by the Bernoulli equation and as the free surface’s shape is undetermined, it must be taken into account when solving the problem. For more information about this subject see: Bhimsen & Shivamoggi [1], Batchelor [2]. Different methods have been proposed to solve this type of problem numerically, such as the boundary integral equation method, series truncation method, relaxation method and Hilbert transformation. Various researchers have studied the free-surface topics numerically considering the influence of gravity, surface tension or both, such as Doak and Vanden-Broeck [3], studied the problem of flow past a vertical flat plate in a channel by using series truncation, Laiadi and Merzougui [4], solved the problem of the flows over successive obstacles.

Several authors have successfully used the Hilbert method in tandem with the perturbation approach, for example: M. B. Abd-el-Malek & S. N. Hanna [5], resolved the issue of a flow over a ramp with gravity effect, M. B & Abd-el-Malek [6], found an Approximate solution of gravity-affected flow from planar sluice gate, M. M. Bounif and A. Gasmi [7], determined the numerical solution for free surface flow over a step and others.

In this paper, we have considered the issue of two dimensional, irrotational and stationary flow of a fluid passing over a triangular hollow at the bottom of a channel. It is believed that the liquid is both inviscid and incompressible. In this work, gravity and viscosity are neglected in order to study a free-surface flow dominated by surface tension. Under these conditions, surface tension governs the interface dynamics and controls the free-surface shape through curvature effects. This simplification permits analytical treatment. However, since the flow is assumed inviscid and irrotational, viscous boundary layers and possible vortical motions inside the cavity are not described in the present formulation. Under the stipulated assumptions, we derive simplifications for the governing Bernoulli’s equation, enabling the application of complex variable theory, specifically the properties inherent to analytical functions.

The objective of this study is to find the numerical solution to the specified problem, characterized by high Weber number, by employing the Hilbert method for a mixed boundary value problem within this novel plane, thus finding a system of nonlinear integral equations and the perturbation technique

to solve the obtained system.

This paper is organized according to this plan: First, section 2 explains the mathematical formulation of the problem in dimensionless variables. Second, in section 3 we derive the approximate equations of the problem. Next, section 4 we find the numerical solution by using the perturbation technique. Then, section 5 provides a graphical representation and discussion. Finally, this study is concluded in section 6.

2. PROBLEM FORMULATION

Let us examine the movement of two-dimensional, irrotational and steady flow of an incompressible and non-viscous fluid. The influence of surface tension is taken into account. Far upstream and downstream, the flow is proposed as uniform with constant flux: $U_1 h_1 = U_2 h_2$, where $U_i, i = 1, 2$ and $h_i, i = 1, 2$ are velocities and depths of the flow upstream and downstream, respectively. The bottom is composed of inclined walls BC, CD and horizontal walls AB, DE , the lengths of the step-inclined walls BC, CD and its inclination angles are indicated by the numbers r, d, γ and δ respectively. As reference, we designate the point C as the origin in the (x', y') -plane, The x' -axis from left to right and the y' -axis is vertically at the point C (see Figure 1) .

We introduce the two harmonic functions ψ' and ϕ' . Next, we introduce the complex potential

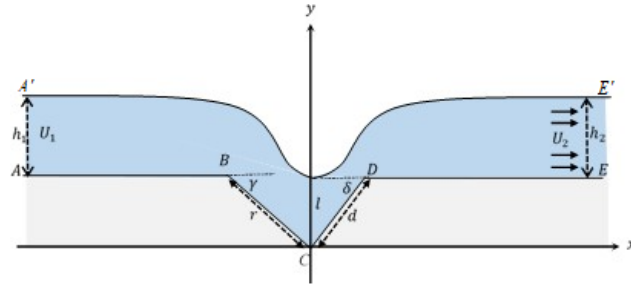


FIGURE 1. Sketch of the flow and of the coordinates.

function:

$$W' = \phi' + i\psi',$$

where ψ' and ϕ' are the stream function and the potential function respectively.

Within the flow space, the function $W'(z')$ represents an analytical function of $z' = x' + iy'$.

The x' and y' directions' velocity components are denoted by u' and v' , then:

$$\frac{dW'}{dz'} = u' - iv' = q' e^{-i\theta}. \quad (2.1)$$

Under the assumption of irrotationality and incompressibility, the 2D steady motion of an inviscid fluid is governed by Laplace's equation for the velocity potential in the flow domain [2],

$$\Delta\phi' = 0,$$

When the pressure is uniform, Bernoulli's equation along the free surface $A'E'$ has the following form:

$$\frac{1}{2}q'^2 + \frac{p'}{\rho} = \frac{1}{2}U_1^2 + \frac{p_a}{\rho}, \quad (2.2)$$

where ρ is the fluid's density and p is the fluid pressure.

Using the variables without dimensions, we make the analysis simpler by selecting h_1 as the unit length and U_1 as the unit velocity. The new dimensionless variables are as follows

$$z = \frac{z'}{h_1}, \quad q = \frac{q'}{U_1}, \quad W = \frac{W'}{U_1 h_1}. \quad (2.3)$$

From Sekhri et al [8], equation (2.2) can be written in dimensionless form

$$q^2 + \frac{2}{\alpha} \left| \frac{\partial \theta}{\partial \phi} \right| q = 1, \quad (2.4)$$

where the Weber number, denoted by α , is defined by:

$$\alpha = \frac{\rho U_1^2 h_1}{T}. \quad (2.5)$$

T represent the surface tension.

By using (2.3), the complex conjugate velocity in (2.1) has the following form:

$$\zeta = \frac{dW}{dz} = \frac{1}{U_1} \frac{dW'}{dz'} = \frac{q'}{U_1} e^{-i\theta} = q e^{-i\theta}. \quad (2.6)$$

Moreover, when utilizing the logarithmic hodograph variable

$$\omega = \ln \zeta = \ln q - i\theta. \quad (2.7)$$

from (2.6) and (2.7), we get

$$z = \int e^{-\omega} dW. \quad (2.8)$$

For the streamline $ABCDE$, we choose $\psi = 0$. $\psi = 1$ on the streamline $A'E'$ can be inferred from the choosing of dimensionless variables.

The flow space in the W -plane is mapped onto the top half of an auxiliary λ -plane using the transformation of Schwarz-Christoffel, see Figures 2 and 3.

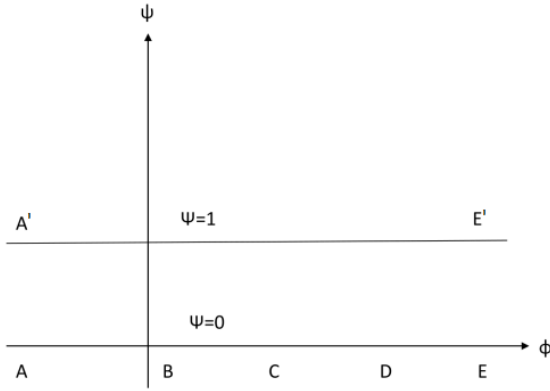


FIGURE 2. The normalized complex a potential-plane.

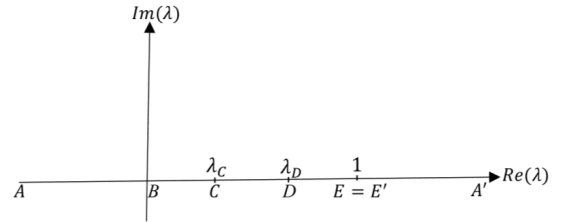


FIGURE 3. The upper a half plane, λ -plane.

The obtained function is

$$W(\lambda) = -\frac{1}{\pi} \ln(1 - \lambda). \quad (2.9)$$

We now introduce the transformation of Hilbert for a mixed boundary-value problem on the λ -plane, which allows us to describe W as a function of λ , the following provides the Hilbert problem's general solution:

$$\chi(\lambda) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{Im[\chi(\tau)]}{\tau - \lambda} d\tau + \sum_{j=0}^{\infty} R_j \lambda^j. \quad (2.10)$$

where R_j and $p.v.$ represent real constants and the integral's principal value respectively.

We can derive a relation between $\omega(\lambda)$ and $\chi(\lambda)$ by introducing an auxiliary function $H(\lambda)$. In our

issue, we choose:

$$H(\lambda) = \begin{cases} \sqrt{1-\lambda}, & \lambda < 1, \\ -i\sqrt{\lambda-1}, & \lambda > 1. \end{cases} \quad (2.11)$$

With

$$\chi(\lambda) = \frac{\omega(\lambda)}{H(\lambda)} = \frac{\ln q(\lambda) - i\theta(\lambda)}{H(\lambda)} = U(\lambda) + iV(\lambda), \quad (2.12)$$

the value of θ is

$$\theta(\lambda) = \begin{cases} 0 & \lambda < 0, \\ -\gamma & 0 < \lambda < \lambda_c, \\ \delta & \lambda_c < \lambda < \lambda_d, \\ 0 & \lambda_d < \lambda < 1, \\ \theta(\lambda) & \lambda > 1. \end{cases} \quad (2.13)$$

Using (2.11), (2.12) and (2.13), we have

$$Im(\chi(\lambda)) = \begin{cases} 0 & \lambda < 0, \\ \frac{\gamma}{\sqrt{1-\lambda}} & 0 < \lambda < \lambda_c, \\ -\frac{\delta}{\sqrt{1-\lambda}} & \lambda_c < \lambda < \lambda_d, \\ 0 & \lambda_d < \lambda < 1, \\ \frac{\ln q(\lambda)}{\sqrt{\lambda-1}} & \lambda > 1. \end{cases} \quad (2.14)$$

We look at the situation upstream. Thus, (2.10) takes the form

$$\chi(\lambda) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{Im[\chi(\tau)]}{\tau - \lambda} d\tau, \quad (2.15)$$

from (2.12) and (2.15), we obtain the equations of $U(\lambda)$ and $V(\lambda)$:

$$U(\lambda) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{V(\tau)}{\tau - \lambda} d\tau. \quad (2.16)$$

And

$$V(\lambda) = \frac{-1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{U(\tau)}{\tau - \lambda} d\tau. \quad (2.17)$$

We create a system of integral equations by using (2.16) and (2.17). The equations below are all that we need for our study:

$$\frac{\theta(\lambda)}{\sqrt{\lambda-1}} = \frac{1}{\pi} \int_1^{\infty} \frac{\ln q(\tau)}{(\tau - \lambda)\sqrt{\tau-1}} d\tau + \frac{\gamma}{\pi} \int_0^{\lambda_c} \frac{1}{(\tau - \lambda)\sqrt{1-\tau}} d\tau - \frac{\delta}{\pi} \int_{\lambda_c}^{\lambda_d} \frac{1}{(\tau - \lambda)\sqrt{1-\tau}} d\tau, \quad \lambda > 1, \quad (2.18)$$

and

$$\frac{\ln(q_{k_i}(\lambda))}{\sqrt{1-\lambda}} = \frac{1}{\pi} \int_1^{\infty} \frac{\ln q(\tau)}{(\tau - \lambda)\sqrt{\tau-1}} d\tau + \frac{\gamma}{\pi} \int_0^{\lambda_c} \frac{1}{(\tau - \lambda)\sqrt{1-\tau}} d\tau - \frac{\delta}{\pi} \int_{\lambda_c}^{\lambda_d} \frac{1}{(\tau - \lambda)\sqrt{1-\tau}} d\tau, \quad \lambda < 1, \quad (2.19)$$

where q_{k_i} , $i = 1, 2, 3, 4$ are defined in $\lambda < 1$.

By integration, (2.18) and (2.19) take the forms

$$\begin{aligned} \theta(\lambda) = & \frac{\sqrt{\lambda-1}}{\pi} \int_1^{\infty} \frac{\ln q(\tau)}{(\tau - \lambda)\sqrt{\tau-1}} d\tau + \frac{2\gamma}{\pi} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right) \\ & - \frac{2\delta}{\pi} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \quad \lambda > 1, \end{aligned}$$

and

$$\ln(q_{k_i}(\lambda)) = \frac{\sqrt{1-\lambda}}{\pi} \int_1^\infty \frac{\ln q(\tau)}{(\tau-\lambda)\sqrt{\tau-1}} d\tau + \frac{1}{\pi} \ln \left(\frac{\left[\frac{(\sqrt{1-\lambda_c}-\sqrt{1-\lambda})(1+\sqrt{1-\lambda})}{(\sqrt{1-\lambda_c}+\sqrt{1-\lambda})(1-\sqrt{1-\lambda})} \right]^\gamma}{\left[\frac{(\sqrt{1-\lambda_d}-\sqrt{1-\lambda})(\sqrt{1-\lambda_c}+\sqrt{1-\lambda})}{(\sqrt{1-\lambda_d}+\sqrt{1-\lambda})(\sqrt{1-\lambda_c}-\sqrt{1-\lambda})} \right]^\delta} \right), \quad \lambda < 1$$

using (2.8) and (2.9) to find the coordinates of z on $A'E'$, we get:

$$z(\lambda) = z_\infty - \frac{1}{\pi} \int_\lambda^\infty \frac{e^{i\theta(\tau)}}{(1-\tau)q(\tau)} d\tau, \quad \lambda > 1,$$

dividing imaginary and real part, we obtain:

$$x(\lambda) = x_\infty - \frac{1}{\pi} \int_\lambda^\infty \frac{\cos \theta(\tau)}{(1-\tau)q(\tau)} d\tau, \quad \lambda > 1,$$

$$y(\lambda) = 1 + l - \frac{1}{\pi} \int_\lambda^\infty \frac{\sin \theta(\tau)}{(1-\tau)q(\tau)} d\tau, \quad \lambda > 1.$$

Along BC' , we have ($0 < \lambda < \lambda_c$, $q = q_{k_2}$ and $\theta = -\gamma$), then we obtain:

$$r = \frac{1}{\pi} \int_0^{\lambda_c} \frac{1}{(1-\tau)q_{k_2}(\tau)} d\tau. \quad (2.20)$$

Along CD , we have ($\lambda_c < \lambda < \lambda_d$, $q = q_{k_3}$ and $\theta = \delta$), then we obtain

$$d = \frac{1}{\pi} \int_{\lambda_c}^{\lambda_d} \frac{1}{(1-\tau)q_{k_3}(\tau)} d\tau, \quad (2.21)$$

using (2.21), we get:

$$l = \frac{\sin \delta}{\pi} \int_{\lambda_c}^{\lambda_d} \frac{1}{(1-\tau)q_{k_3}(\tau)} d\tau. \quad (2.22)$$

3. THE APPROXIMATE EQUATIONS

The solution to equation (2.4) is provided, when the Weber number is high and first-order Taylor development is applied, the result is:

$$q(\lambda) \approx 1 - \frac{1}{\alpha} \left| \frac{\partial \theta}{\partial \phi} \right|,$$

with

$$\left| \frac{\partial \theta}{\partial \phi} \right| = \pi(\lambda - 1) \left| \frac{\partial \theta}{\partial \lambda} \right|, \quad \lambda > 1.$$

Then

$$q(\lambda) \approx 1 - \frac{\pi}{\alpha} (\lambda - 1) \left| \frac{\partial \theta}{\partial \lambda} \right|, \quad \lambda > 1,$$

$$\ln(q(\lambda)) \approx -\frac{\pi}{\alpha} (\lambda - 1) \left| \frac{\partial \theta}{\partial \lambda} \right|, \quad \lambda > 1,$$

$$\frac{1}{q(\lambda)} \approx 1 + \frac{\pi}{\alpha} (\lambda - 1) \left| \frac{\partial \theta}{\partial \lambda} \right|, \quad \lambda > 1.$$

Slight interior degrees of inclination cause a slight variance in θ , As a result, θ can approximate $\sin \theta$, while unity can approximate $\cos \theta$, we obtain the approximate equations

$$q(\lambda) \approx 1 - \frac{\pi}{\alpha} (\lambda - 1) \frac{\partial \theta}{\partial \lambda}, \quad \lambda > 1,$$

$$\begin{aligned}
\theta(\lambda) &\approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^\infty \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta}{\partial \tau}(\tau) d\tau + \frac{2\gamma}{\pi} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right) \\
&\quad - \frac{2\delta}{\pi} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \quad \lambda > 1, \\
x(\lambda) &\approx x_\infty - \frac{1}{\pi} \int_\lambda^\infty \frac{1}{1-\tau} d\tau - \frac{\theta(\lambda)}{\alpha}, \quad \lambda > 1, \\
y(\lambda) &\approx 1+l - \frac{1}{\pi} \int_\lambda^\infty \frac{\theta(\tau)}{1-\tau} d\tau - \frac{\theta^2(\lambda)}{2\alpha}, \quad \lambda > 1, \\
l &\approx \frac{\delta}{\pi} \int_{\lambda_c}^{\lambda_d} \frac{1}{(1-\tau)q_3(\tau)} d\tau.
\end{aligned}$$

Along the solid boundaries BC and CD , we can approximate q_{k_i} by one, (2.20), (2.21) and (2.22) become:

$$r \approx \frac{1}{\pi} \ln \left(\frac{1}{1-\lambda_c} \right), \quad d \approx \frac{1}{\pi} \ln \left(\frac{1-\lambda_c}{1-\lambda_d} \right), \quad \text{and, the height } l \approx \frac{\delta}{\pi} \ln \left(\frac{1-\lambda_c}{1-\lambda_d} \right).$$

The perturbation method is used to solve this resulting nonlinear system.

4. PERTURBATION TECHNIQUE

This series represents the expansion of $X(\lambda)$ in terms of the tiny parameters γ and δ :

$$X(\lambda) = X_0(\lambda) + \sum_{m=1}^{\infty} (\gamma^m X_{m,\gamma}(\lambda) + \delta^m X_{m,\delta}(\lambda)), \quad (4.1)$$

in this case, $X(\lambda)$ represents $q(\lambda)$, $\theta(\lambda)$, $x(\lambda)$ and $y(\lambda)$.

4.1. Zeroth-order approximation. The situation of flow is corresponding to this zeroth-order approximation ($\gamma = \delta = 0$). Then,

$$q_0(\lambda) \approx 1 - \frac{\pi}{\alpha}(\lambda-1) \frac{\partial \theta_0}{\partial \lambda}(\lambda), \quad \lambda > 1, \quad (4.2)$$

$$\theta_0(\lambda) \approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^\infty \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta_0}{\partial \tau}(\tau) d\tau, \quad \lambda > 1, \quad (4.3)$$

$$x_0(\lambda) \approx x_\infty - \frac{1}{\pi} \int_\lambda^\infty \frac{1}{1-\tau} d\tau - \frac{\theta_0(\lambda)}{\alpha}, \quad \lambda > 1, \quad (4.4)$$

$$y_0(\lambda) \approx 1+l - \frac{1}{\pi} \int_\lambda^\infty \frac{\theta_0(\tau)}{1-\tau} d\tau - \frac{\theta_0^2(\lambda)}{2\alpha}, \quad \lambda > 1, \quad (4.5)$$

4.2. First-order approximation. The first order's can be found by using the series (4.1), as well as the equations (4.2), (4.3), (4.4) and (4.5). we get

$$\begin{aligned}
q_{1,\gamma}(\lambda) &\approx \frac{-\pi}{\alpha}(\lambda-1) \frac{\partial \theta_{1,\gamma}}{\partial \lambda}(\lambda), \quad \lambda > 1, \\
q_{1,\delta}(\lambda) &\approx \frac{-\pi}{\alpha}(\lambda-1) \frac{\partial \theta_{1,\delta}}{\partial \lambda}(\lambda), \quad \lambda > 1.
\end{aligned}$$

$$\theta_{1,\gamma}(\lambda) \approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^\infty \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta_{1,\gamma}}{\partial \tau}(\tau) d\tau + \frac{2}{\pi} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right), \quad \lambda > 1 \quad (4.6)$$

$$\theta_{1,\delta}(\lambda) \approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^\infty \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta_{1,\delta}}{\partial \tau}(\tau) d\tau - \frac{2}{\pi} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{1-\lambda_c}\sqrt{1-\lambda_d}} \right), \quad \lambda > 1 \quad (4.7)$$

$$\begin{aligned}
x_{1,\gamma}(\lambda) &\approx \frac{-1}{\alpha} \theta_{1,\gamma}(\lambda), \quad \lambda > 1, \\
x_{1,\delta}(\lambda) &\approx \frac{-1}{\alpha} \theta_{1,\delta}(\lambda), \quad \lambda > 1. \\
y_{1,\gamma}(\lambda) &\approx \frac{-1}{\pi} \int_{\lambda}^{\infty} \frac{1}{1-\tau} \theta_{1,\gamma}(\tau) d\tau - \frac{1}{2\alpha} \theta_{1,\gamma}^2(\lambda), \quad \lambda > 1. \\
y_{1,\delta}(\lambda) &\approx \frac{-1}{\pi} \int_{\lambda}^{\infty} \frac{1}{1-\tau} \theta_{1,\delta}(\tau) d\tau - \frac{1}{2\alpha} \theta_{1,\delta}^2(\lambda), \quad \lambda > 1.
\end{aligned}$$

4.3. Nth-order approximation. For $n \geq 2$, we get

$$\begin{aligned}
q_{n,\gamma}(\lambda) &\approx \frac{-\pi}{\alpha} (\lambda - 1) \frac{\partial \theta_{n,\gamma}(\lambda)}{\partial \lambda}, \quad \lambda > 1, \\
q_{n,\delta}(\lambda) &\approx \frac{-\pi}{\alpha} (\lambda - 1) \frac{\partial \theta_{n,\delta}(\lambda)}{\partial \lambda}, \quad \lambda > 1. \\
\theta_{n,\gamma}(\lambda) &\approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^{\infty} \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta_{n,\gamma}}{\partial \tau}(\tau) d\tau, \quad \lambda > 1, \\
\theta_{n,\delta}(\lambda) &\approx \frac{-\sqrt{\lambda-1}}{\alpha} \int_1^{\infty} \frac{\sqrt{\tau-1}}{\tau-\lambda} \frac{\partial \theta_{n,\delta}}{\partial \tau}(\tau) d\tau, \quad \lambda > 1, \\
x_{n,\gamma}(\lambda) &\approx \frac{-1}{\alpha} \theta_{n,\gamma}(\lambda), \quad \lambda > 1, \quad \text{and} \quad x_{n,\delta}(\lambda) \approx \frac{-1}{\alpha} \theta_{n,\delta}(\lambda), \quad \lambda > 1.
\end{aligned}$$

4.4. Solution of the problem. solution of the zeroth-order is given as: the direction of velocity Considering the horizontal orientation

$$\theta_0(\lambda) \approx 0, \quad \lambda > 1,$$

the flow's velocity

$$q_0(\lambda) \approx 1, \quad \lambda > 1,$$

on the other hand, we have

$$x_0(\lambda) \approx x_{\infty} - \frac{1}{\pi} \int_{\lambda}^{\infty} \frac{1}{1-\tau} d\tau - \frac{1}{\alpha} \theta_0(\lambda), \quad \lambda > 1,$$

at the point C , we have ($\lambda = \lambda_c, x = 0$), therefore

$$x_{\infty} = \frac{1}{\pi} \int_{\lambda_c}^{\infty} \frac{1}{1-\tau} d\tau,$$

Then

$$x_0(\lambda) \approx \frac{-1}{\pi} \ln \left(\frac{\lambda - 1}{1 - \lambda_c} \right), \quad \lambda > 1,$$

and

$$y_0(\lambda) \approx 1 + l, \quad \lambda > 1.$$

For high Weber number α , the first term in (4.6) and (4.7) may be disregarded and we obtain the solution to first-order in γ and δ

$$\begin{aligned}
\theta_{1,\gamma}(\lambda) &\approx \frac{2}{\pi} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right), \quad \lambda > 1, \\
\theta_{1,\delta}(\lambda) &\approx \frac{-2}{\pi} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \quad \lambda > 1, \\
x_{1,\gamma}(\lambda) &\approx \frac{-2}{\pi\alpha} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right), \quad \lambda > 1,
\end{aligned}$$

$$\begin{aligned}
x_{1,\delta}(\lambda) &\approx \frac{2}{\pi\alpha} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \quad \lambda > 1, \\
y_{1,\gamma}(\lambda) &\approx \frac{4(1-\sqrt{1-\lambda_c})}{\pi^2(1-\lambda_c)^{\frac{1}{4}}} \left(\arctan \left(\frac{\sqrt{\lambda-1}}{(1-\lambda_c)^{\frac{1}{4}}} \right) - \frac{\pi}{2} \right), \quad \lambda > 1, \\
y_{1,\delta}(\lambda) &\approx \frac{4(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\pi^2((1-\lambda_c)(1-\lambda_d))^{\frac{1}{4}}} \left(\arctan \left(\frac{\sqrt{\lambda-1}}{((1-\lambda_c)(1-\lambda_d))^{\frac{1}{4}}} \right) - \frac{\pi}{2} \right), \quad \lambda > 1,
\end{aligned}$$

Using an approximate first-order solution, and for $\lambda > 1$, we arrive at:

$$\begin{aligned}
q(\lambda) &\approx 1 + \frac{\gamma\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)(\lambda-1-\sqrt{1-\lambda_c})}{\alpha\lambda(\lambda-\lambda_c)} - \frac{\delta\sqrt{\lambda-1}}{\alpha} \\
&\quad \frac{(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})(\lambda-1-\sqrt{(\lambda_c-1)(\lambda_d-1)})}{((\lambda-1+\sqrt{(\lambda_c-1)(\lambda_d-1)})^2+(\lambda-1)(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})^2)}, \\
\theta(\lambda) &\approx \frac{2\gamma}{\pi} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right) - \frac{2\delta}{\pi} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \\
x(\lambda) &\approx \frac{1}{\pi} \ln \left(\frac{1-\lambda_c}{\lambda-1} \right) - \frac{2\gamma}{\pi\alpha} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_c}-1)}{\lambda-1+\sqrt{1-\lambda_c}} \right) \\
&\quad + \frac{2\delta}{\pi\alpha} \arctan \left(\frac{\sqrt{\lambda-1}(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\lambda-1+\sqrt{(1-\lambda_c)(1-\lambda_d)}} \right), \\
y(\lambda) &\approx 1 + l + \frac{4\gamma(1-\sqrt{1-\lambda_c})}{\pi^2(1-\lambda_c)^{\frac{1}{4}}} \left(\arctan \left(\frac{\sqrt{\lambda-1}}{(1-\lambda_c)^{\frac{1}{4}}} \right) - \frac{\pi}{2} \right) \\
&\quad + \frac{4\delta(\sqrt{1-\lambda_d}-\sqrt{1-\lambda_c})}{\pi^2((1-\lambda_c)(1-\lambda_d))^{\frac{1}{4}}} \left(\arctan \left(\frac{\sqrt{\lambda-1}}{((1-\lambda_c)(1-\lambda_d))^{\frac{1}{4}}} \right) - \frac{\pi}{2} \right)
\end{aligned}$$

5. RESULTS AND DISCUSSIONS

5.1. Results for $\gamma \neq 0$ and $\delta \neq 0$. In this case, the flow configuration as is shown in figure 1, models flow passing over a triangular hollow at the bottom of a channel. Using the previous method illustrated in section 4 and after calculation, the shape of the free surface with the fixed values of the Weber number $\alpha = 10000$, the depth of the hollow $l = 0.05$ and the angles $\gamma = \frac{\pi}{8}$ and $\delta = \frac{\pi}{8}$ is shown in figure 4.a.

The free surface profiles of the flow are plotted in figure 4.b, for different values of the depth l , for a fixed values of $\alpha = 200$, $\gamma = \frac{\pi}{8}$ and $\delta = \frac{\pi}{8}$.

As it appears in this figure, we see that the results show that as the depth l increases, the elevation of the free surface also increases, indicating a correlation between the depth of the hollow and the variation of the surface profile.

Figures 4.c, 4.d and 4.e illustrate the free surface profiles for various values of the angles γ and δ , at a fixed $\alpha = 10000$ and $l = 0.1$.

As shown in figure 4.c, increasing the value of the angle γ causes decreases in the radius of curvature of the free surface above the hollow.

The variation of the angle δ causes changes in the height of the free surface and increasing of the flow's velocity at higher angles causes the free surface to rise significantly see figure 4.d. Thus, the flow behavior can be affected by variations in the shape of the free surface as a function of δ . The figure 4.e present the variation of the angles in the symmetrical case when $\gamma = \delta$ on the free surface forms.

The following table presents some values of λ_c and λ_d for different angles γ , δ and depth l .

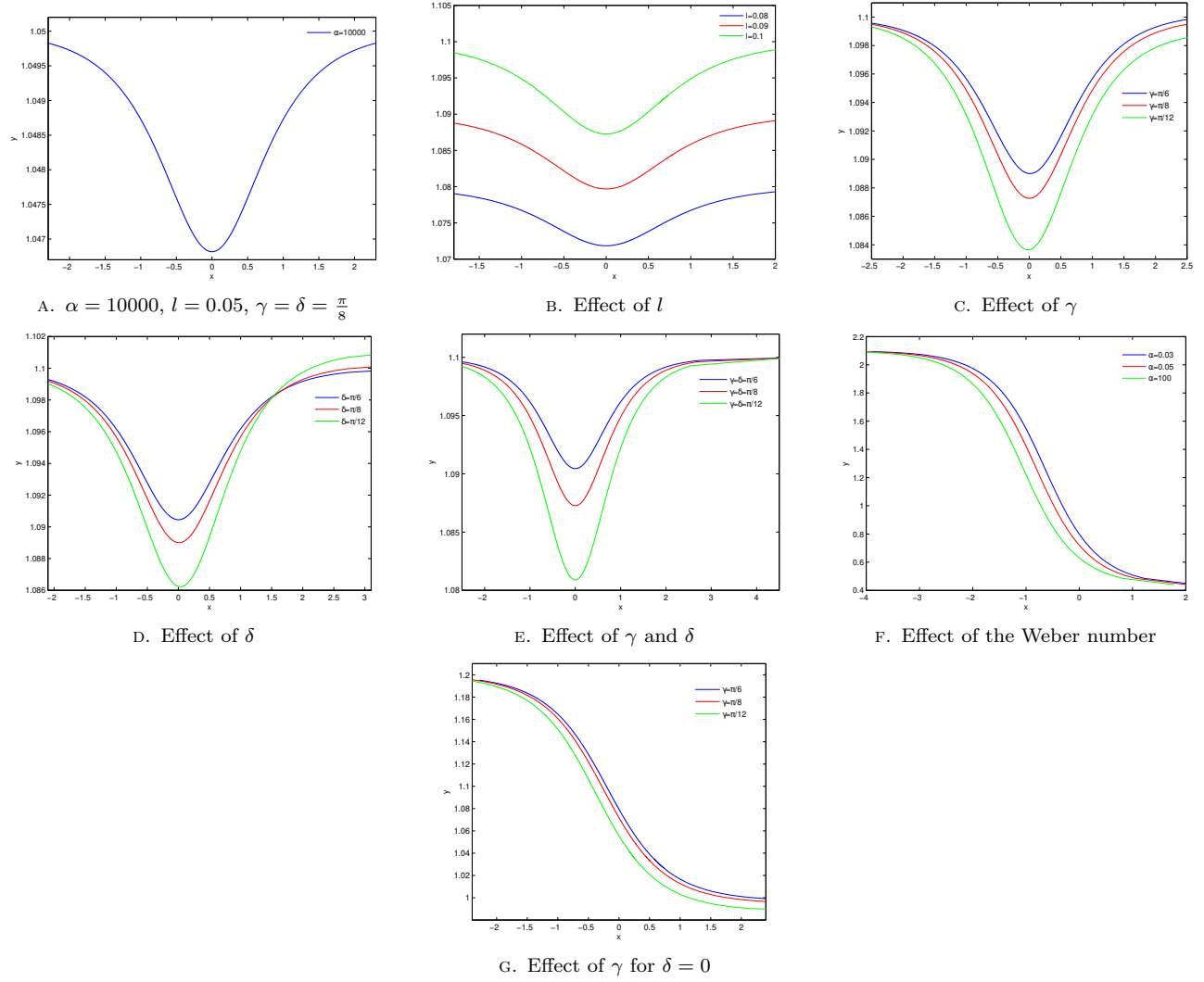


FIGURE 4. Free surface profiles for different values of the Weber number, depth, and inclination angles: (a) the shape of the free surface for $\alpha = 10000$, $l = 0.05$, and $\gamma = \delta = \frac{\pi}{8}$; (b) the effect of the depth of the hollow l for $\alpha = 200$ and $\gamma = \delta = \frac{\pi}{8}$; (c) the effect of the angle γ for $\alpha = 10000$, $l = 0.1$, and $\delta = \frac{\pi}{8}$; (d) the effect of the angle δ for $\alpha = 10000$, $l = 0.1$, and $\gamma = \frac{\pi}{6}$; (e) the effect of the angles γ and δ for $\alpha = 1000$ and $l = 0.1$; (f) the effect of the Weber number for $l = 1.1$ and $\gamma = \frac{\pi}{6}$; (g) the effect of the angle γ for $\alpha = 10000$, $l = 0.2$, and $\delta = 0$.

TABLE 1. The values of λ_c and λ_d for various angles γ , δ , and depth l .

γ	δ	l	λ_c	λ_d
$\frac{\pi}{6}$	$\frac{\pi}{6}$	0.50	0.9502	0.9975
		1.30	0.9996	1.0000
$\frac{\pi}{8}$	$\frac{\pi}{8}$	0.05	0.3297	0.5507
		1.02	0.9997	1.0000
$\frac{\pi}{12}$	$\frac{\pi}{12}$	0.30	0.9727	0.9993
		0.80	0.9999	1.0000

5.2. Results for $\delta = 0$ or $\gamma = 0$. For the case when $\delta = 0$ and several values of γ , the problem is reduced to studying flow over a descending step and the figure 4.f illustrates the effect of different values of the Weber number α on the free-surface profiles. It can be noted from this figure that as surface tension increases, the curvature of the free surface decreases. This is consistent with the theoretical aspect. Figure 4.g shows the effect of different values of the inclination angle γ on the free-surface profiles for $\delta = 0$. It highlights how varying the angle can influence the shape and behavior of the free surface. Finally, when $\gamma = 0$ the flow configuration reduce to a problem of flow over a step. This problem is treated numerically in previous paper presented by Bounif and Gasmi [7].

6. CONCLUSION

The study formulates the flow passing over a trapezoidal hollow at the bottom of a channel as a system of nonlinear integral equations and uses perturbation techniques to derive approximate solutions at high Weber numbers. The results illustrate the free surface profiles and show the impact of parameters like inclination angles, depth, and the Weber number on the flow. The findings highlight the effectiveness of the approximate scheme and provide valuable insights into flow characteristics, demonstrating how these parameters influence the free-surface profiles. We emphasize that the present analysis neglects gravity and viscosity; under these conditions the surface tension dominates the flow dynamics, but the inviscid and irrotational assumptions preclude the observation of viscous boundary layers and possible vortical structures within the hollow.

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Direct and inverse problems for Barenblatt-Zhel'tov-Kochina type fractional equations

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Abstract. In the present work, we consider a Cauchy problem for a nonhomogeneous fractional-order equation of the Barenblatt-Zhel'tov-Kochina type with the Caputo derivative. A solution to the problem is constructed by means of the Fourier method. The continuity of the obtained solution is justified using the convergence properties of functional series. In addition, the uniqueness of the solution is proved. Throughout the analysis, essential use is made of the properties of the Mittag-Leffler function. Moreover, the inverse problem of determining the source function is also investigated, and the existence and uniqueness of its solution are rigorously established.

Keywords: Cauchy problem, Caputo derivative, Mittag-Leffler function, direct problem, inverse problem

MSC (2020): 35R11, 34A12

1. INTRODUCTION AND PROBLEM STATEMENT

Let $A : H \rightarrow H$ be a positive self-adjoint unbounded operator on the separable Hilbert space H , possessing a complete set of orthonormal eigenfunctions $\{v_k\}$ with corresponding positive eigenvalues $\{\lambda_k\}$. By reindexing as needed, we may assume that the eigenvalues satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow +\infty.$$

Let $C((a, b); H)$ denote the set of continuous functions $u(t)$ on the interval $t \in (a, b)$ with values in H . Now, recall the concept of the degree of an operator A in a Hilbert space H . Let τ be an arbitrary real number. We introduce the degree of the operator A in H as follows:

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k,$$

where $h_k = (h, v_k)$ are the Fourier coefficients of the element $h \in H$.

Obviously, the domain of this operator has the form:

$$D(A^\tau) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 < \infty\}.$$

For elements $h, g \in D(A^\tau)$ we introduce the inner product and the norm as follows:

$$(h, g)_\tau = \sum_{k=1}^{\infty} \lambda_k^{2\tau} (h_k, g_k), \quad h, g \in H,$$

$$\|h\|_\tau^2 = \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 = \|A^\tau h\|^2.$$

Together with this inner product and norm $D(A^\tau)$ turns into a Hilbert space.

The Caputo fractional derivative of order $0 < \rho < 1$ is defined as follows (see, [1]):

$$D_t^\rho h(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{h'(\xi)}{(t-\xi)^\rho} d\xi, \quad t > 0.$$

Consider the following problems:

Direct problem. Find a function $u(t)$ that satisfies the Cauchy problem for fractional equations of Barenblatt-Zhel'tov-Kochina type:

$$\begin{cases} D_t^\rho u(t) + A^a(D_t^\rho u(t)) + A^b u(t) = f(t), & 0 < t \leq T; \\ u(+0) = \varphi, \end{cases} \quad (1.1)$$

where $\varphi \in H$, $f(t) \in C([0; T], H)$, $0 < \rho < 1$ and $a, b > 0$.

Let $AC[0, T]$ be the set of absolutely continuous functions defined on $[0, T]$ and let $AC([0, T]; H)$ stand for the space of absolutely continuous functions $u(t)$ with values in H .

Definition 1.1. A function $u(t) \in AC([0, T]; H)$ with the properties $D_t^\rho u(t)$, $A^a(D_t^\rho u(t))$, $A^b u(t) \in C((0, T]; H)$ and satisfying conditions (1.1) is called the solution of the Cauchy problem (1.1).

In this paper, we study not only the direct problem but also the inverse problem. In particular, let us assume that in problem (1.1) the function $f(t)$ is unknown and independent of t . Under this assumption, we investigate the inverse problem of determining the unknown f , and for this purpose, we employ the following additional condition:

$$\int_0^T u(\tau) \omega(\tau) d\tau = \psi, \quad 0 < \tau \leq T, \quad (1.2)$$

where $\omega(t)$ and ψ are given functions, and their specific conditions will be provided later.

Inverse problem. Determine the pair of functions $\{u(t), f\}$ that satisfies problem (1.1) subject to the additional condition (1.2).

Definition 1.2. If the functions $u(t) \in AC([0, T]; H)$ and $f \in H$ have the properties $D_t^\rho u(t)$, $A^a(D_t^\rho u(t))$, $A^b u(t) \in C((0, T]; H)$ and satisfy conditions (1.1) and (1.2), then the pair of functions $\{u(t), f\}$ is called the solution of the inverse problem (1.1) and (1.2).

Let us consider the following equation:

$$\frac{\partial^k u}{\partial t^k} + \Delta \left(\frac{\partial^k u}{\partial t^k} \right) - \nu^2 \Delta u = 0, \quad (1.3)$$

where $\Delta = \sum_{k=1}^N \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator.

If $k = 1$, then this equation is called a partial differential equation of the Barenblatt-Zhel'tov-Kochina type; if $k = 2$, then it is called a partial differential equation of the Boussinesq type. Equations of the form (1.3) are encountered in modeling various processes. For example, equations of the Barenblatt-Zhel'tov-Kochina type are the basic equations for the filtration of homogeneous liquids. This was first noticed in the work of G. I. Barenblatt, Yu. P. Zhel'tov and I. N. Kochina [2]. The article [2] is devoted to the stabilization of solutions to the stochastic Barenblatt-Zhel'tov-Kochina equation. Later, such equations were studied in the papers [3, 4, 5, 6, 7]. Numerous other related works are discussed in the paper, and their results are compared with the corresponding theorems.

For example, in [7], the magneto-hydrodynamic equilibrium of magnetic stars with a poloidal-toroidal mixed magnetic field is analyzed using non-classical mathematical physics equations. The model is constructed and its consistency with observational data is examined. As a result, a general solution describing the equilibrium in the magnetosphere is obtained.

The present paper consists of four sections: in Section (2), the main properties of the Mittag-Leffler function required for the forward and inverse problems are presented. In Section (3), a complete proof of the existence and uniqueness of the solution to the forward problem is provided. Moreover, this section discusses research on problems similar to those studied in this work. Section (4) is devoted to the inverse problem, where the existence and uniqueness of its solution are established.

2. PRELIMINARIES

The Mittag-Leffler function $E_{\rho, \mu}(t)$ has the form

$$E_{\rho, \mu}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\rho n + \mu)},$$

where $\rho > 0$ and μ is a complex number.

We recall the following properties of the Mittag-Leffler function:

Lemma 2.1. *Let $0 < \rho < 1$, $\lambda > 0$. Then the following equality holds:*

$$D_t^\rho [t^\rho E_{\rho, \rho+1}(-\lambda t^\rho)] = E_{\rho, 1}(-\lambda t^\rho).$$

Lemma (2.1) has been proved in [8].

Lemma 2.2. *Let $0 < \rho < 1$, $\lambda > 0$. Then the following equality holds:*

$$D_t^\rho [E_{\rho, 1}(-\lambda t^\rho)] = -\lambda E_{\rho, 1}(-\lambda t^\rho).$$

Lemma (2.2) has been proved in [8].

Lemma 2.3. *Let $0 < \rho < 1$, $\lambda > 0$ and $\mu > 0$. Then the following equality holds:*

$$\int_0^t s^{\mu-1} E_{\rho, \mu}(-\lambda s^\rho) ds = t^\mu E_{\rho, \mu+1}(-\lambda t^\rho).$$

Lemma (2.3) has been proved in [9], p. 100.

We use the following estimates of the Mittag-Leffler function (see, [10] p. 134-136):

$$0 < E_{\rho, \mu}(-t) < 1, \quad t > 0, \quad (2.1)$$

$$0 < E_{\rho, \mu}(-t) \leq \frac{C}{1+t}, \quad t > 0, \quad (2.2)$$

$$E_{\rho, \mu}(-t) = \frac{1}{t} \left(1 + O\left(\frac{1}{t^2}\right) \right), \quad t > 1, \quad (2.3)$$

where $\rho > 0$, and μ is a complex number.

We also use an estimate with a positive number λ and $0 < \varepsilon < 1$ (see, e.g. [11]):

$$|t^{\rho-1} E_{\rho, \rho}(-\lambda t^\rho)| \leq \frac{C t^{\rho-1}}{1 + (\lambda t^\rho)^2} \leq C \lambda^{\varepsilon-1} t^{\varepsilon\rho-1}, \quad t > 0. \quad (2.4)$$

3. CAUCHY PROBLEM

Theorem 3.1. *Let $0 < \rho < 1$. If 1) $b \leq a$, $\varphi \in D(A^b)$, $f \in C([0, T]; H)$ or 2) $b > a$, $\varphi \in D(A^a)$, $f \in C([0, T]; D(A^\varepsilon))$, then the problem (1.1) has a unique solution and this solution has the form:*

$$u(t) = \sum_{k=1}^{\infty} \left[\varphi_k E_{\rho, 1}(-\mu_k t^\rho) + \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho, \rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right] v_k, \quad (3.1)$$

where φ_k are the Fourier coefficients of the function φ and $\mu_k = \frac{\lambda_k^b}{1 + \lambda_k^a}$.

We note that problems similar to (1.1), in the case $a = 1, b = 1$ were studied in [8]. In this work, if $\varphi \in D(A)$, $f \in H$, the existence and uniqueness of the solution are proved. When $a = 2, b = 1$, problem (1.1) is referred to as a Benney-Luke type problem. A problem of this type has been studied in the work [12]. In this work, the existence and uniqueness of the solution of the direct problem are proved for $\varphi \in D(A)$ and $f \in H$, while the existence and uniqueness of the inverse problem are shown for $\varphi, \psi \in D(A^2)$. In the case $a = 0$ and $b = 1$, the equation presented in problem (1.1) was investigated in [13] as a nonlocal problem. When $f = 0$, the problem (1.1) was studied in [14]. In the problem of the paper [14], if $c = \min(a, b)$ and $\varphi \in D(A^c)$, then the problem admits a unique solution.

We focus in particular on paper [15], which was carried out by M. Ruzhansky, D. Serikbaev, N. Tokmagambetov, and B. T. Torebek, as it is closely related to our work. In the paper [15], the following pseudo-parabolic equation is studied:

$$D_t^\rho [u(t) + \mathcal{L}u(t)] + \mathcal{M}u(t) = f(t), \quad 0 < t < T < \infty, \quad (3.2)$$

with the Cauchy condition

$$u(0) = \varphi \in H.$$

Here, $\mathcal{L}$ and $\mathcal{M}$ are self-adjoint operators on a separable Hilbert space H . In this work, the authors investigate the existence and uniqueness of solutions to the direct problem and show that the solution belongs to the following space:

$$u \in L_2([0, T]; D(\mathcal{M})) \cap W^\rho([0, T]; D(\mathcal{L})), \quad 0 < \rho < 1,$$

when $\varphi \in D(\mathcal{L}) \cap D(\mathcal{M})$, $f \in W^1([0, T]; H)$.

For $0 < \rho \leq 1$, the space $W^\rho([0, T]; X)$ consists of all L_2 -integrable functions $g : [0, T] \rightarrow X$ whose fractional derivative $D_t^\rho g$ is also L_2 -integrable.

Now we will compare the results obtained in [15] with the findings of our study, highlighting the similarities and differences between the two works. The operators $\mathcal{L}$ and $\mathcal{M}$ are used instead of the operators A^a and A^b appearing in equation (1.1). The difference between problem (1.1) and (3.2) is that the problem (1.1) is studied depending on the power of the operator A . In [15], the problem is studied in two cases. In the first case, for $1/2 < \rho < 1$, it is assumed that $\varphi \in D(\mathcal{L}) \cap D(\mathcal{M})$ and $f \in L_2([0, T]; H) \cap L_2([0, T]; H_{L, M}^{-1, 1})$. In the second case, for $0 < \rho < 1$, they require more conditions on the source term and $f \in W^1([0, T]; H)$. However, in our work, the problem is studied for $0 < \rho < 1$ without imposing any additional conditions.

Another distinction is that in [15], the variable t is taken into account only almost everywhere, which implies that $f(t) \in W^1([0, T]; H)$. In contrast, in the present work, t is considered at every point, ensuring the continuity of the function $f(t)$.

Next, we compare the inverse problem considered in our work with the inverse problem studied in [15]. In both works, the inverse problem is studied in order to determine the pair $\{u(t), f\}$. In [15], the additional condition $u(T) = \psi$ is imposed, whereas in the present paper the overdetermination condition is given in the form of (1.2). However, in both works it is assumed that $f \in H$ and $\varphi, \psi \in D(A^c)$, where $c = \max(a, b)$.

Now, we present the proof of Theorem (3.1):

Proof: Due to the completeness of the system $\{v_k\}$ in H , an arbitrary solution of (1.1) can be written in the form:

$$u(t) = \sum_{k=1}^{\infty} T_k(t) v_k. \quad (3.3)$$

Substituting into the first equation of (1.1) we have the following equation:

$$D_t^\rho T_k(t) (1 + \lambda_k^a) + T_k(t) \lambda_k^b = f_k(t),$$

Therefore,

$$D_t^\rho T_k(t) + \mu_k T_k(t) = \frac{f_k(t)}{1 + \lambda_k^a}, \quad \mu_k = \frac{\lambda_k^b}{1 + \lambda_k^a}.$$

Using the second condition of (1.1), we obtain the following problem:

$$\begin{cases} D_t^\rho T_k(t) + \mu_k T_k(t) = \frac{f_k(t)}{1 + \lambda_k^a}, \\ T_k(+0) = \varphi_k. \end{cases} \quad (3.4)$$

This problem has a unique solution (see, [16] p. 231):

$$T_k(t) = \varphi_k E_{\rho, 1}(-\mu_k t^\rho) + \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho, \rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta. \quad (3.5)$$

We transform (3.5) into equality (3.3). Then, we obtain the formal solution (3.1).

To prove uniqueness, it suffices to consider the homogeneous equation (1.1) with $f_k(t) = 0$ and $\varphi_k = 0$ for all k . Then from (3.5), it immediately follows that $T_k(t) \equiv 0$ for all $k \geq 1$. Taking into

account the definition of $T_k(t)$ and the completeness of the system $\{v_k\}$, we conclude that $u(t) \equiv 0$. This establishes the uniqueness of the solution.

It remains to prove that the constructed formal solution satisfies all the requirements of Definition (1.1). We will analyze the proof of the theorem separately for the cases $b \leq a$ and $b > a$.

1) Let $a \geq b$.

Let $S_n(t)$ denote the partial sums of (3.1):

$$S_n(t) = \sum_{k=1}^n \left[\varphi_k E_{\rho,1}(-\mu_k t^\rho) + \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right] v_k,$$

By applying the estimate (2.1) for the Mittag-Leffler function together with the Parseval equality, we obtain the estimate below:

$$\begin{aligned} \|S_n(t)\|^2 &= \sum_{k=1}^n \left| \varphi_k E_{\rho,1}(-\mu_k t^\rho) + \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right|^2 \\ &\leq 2 \sum_{k=1}^n |\varphi_k E_{\rho,1}(-\mu_k t^\rho)|^2 + 2 \sum_{k=1}^n \left| \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right|^2. \end{aligned}$$

Using the estimate (2.4), we have:

$$\begin{aligned} \|S_n(t)\|^2 &\leq 2 \sum_{k=1}^n |\varphi_k|^2 + 2C \sum_{k=1}^n \left(\frac{1}{1 + \lambda_k^a} \int_0^t \mu_k^{\varepsilon-1} \eta^{\varepsilon\rho-1} |f_k(t - \eta)| d\eta \right)^2 \\ &\leq 2 \sum_{k=1}^n |\varphi_k|^2 + 2C \sum_{k=1}^n \frac{\lambda_k^{2b(\varepsilon-1)}}{(1 + \lambda_k^a)^{2\varepsilon}} \left(\int_0^t \eta^{\varepsilon\rho-1} |f_k(t - \eta)| d\eta \right)^2 \\ &\leq 2 \sum_{k=1}^n |\varphi_k|^2 + 2C \sum_{k=1}^n \lambda_k^{-2(a\varepsilon+b(1-\varepsilon))} \left(\int_0^t \eta^{\varepsilon\rho-1} |f_k(t - \eta)| d\eta \right)^2. \end{aligned}$$

Applying the generalized Minkowski inequality, we obtain the following estimate:

$$\begin{aligned} \|S_n(t)\|^2 &\leq 2 \sum_{k=1}^n |\varphi_k|^2 + \frac{2C}{\lambda_1^{2(a\varepsilon+b(1-\varepsilon))}} \left(\int_0^t \eta^{\varepsilon\rho-1} \left(\sum_{k=1}^n |f_k(t - \eta)|^2 \right)^{\frac{1}{2}} d\eta \right)^2 \\ &\leq 2 \sum_{k=1}^n |\varphi_k|^2 + C_1 \max_{t \in [0, T]} \|f(t)\|^2. \end{aligned}$$

According to the above expression, if $\varphi \in H$, $f(t) \in C([0, T]; H)$ then $u(t) \in AC([0, T]; H)$.

Now let us estimate $A^b u(t)$. Using the Parseval equality, the estimate is as follows:

$$\begin{aligned} \|A^b S_n(t)\|^2 &\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k E_{\rho,1}(-\mu_k t^\rho)|^2 \\ &\quad + 2 \sum_{k=1}^n \lambda_k^{2b} \left| \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right|^2. \end{aligned}$$

Next, similarly to the above, using the properties of the Mittag-Leffler function and the generalized Minkowski inequality, we obtain the estimate below:

$$\|A^b S_n(t)\|^2 \leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + \frac{2C}{\lambda_1^{2\varepsilon(a-b)}} \left(\int_0^t \eta^{\varepsilon\rho-1} \left(\sum_{k=1}^n |f_k(t - \eta)|^2 \right)^{1/2} d\eta \right)^2$$

$$\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + C_1 \max_{t \in [0, T]} \|f(t)\|^2.$$

Thus, if $\varphi \in D(A^b)$ and $f(t) \in C([0, T]; H)$, then $A^b u(t) \in C([0, T]; H)$.

Next, we estimate the term $A^a(D_t^\rho u(t))$. To show this, we first carry out the necessary calculations:

$$\begin{aligned} G(t) &= \int_0^t \eta^{\rho-1} E_{\rho, \rho}(-\mu_k \eta^\rho) f_k(t-\eta) d\eta = \int_0^t (t-\eta)^{\rho-1} E_{\rho, \rho}(-\mu_k(t-\eta)^\rho) f_k(\eta) d\eta \\ &= -f_k(\eta)(t-\eta)^\rho E_{\rho, \rho+1}(-\mu_k(t-\eta)^\rho) \Big|_0^t + \int_0^t (t-\eta)^\rho E_{\rho, \rho+1}(-\mu_k(t-\eta)^\rho) f'_k(\eta) d\eta \\ &= f_k(0)t^\rho E_{\rho, \rho+1}(-\mu_k t^\rho) + \int_0^t (t-\eta)^\rho E_{\rho, \rho+1}(-\mu_k(t-\eta)^\rho) f'_k(\eta) d\eta = g_1(t) + g_2(t). \end{aligned}$$

By using Lemma (2.1), we obtain the following expression:

$$D_t^\rho g_1(t) = D_t^\rho (f_k(0)t^\rho E_{\rho, \rho+1}(-\mu_k t^\rho)) = f_k(0)E_{\rho, 1}(-\mu_k t^\rho),$$

and

$$\begin{aligned} D_t^\rho g_2(t) &= \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{1}{(t-s)^\rho} \int_0^s (s-\eta)^{\rho-1} E_{\rho, \rho}(-\mu_k(s-\eta)^\rho) f'_k(\eta) d\eta ds \\ &= \frac{1}{\Gamma(1-\rho)} \int_0^t f'_k(\eta) d\eta \int_\eta^t \frac{(s-\eta)^{\rho-1}}{(t-s)^\rho} E_{\rho, \rho}(-\mu_k(s-\eta)^\rho) ds \\ &= \frac{1}{\Gamma(1-\rho)} \int_0^t f'_k(\eta) \left(\sum_{k=0}^{\infty} \frac{(-\mu_k)^k}{\Gamma(\rho k + \rho)} \int_\eta^t \frac{(s-\eta)^{\rho k + \rho - 1}}{(t-s)^\rho} ds \right) d\eta \\ &= \frac{1}{\Gamma(1-\rho)} \int_0^t f'_k(\eta) \left(\sum_{k=0}^{\infty} \frac{(-\mu_k)^k}{\Gamma(\rho k + \rho)} (t-\eta)^{\rho k} B(\rho k + \rho, 1-\rho) \right) d\eta \\ &= \int_0^t f'_k(\eta) E_{\rho, 1}(-\mu_k(t-\eta)^\rho) d\eta = f_k(\eta) E_{\rho, 1}(-\mu_k(t-\eta)^\rho) \Big|_0^t - \int_0^t f_k(\eta)(t-\eta)^{\rho-1} E_{\rho, \rho}(-\mu_k(t-\eta)^\rho) d\eta \\ &= f_k(t) - f_k(0)E_{\rho, 1}(-\mu_k t^\rho) - \int_0^t f_k(\eta)(t-\eta)^{\rho-1} E_{\rho, \rho}(-\mu_k(t-\eta)^\rho) d\eta. \end{aligned}$$

As a result, we obtain the resulting equality:

$$D_t^\rho G(t) = f_k(t) - \int_0^t f_k(\eta)(t-\eta)^{\rho-1} E_{\rho, \rho}(-\mu_k(t-\eta)^\rho) d\eta.$$

To this end, we apply Lemma (2.2), which allows us to derive the required bound:

$$\begin{aligned} \|A^a(D_t^\rho S_n(t))\|^2 &\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + 2 \sum_{k=1}^n \left| D_t^\rho \left(\int_0^t \eta^{\rho-1} E_{\rho, \rho}(-\mu_k \eta^\rho) f_k(t-\eta) d\eta \right) \right|^2 \\ &\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + 4 \sum_{k=1}^n |f_k(t)|^2 + C \max_{t \in [0, T]} \|f(t)\|^2 \\ &\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + 4 \|f(t)\|^2 + C \max_{t \in [0, T]} \|f(t)\|^2. \end{aligned}$$

From this, if $\varphi \in D(A^b)$ and $f(t) \in C([0, T]; H)$, we obtain $A^a(D_t^\rho u(t)) \in C([0, T]; H)$.

Estimating $D_t^\rho u(t)$ in the same way as above, we obtain the estimate below:

$$\|D_t^\rho S_n(t)\|^2 \leq 2 \sum_{k=1}^n |\varphi_k|^2 + C_1 \|f(t)\|^2 + C_2 \max_{t \in [0, T]} \|f(t)\|^2.$$

From the above expression, for $\varphi \in H$ and $f(t) \in C([0, T]; H)$, $D_t^\rho u(t)$ belongs to $C([0, T]; H)$.

2) Let $a < b$.

Next, an estimate of $A^b u(t)$ is obtained by applying the Parseval equality:

$$\begin{aligned} \|A^b S_n(t)\|^2 &= 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k E_{\rho,1}(-\mu_k t^\rho)|^2 \\ &+ 2 \sum_{k=1}^n \lambda_k^{2b} \left| \frac{1}{1 + \lambda_k^a} \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right|^2. \end{aligned}$$

Then, using (2.2), (2.4), and the generalized Minkowski inequality, we obtain the following estimate:

$$\begin{aligned} \|A^b S_n(t)\|^2 &\leq 2 \sum_{k=1}^n \lambda_k^{2b} \left| \frac{C}{1 + \mu_k t^\rho} \right|^2 |\varphi_k|^2 + 2 \sum_{k=1}^n \lambda_k^{2b} \left(\frac{1}{1 + \lambda_k^a} \int_0^t C \mu_k^{\varepsilon-1} \eta^{\varepsilon\rho-1} |f_k(t - \eta)| d\eta \right)^2 \\ &\leq \frac{C}{t^{2\rho}} \sum_{k=1}^n \lambda_k^{2a} |\varphi_k|^2 + 2C \sum_{k=1}^n \lambda_k^{2\varepsilon(b-a)} \left(\int_0^t \eta^{\varepsilon\rho-1} |f_k(t - \eta)| d\eta \right)^2 \\ &\leq 2 \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2 + 2C_1 \left(\int_0^t \eta^{\varepsilon\rho-1} \left(\sum_{k=1}^n \lambda_k^{2\varepsilon(b-a)} |f_k(t - \eta)|^2 \right)^{1/2} d\eta \right)^2 \\ &\leq \frac{C}{t^{2\rho}} \sum_{k=1}^n \lambda_k^{2a} |\varphi_k|^2 + C_2 \max_{t \in [0, T]} \|f(t)\|_{\varepsilon(b-a)}^2, \quad t > 0. \end{aligned}$$

It follows that, if $\varphi \in D(A^a)$, $f(t) \in C([0, T]; D(A^{\varepsilon(b-a)}))$, the term $A^b u(t)$ belongs to the space $C((0, T]; H)$.

The term $A^a(D_t^\rho u(t))$ is next estimated using Lemma (2.2) and the Parseval equality:

$$\begin{aligned} \|A^a(D_t^\rho S_n(t))\|^2 &\leq 2 \sum_{k=1}^n \lambda_k^{2a} |-\mu_k \varphi_k E_{\rho,1}(-\mu_k t^\rho)|^2 + 2 \sum_{k=1}^n \left| D_t^\rho \left(\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\mu_k \eta^\rho) f_k(t - \eta) d\eta \right) \right|^2. \end{aligned}$$

Using the above properties of the Mittag-Leffler function and the generalized Minkowski inequality, we obtain an estimate of the form:

$$\begin{aligned} \|A^a(D_t^\rho S_n(t))\|^2 &\leq 2 \sum_{k=1}^n \lambda_k^{2a} \left| \mu_k \varphi_k \frac{C_1}{1 + \mu_k t^\rho} \right|^2 + 4 \sum_{k=1}^n |f_k(t)|^2 + 4C \max_{t \in [0, T]} \|f(t)\|^2 \\ &\leq \frac{C_1}{t^{2\rho}} \sum_{k=1}^n \lambda_k^{2a} |\varphi_k|^2 + 4\|f(t)\|^2 + C_2 \max_{t \in [0, T]} \|f(t)\|^2, \quad t > 0. \end{aligned}$$

Based on the above result, assuming that $\varphi \in D(A^a)$ and $f(t) \in C([0, T]; H)$, the term $A^a(D_t^\rho u(t))$ belongs to the space $C((0, T]; H)$.

Estimating $D_t^\rho u(t)$ in the same way as above, we obtain the following estimate:

$$\|D_t^\rho S_n(t)\|^2 \leq \frac{C_1}{t^{2\rho}} \sum_{k=1}^n |\varphi_k|^2 + C_2 \|f(t)\|^2 + C_3 \max_{t \in [0, T]} \|f(t)\|^2, \quad t > 0.$$

Assuming that $\varphi \in H$, $f(t) \in C([0, T]; H)$, it follows that $D_t^\rho u(t)$ is an element of the space $C((0, T]; H)$.

Thus Theorem (3.1) is fully proved. $\square$

4. INVERSE PROBLEM

Theorem 4.1. *Let $0 < \rho < 1$. If $c = \max(a, b)$, $\varphi, \psi \in D(A^c)$, and the function $\omega(t)$ is sign-preserving on the interval $(0, T]$, then the inverse problem (1.1)–(1.2) admits a unique solution $\{u(t), f\}$, and this solution has the form:*

$$u(t) = \sum_{k=1}^{\infty} \left[\varphi_k E_{\rho,1}(-\mu_k t^\rho) + \frac{f_k}{1 + \lambda_k^a} t^\rho E_{\rho,\rho+1}(-\mu_k t^\rho) \right] v_k \quad (4.1)$$

and

$$f = \sum_{k=1}^{\infty} f_k v_k, \quad (4.2)$$

where

$$f_k = \left(\frac{\psi_k - \varphi_k \int_0^T \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau}{\int_0^T \omega(\tau) \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau} \right) (1 + \lambda_k^a). \quad (4.3)$$

Proof: Existence of the solution.

By (1.2) we have the following equality:

$$\sum_{k=1}^{\infty} \left[\int_0^T \varphi_k \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau + \frac{f_k}{1 + \lambda_k^a} \int_0^T \omega(\tau) \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau \right] v_k = \psi$$

If the ψ function is expanded into a series with respect to $\{v_k\}$, then we obtain the following equality:

$$\begin{aligned} \int_0^T \varphi_k \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau + \frac{f_k}{1 + \lambda_k^a} \int_0^T \omega(\tau) \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau &= \psi_k, \\ \sigma_k(T) \cdot f_k &= \left(\psi_k - \varphi_k \int_0^T \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau \right) (1 + \lambda_k^a), \end{aligned}$$

where

$$\sigma_k(T) = \int_0^T \omega(\tau) \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau.$$

According to the theorem, if the function $\omega(t)$ is sign-preserving, then $\sigma_k(T) \neq 0$ for all $k \in \mathbb{N}$. To prove this, we derive a lower bound for the function $\sigma_k(T)$ by using lemma (2.3):

$$\begin{aligned} |\sigma_k(T)| &= \left| \int_0^T \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) \omega(\tau) d\tau \right| \geq \min_{\tau \in (0,T]} |\omega(\tau)| \int_0^T \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau \\ &= \min_{\tau \in (0,T]} |\omega(\tau)| \int_0^T \frac{1}{\mu_k} (1 - E_{\rho,1}(-\mu_k \tau^\rho)) d\tau = \min_{\tau \in (0,T]} |\omega(\tau)| \cdot \frac{T}{\mu_k} (1 - E_{\rho,2}(-\mu_k T^\rho)). \end{aligned}$$

We consider the following two cases:

1) Let $a \geq b$. In this case, $\mu_k \rightarrow 0$ and we use the following estimate for the Mittag-Leffler function:

$$E_{\rho,2}(-\mu_k T^\rho) = \sum_{k=0}^{\infty} \frac{(-\mu_k T^\rho)^k}{\Gamma(\rho k + 2)} = 1 - \frac{\mu_k T^\rho}{\Gamma(\rho + 2)} + O(\mu_k^2 T^{2\rho}).$$

Then we obtain the following expression:

$$|\sigma_k(T)| \geq \min_{\tau \in (0,T]} |\omega(\tau)| \cdot \frac{T}{\mu_k} \cdot \frac{\mu_k T^\rho}{\Gamma(\rho + 2)} = C T^{\rho+1}. \quad (4.4)$$

2) Let $a < b$. In this case, $\mu_k \rightarrow \infty$ and we obtain the following lower bound for $\sigma_k(T)$ by applying (2.3):

$$|\sigma_k(T)| \geq \frac{C_0}{\mu_k} \quad (4.5)$$

It should be emphasized that if $\omega_k(t) = 0$ or changes its sign on the interval $(0, T)$, then $\sigma_k(T)$ may be equal to zero. In this case, the problem (1.1)–(1.2) may have infinitely many solutions or may not have a solution.

It remains to prove that the constructed formal solution satisfies all the requirements of Definition 1.2. We will analyze the proof of the theorem separately for the cases $a \geq b$ and $a < b$.

1) Let $a \geq b$.

Let F_n denote the partial sums of (4.2):

$$F_n = \sum_{k=1}^n \left[\left(\frac{\psi_k - \varphi_k \int_0^T \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau}{\int_0^T \omega(\tau) \tau^\rho E_{\rho,\rho+1}(-\mu_k \tau^\rho) d\tau} \right) (1 + \lambda_k^a) \right] v_k. \quad (4.6)$$

Then by applying the Parseval equality and (4.4), we derive the following estimate:

$$\begin{aligned} \|F_n\|^2 &= \sum_{k=1}^n \left| \frac{\psi_k - \varphi_k \int_0^T \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau}{\sigma_k(T)} \right|^2 |1 + \lambda_k^a|^2 \\ &\leq 2 \sum_{k=1}^n \left| \frac{\psi_k}{CT^{\rho+1}} \right|^2 |1 + \lambda_k^a|^2 + 2 \sum_{k=1}^n \left| \frac{\varphi_k T \max_{\tau \in (0,T]} \omega(\tau)}{CT^{\rho+1}} \right|^2 |1 + \lambda_k^a|^2 \\ &= \frac{C_1}{T^{2\rho+2}} \sum_{k=1}^n |\psi_k|^2 |1 + \lambda_k^a|^2 + \frac{C_2}{T^{2\rho}} \sum_{k=1}^n |\varphi_k|^2 |1 + \lambda_k^a|^2. \end{aligned}$$

Thus if $\varphi, \psi \in D(A^a)$, series (4.2) is convergent.

2) Let $a < b$. Then the following estimate of (4.6) is obtained by applying (4.5) together with the Parseval equality:

$$\begin{aligned} \|F_n\|^2 &= \sum_{k=1}^n \left| \frac{\psi_k - \varphi_k \int_0^T \omega(\tau) E_{\rho,1}(-\mu_k \tau^\rho) d\tau}{\frac{C_0}{\mu_k}} \right|^2 |1 + \lambda_k^a|^2 \\ &\leq 2 \sum_{k=1}^n \left| \frac{\mu_k \psi_k}{C_0} \right|^2 |1 + \lambda_k^a|^2 + 2 \sum_{k=1}^n \left| \frac{\mu_k \varphi_k T \max_{\tau \in (0,T]} \omega(\tau)}{C_0} \right|^2 |1 + \lambda_k^a|^2 \\ &= C_1 \sum_{k=1}^n \lambda_k^{2b} |\psi_k|^2 + C_2 T \sum_{k=1}^n \lambda_k^{2b} |\varphi_k|^2. \end{aligned}$$

Thus if $\varphi, \psi \in D(A^b)$, series (4.2) is convergent.

Uniqueness. Suppose by contradiction that we have two solutions: $\{u_1(t), f_1\}$ and $\{u_2(t), f_2\}$. It is required to prove $u(t) = u_1(t) - u_2(t) = 0$ and $f = f_1 - f_2 = 0$. Since the problem is linear, to determine $u(t)$ and f we have the problem:

$$D_t^\rho u(t) + A^a(D_t^\rho u(t)) + A^b u(t) = f, \quad (4.7)$$

$$u(+0) = 0,$$

$$\int_0^T u(\tau) \omega(\tau) d\tau = 0 \quad (4.8)$$

Let $u(t)$ be a solution to this problem and $u_k(t) = (u(t), v_k)$. From equation (4.7), taking into account that the operator A is self-adjoint and $\{v_k\}$ is a system of eigenfunctions, we obtain

$$\sum_{k=1}^{\infty} \left[D_t^\rho u_k(t) + \lambda_k^a D_t^\rho u_k(t) + \lambda_k^b u_k(t) \right] v_k = f.$$

Using condition (4.7), we arrive at the following Cauchy problem:

$$\begin{cases} D_t^\rho u_k(t) + \frac{\lambda_k^b}{1 + \lambda_k^a} u_k(t) = \frac{1}{1 + \lambda_k^a} f_k, \\ u_k(0+) = 0. \end{cases}$$

The solution of this problem has the following form (see [16]):

$$u_k(t) = \frac{f_k}{1 + \lambda_k^a} t^\rho E_{\rho, \rho+1}(-\mu_k t^\rho).$$

Using the additional condition (4.8), we obtain

$$\int_0^T u(\tau) \omega(\tau) d\tau = \frac{f_k}{1 + \lambda_k^a} \int_0^T \tau^\rho E_{\rho, \rho+1}(-\mu_k \tau^\rho) \omega(\tau) d\tau = 0.$$

Since

$$\tau^\rho E_{\rho, \rho+1}(-\mu_k \tau^\rho) \omega(\tau) \neq 0 \quad \text{for } \tau > 0, \quad 0 < \rho < 1,$$

it follows that $f_k = 0$ for all $k \geq 1$. Since $\{v_k\}$ forms a complete orthonormal system in H , we obtain $f = 0$. From $f_k = 0$ it follows that $u_k(t) = 0$ for all $k \geq 1$. Again, using the completeness of $\{v_k\}$ in H , we conclude that $u(t) \equiv 0$.

Theorem (4.1) is completely proved. $\square$

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New fixed point theorems for nonlinear contractive mappings of integral type involving ω -distance and its applications

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Abstract. This paper introduces a new class of contractive mappings of integral type involving the concept of ω -distance in complete metric spaces. We establish a fixed point theorem for these mappings, providing sufficient conditions for the existence and uniqueness of fixed points. We also provide an example to illustrate the validity of our results and discuss applications of this fixed point theorem in solving integral equations, demonstrating its practical relevance.

Keywords: Fixed point theorem; Integral-type contraction; ω -distance; Complete metric space; Integral equation

MSC (2020): 47H10, 54H25

1. INTRODUCTION

Fixed point theory is a cornerstone of nonlinear analysis, providing powerful tools for solving a wide range of problems in mathematics, engineering, and other disciplines. Its central concept revolves around finding solutions to equations of the form $T(x) = x$, where T is a mapping from a set to itself, and x is a fixed point of T .

The Banach contraction principle, established by S. Banach in 1922, is one of the fundamental results in fixed point theory. It states that a contraction mapping on a complete metric space has a unique fixed point. It is well known that every Banach contraction mapping is continuous; however, Kannan [1] showed that a discontinuous function satisfying their contractive condition has a unique fixed point. Among these generalizations, we must quote the fixed point results due to Prasad [2] wherein the respective author proved an analogous version of fixed point theorem due to Kannan via conditional contractive mappings and some other relational metrical notions. The Banach contraction principle has been extensively generalized and extended in various directions. Ali et al. [3] generalized classical Banach contraction principle in the setup of quasi-metric spaces. They introduced Suzuki-type contractions of quasi-metric spaces and proved some fixed point results. In [4] a family of hybrid contractions, termed Jaggi-type hybrid $(\aleph - \tau)$ -contractive mapping is proposed in metric space equipped with a graph and new conditions under which the mapping is a Picard operator are studied and also in [5] the authors established the existence and uniqueness of a common coupled fixed point result in the framework of complete symmetric S_b -metric space.

On another point of note, recently in [6] the author prove relational analogue of the Banach contraction principle in the settings of Menger probabilistic metric spaces under a Hadžić-type t -norm. Also, in [7] the authors proved an analogous version of fixed point theorem due to Suzuki using R -sequential limit property and some other relation theoretic metrical variants. For other related results, we refer to the references [8, 9, 10, 11, 12, 13].

One such direction involves generalizing the notion of contractive mappings. In particular, integral-type contractions, initiated by Branciari [11], have gained considerable attention. These contractions replace the usual metrical inequality with an integral inequality, offering a more general framework. Branciari's work sparked further research, leading to numerous fixed point theorems involving integral-type contractions under various conditions (see [14, 15, 16, 17]).

Another important generalization involves weakening the metric itself. The concept of ω -distance, introduced by Kada, Suzuki, and Takahashi [18], replaces the metric with a function satisfying weaker properties, such as lacking symmetry or the triangle inequality. This allows for the study of fixed points in spaces where the standard metric assumptions are not satisfied. Recently, in [19] the author established fixed point theorems for generalized nonlinear contractive mappings using the concept of ω -distance on metric spaces endowed with an arbitrary binary relation. We refer to some notable

references discussing fixed point theorems for nonlinear contractive mappings involving ω -distance [20, 21, 22].

The combination of integral-type contractions and ω -distance provides a powerful framework for studying fixed points in a more general setting. This paper aims to contribute to this area by introducing a new class of contractive mappings of integral type involving ω -distance and establishing corresponding fixed point theorems. We also explore applications of these theorems in solving integral equations.

2. PRELIMINARIES

In this section, we present some basic definitions and results that are necessary for our discussion.

Definition 2.1. Let (X, d) be a metric space. A function $\omega : X \times X \rightarrow [0, \infty)$ is called an ω -distance on X if the following conditions hold:

- (i) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$, for all $x, y, z \in X$;
- (ii) $\omega(\cdot, y) : X \rightarrow [0, \infty)$ is lower semicontinuous for all $y \in X$; that is, if $\{x_n\}$ converges to x , then $\omega(x, y) \leq \liminf_{n \rightarrow \infty} \omega(x_n, y)$;
- (iii) For any $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$, for all $x, y, z \in X$.

Example 2.1. The metric d itself is an ω -distance.

Example 2.2. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. Define $\omega(x, y) = \max\{y - x, 0\}$. Then ω is an ω -distance on X .

Definition 2.2. Let Ψ be the family of mappings $\psi : [0, \infty) \rightarrow [0, \infty)$ which is Lebesgue integrable on each compact subset of $[0, \infty)$ such that for each $\epsilon > 0$, $\int_0^\epsilon \psi(t)dt > 0$.

Definition 2.3. Let (X, d) be a metric space and $T : X \rightarrow X$. T is called a *Branciari contraction* if there exists $k \in (0, 1)$ such that for all $x, y \in X$,

$$\int_0^{d(Tx, Ty)} \psi(t)dt \leq k \int_0^{d(x, y)} \psi(t)dt,$$

where $\psi \in \Psi$.

Lemma 2.1. Let X be a metric space with metric d and let ω be an ω -distance on X . Let $\{x_n\}$ and $\{y_n\}$ be sequences in X , let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to 0, and let $x, y, z \in X$. Then the following hold:

- (i) If $\omega(x_n, y) \leq \alpha_n$ and $\omega(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $\omega(x, y) = 0$ and $\omega(x, z) = 0$, then $y = z$;
- (ii) If $\omega(x_n, y_n) \leq \alpha_n$ and $\omega(x_n, z_n) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z ;
- (iii) If $\omega(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $n > m$, then $\{x_n\}$ is a Cauchy sequence;
- (iv) If $\omega(x, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

3. MAIN RESULTS

Now, we introduce our new class of contractive mappings and establish the corresponding fixed point theorems.

Definition 3.1. Let (X, d) be a metric space and ω be an ω -distance on X . Let $T : X \rightarrow X$ be a mapping. We say that T is a *generalized ω -integral type contraction* if there exists a function $\psi \in \Psi$, and there exists $k \in (0, 1)$ such that for all $x, y \in X$,

$$\int_0^{\omega(Tx, Ty)} \psi(t) dt \leq k \int_0^{M(x, y)} \psi(t) dt,$$

where

$$M(x, y) = \max \left\{ \omega(x, y), \omega(x, Tx), \omega(y, Ty), \frac{\omega(x, Ty) + \omega(y, Tx)}{2} \right\}.$$

Theorem 3.1. Let (X, d) be a complete metric space and ω be an ω -distance on X . Let $T : X \rightarrow X$ be a generalized ω -integral type contraction. Then T has a unique fixed point in X .

Proof: Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by $x_{n+1} = T(x_n)$ for all $n \geq 0$. Consider $\omega(x_{n+1}, x_n) = \omega(Tx_n, Tx_{n-1})$. By the definition of T , we have

$$\int_0^{\omega(Tx_{n-1}, Tx_n)} \psi(t) dt \leq k \int_0^{M(x_{n-1}, x_n)} \psi(t) dt,$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max \left\{ \omega(x_{n-1}, x_n), \omega(x_{n-1}, Tx_{n-1}), \omega(x_n, Tx_n), \frac{\omega(x_{n-1}, Tx_n) + \omega(x_n, Tx_{n-1})}{2} \right\} \\ &= \max \left\{ \omega(x_{n-1}, x_n), \omega(x_{n-1}, x_n), \omega(x_n, x_{n+1}), \frac{\omega(x_{n-1}, x_{n+1}) + \omega(x_n, x_n)}{2} \right\} \\ &= \max \left\{ \omega(x_{n-1}, x_n), \omega(x_n, x_{n+1}), \frac{\omega(x_{n-1}, x_{n+1}) + 0}{2} \right\} \\ &\leq \max \left\{ \omega(x_{n-1}, x_n), \omega(x_n, x_{n+1}), \frac{\omega(x_{n-1}, x_n) + \omega(x_n, x_{n+1})}{2} \right\}. \end{aligned}$$

Let $a_n = \omega(x_n, x_{n+1})$. Then,

$$\int_0^{a_n} \psi(t) dt \leq k \int_0^{\max\{a_{n-1}, a_n, \frac{a_{n-1} + a_n}{2}\}} \psi(t) dt. \quad (3.1)$$

If $a_n \leq a_{n-1}$ for all $n \geq N$, then $\max\{a_{n-1}, a_n\} = a_{n-1}$ and

$$\int_0^{a_n} \psi(t) dt \leq k \int_0^{a_{n-1}} \psi(t) dt.$$

Let $I_n = \int_0^{a_n} \psi(t) dt$. Then $I_n \leq k I_{n-1}$. This implies $I_n \rightarrow 0$. Since $\psi \geq 0$ and $\int_0^r \psi(t) dt$ is strictly increasing, then $a_n \rightarrow 0$.

On the other hand, if $a_n > a_{n-1}$ for infinitely many n , let n be such that $a_n > a_{n-1}$. Then $\max\{a_{n-1}, a_n\} = a_n$ and

$$\int_0^{a_n} \psi(t) dt \leq k \int_0^{a_n} \psi(t) dt.$$

Hence

$$(1 - k) \int_0^{a_n} \psi(t) dt \leq 0.$$

Since $1 - k > 0$ and $\int_0^{a_n} \psi(t) dt \geq 0$, then $\int_0^{a_n} \psi(t) dt = 0$. Since $\psi \geq 0$, this means $a_n = 0$. But we assumed $a_n > a_{n-1} \geq 0$. So this is a contradiction unless $a_{n-1} = 0$ and $a_n = 0$. Thus, we cannot have $a_n > a_{n-1}$ with $a_n > 0$.

Therefore, it must be that $a_n \leq a_{n-1}$ for all n sufficiently large. This implies that

$$\lim_{n \rightarrow \infty} \omega(x_n, x_{n+1}) = 0.$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. Since $\{a_i\}$ is non-increasing and converges to 0, the sum $\sum_{i=n}^{\infty} a_i$ converges. Let $S_n = \sum_{i=n}^{\infty} a_i$. Then $S_n \rightarrow 0$ as $n \rightarrow \infty$. For $m > n$, we have

$$\begin{aligned}\omega(x_n, x_m) &\leq \omega(x_n, x_{n+1}) + \omega(x_{n+1}, x_{n+2}) + \cdots + \omega(x_{m-1}, x_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \leq \sum_{i=n}^{\infty} \omega(x_i, x_{i+1}) = S_n.\end{aligned}$$

Since $S_n \rightarrow 0$ as $n \rightarrow \infty$, then

$$\lim_{n, m \rightarrow \infty} \omega(x_n, x_m) = 0. \quad (3.2)$$

Let $\epsilon > 0$ and δ denote the number in the property (iii) of ω -distance. It follows from (3.2) that there exists $N \in \mathbb{N}$ satisfying

$$\omega(x_N, x_n) < \delta \quad \text{and} \quad \omega(x_N, x_m) < \delta, \quad \forall m, n \geq N,$$

which together with the property (iii) of ω -distance yields that

$$d(x_n, x_m) < \epsilon, \quad \forall m, n \geq N.$$

This shows that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, $\{x_n\}$ converges to some $x \in X$. That is,

$$\lim_{n \rightarrow \infty} x_n = x.$$

Now, we prove that x is a fixed point of T . Observe that (3.2) guarantees that for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ satisfying

$$\omega(x_n, x_m) < \epsilon, \quad \forall m, n \geq N_\epsilon,$$

which together with the property (iii) of ω -distance and $\lim_{n \rightarrow \infty} x_n = x$ yields that

$$0 \leq \omega(x_n, x) \leq \liminf_{n \rightarrow \infty} \omega(x_n, x_m) \leq \epsilon, \quad \forall n \geq N_\epsilon,$$

which gives that

$$\lim_{n \rightarrow \infty} \omega(x_n, x) = 0. \quad (3.3)$$

Also, since $\omega(\cdot, y)$ is lower semi-continuous, we have

$$0 \leq \omega(x, x) \leq \liminf_{n \rightarrow \infty} \omega(x_n, x) = 0.$$

Hence, $\omega(x, x) = 0$. On the other hand,

$$\omega(x, Tx) \leq \liminf_{n \rightarrow \infty} \omega(x_n, Tx) = \liminf_{n \rightarrow \infty} \omega(Tx_{n-1}, Tx) \leq \liminf_{n \rightarrow \infty} M(x_{n-1}, x).$$

It is easy to see that $M(x_{n-1}, x) \rightarrow \omega(x, Tx)$ as $n \rightarrow \infty$. Since $k \in (0, 1)$, then

$$\omega(x, Tx) = 0.$$

Now, since $\omega(x, Tx) = 0$ and $\omega(x, x) = 0$, in view of Lemma 2.1 we deduce that $Tx = x$ and x is a fixed point of T .

To prove uniqueness, suppose y is another fixed point of T , i.e., $Ty = y$ and $x \neq y$. Then,

$$\begin{aligned}M(x, y) &= \max \left\{ \omega(x, y), \omega(x, Tx), \omega(y, Ty), \frac{\omega(x, Ty) + \omega(y, Tx)}{2} \right\} \\ &= \max \left\{ \omega(x, y), \omega(x, x), \omega(y, y), \frac{\omega(x, y) + \omega(y, x)}{2} \right\} \\ &= \max \left\{ \omega(x, y), \frac{\omega(x, y) + \omega(y, x)}{2} \right\}.\end{aligned}$$

Now, we consider the following two cases:

Case 1. If $\omega(y, x) \leq \frac{\omega(x, y) + \omega(y, x)}{2} \leq \omega(x, y)$, then $M(x, y) = \omega(x, y)$. We have

$$\int_0^{\omega(x, y)} \psi(t) dt = \int_0^{\omega(Tx, Ty)} \psi(t) dt \leq k \int_0^{M(x, y)} \psi(t) dt = k \int_0^{\omega(x, y)} \psi(t) dt < \int_0^{\omega(x, y)} \psi(t) dt,$$

which is a contradiction.

Case 2. If $\omega(x, y) \leq \frac{\omega(x, y) + \omega(y, x)}{2} < \omega(y, x)$, then $M(y, x) = \omega(y, x)$. We have

$$\begin{aligned} \int_0^{\omega(x, y)} \psi(t) dt &< \int_0^{\omega(y, x)} \psi(t) dt = \int_0^{\omega(Ty, Tx)} \psi(t) dt \leq k \int_0^{M(y, x)} \psi(t) dt \\ &= k \int_0^{\omega(y, x)} \psi(t) dt < \int_0^{\omega(y, x)} \psi(t) dt, \end{aligned}$$

which is a contradiction. Hence, in two cases, $\omega(x, y) = 0$, which together with $\omega(x, x) = 0$ and Lemma 2.1 implies that $y = x$. This completes the proof. $\square$

Example 3.1. Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. Define $\omega(x, y) = \max\{y - x, 0\}$. Then ω is an ω -distance on X . Define $T : X \rightarrow X$ by $T(x) = x/4$ for all $x \in X$. Let $\psi(t) = 1$ for all $t \in [0, \infty)$. Then ψ is a Lebesgue integrable function. We need to show that

$$\int_0^{\omega(Tx, Ty)} \psi(t) dt \leq k \int_0^{M(x, y)} \psi(t) dt$$

for some $k \in (0, 1)$. We have

$$\omega(Tx, Ty) = \max\{Ty - Tx, 0\} = \max\{y/4 - x/4, 0\} = \frac{1}{4} \max\{y - x, 0\} = \frac{1}{4} \omega(x, y),$$

and

$$\begin{aligned} M(x, y) &= \max \left\{ \omega(x, y), \omega(x, Tx), \omega(y, Ty), \frac{\omega(x, Ty) + \omega(y, Tx)}{2} \right\} \\ &= \max \left\{ \max\{y - x, 0\}, \max\{-\frac{3}{4}x, 0\}, \max\{-\frac{3}{4}y, 0\}, \frac{\max\{\frac{y}{4} - x, 0\} + \max\{\frac{x}{4} - y, 0\}}{2} \right\}. \end{aligned}$$

If $y > x$, then $M(x, y) = y - x$. Hence,

$$\int_0^{\omega(Tx, Ty)} \psi(t) dt = \frac{1}{4} \omega(x, y) = \frac{1}{4} M(x, y) = \frac{1}{4} \int_0^{M(x, y)} \psi(t) dt.$$

Thus, T satisfies the generalized ω -integral type contraction with $k = 1/4$. Therefore, by Theorem 3.1, T has a unique fixed point in X .

4. APPLICATION TO INTEGRAL EQUATIONS

In this section, we apply our fixed point theorem to prove the existence and uniqueness of solutions to a class of nonlinear integral equations.

Integral equations represent a fundamental class of mathematical problems that have found widespread use in fields ranging from physics and engineering to economics and biology. They offer an alternative formulation to differential equations for many physical phenomena and often provide a more natural and efficient way to model certain problems.

Consider the following integral equation:

$$x(t) = \int_0^\tau K(t, s, x(s)) ds, \quad \text{for } t \in [0, \tau],$$

where $K : [0, \tau] \times [0, \tau] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Let $X = C([0, \tau], \mathbb{R})$ be the space of continuous real-valued functions on $[0, \tau]$, equipped with the metric

$$d(x, y) = \max_{t \in [0, \tau]} |x(t) - y(t)|.$$

Then (X, d) is a complete metric space. Define an operator $T : X \rightarrow X$ by

$$T(x)(t) = \int_0^\tau K(t, s, x(s)) ds.$$

It is easy to see that a solution to this integral equation is a fixed point of the operator T .

Assume that there exists $c \in (0, 1)$ such that for all $t, s \in [0, \tau]$ and $u, v \in \mathbb{R}$,

$$|K(t, s, u) - K(t, s, v)| \leq \frac{c}{\tau} |u - v|.$$

We will show that T is a generalized ω -integral type contraction with $\omega(x, y) = d(x, y)$ and $\psi(t) = \frac{1}{2}t$. Then,

$$\begin{aligned} d(Tx, Ty) &= \max_{t \in [0, \tau]} |T(x)(t) - T(y)(t)| = \max_{t \in [0, \tau]} \left| \int_0^\tau K(t, s, x(s)) - K(t, s, y(s)) ds \right| \\ &\leq \max_{t \in [0, \tau]} \int_0^\tau |K(t, s, x(s)) - K(t, s, y(s))| ds \\ &\leq \max_{t \in [0, \tau]} \int_0^\tau \frac{c}{\tau} |x(s) - y(s)| ds = \frac{c}{\tau} \int_0^\tau d(x, y) ds = cd(x, y). \end{aligned}$$

Therefore,

$$\int_0^{d(Tx, Ty)} \frac{1}{2} t dt \leq \sqrt{c} \int_0^{d(x, y)} \frac{1}{2} t dt.$$

Since $d(x, y) = \omega(x, y)$, this shows that T is a generalized ω -integral type contraction with $\psi(t) = \frac{1}{2}t$ and $k = \sqrt{c} \in (0, 1)$.

By Theorem 3.1, T has a unique fixed point in X , which implies that the integral equation (4) has a unique solution in $C([0, \tau], \mathbb{R})$.

CONCLUSION

In this paper, we have introduced a new class of contractive mappings of integral type involving ω -distance and established several fixed point theorems for these mappings. This work opens up avenues for further research, including:

- Exploring different types of ω -distance and their impact on fixed point results;
- Investigating applications in other areas of mathematics and engineering, such as differential equations, optimization, and image processing;
- Developing numerical methods for approximating the fixed points of these mappings.

By combining the power of integral-type contractions and ω -distance, we can continue to expand the scope of fixed point theory and its applications.

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Sequential and computational approach for obtaining the quadruples of a Diophantine System

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Abstract. This manuscript is concerned with obtaining non-negative and non-zero sequential and computational solutions to the Diophantine problem, which states that the sum of four non-negative, non-zero integers, the sum of the squares of these integers, and their cubic sum are perfect squares. Hitherto, no sequential pattern has been discovered on this particular Diophantine form of four unknowns under specific conditions. An open question is further addressed at the end.

Keywords: Diophantine problem, Sequential solution, Cubic sum

MSC (2020): 11D25 , 11D72

1. INTRODUCTION

The paper is primarily concerned with the problem of obtaining four unknowns, provided these unknowns are non-negative, non-zero integers. In detail, the unknowns are associated with the condition that the sum of these unknowns, their squares, and cubes is a square of a non-negative non-zero integer. Two different conditions are presumed for the outcomes. These conditions have not been explored yet in the prevailing scientific literature.

Let us consider a Diophantine system consisting of four unknowns $\alpha, \beta, \gamma, \delta$ associated with the following three concurrent equations, where there are some non-zero, non-negative integers a, b, c .

$$\alpha + \beta + \gamma + \delta = a^2 \quad (1.1)$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = b^2 \quad (1.2)$$

$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = c^2 \quad (1.3)$$

Our prior intention is to find the sequential pattern of the desired solution to the aforementioned problems. Back in 1795, a gentleman named *J. Saul* obtained an integral solution to a simultaneous system that states the sum of two variables, their squares, and their cubes are a perfect square. Based on his remarkable study, *A. Bremner* [1] exhibits all the integral solutions to the problems stated by *J. Saul*. In the problem stated above, when it is considered, the problem turns out to be fully solved by *A. Bremner* [1]. Recently, in the year 2020, *A. Choudhry* [2] extended the problem for three positive integers and found infinitely many parametric solutions. As surveyed till now, no studies have been conducted to attain the solution by addressing one more unknown in the simultaneous system of equations considered by *A. Choudhry* [2].

2. QUADRUPLES FOR THE PRESCRIBED DIOPHANTINE SYSTEM

We will now solve the simultaneous Diophantine system of equations represented by equations (1.1), (1.2), and (1.3), assuming two unknowns are equal.

Let us consider, $\beta = \gamma = \kappa$, where κ is a non-negative non-zero integer. Then (1.1), (1.2), and (1.3) can be rewritten as follows.

$$\alpha + 2\kappa + \delta = a^2 \quad (2.1)$$

$$\alpha^2 + 2\kappa^2 + \delta^2 = b^2 \quad (2.2)$$

$$\alpha^3 + 2\kappa^3 + \delta^3 = c^2 \quad (2.3)$$

Since a, b, c are non-zero non-negative integers, thus there exist some non-zero non-negative integers θ, ψ, p, q, r such that

$$a = \theta + \psi - p \quad (2.4)$$

$$b = \theta^2 - 2p\theta + \psi^2 + q \quad (2.5)$$

$$c = \theta^3 + \psi^3 - 3\theta p(\theta - p) - r \quad (2.6)$$

Our concern is to show whether the values depicted through equations (2.4), (2.5) and (2.6) justify the desired demand of the sequential pattern of the solution.

On solving equations (2.1) and (2.2) by substituting the values of a, b and $q = p^2$, we have obtained a quadratic form of the equation in terms of δ and κ .

$$\begin{aligned} &\delta^2 + (2\kappa - \theta^2 - p^2 + 2\theta p - \psi^2 - 2\theta\psi + 2p\psi)\delta + 2\theta^2\psi^2 + 2p^2\psi^2 - 4\theta p\psi^2 + 3\kappa^2 - 4\theta\psi\kappa \\ &+ 4p\psi\kappa + 2\psi^3\theta - 2\psi^3p - 2\psi^2\kappa + 2\theta^3\psi - 2p^3\psi - 6\theta^2p\psi + 6\theta p^2\psi - 2\kappa\theta^2 - 2\kappa p^2 + 4\kappa\theta p = 0 \end{aligned} \quad (2.7)$$

Without loss of generality, on further substituting $\kappa = \theta p - \psi p$ in equation (2.7), we have

$$\begin{aligned} &\delta^2 + (4\theta p - \theta^2 - p^2 - \psi^2 - 2\theta\psi)\delta + (2\theta^2\psi^2 + p^2\psi^2 + 7\theta^2p^2 - 8\theta^2p\psi + 2\psi^3\theta - 2\theta p\psi^2 \\ &+ 2\theta^3\psi - 2\theta^3p - 2\theta p^3) = 0 \end{aligned} \quad (2.8)$$

It is clearly observed that $(\theta - p)^2$ and ψ^2 satisfies the equation (2.8) while $p = \psi$ is considered.

From equation (2.1), we can also obtain the value of α by substituting the assumed value of $\kappa = \theta p - \psi p$ and $\delta = (\theta - p)^2$. Equation (2.1) leads to $\alpha = (\theta + \psi - p)^2 - 2(\theta p - \psi p) - (\theta - p)^2$ and finally $\alpha = \psi^2$ can be obtained.

While the values of κ, δ and α are substituted in L.H.S. of equation (2.2), it is found to be equal to b^2 that implies that our assumption for κ is correct.

Now, to find the condition under which equation (2.6) is valid to justify the modified Diophantine system of equations portrayed through equation (2.1), (2.2) and (2.3). Putting the values of α, κ, δ and c in equation (2.3), we have

$$\psi^6 + 2(\theta p - \psi p)^3 + (\theta - p)^6 = (\theta^3 + \psi^3 - 3\theta p(\theta - p) - r)^2 \quad (2.9)$$

On solving equation (2.9), we have obtained that $r = \psi^3$.

It shows cases the clarity of our assumption. Hence, we can arrive at a sequential pattern of solution as $\langle \alpha, \beta, \gamma, \delta, a, b, c \rangle = \langle \psi^2, \theta p - \psi p, \theta p - \psi p, (\theta - p)^2, \theta + \psi - p, \theta^2 - 2\theta p + \psi^2 + p^2, \theta^3 + \psi^3 - 3\theta p(\theta - p) - p^3 \rangle$ provided $p = \psi$ i.e.

$$\langle \alpha, \beta, \gamma, \delta, a, b, c \rangle = \langle p^2, \theta p - p^2, \theta p - p^2, (\theta - p)^2, \theta, \theta^2 - 2\theta p + 2p^2, \theta^3 + p^3 - 3\theta p(\theta - p) - p^3 \rangle \quad (2.10)$$

It is the desired sequential pattern, and the required quadruple is

$$\langle \alpha, \beta, \gamma, \delta \rangle = \langle p^2, \theta p - p^2, \theta p - p^2, (\theta - p)^2 \rangle.$$

2.1. Particular Cases.

2.1.1. When $\theta = 2p$ is considered, a special pattern of sequence is acquired as follows. $\langle \alpha, \beta, \gamma, \delta, a, b, c \rangle = \langle p^2, p^2, p^2, p^2, 2p, 2p^2, 2p^3 \rangle$. This is valid for all positive non-trivial integral values of p .

Illustration: For instance, $(1, 1, 1, 1, 2, 2, 2)$, $(4, 4, 4, 4, 8, 16)$, $(9, 9, 9, 9, 18, 54)$ and $(16, 16, 16, 16, 32, 128)$ lie under this pattern. Thus, when $\alpha = \beta = \gamma = \delta$, an infinite symmetric pattern of non-trivial solutions is obtained.

2.1.2. When $p = 1$ and $\theta = n$, (n is a Natural number) is considered, in that case, another sequential pattern has been obtained $\langle \alpha, \beta, \gamma, \delta, a, b, c \rangle = \langle 1, n - 1, n - 1, (n - 1)^2, n, n^2 - 2n + 2, n^3 - 3n(n - 1) \rangle$ This is valid for all integral values of n greater than 1.

Illustration: Examples include $(1, 1, 1, 1, 2, 2, 2)$, $(1, 2, 2, 4, 3, 5, 9)$, $(1, 3, 3, 9, 4, 10, 28)$ and $(1, 4, 4, 16, 5, 17, 65)$. This gives infinitely many non-symmetric solutions with $\alpha \leq \beta = \gamma \leq \delta$.

2.1.3. When $p = n^3, \theta = n^4$, the sequential pattern is reduce to a new pattern such as $\langle \alpha, \beta, \gamma, \delta, a, b, c \rangle = \langle n^6, n^7 - n^6, n^7 - n^6, (n^4 - n^3)^2, n^4, n^8 - 2n^7 + 2n^6, n^{12} - 3n^{11} + 3n^{10} \rangle$. This is valid for all non-zero non-negative values of $n \geq 2$.

Illustration: Examples include (64, 64, 64, 64, 16, 128, 1024) and (729, 1458, 1458, 2916, 81, 3645, 177147). Hence this pattern yields infinitely many non-symmetric solutions of the prescribed system.

3. WHETHER SOLUTIONS EXIST FOR $\alpha \neq \beta \neq \gamma \neq \delta$

We have also found some special solutions that do not hold the sequential pattern of the solution of the equation represented by (2.10). However, through an analytical approach, these solutions cannot be acquired. A computational approach has been employed to obtain such a type of solution, where $\alpha < \beta < \gamma < \delta$. The whole set of solutions ranging from 1 to 500 is as follows.

α	β	γ	δ	a	b	c	α	β	γ	δ	a	b	c	α	β	γ	δ	a	b	c
9	10	60	90	13	109	973	30	57	168	474	27	507	10557	60	241	282	378	31	533	9521
10	44	57	58	13	93	683	33	205	441	477	34	682	14246	62	73	266	440	29	523	10229
10	74	98	142	18	188	2052	34	82	98	110	18	172	1692	67	209	233	275	28	422	6548
14	58	122	130	18	188	2052	36	40	240	360	26	436	7784	88	96	232	260	26	372	5624
17	91	299	377	28	490	9004	40	176	228	232	26	372	5464	98	212	305	346	31	517	8959
18	84	129	210	21	261	3465	42	66	108	145	19	197	2161	125	241	305	485	34	634	12586
22	24	58	65	13	93	703	46	170	202	258	26	372	5516	186	234	258	478	34	620	12068
22	73	130	136	19	203	2261	49	98	170	212	23	293	3937	222	354	370	498	38	748	15148
22	161	266	280	27	419	6705	52	190	233	254	27	397	6003	240	306	313	366	35	619	11053
29	109	301	461	30	562	11250	57	102	222	348	27	429	7371	253	295	341	407	36	658	12204

Thus, the solution exists, but there may not exist sequential pattern for such a type of solution.

The computational search was performed by running through all ordered quadruples $1 \leq \alpha < \beta < \gamma < \delta \leq 500$ and retaining only those for which $\alpha + \beta + \gamma + \delta$, $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$, and $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$ are perfect squares. The corresponding values of a , b , and c are the square roots of these three sums.

4. SOME SIGNIFICANT REMARKS BASED ON THE PRESCRIBED DIOPHANTINE SYSTEM

4.1. **Remark.** The following Diophantine system of equations, having three non-trivial

$$\begin{aligned}\alpha + n\beta + \delta &= a^2 \\ \alpha^2 + n\beta^2 + \delta^2 &= b^2 \\ \alpha^3 + n\beta^3 + \delta^3 &= c^2\end{aligned}$$

unknowns α, β, δ with positive integers a, b, c, n has a sequential pattern of solutions if n is a positive even number.

4.2. **Remark.** For all symmetric and non-symmetric solutions of the prescribed Diophantine system exhibited by equations (1.1), (1.2) and (1.3); the following specific relations have been acquired:

- (1) b greater than or equal to half of the square of a .
- (2) c greater than or equal to one-fourth of the cube of a .
- (3) Square of b is equal to the fourth power of a minus twice the sum of the product of all integers $\alpha, \beta, \gamma, \delta$ twice at a time.
- (4) Sixth power of a is equal to the square of c , plus thrice the product of the square of a and the square of b , minus three times the sum of the products of two terms of $\alpha, \beta, \gamma, \delta$ and sums of two same terms selected.

4.3. **Remark.** In the sequential pattern of solution $\langle \alpha, \beta, \gamma, \delta, z_1, z_2, z_3, z_4, \dots, z_n \rangle$ of the following Diophantine system

$$\begin{aligned}\alpha + \beta + \gamma + \delta &= z_1^2 \\ \alpha^2 + \beta^2 + \gamma^2 + \delta^2 &= z_2^2 \\ &\dots \\ \alpha^n + \beta^n + \gamma^n + \delta^n &= z_n^2\end{aligned}$$

there always exists a term of the form $2p^{2n}$ in z_{2n} , where p is a non-zero non-negative integer and n is a natural number.

4.4. **Remark.** The following system of m -Diophantine equations with $2k$ non-zero non-negative variables $\alpha, \beta_1, \beta_2, \dots, \beta_n, \gamma$ and non-zero non-negative integers $z_1, z_2, z_3, z_4, \dots, z_m$

$$\begin{aligned}\alpha + \beta_1 + \beta_2 + \dots + \beta_n + \gamma &= z_1^2 \\ \alpha^2 + \beta_1^2 + \beta_2^2 + \dots + \beta_n^2 + \gamma^2 &= z_2^2 \\ &\dots \\ \alpha^m + \beta_1^m + \beta_2^m + \dots + \beta_n^m + \gamma^m &= z_m^2\end{aligned}$$

have the generalized sequential pattern of a non-symmetric solution

$$\langle \alpha, \underbrace{\beta_1, \beta_2, \beta_3, \beta_4, \dots, \beta_n}_{(2k-2) \text{ times}}, \gamma \rangle = \langle p^2, \underbrace{\theta p - p^2, \theta p - p^2, \theta p - p^2, \theta p - p^2, \dots, \theta p - p^2}_{(2k-2) \text{ times}}, (\theta - p)^2 \rangle, \text{ where } k \text{ is}$$

greater than or equal to 1 and θ, p be any non-zero non-negative integers.

Open Questions It would be challenging and interesting to determine whether there exist any non-integral rational or irrational solutions or any sequential non-integral patterns. Further, it would be useful for comparison if the solutions to the prescribed problem portrayed by (1.1), (1.2), and (1.3) with four distinct unknowns can be acquired by employing an elliptic curve.

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Symbolic Dynamics and Markov Structure for Tent-Type Maps

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Abstract. We study a parameter-dependent family of tent-type maps and focus on the distinguished parameter value $\omega_0 = \frac{1+\sqrt{5}}{2}$. After normalizing the dynamics to the unit interval, we show that the backward orbit of the unique turning point generates a Markov partition with a Fibonacci-type combinatorial structure. This yields an explicit symbolic model: the normalized map is (up to a standard countable exceptional set) topologically conjugate to the golden mean shift. We also determine an invariant probability measure that is absolutely continuous with respect to Lebesgue measure by solving the transfer-operator fixed point equation for the corresponding density. Finally, using the Markov/symbolic description, we compute escape rates for Markov holes.

Keywords: chaotic map; tent map; Markov property; symbolic dynamics; escape rate

MSC (2020): 37B10, 37E05

1. INTRODUCTION

Piecewise monotone interval maps provide a classical source of chaotic dynamical systems while also offering a tractable framework in which geometric dynamics can be translated into symbolic and combinatorial models. Among them, tent maps and their variants occupy a central position: they exhibit rich topological behavior, admit explicit inverse branches, and often allow the construction of Markov partitions and symbolic codings. Such codings are particularly useful for studying invariant measures, periodic points, entropy, and quantitative features of open systems such as escape rates. In this paper we consider the tent family $T_\omega : \mathbb{R} \rightarrow \mathbb{R}$, $\omega > 0$, defined by

$$T_\omega(x) = \begin{cases} \omega x, & x \leq \frac{1}{2}, \\ \omega(1-x), & x > \frac{1}{2}, \end{cases}$$

and focus on the chaotic regime $1 < \omega < 2$. Hasfura-Buenaga and Lynch [1] proved that in this range the map is chaotic in the sense of Devaney [2]: it is topologically transitive, periodic points are dense, and the system has sensitive dependence on initial conditions. For our purposes it is convenient to restrict to a forward invariant interval and then normalize the dynamics to the unit interval. Indeed, for $1 < \omega < 2$ the map leaves invariant

$$I_\omega = \left[\omega - \frac{\omega^2}{2}, \frac{\omega}{2} \right],$$

and via the affine rescaling

$$h_\omega : [0, 1] \rightarrow I_\omega, \quad h_\omega(x) = \left(\omega - \frac{\omega^2}{2} \right) (1-x) + \frac{\omega}{2} x,$$

we obtain the conjugate map

$$\widehat{T}_\omega = h_\omega^{-1} \circ T_\omega \circ h_\omega : [0, 1] \rightarrow [0, 1],$$

which is piecewise linear and expanding with a single turning point. This normalization fixes the phase space to $[0, 1]$ and makes Markov and symbolic constructions transparent.

A fundamental theme in one-dimensional dynamics is that a Markov partition provides a bridge from interval dynamics to symbolic dynamics. If a map admits a Markov partition, then its dynamics can be modeled by a shift space whose admissibility rules are encoded by a transition matrix. This viewpoint allows one to import tools from symbolic dynamics and Perron–Frobenius theory, including

Parry measures and spectral methods (see, e.g., [3, 4]). In particular, for expansive Markov maps the coding map intertwines the dynamics and, outside a small exceptional set, yields a conjugacy with a subshift of finite type.

The central object of the present paper is the parameter value

$$\omega_0 = \frac{1 + \sqrt{5}}{2},$$

for which the partition generated by the backward orbit of the turning point forms a Markov partition with a Fibonacci-type combinatorial structure. More precisely, the primary partition $\mathcal{P}_1 = \{[0, c], [c, 1]\}$, where $c = (3 - \sqrt{5})/2$, generates a refining sequence of partitions $\{\mathcal{P}_n\}_{n \geq 1}$ whose endpoints belong to the backward orbit $\{0, c, \bar{T}_{\omega_0}^{-1}c, \dots, \bar{T}_{\omega_0}^{-n}c, 1\}$. Each $\mathcal{P}_n$ consists of intervals of exactly two lengths (“small” and “big”), and the numbers of small and big intervals follow the Fibonacci recursion. This structure leads naturally to the golden mean shift, i.e. the subshift of finite type on $\{0, 1\}$ with transition matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

equivalently the shift obtained by forbidding the word 00. We construct an explicit coding and show that $\bar{T}_{\omega_0}$ is topologically conjugate (up to the standard countable exceptional set) to this shift.

Since $\bar{T}_{\omega}$ does not preserve Lebesgue measure in general, a further goal is to determine a natural invariant probability measure. For the piecewise linear expanding map $\bar{T}_{\omega_0}$ we compute an absolutely continuous invariant measure using the transfer (Perron–Frobenius) operator. On the symbolic side, irreducible subshifts of finite type carry a unique measure of maximal entropy, the Parry measure [4], and we relate the interval invariant measure to the symbolic Parry measure via the conjugacy.

Finally, we apply the symbolic Markov structure to open dynamical systems. Given a hole $H \subset [0, 1]$ (typically a union of elements of $\mathcal{P}_n$), one studies the survivor sets of points whose orbits avoid H up to time n and defines the exponential escape rate from the decay of their measures. For Markov holes, escape rates can be expressed in terms of Perron roots (or, equivalently, topological entropies) of subshifts obtained by adding the forbidden words corresponding to the hole (see, e.g., [5, 3]). We also discuss local escape rates for shrinking holes, which distinguish between periodic and non-periodic points and admit explicit formulas in the symbolic setting.

Tent and Generalized Luroth Series (GLS) maps possess a unique combination of ergodicity, strong mixing, positive Lyapunov exponent, and robust chaos, which makes them particularly effective in engineering applications [6]. The identity $\lambda_{\text{GLS}} = H_{\text{GLS}}$ provides a rigorous bridge between dynamical systems and information theory, forming the theoretical basis for Shannon-optimal lossless compression and arithmetic coding schemes. Their symbolic dynamics and finite-precision implementability enable secure cryptographic protocols, pseudo-random number generation, and joint compression-encryption mechanisms. Furthermore, GLS-based neurons exploiting topological transitivity and universal orbits form the mathematical foundation of Neurochaos Learning architectures for efficient classification tasks [6].

Outline of the paper. Section 2 introduces the family of tent-type maps and establishes the existence and uniqueness of an absolutely continuous invariant measure for $\hat{T}_{\omega_0}$. Section 3 proves that the backward orbit of the turning point generates a sequence of Markov partitions $\mathcal{P}_n$, and establishes their explicit Fibonacci-type structure (Theorems 3.9 and 3.10). Section 4 develops the symbolic dynamics associated with these partitions and proves (semi-)conjugacy with the golden mean shift, together with the description of the corresponding Parry measure. Section 5 studies periodic points and derives escape rates through Markov holes, including local escape rates for shrinking holes.

2. THE FAMILY OF TENT-TYPE MAPS

Let $T_{\omega} : \mathbb{R} \rightarrow \mathbb{R}$ be the family of tent maps defined by (see Figure 1)

$$T_{\omega}(x) = \begin{cases} \omega x, & x \leq \frac{1}{2}, \\ \omega(1 - x), & x > \frac{1}{2}, \end{cases} \quad \omega > 0.$$

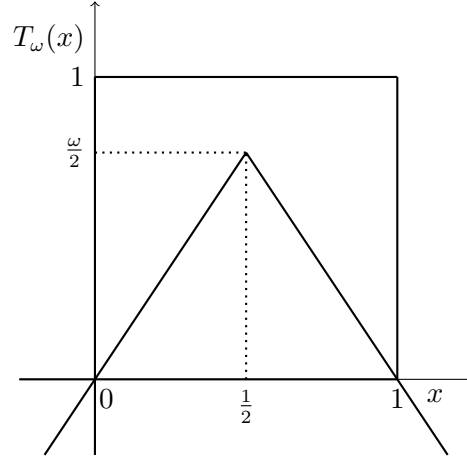


Figure 1

This family was studied by Hasfura–Buenaga and Lynch [1] in the parameter range $1 < \omega < 2$. They proved that for $1 < \omega < 2$ the map T_ω is chaotic in the sense of Devaney (see [2],[7]), namely:

- (1) T_ω is topologically transitive, i.e. for any nonempty open sets $U, V \subset \mathbb{R}$ there exists $n \geq 1$ such that $T_\omega^n(U) \cap V \neq \emptyset$;
- (2) the set of periodic points of T_ω is dense;
- (3) T_ω has sensitive dependence on initial conditions.

For $1 < \omega < 2$ the map T_ω leaves invariant the interval

$$I_\omega = \left[\omega - \frac{\omega^2}{2}, \frac{\omega}{2} \right] \subset [0, 1], \text{ i.e. } T_\omega(I_\omega) \subseteq I_\omega.$$

Hence, throughout this paper we study the restriction

$$T_\omega|_{I_\omega} : I_\omega \rightarrow I_\omega.$$

To normalize the dynamics to the unit interval, we introduce the affine homeomorphism

$$h_\omega : [0, 1] \rightarrow I_\omega, \quad h_\omega(x) = \left(\omega - \frac{\omega^2}{2} \right) (1 - x) + \frac{\omega}{2} x,$$

and define the conjugate map $\hat{T}_\omega : [0, 1] \rightarrow [0, 1]$ as

$$\hat{T}_\omega := h_\omega^{-1} \circ T_\omega \circ h_\omega.$$

A direct computation shows that $\hat{T}_\omega$ is a piecewise linear expanding map (see Figure 2)

$$\hat{T}_\omega(x) = \begin{cases} \omega x + 2 - \omega, & 0 \leq x \leq \frac{\omega - 1}{\omega}, \\ -\omega x + \omega, & \frac{\omega - 1}{\omega} < x \leq 1. \end{cases}$$

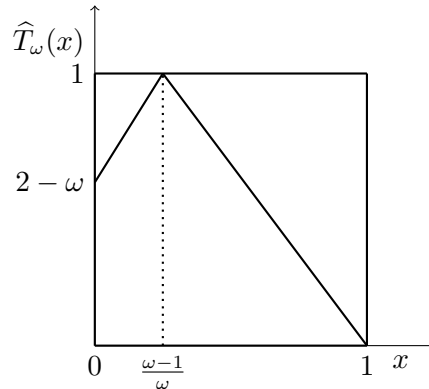


Figure 2

We determine the values of the parameter ω for which a Markov partition can be constructed on the interval $[0, 1]$ for the map $\hat{T}_\omega$. Let $f : [a, b] \rightarrow [a, b]$ be a map.

Definition 2.1. (see, for instance, [8]). Let $\mathcal{P} = \{I_0, I_1, \dots, I_n\}$ be a partition of the interval $[a, b]$ with endpoints $a = x_0 < x_1 < \dots < x_n = b$. The partition $\mathcal{P}$ is called a **Markov partition** if $f(x_i) = x_j$, for every i , that is, if the set of boundary points of the partition elements is invariant under the action of f .

We choose the initial partition

$$\mathcal{P}_0 = \left\{ I_0 = \left[0, \frac{\omega-1}{\omega} \right], I_1 = \left[\frac{\omega-1}{\omega}, 1 \right] \right\}.$$

The boundary points of $\mathcal{P}_0$ are given by

$$x_0 = 0, \quad x_1 = \frac{\omega-1}{\omega}, \quad x_2 = 1.$$

A direct computation shows that

$$\hat{T}_\omega(0) = 2 - \omega, \quad \hat{T}_\omega\left(\frac{\omega-1}{\omega}\right) = 1, \quad \hat{T}_\omega(1) = 0.$$

For the partition $\mathcal{P}_0$ to be a Markov partition for the map $\hat{T}_\omega$, the invariance condition for the boundary points must be satisfied. In particular, we require $\hat{T}_\omega(0) = \omega^{-1}(\omega-1)$. Solving the equation $2 - \omega = \frac{\omega-1}{\omega}$, we obtain the unique admissible value $\omega_0 = \frac{1+\sqrt{5}}{2}$.

In the present work we focus on the distinguished parameter value $\omega_0 := \frac{1+\sqrt{5}}{2}$, for which the partition generated by the backward orbit of the turning point forms a Markov partition. In this case the normalized map takes the form (see Figure 3)

$$\hat{T}_{\omega_0}(x) = \begin{cases} \frac{1+\sqrt{5}}{2}x + \frac{3-\sqrt{5}}{2}, & 0 \leq x \leq \frac{3-\sqrt{5}}{2}, \\ \frac{1+\sqrt{5}}{2}(1-x), & \frac{3-\sqrt{5}}{2} < x \leq 1. \end{cases}$$

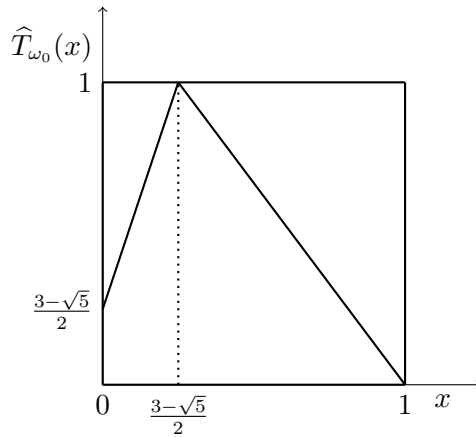


Figure 3

The map $\hat{T}_\omega$ does not preserve the Lebesgue measure λ . Indeed, for a general measurable set $A \subset [0, 1]$ one has

$$\lambda(A) \neq \lambda(\hat{T}_\omega^{-1}(A)).$$

Therefore, it is natural to seek an invariant probability measure that is absolutely continuous with respect to λ .

Lemma 2.1. *For the map $\hat{T}_{\omega_0}$ there exists a unique invariant probability measure μ that is absolutely continuous with respect to the Lebesgue measure λ .*

Proof: Since $\widehat{T}_{\omega_0}$ is a piecewise linear expanding map with constant slope φ on each monotonicity interval, the existence of an absolutely continuous invariant measure follows from the classical theory of Perron–Frobenius (transfer) operators for expanding interval maps (see [9],[10]).

Let L denote the Perron–Frobenius operator associated with $\widehat{T}_{\omega_0}$, acting on integrable functions h by

$$(Lh)(x) = \sum_{y \in \widehat{T}_{\omega_0}^{-1}(x)} \frac{h(y)}{|\widehat{T}'_{\omega_0}(y)|}.$$

Denoting $c := (3 - \sqrt{5})/2$ and $\omega_0 := (1 + \sqrt{5})/2$, the operator L satisfies

$$(Lh)(x) = \begin{cases} \omega_0^{-1} h\left(\frac{\omega_0 - x}{\omega_0}\right), & x \in [0, c), \\ \omega_0^{-1} \left[h\left(\frac{x - c}{\omega_0}\right) + h\left(\frac{\omega_0 - x}{\omega_0}\right) \right], & x \in [c, 1]. \end{cases}$$

Solving the fixed point equation $Lh = h$ under the normalization $\int_0^1 h(x) dx = 1$, we obtain the invariant density

$$h(x) = \begin{cases} \frac{5 + \sqrt{5}}{10}, & 0 \leq x \leq c, \\ \frac{3\sqrt{5} + 5}{10}, & c < x \leq 1. \end{cases}$$

Defining $\mu(A) = \int_A h(x) dx$ for measurable sets $A \subset [0, 1]$, it follows directly from $Lh = h$ that μ is invariant under $\widehat{T}_{\omega_0}$, i.e.

$$\mu(A) = \mu(\widehat{T}_{\omega_0}^{-1}(A)).$$

This completes the proof of Lemma 2.1. □

3. CONSTRUCTION OF A MARKOV PARTITION FOR $\widehat{T}_{\omega_0}$

In this section we construct a Markov partition for the normalized tent-type map $\widehat{T}_{\omega_0} : [0, 1] \rightarrow [0, 1]$, where

$$\omega_0 = \frac{1 + \sqrt{5}}{2}, \quad c := \frac{3 - \sqrt{5}}{2}.$$

The partition is generated by the backward orbit of the corner point c and yields a symbolic model conjugate to the golden mean shift.

3.1. Basic definitions. The **golden mean** is the irrational number $\omega_0 := \frac{1 + \sqrt{5}}{2}$, which admits the purely periodic simple continued fraction expansion

$$\omega_0 = 1 + \frac{1}{1 + \frac{1}{1 + \dots}} = [1; 1, 1, 1, \dots].$$

Denoting by

$$\frac{p_n}{q_n} := [1; \underbrace{1, 1, \dots, 1}_{n \text{ times}}]$$

the n -th convergent of this expansion, the numerators and denominators satisfy the recursions

$$p_{-1} = 1, \quad p_0 = 1, \quad p_n = p_{n-1} + p_{n-2},$$

$$q_{-1} = 0, \quad q_0 = 1, \quad q_n = q_{n-1} + q_{n-2}, \quad n \geq 1.$$

It is easy to see that $q_n = q_{n-1}$, $n \in \mathbb{N}$. Using the recursive relation for q_n we get the sequence the Fibonacci numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

Remark 3.1. An important arithmetic property of the Fibonacci sequence concerns the parity of its terms. In particular, the parity follows a periodic pattern of period three: for every integer $n \geq 0$, the Fibonacci number q_{3n} is even, while the two successive terms q_{3n+1} and q_{3n+2} are odd. As a result, the sequence alternates regularly according to the repeating pattern *odd, odd, even*.

Next we give necessary definitions from theory of Markov partitions, Expanding Markov maps their natural symbolic dynamics.

Definition 3.1. (see [11]) Let $I = [a, b] \subset \mathbb{R}$ be a closed interval. A finite collection of closed intervals $\{I_0, \dots, I_{N-1}\}$ is called a **partition** of I if

- (1) $I = \bigcup_{i=0}^{N-1} I_i$;
- (2) $\text{int}(I_i) \cap \text{int}(I_j) = \emptyset$ for $i \neq j$.

Definition 3.2. (see [11]) A map $f : I \rightarrow I$ is called a **Markov map** if there exists a partition $\{I_0, \dots, I_{N-1}\}$ of I such that for every pair (i, j) , $0 \leq i, j \leq N-1$,

- (1) either $f(\text{int } I_i) \cap \text{int } I_j = \emptyset$,
- (2) or $\text{int } I_j \subset f(\text{int } I_i)$.

Such a partition is called a **Markov partition** for f .

We note that the last definitions of Markov partitions for the class piecewise-expanding maps are equivalent.

Definition 3.3. (see [11]) A Markov map $f : I \rightarrow I$ is called **expanding** if f is C^1 on each $\text{int}(I_i)$ and there exists a constant $\lambda > 1$ such that

$$|f'(x)| \geq \lambda \quad \text{for all } x \in \text{int}(I_i), \quad i = 0, \dots, N-1.$$

Definition 3.4. (see [11]) Let $f : I \rightarrow I$ be a Markov map with Markov partition $\{I_0, \dots, I_{N-1}\}$. The **topological transition matrix** $A = (A_{ij})_{0 \leq i, j \leq N-1}$ is defined by

$$A_{ij} = \begin{cases} 1, & \text{int}(I_j) \subset f(\text{int}(I_i)), \\ 0, & f(\text{int}(I_i)) \cap \text{int}(I_j) = \emptyset. \end{cases}$$

Equivalently, $A_{ij} = 1$ if and only if the transition from I_i to I_j is allowed.

Definition 3.5. (see [11]) A word $x = (x_0 x_1 x_2 \dots) \in \{0, 1, \dots, N-1\}^{\mathbb{N}}$ is called **admissible** if $A_{x_n x_{n+1}} = 1$ for all $n \geq 0$. Let Σ_A denote the set of all admissible sequences and let $\sigma : \Sigma_A \rightarrow \Sigma_A$ be the left shift map

$$\sigma(x_0 x_1 x_2 \dots) = (x_1 x_2 x_3 \dots).$$

The following theorem shows a connection between symbolic representation of points from interval I and the basic intervals of the Markov partition.

Theorem 3.1. (see [11]) Let $f : I \rightarrow I$ be an expanding Markov map with Markov partition $\{I_0, \dots, I_{N-1}\}$ and transition matrix A . Then for every admissible sequence $x = (x_0 x_1 \dots) \in \Sigma_A$ there exists a unique point $\pi(x) \in I$ such that

$$f^n(\pi(x)) \in I_{x_n} \quad \text{for all } n \geq 0.$$

Moreover, the coding map $\pi : \Sigma_A \rightarrow I$ is onto and satisfies

$$\pi \circ \sigma = f \circ \pi.$$

3.2. The natural partition and its refinements. We start from the natural (two-interval) partition (see Figure 4)

$$\mathcal{P}_1 = \{I^s = [0, c], \quad I^b = [c, 1]\}, \quad c = \frac{3 - \sqrt{5}}{2}.$$

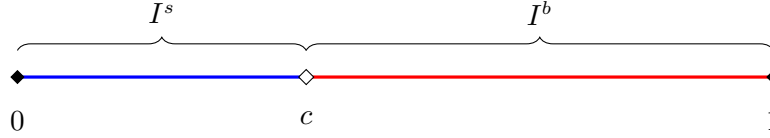


Figure 4

We refer to I^s as the small interval and I^b as the big interval. For $n \geq 1$, define the refinement

$$\mathcal{P}_{n+1} := \mathcal{P}_n \vee \widehat{T}_{\omega_0}^{-1}(\mathcal{P}_n),$$

i.e. $\mathcal{P}_{n+1}$ is obtained by subdividing each atom of $\mathcal{P}_n$ using the preimages of its endpoints under $\widehat{T}_{\omega_0}$. Equivalently, the set of endpoints of $\mathcal{P}_n$ is contained in

$$\{0, c, \widehat{T}_{\omega_0}^{-1}(c), \widehat{T}_{\omega_0}^{-2}(c), \dots, \widehat{T}_{\omega_0}^{-n}(c), 1\}.$$

Remark 3.2. Every interval of $\mathcal{P}_n$ has endpoints taken from the finite backward orbit of c up to depth n (together with 0 and 1).

3.3. Markov partitions and Fibonacci structure.

The following result describes the exact structure of the partitions $\mathcal{P}_n$ generated by the backward orbit of the critical point. It establishes their Markov property and the explicit refinement rule, which produces two interval types with Fibonacci-type growth. This structure forms the basis for the symbolic and matrix description of the system.

Theorem 3.2. 1. For every $n \geq 1$ the system of points $\{0, c, \widehat{T}_{\omega_0}^{-1}(c), \dots, \widehat{T}_{\omega_0}^{-n}(c), 1\}$ determine a Markov partition $\mathcal{P}_n$.

2. Each Markov partition $\mathcal{P}_n$ contain two type of intervals: big intervals $\{I_i^b, 1 \leq i \leq q_{n-1}\}$ with $|I_i^b| = \frac{\sqrt{5}-1}{2}\rho^{-(n-1)}$ and small intervals $\{I_i^s, 1 \leq i \leq q_{n-2}\}$ with $|I_i^s| = \frac{3-\sqrt{5}}{2}\rho^{-(n-1)}$

3. Under the refinement from the partition $\mathcal{P}_n$ to $\mathcal{P}_{n+1}$, each big interval I_i^b is subdivided into two subintervals, one of which becomes a big interval of the partition $\mathcal{P}_{n+1}$, while the other becomes a small interval. Moreover, every small interval of the partition $\mathcal{P}_n$ becomes a big interval in the partition $\mathcal{P}_{n+1}$.

Proof: (1) The partition $\mathcal{P}_n$ of the interval $[0, 1]$ is generated by the points

$$c, \widehat{T}^{-1}c, \dots, \widehat{T}^{-n}c, \quad c = \frac{3-\sqrt{5}}{2}.$$

Then, the partition $\mathcal{P}_n$ can be written as

$$\mathcal{P}_n = \{I_i^n = [\widehat{T}^{-k}x, \widehat{T}^{-l}x] : 0 \leq k, l \leq n\}.$$

These points subdivide $[0, 1]$ into finitely many intervals with pairwise disjoint interiors, hence $\mathcal{P}_n$ is a partition in the sense of Definition 3.1. Moreover, for any $I_i^n \in \mathcal{P}_n$,

$$\widehat{T}(I_i^n) = [\widehat{T}^{-(k-1)}x, \widehat{T}^{-(l-1)}x],$$

and therefore the Markov property holds: for any $I_i^n, I_j^n \in \mathcal{P}_n$, either $f(\text{int } I_i^n) \cap \text{int } I_j^n \neq \emptyset$ or $\text{int } I_j^n \subseteq f(\text{int } I_i^n)$. Hence, by Definition 3.2, $\mathcal{P}_n$ is a Markov partition.

(2) We prove items (2) and (3) of the theorem by mathematical induction. For $n = 1$, the statements are readily verified. Assume that they hold for $n = k$.

The partition $\mathcal{P}^{n+1}$ is obtained from $\mathcal{P}^n$ via the refinement

$$\mathcal{P}^{n+1} = \mathcal{P}^n \vee \widehat{T}^{-1}\mathcal{P}^n,$$

that is, by taking the preimages $\widehat{T}^{-1}I_i^n$ of all intervals $I_i^n \in \mathcal{P}^n$. Since each inverse branch of $\widehat{T}$ contracts lengths by the factor $\rho = \frac{1+\sqrt{5}}{2}$, every interval is scaled by ϕ^{-1} and new intervals I_i^{n+1} are created.

More precisely, the lengths of the intervals in $\mathcal{P}^{n+1}$ take only two possible values:

$$\lambda(I^{n+1}) = \frac{3 - \sqrt{5}}{2} \cdot \rho^{-n} \quad \text{or} \quad \lambda(I^{n+1}) = \frac{\sqrt{5} - 1}{2} \cdot \rho^{-n}.$$

Hence, the partition $\mathcal{P}^{n+1}$ consists of exactly two types of intervals. Moreover, the length of a big interval at level $n + 1$ coincides with the length of a small interval at level n , that is,

$$\lambda(I_b^{n+1}) = \lambda(I_s^n).$$

$$\lambda(I_b^{n+1}) + \lambda(I_s^{n+1}) = \frac{3 - \sqrt{5}}{2} \cdot \rho^{-n} + \frac{\sqrt{5} - 1}{2} \cdot \rho^{-n} = \frac{\sqrt{5} - 1}{2} \cdot \rho^{-(n-1)} = \lambda(I_b^n).$$

Consequently, when passing from $\mathcal{P}^n$ to $\mathcal{P}^{n+1}$, each small interval I_s^n remains unchanged, while each big interval I_b^n is subdivided into one big and one small interval. As a result, the number of big intervals satisfies the recursion

$$q_n = q_{n-1} + q_{n-2},$$

while the number of small intervals equals q_{n-1} . This completes the inductive step. $\square$

Recall that the Fibonacci sequence of numbers q_n alternates regularly according to the repeating pattern *odd, odd, even*. Also, the intervals of each Markov partition $\{\mathcal{P}_n, n \geq 1\}$ enumerated in positive direction from 0 to 1.

Next theorem describes the Fibonacci structure and location of intervals Markov partitions $\{\mathcal{P}_n, n \geq 1\}$.

Theorem 3.3. *Let $\{\mathcal{P}_n, n \geq 1\}$ the sequence of Markov partitions associated by map $\hat{T}_{\omega_0}$. Then the small and big intervals are arranged according to the following three distinct cases:*

- (1) *If, for the n -th partition, the total number q_n of intervals is even, then the first interval is small, while the last interval is big.*
- (2) *For the $(n + 1)$ -st partition, the first two intervals are big, whereas the last interval is small.*
- (3) *For the $(n + 2)$ -nd partition, both the first two and the last two intervals are big.*

In all three cases, every remaining small interval appears exclusively in consecutive pairs, while the big intervals occur only in blocks consisting of either two or four consecutive intervals.

Proof: For the partitions $\mathcal{P}_n$ with $n = 3, 4, 5$, the validity of the stated properties is illustrated by the diagrams above.

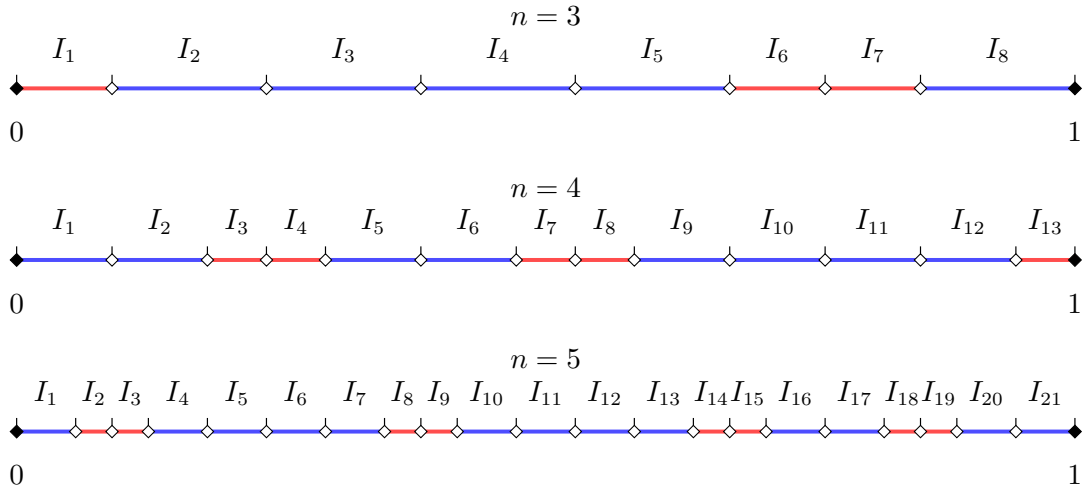


Figure 5

We now proceed by induction. Assume that the statements hold for $\mathcal{P}_n$ with $n = k$, where q_k is even. We prove that they also hold for $n = k + 1$.

Consider the partition $\mathcal{P}_{k+1}$. The subcollection of intervals contained in the subinterval $[\frac{3-\sqrt{5}}{2}, 1]$ is obtained from the intervals of $\mathcal{P}_k$ by reversing their order. Consequently, the last interval is small, while the first interval starting at $\frac{3-\sqrt{5}}{2}$ is big. On the other hand, the subcollection of intervals contained in $[0, \frac{3-\sqrt{5}}{2}]$ inherits the order of the intervals of $\mathcal{P}_k$ lying in $[\frac{3-\sqrt{5}}{2}, 1]$. In particular, the first two intervals and the interval immediately to the left of $\frac{3-\sqrt{5}}{2}$ are big. Since the relative order of the remaining intervals is preserved and big intervals appear on both sides of the point $\frac{3-\sqrt{5}}{2}$, the configuration of small and big intervals is maintained. This corresponds precisely to case (b) of the theorem.

Next, assume that the statements hold for $\mathcal{P}_n$ with $n = k + 1$, where q_k is even, and consider $\mathcal{P}_{k+2}$. The intervals of $\mathcal{P}_{k+2}$ contained in $[\frac{3-\sqrt{5}}{2}, 1]$ are obtained by reversing the order of the intervals of $\mathcal{P}_{k+1}$. Hence, the last interval is big, while the first interval starting at $\frac{3-\sqrt{5}}{2}$ is small. For the subinterval $[0, \frac{3-\sqrt{5}}{2}]$, the intervals inherit the order of those intervals of $\mathcal{P}_{k+1}$ lying in $[\frac{3-\sqrt{5}}{2}, 1]$. Thus, the first interval and the interval immediately to the left of $\frac{3-\sqrt{5}}{2}$ are small. Since the relative positions of the remaining intervals are unchanged and small intervals occur on both sides of $\frac{3-\sqrt{5}}{2}$, the pattern of small and big intervals is preserved. This corresponds to case (c) of the theorem.

Finally, assume that the statements hold for $\mathcal{P}_n$ with $n = k + 2$, where q_k is even, and consider $\mathcal{P}_{k+3}$. In this case, the intervals of $\mathcal{P}_{k+3}$ contained in $[\frac{3-\sqrt{5}}{2}, 1]$ are obtained by reversing the order of the intervals of $\mathcal{P}_{k+2}$. As a result, the last interval and the first two intervals starting at $\frac{3-\sqrt{5}}{2}$ are big. For the subinterval $[0, \frac{3-\sqrt{5}}{2}]$, the order of the corresponding intervals of $\mathcal{P}_{k+2}$ is preserved, and the first interval together with the two intervals immediately to the left of $\frac{3-\sqrt{5}}{2}$ are big. Since the mutual arrangement of the remaining intervals is unchanged and pairs of big intervals appear on both sides of $\frac{3-\sqrt{5}}{2}$, the configuration of small and big intervals is preserved. This is exactly case (a) of the theorem.

In conclusion, the arrangement of small and big intervals in the partition $\mathcal{P}_n$ repeats cyclically according to the three cases stated in item (4) of Theorem 3.3. $\square$

3.4. Transition matrix for partition $\mathcal{P}_1$. Label the intervals of $\mathcal{P}_1$ by

$$I^s \leftrightarrow 0, \quad I^b \leftrightarrow 1.$$

A direct inspection of $\widehat{T}_{\omega_0}$ shows that $\widehat{T}_{\omega_0}(I^s)$ intersects only I^b (so the transition $0 \rightarrow 0$ is forbidden), whereas $\widehat{T}_{\omega_0}(I^b)$ intersects both I^s and I^b . Hence the transition matrix is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

4. SYMBOLIC CODING AND CONJUGACY

Let (X, T) be a dynamical system, where X is a measurable space and $T : X \rightarrow X$ is a measurable map. Assume that X is partitioned into finitely many pairwise disjoint measurable sets

$$X = P_0 \cup \dots \cup P_{m-1}, \quad P_i \cap P_j = \emptyset \text{ for } i \neq j.$$

For $x \in X$ we define its *itinerary* (symbolic name) by

$$\omega(x) = (\omega_0, \omega_1, \dots) \in \{0, 1, \dots, m-1\}^{\mathbb{N}}, \quad \omega_n = i \iff \widehat{T}^n(x) \in P_i, \quad n \geq 0.$$

This gives the coding map

$$\pi : X \rightarrow \{0, 1, \dots, m-1\}^{\mathbb{N}}, \quad \pi(x) = \omega(x).$$

Let σ denote the left shift on $\{0, 1, \dots, m-1\}^{\mathbb{N}}$,

$$\sigma(\omega_0, \omega_1, \omega_2, \dots) = (\omega_1, \omega_2, \omega_3, \dots).$$

By construction,

$$\pi \circ \widehat{T} = \sigma \circ \pi.$$

The image $\Sigma := \pi(X)$ is the *symbolic dynamical system* associated to (X, T) .

Let $\mathcal{A} = \{0, 1, \dots, m-1\}$. The *full shift* over $\mathcal{A}$ is $\mathcal{A}^{\mathbb{N}}$ with the shift map σ . A *subshift* is a closed, shift-invariant subset $\Sigma \subseteq \mathcal{A}^{\mathbb{N}}$, i.e. $\sigma(\Sigma) = \Sigma$. Equivalently, there exists a (possibly infinite) family $\mathcal{F}$ of finite words (forbidden blocks) such that Σ consists of all sequences in $\mathcal{A}^{\mathbb{N}}$ which do not contain any word from $\mathcal{F}$ as a contiguous block.

Lemma 4.1 (Conjugacy to the golden mean shift). *Let $\omega_0 = \frac{1+\sqrt{5}}{2}$ and let $\widehat{T} = \widehat{T}_{\omega_0} : [0, 1] \rightarrow [0, 1]$. Consider the Markov partition $\mathcal{P}_1 = \{I^s = [0, c], I^b = [c, 1]\}$, where $c = \frac{3-\sqrt{5}}{2}$, and label $I^s \leftrightarrow 0$, $I^b \leftrightarrow 1$. Then the associated symbolic space is the one-sided golden mean shift*

$$\Sigma_A = \{x \in \{0, 1\}^{\mathbb{N}} : \text{the block } 00 \text{ does not occur}\}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and the systems $([0, 1], \widehat{T})$ and (Σ_A, σ) are topologically conjugate.

Proof: Since $\widehat{T}$ is an expanding Markov map with Markov partition $\mathcal{P}_1$ and transition matrix A , Theorem 3.1 (coding for expanding Markov maps) provides a continuous surjection $\pi : \Sigma_A \rightarrow [0, 1]$ such that $\pi \circ \sigma = \widehat{T} \circ \pi$. Moreover, expansivity implies that for each admissible sequence $x = (x_0 x_1 \dots) \in \Sigma_A$ there is a unique point $\pi(x) \in [0, 1]$ whose itinerary equals x . Hence π is bijective and therefore a homeomorphism, proving topological conjugacy.

In particular, the transition $0 \rightarrow 0$ is forbidden (equivalently, the block 00 cannot occur), while all other transitions are allowed, which yields the golden mean shift. $\square$

4.1. Parry measure and the invariant measure. Let A be the transition matrix of an irreducible one-step shift of finite type Σ_A and let $\theta > 0$ be its Perron eigenvalue. Let v and u be positive Perron eigenvectors:

$$Av = \theta v, \quad u^T A = \theta u^T, \quad u^T v = 1.$$

Define the stochastic matrix $P = (P_{ij})$ by

$$P_{ij} = \frac{A_{ij}v_j}{\theta v_i}, \quad 0 \leq i, j \leq N-1,$$

and the stationary distribution $\pi = (\pi_i)$ by $\pi_i = u_i v_i$. Then the *Parry measure* μ on Σ_A is the Markov measure given by

$$\mu([x_0 x_1 \dots x_{n-1}]) = \pi_{x_0} P_{x_0 x_1} \dots P_{x_{n-2} x_{n-1}}, \quad n \geq 1,$$

where $[x_0 \dots x_{n-1}]$ denotes the cylinder set.

In our case,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \theta = \frac{1+\sqrt{5}}{2}.$$

A convenient choice of normalized Perron eigenvectors yields

$$\pi_0 = \mu([0]) = \frac{5-\sqrt{5}}{10}, \quad \pi_1 = \mu([1]) = \frac{5+\sqrt{5}}{10},$$

and the corresponding transition matrix is

$$P = \begin{pmatrix} 0 & 1 \\ \frac{3-\sqrt{5}}{2} & \frac{\sqrt{5}-1}{2} \end{pmatrix}.$$

Finally, the $\widehat{T}$ -invariant probability measure on $[0, 1]$ is obtained as the push-forward of μ under the conjugacy map π^{-1} from Lemma 4.1.

4.2. Periodic points. In this section we study periodic points of $\widehat{T} = \widehat{T}_{\omega_0}$ on $[0, 1]$. Since $\widehat{T}$ is conjugate to the one-sided subshift of finite type (Σ_A, σ) (Lemma 4.1), periodic orbits of $\widehat{T}$ correspond to periodic sequences in Σ_A . More precisely, if $\pi : \Sigma_A \rightarrow [0, 1]$ denotes the conjugacy, then

$$\widehat{T}^m(\pi(s)) = \pi(\sigma^m(s)) \quad \text{for all } s \in \Sigma_A, m \geq 1.$$

Hence, if s is σ -periodic with (prime) period m , then $\pi(s)$ is $\widehat{T}$ -periodic with (prime) period m .

4.3. An explicit example: period 3.

Example 4.1 (Period-3 orbits). For the golden mean shift $\Sigma_A \subset \{0, 1\}^{\mathbb{N}}$ (no block 00), the σ -periodic sequences of period 3 are obtained from admissible words of length 3 whose infinite repetition is admissible. Up to cyclic rotation, the admissible period-3 sequences are

$$s_1 = (011)^\infty, \quad s_2 = (110)^\infty, \quad s_3 = (101)^\infty,$$

and, in addition, the fixed point

$$s_4 = 1^\infty$$

(which is periodic for every period, but has prime period 1).

Let $\rho = \frac{1+\sqrt{5}}{2}$ and $c = \frac{3-\sqrt{5}}{2}$. Recall that the conjugacy π is determined by the inverse branches:

$$\pi(s) = \begin{cases} \frac{\pi(\sigma(s)) - c}{\rho}, & s_0 = 0, \\ \frac{\rho - \pi(\sigma(s))}{\rho}, & s_0 = 1. \end{cases} \quad (*)$$

(Here we use the explicit linear formula for $\widehat{T}$ on each Markov interval.)

To compute the points corresponding to $s_1 = (011)^\infty$, $s_2 = (110)^\infty$, $s_3 = (101)^\infty$, set

$$x = \pi((011)^\infty), \quad y = \pi((110)^\infty), \quad z = \pi((101)^\infty).$$

Using $(*)$ and the relations $\sigma((011)^\infty) = (110)^\infty$, $\sigma((110)^\infty) = (101)^\infty$, $\sigma((101)^\infty) = (011)^\infty$, we obtain the system

$$\begin{cases} x = \frac{y - c}{\rho}, \\ y = \frac{\rho - z}{\rho}, \\ z = \frac{\rho - x}{\rho}. \end{cases}$$

Solving yields

$$x = 0, \quad y = c = \frac{3 - \sqrt{5}}{2}, \quad z = 1.$$

Finally, for the fixed point $s_4 = 1^\infty$ we have

$$t := \pi(1^\infty) = \frac{\rho - t}{\rho},$$

hence $t = \frac{\sqrt{5}-1}{2}$. Therefore, $\widehat{T}$ has periodic points

$$0, \quad \frac{3 - \sqrt{5}}{2}, \quad 1, \quad \frac{\sqrt{5} - 1}{2},$$

where the first three belong to a period-3 orbit, while $\frac{\sqrt{5}-1}{2}$ is a fixed point.

4.4. Uniqueness of short-period points in refined cylinders. Let $\mathcal{P}_n$ be the n -th Markov refinement and let $\{I_n^1, \dots, I_n^{q_n}\}$ denote the collection of its interval elements. Each I_n^i corresponds to a cylinder of length n in Σ_A .

Lemma 4.2. *For each $n \in \mathbb{N}$ and each element I_n^i of $\mathcal{P}_n$, the interval I_n^i contains at most one periodic point whose (prime) period is $\leq n$.*

Proof: The map $\hat{T}$ is strictly monotone and expanding on the interior of every $J \in \mathcal{P}_n$. Hence $\hat{T}^k$ is strictly monotone on $\text{int}(J)$ for every $1 \leq k \leq n$. If $x, y \in J$ were two distinct points with $\hat{T}^k(x) = x$ and $\hat{T}^k(y) = y$ for some $k \leq n$, then by monotonicity $\hat{T}^k - \text{Id}$ would have at least two zeros in $\text{int}(J)$, forcing it to change sign, which is impossible because $\hat{T}^k$ is strictly monotone on $\text{int}(J)$. Therefore, J contains at most one such periodic point. $\square$

Lemma 4.3. *The set of periodic points of $\hat{T}$ is dense in $[0, 1]$.*

Proof: Let $(a, b) \subset [0, 1]$ be a nonempty open interval. Choose n large enough so that $\mathcal{P}_n$ has an element $I_n^i \subset (a, b)$ (the mesh of $\mathcal{P}_n$ tends to 0 since $\hat{T}$ is expanding). The interval I_n^i corresponds to a cylinder $[w]$ of length n in Σ_A . Pick an admissible periodic sequence in $[w]$, e.g. by repeating w and, if necessary, modifying finitely many symbols to respect admissibility (no 00 occurs). Its image under the conjugacy π is a periodic point of $\hat{T}$ lying in $I_n^i \subset (a, b)$. Hence periodic points are dense in $[0, 1]$. $\square$

5. ESCAPE RATES THROUGH MARKOV HOLES

5.1. Survivor sets and exponential escape rate.

Definition 5.1 (see [12]). Let $(\hat{M}, T)$ be a dynamical system and let $A \subset \hat{M}$ be a measurable *hole*. For $n \geq 1$ the n -step survivor set is

$$\Omega_n(A) := \{x \in \hat{M} : T^k(x) \notin A \text{ for all } 0 \leq k < n\}.$$

Definition 5.2 (see [12]). Assume λ is a reference measure (e.g. Lebesgue measure on $[0, 1]$ or an invariant probability measure). The (exponential) escape rate into A is

$$\rho(A) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log \lambda(\Omega_n(A)),$$

provided the limit exists.

For later use, define the set of points that fall into the hole exactly at time n :

$$\Theta_n(A) := \Omega_n(A) \setminus \Omega_{n+1}(A) = \{x : T^k(x) \notin A \text{ } (0 \leq k < n), \text{ and } T^n(x) \in A\}.$$

Lemma 5.1 (see [12]). *Assume $\sum_{n \geq 0} \lambda(\Theta_n(A)) = 1$ and that $\rho(A)$ exists. Then*

$$\rho(A) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log \lambda(\Theta_n(A)).$$

Proof: Since $\Omega_n(A) = \bigsqcup_{k \geq n} \Theta_k(A)$, we have $\lambda(\Omega_n(A)) = \sum_{k \geq n} \lambda(\Theta_k(A))$. The assumption $\sum_{k \geq 0} \lambda(\Theta_k(A)) = 1$ ensures that the tail sums have the same exponential decay rate as $\lambda(\Theta_n(A))$, yielding the claim. $\square$

5.2. Escape rate via entropy of subshifts. We consider Markov holes defined by cylinders in the symbolic space. Let Σ_F be an irreducible subshift of finite type with transition matrix A_F and Perron eigenvalue $\theta > 1$ (so that $h_{\text{top}}(\Sigma_F) = \log \theta$). Let G be an additional finite family of forbidden words and write $\Sigma_{F \cup G}$ for the resulting subshift with Perron eigenvalue Λ .

Theorem 5.1 (see [5]). *Let H_G be the hole corresponding to forbidding the family G (equivalently, the set of sequences that enter the forbidden cylinder). Then the escape rate exists and satisfies*

$$\rho(H_G) = \log\left(\frac{\theta}{\Lambda}\right) > 0,$$

where $\log \theta = h_{\text{top}}(\Sigma_F)$ and $\log \Lambda = h_{\text{top}}(\Sigma_{F \cup G})$.

5.3. Examples of Markov holes. We illustrate Theorem 5.1 for the golden mean shift

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \theta = \frac{1 + \sqrt{5}}{2}.$$

Fix $N = 3$. The admissible cylinders of length 3 are

$$[010], [011], [101], [110], [111].$$

Each cylinder corresponds (under conjugacy) to one element of the Markov refinement $\mathcal{P}_3$.

Example 5.1. Consider the hole $H = [010]$ (i.e. we forbid the word 010). To describe the perturbed shift as a one-step SFT, we use states given by admissible words of length 2:

$$01, 10, 11.$$

Appending a symbol produces a transition; we keep the transition if the resulting length-3 word is admissible and not forbidden. This yields the 3×3 transition matrix

$$A_{010} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let Λ be the Perron eigenvalue of A_{010} . Then

$$\rho([010]) = \log\left(\frac{\theta}{\Lambda}\right).$$

(One may compute numerically $\Lambda \approx 1.4656$, hence $\rho([010]) \approx 0.102$.)

5.4. Local escape rate. Let $\alpha = \alpha_0 \alpha_1 \alpha_2 \cdots \in \Sigma_F$. For $n \geq 1$ set $w^n = \alpha_0 \cdots \alpha_{n-1}$ and $H_n = [w^n]$. The local escape rate around α is

$$\rho(\alpha) = \lim_{n \rightarrow \infty} \frac{\rho(H_n)}{\mu(H_n)},$$

provided the limit exists.

Theorem 5.2 (see [13]). *Let Σ_F be an irreducible subshift of finite type with Perron eigenvalue $\theta > 1$ and Parry measure μ . Then*

$$\rho(\alpha) = \begin{cases} 1, & \text{if } \alpha \text{ is non-periodic,} \\ 1 - \theta^{-m}, & \text{if } \alpha \text{ has prime period } m. \end{cases}$$

Lemma 5.2 (Local escape for shrinking Markov holes). *Let $x \in [0, 1]$ and let $\{A_n(x)\}_{n \geq 1}$ be a nested sequence of Markov intervals such that $\bigcap_{n \geq 1} A_n(x) = \{x\}$. Assume the corresponding symbolic point is $\alpha = \pi^{-1}(x) \in \Sigma_F$. Then:*

(1) *If x is periodic with prime period m , then*

$$\lim_{n \rightarrow \infty} \frac{\rho(A_n(x))}{\mu(A_n(x))} = 1 - \theta^{-m}.$$

(2) *If x is non-periodic, then*

$$\lim_{n \rightarrow \infty} \frac{\rho(A_n(x))}{\mu(A_n(x))} = 1.$$

Proof: Under the conjugacy, each Markov interval $A_n(x)$ corresponds to a cylinder $H_n = [w^n]$ shrinking to α . Therefore

$$\frac{\rho(A_n(x))}{\mu(A_n(x))} = \frac{\rho(H_n)}{\mu(H_n)}.$$

The conclusion follows immediately from Theorem 5.2. $\square$

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On $(O-C)$ -compactness of the space of idempotent probability measures

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Abstract. We establish a connection between connected sets and max-plus connected sets. Then this connection is applied between spaces of ordinary probability measures and idempotent (max-plus) probability measures on a given Tychonoff space. We show that for every Tychonoff space X the space $I_\beta(X)$ is max-plus convex and therefore, path max-plus-connected. Combining these properties with topological arguments we prove that for a Tychonoff space X the space $I_\beta(X)$ is $(O-C)$ -compact. Auxiliary propositions relating path connectedness and max-plus connectedness are established and used in the derivation of the main results.

Keywords: idempotent probability measures, max-plus probability measures, Tychonoff spaces, max-plus convexity, path connectedness, $(O-C)$ -compactness

MSC (2020): 28C15, 54D35, 54C10

1. INTRODUCTION

Beginning in the last decade of the past century, idempotent mathematics has been extensively developed (see, e.g., [1, 2, 3, 4, 5]). In [6], the systematic study of the space of idempotent probability measures on the algebra of continuous functions on a given compact Hausdorff space was initiated and many results similar to those of [7, 8, 9] for usual probability measures were obtained. Further progress was made in [10, 11, 12, 13, 14]. In the paper [12], a modified method proposed in [15] was applied. More advanced results on idempotent probability measures were obtained in subsequent works.

By a *space* we mean a topological T_1 -space, and by a *map* a continuous map. Recall that a Tychonoff space is a space X such that every closed subset $F \subset X$ and any point $x \in X \setminus F$ are functionally separated, i. e., there exists a continuous function $\varphi: X \rightarrow \mathbb{R}$ for which $\varphi(x) = 0$ and $\varphi(y) = 1$ for $y \in F$.

A finite sequence of subsets $M_0, \dots, M_s$ of a set X is [14] a *chain* connecting sets M_0 and M_s , if $M_{i-1} \cap M_i \neq \emptyset$ for all $i = 1, \dots, s$. A collection ω of subsets of a set X is said to be *connected* if for any pair of sets $M, M' \subset X$ there exists a chain ω connecting the sets M and M' . The maximal connected sub-collections of ω are called *components* of ω . A star-finite open cover of a space X is said to be a *finite-component cover* if the number of elements of each component is finite.

1.1. Preliminaries. Let βX be the Stone-Čech compactification of a Tychonoff space X (i.e., the maximal compact extension).

The author of [6] observed the functor $I: \mathcal{Comp} \rightarrow \mathcal{Comp}$ and showed that it is normal. Then in [12] using the construction suggested by A.Ch. Chigogidze [15], it was obtained an extension $I_\beta: \mathcal{Tych} \rightarrow \mathcal{Tych}$. Here, the sign $\mathcal{Comp}$ means the category of compact Hausdorff spaces and their continuous maps, and $\mathcal{Tych}$ the category of Tychonoff spaces and their continuous maps.

Consider the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with two algebraic operations, addition $\oplus$ and multiplication $\odot$, defined by $u \oplus v = \max\{u, v\}$, $u \odot v = u + v$, where $u, v \in \mathbb{R}$.

Let X be a compact Hausdorff space, and let $C(X)$ be the algebra of continuous functions on X with ordinary algebraic operations. We also define operations $\oplus$ and $\odot$ on $C(X)$ by

$$\varphi \oplus \psi = \max\{\varphi, \psi\}, \quad \text{and} \quad \varphi \odot \psi = \varphi + \psi, \quad \text{where} \quad \varphi, \psi \in C(X).$$

Definition 1.1. [6] A functional $\mu: C(X) \rightarrow \mathbb{R}$ is called an *idempotent probability measure* on X if it has the following properties:

- 1I. $\mu(\lambda_X) = \lambda$ for all $\lambda \in \mathbb{R}$, where λ_X is the constant function taking the value λ ;

2I. $\mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi)$ for all $\lambda \in \mathbb{R}$ and $\varphi \in C(X)$;

3I. $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$ for all $\varphi, \psi \in C(X)$.

For a compact space X , we denote by $I(X)$ the set of all idempotent probability measures on X . We consider $I(X)$ as a subspace of $\mathbb{R}^{C(X)}$.

It can be shown that, for any compact spaces X and Y and any continuous map $f: X \rightarrow Y$, the map $I(f): I(X) \rightarrow I(Y)$ defined by

$$I(f)(\mu)(\psi) = \mu(\psi \circ f)$$

is continuous. Moreover, the construction I is a normal functor. Given any idempotent probability measure $\mu \in I(X)$, we define its support as

$$\text{supp } \mu = \cap \{A \subset X : A = \overline{A}, \mu \in I(A)\},$$

thus, the support of an idempotent probability measure μ is the least (by inclusion) closed subset of X on which μ is concentrated.

For each positive integer n , we set

$$I_n(X) = \{\mu \in I(X) : |\text{supp } \mu| \leq n\} \quad \text{and} \quad I_\omega(X) = \bigcup_{n=1}^{\infty} I_n(X).$$

The set $I_\omega(X)$ is dense in $I(X)$ [6]. An idempotent probability measure $\mu \in I_\omega(X)$ is said to be *finitely supported*. Note that if μ is an idempotent probability measure with finite support $\text{supp } \mu = \{x_1, x_2, \dots, x_k\}$, then μ admits a unique representation in the form

$$\mu = \lambda_1 \odot \delta_{x_1} \oplus \lambda_2 \odot \delta_{x_2} \oplus \dots \oplus \lambda_k \odot \delta_{x_k},$$

where $-\infty < \lambda_i \leq 0$ for $i = 1, 2, \dots, k$ and $\lambda_1 \oplus \lambda_2 \oplus \dots \oplus \lambda_k = 0$. Here, as usual, given $x \in X$, by δ_x we denote the functional on $C(X)$ defined by $\delta_x(\varphi) = \varphi(x)$ for $\varphi \in C(X)$ and called the *Dirac measure*. This measure is concentrated at x .

For a Tychonoff space X , let βX be its the maximal compact (i. e. the Stone-Čech) extension. Following [12], consider the set

$$I_\beta(X) = \{\mu \in I(\beta X) : \text{supp } \mu \subset X\}.$$

Since the support $\text{supp } \mu$ of $\mu \in I_\beta(X)$ is compact, μ is called an idempotent measure with compact support. $I_\beta(X)$ is considered as a subspace of $I(\beta X)$.

Note that a functional $\mu: C(X) \rightarrow \mathbb{R}$ is a *probability measure* on X if it satisfies the following properties:

1P. $\mu(\lambda_X) = \lambda$ for all $\lambda \in \mathbb{R}$;

2P. $\mu(\lambda\varphi) = \lambda\mu(\varphi)$ for all $\lambda \in \mathbb{R}$ and $\varphi \in C(X)$;

3P. $\mu(\varphi + \psi) = \mu(\varphi) + \mu(\psi)$ for all $\varphi, \psi \in C(X)$.

The sets $P(X)$ of all probability measures and $I(X)$ of all idempotent probability measures are endowed with the topology of pointwise convergence. For a compact Hausdorff space X the topological spaces $P(X)$ and $I(X)$ are compact Hausdorff spaces.

In the case when X is a Tychonoff space we put

$$P_\beta(X) = \{\mu \in P(\beta X) : \text{supp } \mu \subset X\}.$$

Although constructions P and I are not isomorphic functors, in [16] it was shown that the spaces $P(X)$ and $I(X)$ are homeomorphic for any compact metrizable space X .

2. FUNCTOR OF IDEMPOTENT PROBABILITY MEASURES AND $(O-C)$ -COMPACTNESS

In the current work we establish that the above mentioned result in [16] admits extension for wider classes of topological spaces, exactly, for the class of Tychonoff spaces. In [17] the authors shown that the functor I_β transforms Π -complete spaces into Π -complete spaces. We will show that acting at a Tychonoff space X by the functor I_β one gets a $(O-C)$ -compact space $I_\beta(X)$.

Theorem 2.1. *For any Tychonoff space X , the spaces $P_\beta(X)$ and $I_\beta(X)$ are homeomorphic.*

Proof: Let $\mu \in P_\beta(X)$. Then $\text{supp } \mu$ is a compact set. Consider the map

$$z_I^P : P_\omega(\text{supp } \mu) \rightarrow I_\omega(\text{supp } \mu)$$

given [16] by

$$z_I^P \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) = \bigoplus_{i=1}^n \left(\left(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j \right) \odot \delta_{x_i} \right)$$

and the map

$$z_P^I : I_\omega(\text{supp } \mu) \rightarrow P_\omega(\text{supp } \mu)$$

given by

$$z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) = \sum_{i=1}^n \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} \cdot \delta_{x_i}.$$

It is easy to see that

$$\text{supp} \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) = \text{supp} \left(z_I^P \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) \right)$$

and

$$\text{supp} \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) = \text{supp} \left(z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) \right).$$

The maps z_I^P and z_P^I are continuous and mutually inverse maps [16]. Therefore, $z_P^I z_I^P : P_\omega(\text{supp } \mu) \rightarrow P_\omega(\text{supp } \mu)$ and $z_I^P z_P^I : I_\omega(\text{supp } \mu) \rightarrow I_\omega(\text{supp } \mu)$ are the identity maps. Really, for each probability measure

$$\sum_{i=1}^n \alpha_i \delta_{x_i} \in P_\omega(\text{supp } \mu),$$

we have

$$\begin{aligned} z_P^I z_I^P \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) &= z_P^I \left(\bigoplus_{i=1}^n \left(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j \right) \odot \delta_{x_i} \right) = \sum_{i=1}^n \frac{\exp \left(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j \right)}{\sum_{l=1}^n \exp \left(\ln \alpha_l - \bigoplus_{j=1}^n \ln \alpha_j \right)} \delta_{x_i} \\ &= \sum_{i=1}^n \frac{\exp(\ln \alpha_i)}{\exp \left(\bigoplus_{j=1}^n \ln \alpha_j \right)} \cdot \frac{\sum_{l=1}^n \exp(\ln \alpha_l)}{\exp \left(\bigoplus_{j=1}^n \ln \alpha_j \right)} \delta_{x_i} = \sum_{i=1}^n \frac{\alpha_i}{\sum_{l=1}^n \alpha_l} \delta_{x_i} = \sum_{i=1}^n \alpha_i \delta_{x_i}. \end{aligned}$$

For each idempotent probability measure

$$\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \in I_\omega(\text{supp } \mu),$$

we have

$$z_I^P z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) = z_I^P \left(\sum_{i=1}^n \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} \delta_{x_i} \right) = \bigoplus_{i=1}^n \left(\ln \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} - \bigoplus_{l=1}^n \ln \frac{e^{\lambda_l}}{\sum_{j=1}^n e^{\lambda_j}} \right) \odot \delta_{x_i}$$

$$\begin{aligned}
&= \bigoplus_{i=1}^n \left(\ln e^{\lambda_i} - \ln \sum_{j=1}^n e^{\lambda_j} - \bigoplus_{l=1}^n \left(\ln e^{\lambda_l} - \ln \sum_{j=1}^n e^{\lambda_j} \right) \right) \odot \delta_{x_i} \\
&= \bigoplus_{i=1}^n \left(\lambda_i - \ln \sum_{j=1}^n e^{\lambda_j} - \bigoplus_{l=1}^n \lambda_l + \ln \sum_{j=1}^n e^{\lambda_j} \right) \odot \delta_{x_i} = \bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i}.
\end{aligned}$$

Theorem 2.1 is proved. $\square$

A nonempty set L is said to be a *max-plus linear space* over a field $\mathbb{K}$ if it is linear with respect to operations $\oplus$ and $\odot$. Here $\oplus$ is the new addition, and $\odot$ the multiplication by scalar. A subset A of a max-plus linear space L over a field $\mathbb{K}$ is said to be a *max-plus linear subspace* of L if

- (1) $c_X \in A$ for each $c \in \mathbb{K}$;
- (2) $\lambda \odot \varphi \in A$ for any $\lambda \in \mathbb{K}$ and $\varphi \in A$;
- (3) $\varphi \oplus \psi \in A$ for any $\varphi, \psi \in A$.

We note that the first condition can be replaced by the requirement $0_X \in L$. Then condition 2 yields condition 1.

A subset $A \subset L$ is called *max-plus-convex* if for every pair $x_1, x_2 \in A$ we have $\lambda_1 \odot x_1 \oplus \lambda_2 \odot x_2 \in A$ for all $\lambda_1, \lambda_2 \in \mathbb{R}_{\max}$ with $\lambda_1 \oplus \lambda_2 = 0$. In other words,

$$[x_1, x_2]_{\oplus} = \{\lambda_1 \odot x_1 \oplus \lambda_2 \odot x_2 : \lambda_1, \lambda_2 \in \mathbb{R}_{\max} \text{ with } \lambda_1 \oplus \lambda_2 = 0\} \subset A.$$

A subset A of a topological space X is *path max-plus-connected* when, for every pair of points $a, b \in A$, there exists a continuous function $f: [-\infty, 0]_{\oplus} \rightarrow A$ such that $f(-\infty) = a$ and $f(0) = b$.

It is easy to verify that the following statement is faithful.

Proposition 2.1. *Every max-plus-convex set is a path max-plus-connected set.*

Theorem 2.2. *Let X be a set in a linear space L , and Y a set in a max-plus linear space M . Moreover, let X and Y be homeomorphic. Then Y is max-plus connected if and only if X is path connected.*

Proof: Let $f: X \rightarrow Y$ is a homeomorphism, X a path connected set in a linear space, and $y_1, y_2 \in Y$ an arbitrary couple of points. Let $x_1, x_2 \in X$ be corresponding preimages of y_1, y_2 , i. e. $f(x_i) = y_i$, $i = 1, 2$.

Consider any continuous map $g: [0, 1] \rightarrow X$ with $g(0) = x_1, g(1) = x_2$. The exponential function $e: [-\infty, 0] \rightarrow [0, 1]$, $e(t) = e^t$, is a homeomorphism and satisfies equalities $e(-\infty) = 0$ and $e(0) = 1$. It reminds to remark that $h = f \circ g \circ e: [-\infty, 0]_{\oplus} \rightarrow Y$ is a max-plus path such that $h(-\infty) = y_1$ and $h(0) = y_2$.

Let Y be a max-plus connected set. Take a max-plus path $H: [-\infty, 0]_{\oplus} \rightarrow Y$ with $H(-\infty) = y_1, H(0) = y_2$, and any pair of points $x_1, x_2 \in X$. Suppose that $f(x_i) = y_i, i = 1, 2$. Then the composition $f^{-1} \circ H \circ \ln: [0, 1] \rightarrow X$ is the searching path. Here $\ln: [0, 1] \rightarrow [-\infty, 0]_{\oplus}$ is the logarithmic function.

Theorem 2.2 is proved. $\square$

It is well known [6] for a compact Hausdorff space X the set $I(X)$ is a max-plus convex. We affirm the following statement.

Proposition 2.2. *For a Tychonoff space X the set $I_{\beta}(X)$ is max-plus convex.*

Proof: The fact that a finite union of compact sets is also compact, completes the proof. $\square$

Definition 2.1. [14]. A topological space X is called (O-C)-compact if every clopen cover of X admits a finite subcover.

Since each connected space is (O-C)-compact [14] according to Theorem 2.2 we have the following result.

Proposition 2.3. *A max-plus connected space is (O-C)-compact.*

Theorem 2.2, Propositions 2.2, 2.3 imply the following main result of the paper.

Theorem 2.3. *For any Tychonoff space X the space $I_{\beta}(X)$ is (O-C)-compact.*

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Uzbek Mathematical Journal
 2026, Volume 70, Issue 3, pp. 67–78
 DOI: 10.29229/uzmj.2026-3-8

Optimal quadrature formula for approximate solution of a singular integral equation with the Hilbert kernel in the space $L_2^{(3)}(0, 2\pi)$

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Abstract. In this article, an optimal quadrature formula for the approximate calculation of the solution of a Fredholm-type singular integral equation of the second kind with the Hilbert kernel in the space $L_2^{(3)}(0, 2\pi)$ is constructed. In the proposed quadrature formula, the Sard problem arises, where the interval $[0, 2\pi]$ is evenly divided into any N points, and this problem is solved using the Sobolev method. For this purpose, an analytical representation of the optimal coefficients of the quadrature formula is obtained by minimizing the norm of the error functional in the conjugate space $L_2^{(3)*}(0, 2\pi)$.

Keywords: Hilbert space; singular integral; error functional; optimal quadrature formula; singular integral equations; Hilbert kernel

MSC (2020): 65D32, 65R20

1. INTRODUCTION

Singular integral equations arise in the mathematical formulation of problems in physics, mechanics, engineering, and numerous other fields. One such equation is the following Fredholm-type singular integral equation of the second kind:

$$a \cdot g(t) + \frac{b}{2\pi} \int_0^{2\pi} g(x) \cot \frac{x-t}{2} dx = \varphi(t), \quad (1.1)$$

where $g(t)$ is an unknown function, and $\varphi(t)$ is a known function that satisfies the Lipschitz condition $[1, 2]$. a and b are arbitrary constants, which in general can be complex numbers.

This kernel

$$\cot \frac{x-t}{2}$$

is known as the Hilbert kernel. It is defined in the sense of the Cauchy principal value with properties similar to a singular Cauchy-type integral, having a singularity when $x = t$ in the square where $0 < x < 2\pi$ and $0 < t < 2\pi$.

Using Hilbert transformations, the integral equation (1.1), if $a \neq 0$, has the following solution (see [3], p. 144, [4], p. 115):

$$g(t) = \frac{a}{a^2 + b^2} \varphi(t) - \frac{b}{2\pi(a^2 + b^2)} \int_0^{2\pi} \varphi(x) \cot \frac{x-t}{2} dx + \frac{b^2}{2\pi a(a^2 + b^2)} \int_0^{2\pi} \varphi(x) dx. \quad (1.2)$$

The main problem encountered in the solution of the integral equation (1.1) in the form (1.2) is the calculation of Cauchy's principal value integral

$$\int_0^{2\pi} \varphi(x) \cot \frac{x-t}{2} dx. \quad (1.3)$$

This problem persists even when the expression (1.2) is regularized.

Articles dedicated to the numerical solution of the integral equation (1.1) are scarce. In this work, we develop a numerical method for solving the singular integral (1.3) in the sense of Cauchy's principal value, which appears in the solution of the integral equation (1.1) in the form (1.2). This method is based on a functional approach. Since this method considers functions belonging to the Hilbert space $L_2^{(3)}(0, 2\pi)$, it is considered highly effective and accurate.

2. STATEMENT OF THE PROBLEM

We consider the following quadrature formula

$$\int_0^{2\pi} \varphi(x) \cot \frac{x-t}{2} dx \cong \sum_{\beta=0}^N C_\beta \varphi(x_\beta), \quad (2.1)$$

where C_β are coefficients, x_β are nodal points, and $x_\beta \in [0, 2\pi]$. The functions $\varphi(x)$ belong to the Hilbert space $L_2^{(3)}(0, 2\pi)$, which is defined as

$$L_2^{(3)}(0, 2\pi) = \{\varphi(x) : [0, 2\pi] \rightarrow \mathbb{R}, \varphi''(x) \text{ is absolutely continuous, } \varphi'''(x) \in L_2(0, 2\pi)\}.$$

In this $L_2^{(3)}(0, 2\pi)$ space, the inner product of functions φ and ψ is defined as follows

$$\langle \varphi, \psi \rangle = \int_0^{2\pi} \varphi'''(x) \psi'''(x) dx, \quad (2.2)$$

and the norm of the function φ in this space is calculated using the inner product (2.2) as follows

$$\|\varphi\| = \sqrt{\int_0^{2\pi} \left(\varphi'''(x)\right)^2 dx}. \quad (2.3)$$

The difference between the integral and the quadrature sum in the considered quadrature formula (2.1)

$$\begin{aligned} \int_0^{2\pi} \varphi(x) \cot \frac{x-t}{2} dx - \sum_{\beta=0}^N C_\beta \varphi(x_\beta) &= \int_{-\infty}^{\infty} (\varepsilon_{[0, 2\pi]}(x) \cot \frac{x-t}{2} - \sum_{\beta=0}^N C_\beta \delta(x - x_\beta)) \varphi(x) dx = \\ &= \int_{-\infty}^{\infty} \ell(x) \varphi(x) dx = (\ell, \varphi) \end{aligned} \quad (2.4)$$

is called the *error* of the quadrature formula.

For the derived expression (2.4), the following *error functional* corresponding to the conjugate space $L_2^{(3)*}(0, 2\pi)$ is obtained

$$\ell(x) = \varepsilon_{[0, 2\pi]}(x) \cot \frac{x-t}{2} - \sum_{\beta=0}^N C_\beta \delta(x - x_\beta), \quad (2.5)$$

where $\varepsilon_{[0, 2\pi]}(x)$ is the characteristic function of the interval $[0, 2\pi]$, and $\delta(x)$ is Dirac's delta-function.

According to the definition of the norm of a linear continuous functional, the following is known (see [5], p. 82):

$$\|\ell\|_{L_2^{(3)*}} = \sup_{\|\varphi\| \neq 0} \frac{|(\ell, \varphi)|}{\|\varphi\|}.$$

From this equality, the Cauchy-Schwarz inequality follows

$$|(\ell, \varphi)| \leq \|\ell\|_{L_2^{(3)*}} \|\varphi\|_{L_2^{(3)}}. \quad (2.6)$$

According to the Cauchy-Schwarz inequality, the following conditions hold for the norm (2.3) of the function $\varphi(x)$

$$(\ell, x^\alpha) = 0, \quad \alpha = 0, 1, 2. \quad (2.7)$$

According to the conditions (2.7), the quadrature formula (2.1) is exact for the function $d_2x^2 + d_1x + d_0$. These conditions ensure that the error functional is bounded.

Thus, the upper bound of the error (2.4) of the quadrature formula (2.1) is estimated using the norm of the error functional $\ell(x)$. From this, the following problem arises.

Problem 1. Find the form of the norm of the error functional $\ell(x)$ belonging to the conjugate space $L_2^{(3)*}(0, 2\pi)$.

It can be observed that the error functional depends on the coefficients C_β and the node points x_β . Minimizing the norm of the error functional with respect to both the coefficients C_β and the node points x_β is called the *Nikolsky problem*, and this problem was first studied in Nikolsky's works [6, 7]. The problem of minimizing the norm of the error functional only with respect to the coefficients C_β , while fixing the node points x_β in advance, is called the *Sard problem*, and this problem was first studied in the works of Sard [8, 9].

In this work, we solve the Sard problem. For this, we construct a quadrature formula at nodal points x_β where $x_\beta = h\beta$ ($h = \frac{2\pi}{N}$, $N \in \mathbb{N}$) are equidistantly distributed points.

The coefficients C_β that minimize the norm of the error functional $\ell(x)$, that is

$$\inf_{C_\beta} \|\ell\|_{L_2^{(3)*}(0, 2\pi)} \quad (2.8)$$

are called *optimal coefficients* and are denoted by $\mathring{C}_\beta$. The quadrature formula constructed using these optimal coefficients is called an *optimal quadrature formula*. Therefore, to construct an optimal quadrature formula in the sense of Sard in the space $L_2^{(3)}(0, 2\pi)$, it is necessary to solve the following problem.

Problem 2. Find the optimal coefficients $\mathring{C}_\beta$ that achieve the value in (2.8).

The main purpose of this article is to construct an optimal quadrature formula that calculates the integral equation (1.1) with high accuracy in the space $L_2^{(3)}(0, 2\pi)$. Using the functional method in Hilbert spaces, optimal quadrature formulas were constructed for singular integrals with Cauchy kernels in works [10, 11, 12, 13], and for singular integrals with Hilbert kernels in works [14, 15]. Furthermore, optimal quadrature formulas for regular integrals in various other spaces were constructed using the Sobolev method in works [16, 17].

The remainder of this article is structured as follows. In the third section, the extremal function corresponding to the error functional $\ell(x)$ is determined, and using this function, the expression for the norm of the error functional (2.5) is derived, thus solving Problem 1. In the fourth section, to find the minimum of the quantity $\|\ell\|^2$ with respect to the coefficients C_β , a system of linear equations for the coefficients of the optimal quadrature formula in the Sard sense in the space $L_2^{(3)}(0, 2\pi)$ is formulated and solved using the Sobolev method. In the fifth section, the analytical form of the optimal coefficients for the quadrature formula (2.1) is obtained, thereby solving Problem 2.

3. FINDING THE EXTREMAL FUNCTION AND EXPRESSING THE NORM OF THE ERROR FUNCTIONAL

To solve problem 1, that is, to find the form of the norm of the error functional (2.5), we use the concept of an extremal function.

The function $\psi_\ell(x)$ is called an *extremal function* corresponding to the error functional $\ell(x)$ if the following equality holds (see, [18, 19, 20])

$$(\ell, \psi_\ell) = \|\ell\|_{L_2^{(3)*}} \cdot \|\psi_\ell\|_{L_2^{(3)}}. \quad (3.1)$$

The concept of an extremal function was first introduced by S.L. Sobolev and is defined as follows

$$\psi_\ell = -\ell(x) * G_3(x) + P_2(x),$$

where $G_3(x) = \frac{|x|^5}{2 \cdot 5!}$, and $P_2(x)$ is a second-degree polynomial.

We note that $*$ is the convolution and the convolution of two functions

$$\varphi(x) * \psi(x) = \int_{-\infty}^{\infty} \varphi(x-y)\psi(y)dy = \int_{-\infty}^{\infty} \varphi(y)\psi(x-y)dy$$

is defined by the formula.

Since the space $L_2^3(0, 2\pi)$ we are considering is a Hilbert space, we present the Riesz theorem on the general form of a linear continuous functional.

Theorem 3.1. (*Riesz theorem*) Let H be a Hilbert space. Then for each linear continuous functional $\ell \in H^*$, there exists a unique function ψ_ℓ in the space H itself, called the Riesz element, which satisfies the condition $(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle$ corresponding to this functional, and the following equality holds for this function

$$\|\ell\|_{H^*} = \|\psi_\ell\|_H$$

and conversely.

For any $\psi_\ell \in H$, there exists a unique functional $\ell \in H^*$ that satisfies the conditions $(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle$ and $\|\ell\|_{H^*} = \|\psi_\ell\|_H$.

According to the theorem, the extremal function $\psi_\ell(x)$ is a Riesz element and

$$(\ell, \psi_\ell) = \|\ell\|^2 = \|\psi_\ell\|^2.$$

Based on the theorem, we calculate the square of the norm of the error functional

$$\begin{aligned} \|\ell\|^2 = & - \int_0^{2\pi} \int_0^{2\pi} \frac{|x-y|^5}{2 \cdot 5!} \cot \frac{x-t}{2} \cot \frac{y-t}{2} dx dy + \\ & + 2 \sum_{\beta=0}^N C_\beta \int_0^{2\pi} \frac{|x-h\beta|^5}{2 \cdot 5!} \cot \frac{x-t}{2} dx - \sum_{\beta=0}^N \sum_{\gamma=0}^N C_\beta C_\gamma \frac{|h\beta-h\gamma|^5}{2 \cdot 5!}. \end{aligned} \quad (3.2)$$

Thus, the square of the norm of the error functional $\ell(x)$ has been determined, and problem 1 has been solved.

4. FINDING THE CONDITIONAL MINIMUM OF THE ERROR FUNCTIONAL

Now we will tackle the second problem. The square of the norm of the error functional obtained in (3.2) is a multi-variable function with respect to the coefficients C_β . According to the rule for finding the conditional extremum of a multi-variable function, we will determine its local minimum in conjunction with the conditions (2.7). To achieve this, we construct the Lagrange function:

$$\Phi(C_\beta, d_2, d_1, d_0) = \|\ell\|^2 + 2 \sum_{\alpha=0}^2 d_\alpha (\ell, x^\alpha), \quad \beta = 0, 1, \dots, N.$$

According to (2.7), the following holds true here

$$(\ell, x^\alpha) = \int_0^{2\pi} x^\alpha \cot \frac{x-t}{2} dx - \sum_{\beta=0}^N C_\beta (h\beta)^\alpha = 0, \quad \alpha = 0, 1, 2.$$

By setting the partial derivatives of the resulting function Φ with respect to the coefficients C_β and the unknowns d_2, d_1, d_0 to zero, we obtain the following system of linear equations

$$\sum_{\gamma=0}^N C_\gamma \frac{|h\beta-h\gamma|^5}{2 \cdot 5!} + d_2 (h\beta)^2 + d_1 (h\beta) + d_0 = f_3(h\beta), \quad \beta = 0, 1, \dots, N, \quad (4.1)$$

$$\sum_{\gamma=0}^N C_\gamma (h\gamma)^2 = g_2, \quad (4.2)$$

$$\sum_{\gamma=0}^N C_\gamma (h\gamma) = g_1, \quad (4.3)$$

$$\sum_{\gamma=0}^N C_{\gamma} = g_0, \quad (4.4)$$

where

$$\begin{aligned} f_3(h\beta) &= \int_0^{2\pi} \frac{|x - h\beta|^5}{2 \cdot 5!} \cot \frac{x - t}{2} dx = \\ &= \frac{1}{5!} \left[\sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+5)(2k)!} \left((2\pi - t)^{2k+5} - 2(h\beta - t)^{2k+5} - t^{2k+5} \right) + \right. \\ &\quad + 5(t - h\beta) \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+4)(2k)!} \left((2\pi - t)^{2k+4} - 2(h\beta - t)^{2k+4} + t^{2k+4} \right) + \\ &\quad + 10(t - h\beta)^2 \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+3)(2k)!} \left((2\pi - t)^{2k+3} - 2(h\beta - t)^{2k+3} - t^{2k+3} \right) + \\ &\quad + 10(t - h\beta)^3 \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+2)(2k)!} \left((2\pi - t)^{2k+2} - 2(h\beta - t)^{2k+2} + t^{2k+2} \right) + \\ &\quad + 5(t - h\beta)^4 \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+1)(2k)!} \left((2\pi - t)^{2k+1} - 2(h\beta - t)^{2k+1} - t^{2k+1} \right) + \\ &\quad \left. + 2(t - h\beta)^5 \ln \left| \frac{\sin \frac{t}{2}}{\sin \frac{h\beta - t}{2}} \right| \right], \\ g_2 &= 2 \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+2)(2k)!} \left((2\pi - t)^{2k+2} - t^{2k+2} \right) + \\ &\quad + 4t \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+1)(2k)!} \left((2\pi - t)^{2k+1} + t^{2k+1} \right), \\ g_1 &= 2 \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k}}{(2k+1)(2k)!} \left((2\pi - t)^{2k+1} + t^{2k+1} \right), \\ g_0 &= \int_0^{2\pi} \cot \frac{x - t}{2} dx = 0, \end{aligned}$$

here, B_{2k} are the Bernoulli numbers, and the following formula was used (see [21], p. 224)

$$\int x^p \cot x dx = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k+p)(2k)!} x^{2k+p}, \quad [p \geq 1, |x| < \pi].$$

The system (4.1)-(4.4) consists of $(N+4)$ unknowns and $(N+4)$ linear equations with respect to $C_{\gamma}, \gamma = 0, 1, \dots, N, d_0, d_1, d_2$. To solve this system, we will employ the functional method proposed by Sobolev. For this method, it is crucial to use a discrete analogue of the differential operator d^6/dx^6 . In [22], discrete analogue of the differential operator d^6/dx^6 has been constructed.

Theorem 4.1. *The discrete analog of the differential operator d^6/dx^6 satisfying the equality $D_3(h\beta) * G_3(h\beta) = \delta(h\beta)$ has the following form*

$$D_3(h\beta) = \frac{5!}{h^6} \begin{cases} \sum_{t=1}^2 K_t \cdot q_t^{|\beta|-1}, & |\beta| \geq 2, \\ 1 + \sum_{t=1}^2 K_t, & |\beta| = 1, \\ -2^5 + \sum_{t=1}^2 \frac{K_t}{q_t}, & \beta = 0, \end{cases} \quad (4.5)$$

where $\delta(h\beta)$ is a discrete delta function, $K_1 = \frac{(1-q_1)^7}{E_5(q_1)}$, $K_2 = \frac{(1-q_2)^7}{E_5(q_2)}$, $E_5(x)$ is a fifth-degree Euler-Frobenius polynomial, and q_1 and q_2 are roots of the Euler-Frobenius polynomial $E_4(q)$ satisfying the inequality $|q_t| < 1$, ($t = 1, 2$).

Theorem 4.2. *The discrete operator $D_3(h\beta)$ has the following properties:*

$$\begin{aligned} 1) \quad & D_3(h\beta) * 1 = 0 \\ 2) \quad & D_3(h\beta) * (h\beta) = 0 \\ 3) \quad & D_3(h\beta) * (h\beta)^2 = 0 \\ 4) \quad & hD_3(h\beta) * G_3(h\beta) = \delta(h\beta) \end{aligned} \quad (4.6)$$

The proofs of the theorems are provided in [22].

Since we will be using functions with discrete arguments and operations on them from this point forward, we will introduce some concepts related to these.

Definition 4.1. The function $\varphi(h\beta)$ is a *function of discrete argument* if it is given on some set of integer values of β .

Definition 4.2. The *inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

Definition 4.3. The *convolution* of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

We denote the left side of equation (4.1) as

$$u(h\beta) = \sum_{\gamma=0}^N C_\gamma \frac{|h\beta - h\gamma|^5}{2 \cdot 5!} + d_2(h\beta)^2 + d_1(h\beta) + d_0. \quad (4.7)$$

We transform this expression (4.7) into a convolution form using the concept of discrete argument function mentioned above

$$u(h\beta) = C_\beta * G_3(h\beta) + d_2(h\beta)^2 + d_1(h\beta) + d_0. \quad (4.8)$$

By applying the discrete operator $D_3(h\beta)$ to this expression and using its properties, we obtain

$$C_\beta = hD_3(h\beta) * u(h\beta). \quad (4.9)$$

Thus, to find the coefficients of the optimal quadrature formula, it is necessary to determine the general form of the function $u(h\beta)$ at all integer points. From equation (4.1), we see that $u(h\beta) =$

$f_3(h\beta)$ for $\beta = 0, 1, \dots, N$. Now, assuming $C_\beta = 0$, we need to find the form of the function $u(h\beta)$ from expression (4.8) when $\beta < 0$ and $\beta > N$.

When $\beta < 0$,

$$\begin{aligned} u(h\beta) &= - \sum_{\gamma=0}^N C_\gamma \frac{(h\beta - h\gamma)^5}{2 \cdot 5!} + d_2(h\beta)^2 + d_1(h\beta) + d_0 = \\ &= -\frac{g_0}{240}(h\beta)^5 + \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + p_2(h\beta)^2 - p_1(h\beta) + p_0 + d_2(h\beta)^2 + d_1(h\beta) + d_0, \end{aligned}$$

here, p_2, p_1, p_0 are coefficients that are equal to the following

$$p_2 = \frac{1}{24} \sum_{\gamma=0}^N C_\gamma (h\gamma)^3, \quad p_1 = \frac{1}{48} \sum_{\gamma=0}^N C_\gamma (h\gamma)^4, \quad p_0 = \frac{1}{240} \sum_{\gamma=0}^N C_\gamma (h\gamma)^5.$$

When $\beta > N$,

$$\begin{aligned} u(h\beta) &= \sum_{\gamma=0}^N C_\gamma \frac{(h\beta - h\gamma)^5}{2 \cdot 5!} + d_2(h\beta)^2 + d_1(h\beta) + d_0 = \\ &= \frac{g_0}{240}(h\beta)^5 - \frac{g_1}{48}(h\beta)^4 + \frac{g_2}{24}(h\beta)^3 - p_2(h\beta)^2 + p_1(h\beta) - p_0 + d_2(h\beta)^2 + d_1(h\beta) + d_0, \end{aligned}$$

Thus, the general form of $u(h\beta)$ became as follows:

$$u(h\beta) = \begin{cases} -\frac{g_0}{240}(h\beta)^5 + \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + (d_2 + p_2)(h\beta)^2 + (d_1 - p_1)(h\beta) + (d_0 + p_0), & \beta < 0, \\ f_3(h\beta), & 0 \leq \beta \leq N, \\ \frac{g_0}{240}(h\beta)^5 - \frac{g_1}{48}(h\beta)^4 + \frac{g_2}{24}(h\beta)^3 + (d_2 - p_2)(h\beta)^2 + (d_1 + p_1)(h\beta) + (d_0 - p_0), & \beta > N. \end{cases}$$

Here, to simplify the form of $u(h\beta)$, we introduce the following notations

$$\begin{aligned} d_2 + p_2 &= a_2^-, \quad d_1 - p_1 = a_1^-, \quad d_0 + p_0 = a_0^-, \\ d_2 - p_2 &= a_2^+, \quad d_1 + p_1 = a_1^+, \quad d_0 - p_0 = a_0^+. \end{aligned}$$

From this, we can determine the following

$$\begin{aligned} d_2 &= \frac{a_2^- + a_2^+}{2}, \quad d_1 = \frac{a_1^- + a_1^+}{2}, \quad d_0 = \frac{a_0^- + a_0^+}{2}, \\ p_2 &= \frac{a_2^- - a_2^+}{2}, \quad p_1 = \frac{a_1^- - a_1^+}{2}, \quad p_0 = \frac{a_0^- - a_0^+}{2}. \end{aligned}$$

Then, taking into account that $g_0 = 0$, we write the expression for $u(h\beta)$ as follows

$$u(h\beta) = \begin{cases} \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + a_2^-(h\beta)^2 + a_1^-(h\beta) + a_0^-, & \beta < 0, \\ f_3(h\beta), & 0 \leq \beta \leq N, \\ -\frac{g_1}{48}(h\beta)^4 + \frac{g_2}{24}(h\beta)^3 + a_2^+(h\beta)^2 + a_1^+(h\beta) + a_0^+, & \beta > N. \end{cases} \quad (4.10)$$

Here, $a_2^-, a_1^-, a_0^-, a_2^+, a_1^+, a_0^+$ are unknown quantities. If these unknowns are determined, the general form of $u(h\beta)$ will be established, and the optimal coefficients C_β , $\beta = 0, 1, \dots, N$ can be found using formula (4.9). Now let's find these unknowns.

First, we will determine a_0^- and a_0^+ from expression (4.10).

At $\beta = 0$,

$$a_0^- = f_3(0),$$

and at $\beta = N$,

$$a_0^+ = f_3(hN) + \frac{g_1}{48}(hN)^4 - \frac{g_2}{24}(hN)^3 - a_2^+(hN)^2 - a_1^+(hN)$$

we determine that it is equal to. Then $u(h\beta)$ takes the following form

$$u(h\beta) = \begin{cases} \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + a_2^-(h\beta)^2 + a_1^-(h\beta) + f_3(0), & \beta < 0, \\ f_3(h\beta), & 0 \leq \beta \leq N, \\ -\frac{g_1}{48}((hN)^4 - (h\beta)^4) + \frac{g_2}{24}((h\beta)^3 - (hN)^3) + a_2^+((h\beta)^2 - (hN)^2) + \\ + a_1^+(h\beta - hN) + f_3(hN), & \beta > N. \end{cases} \quad (4.11)$$

Now we will determine the remaining unknowns $a_2^-, a_1^-, a_2^+, a_1^+$. According to our assumption, since $C_\beta = 0$ when $\beta < 0$ and $\beta > N$, therefore based on (4.9)

$$C_\beta = hD_3(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0, 2\pi]$$

$$D_3(h\beta) * u(h\beta) = 0,$$

$$\sum_{\gamma=-\infty}^{\infty} D_3(h\beta - h\gamma)u(h\gamma) = 0,$$

Taking into account expression (4.11), we can rewrite it as follows

$$\begin{aligned} \sum_{\gamma=-\infty}^{-1} D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=N+1}^{\infty} D_3(h\beta - h\gamma)u(h\gamma) = 0, \\ \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma)u(-h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N))u(h(\gamma + N)) = 0, \end{aligned}$$

By substituting the function $u(h\beta)$ into the last expression, we obtain an equation with respect to the unknowns $a_2^-, a_1^-, a_2^+, a_1^+$

$$\begin{aligned} & a_2^- \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma)(h\gamma)^2 - a_1^- \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma)(h\gamma) + \\ & + a_2^+ \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N))h^2(\gamma^2 + 2N\gamma) + a_1^+ \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N))(h\gamma) = \\ & = - \sum_{\gamma=0}^N D_3(h\beta - h\gamma)f_3(h\gamma) - \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma) \left(\frac{g_1}{48}(h\gamma)^4 + \frac{g_2}{24}(h\gamma)^3 + f_3(0) \right) - \\ & - \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N)) \left(-\frac{g_1}{48}(h\gamma)^4 + \left(\frac{g_2 h^3}{24} - \frac{g_1 h^4 N}{12} \right) \gamma^3 + \left(\frac{g_2 h^3 N}{8} - \frac{g_1 h^4 N^2}{8} \right) \gamma^2 + \right. \\ & \quad \left. + \left(\frac{g_2 h^3 N^2}{8} - \frac{g_1 h^4 N^3}{12} \right) \gamma + f_3(hN) \right); \end{aligned} \quad (4.12)$$

To find the unknowns $a_2^-, a_1^-, a_2^+, a_1^+$, we calculate equation (4.12) for $\beta = -1, \beta = -2, \beta = N + 1$, and $\beta = N + 2$, thus obtaining the following system of linear equations

$$\begin{cases} A_{11}a_2^- + A_{12}a_1^- + A_{13}a_2^+ + A_{14}a_1^+ = B_1 \\ A_{21}a_2^- + A_{22}a_1^- + A_{23}a_2^+ + A_{24}a_1^+ = B_2 \\ A_{31}a_2^- + A_{32}a_1^- + A_{33}a_2^+ + A_{34}a_1^+ = B_3 \\ A_{41}a_2^- + A_{42}a_1^- + A_{43}a_2^+ + A_{44}a_1^+ = B_4 \end{cases} \quad (4.13)$$

where

$$A_{11} = \frac{5!}{h^4} \left(-28 + \sum_{i=1}^2 \frac{K_i D_{i2}}{q_i^2} \right);$$

$$\begin{aligned}
A_{12} &= -\frac{5!}{h^5} \left(-30 + \sum_{i=1}^2 \frac{K_i D_{i1}}{q_i^2} \right); \\
A_{13} &= \frac{5!}{h^4} \left(\sum_{i=1}^2 K_i q_i^N (D_{i2} + 2ND_{i1}) \right); \\
A_{14} &= \frac{5!}{h^5} \left(\sum_{i=1}^2 K_i D_{i1} q_i^N \right); \\
A_{21} &= \frac{5!}{h^4} \left(-118 + \sum_{i=1}^2 K_i \left(1 - \frac{1}{q_i^2} + \frac{D_{i2}}{q_i^3} \right) \right); \\
A_{22} &= -\frac{5!}{h^5} \left(-60 + \sum_{i=1}^2 K_i \left(1 - \frac{1}{q_i^2} + \frac{D_{i1}}{q_i^3} \right) \right); \\
A_{23} &= -\frac{5!}{h^5} \left(\sum_{i=1}^2 K_i q_i^{N+1} (D_{i2} + 2ND_{i1}) \right); \\
A_{24} &= \frac{5!}{h^5} \left(\sum_{i=1}^2 K_i D_{i1} q_i^{N+1} \right); \\
A_{31} &= \frac{5!}{h^4} \left(\sum_{i=1}^2 K_i D_{i2} q_i^N \right); \\
A_{32} &= -\frac{5!}{h^5} \left(\sum_{i=1}^2 K_i D_{i1} q_i^N \right); \\
A_{33} &= \frac{5!}{h^4} \left(-28 - 60N + \sum_{i=1}^2 \frac{K_i}{q_i^2} (D_{i2} + 2ND_{i1}) \right); \\
A_{34} &= \frac{5!}{h^5} \left(-30 + \sum_{i=1}^2 \frac{K_i D_{i1}}{q_i^2} \right); \\
A_{41} &= \frac{5!}{h^4} \left(\sum_{i=1}^2 K_i D_{i2} q_i^{N+1} \right); \\
A_{42} &= -\frac{5!}{h^5} \left(\sum_{i=1}^2 K_i D_{i1} q_i^{N+1} \right); \\
A_{43} &= \frac{5!}{h^4} \left(-118 - 120N + \sum_{i=1}^2 K_i \left(\left(1 - \frac{1}{q_i^2} \right) (1 + 2N) + \frac{D_{i2} + 2ND_{i1}}{q_i^3} \right) \right); \\
A_{44} &= \frac{5!}{h^5} \left(-60 + \sum_{i=1}^2 K_i \left(1 - \frac{1}{q_i^2} + \frac{D_{i1}}{q_i^3} \right) \right); \\
B_1 &= -\frac{5!}{h^6} \left(-\frac{g_1 h^4}{3} - g_2 h^3 - 30f_3(0) + \sum_{i=1}^2 K_i \left(F_i + \frac{1}{q_i^2} \left(\frac{D_{i4} g_1 h^4}{48} + \frac{D_{i3} g_2 h^3}{24} + D_{i0} f_3(0) \right) + q_i^N \left(-\frac{D_{i4} g_1 h^4}{48} + \right. \right. \right. \\
&\quad \left. \left. + D_{i3} h^3 \left(\frac{g_2}{24} - \frac{g_1 h N}{12} \right) + D_{i2} h^2 \left(\frac{g_2 h N - g_1 h^2 N^2}{8} \right) + D_{i1} h \left(\frac{g_2 h^2 N^2}{8} - \frac{g_1 h^3 N^3}{12} \right) + D_{i0} f_3(hN) \right) \right) \right); \\
B_2 &= -\frac{5!}{h^6} \left(-\frac{215g_1 h^4}{24} - \frac{19g_2 h^3}{2} - 30f_3(0) + \sum_{i=1}^2 K_i \left(F_i q_i + \left(1 - \frac{1}{q_i^2} \right) \left(\frac{g_1 h^4}{48} + \frac{g_2 h^3}{24} + f_3(0) \right) + \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{q_i^3} \left(\frac{D_{i4}g_1h^4}{48} + \frac{D_{i3}g_2h^3}{24} + D_{i0}f_3(0) \right) + q_i^{N+1} \left(-\frac{D_{i4}g_1h^4}{48} + D_{i3}h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) + \right. \\
& \quad \left. + D_{i2}h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) + D_{i1}h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) + D_{i0}f_3(hN) \right) \Bigg); \\
B_3 = & -\frac{5!}{h^6} \left(\frac{g_1h^4}{3} - 24h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) - 28h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) - 30h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) - 30f_3(hN) + \right. \\
& + \sum_{i=1}^2 K_i \left(\sum_{\gamma=0}^N q_i^{N-\gamma} f_3(h\gamma) + q_i^N \left(\frac{D_{i4}g_1h^4}{48} + \frac{D_{i3}g_2h^3}{24} + D_{i0}f_3(0) \right) + \frac{1}{q_i^2} \left(-\frac{D_{i4}g_1h^4}{48} + \right. \\
& \quad \left. + D_{i3}h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) + D_{i2}h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) + D_{i1}h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) + D_{i0}f_3(hN) \right) \Bigg); \\
B_4 = & -\frac{5!}{h^6} \left(\frac{215g_1h^4}{24} - 228h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) - 118h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) - 60h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) - \right. \\
& - 30f_3(hN) + \sum_{i=1}^2 K_i \left(\sum_{\gamma=0}^N q_i^{N+1-\gamma} f_3(h\gamma) + q_i^{N+1} \left(\frac{D_{i4}g_1h^4}{48} + \frac{D_{i3}g_2h^3}{24} + D_{i0}f_3(0) \right) + \left(1 - \frac{1}{q_i^2} \right) \left(-\frac{g_1h^4}{48} + \right. \\
& \quad \left. + h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) + h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) + h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) + f_3(hN) \right) + \frac{1}{q_i^3} \left(-\frac{D_{i4}g_1h^4}{48} + \right. \\
& \quad \left. + D_{i3}h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) + D_{i2}h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} \right) + D_{i1}h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} \right) + D_{i0}f_3(hN) \right) \Bigg);
\end{aligned}$$

here

$$\begin{aligned}
D_{10} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma}, & D_{11} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma}\gamma, & D_{12} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma}\gamma^2, & D_{13} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma}\gamma^3, & D_{14} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma}\gamma^4, \\
D_{20} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma}, & D_{21} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma}\gamma, & D_{22} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma}\gamma^2, & D_{23} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma}\gamma^3, & D_{24} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma}\gamma^4, \\
F_1 &= \sum_{\gamma=0}^N q_1^{\gamma} f_3(\gamma), & F_2 &= \sum_{\gamma=0}^N q_2^{\gamma} f_3(\gamma).
\end{aligned}$$

We solve the system of linear equations (4.13) using Cramer's method to determine the unknowns $a_2^-, a_1^-, a_2^+, a_1^+$.

Thus, the general form of the function $u(h\beta)$ has been determined.

5. OPTIMAL COEFFICIENTS OF THE QUADRATURE FORMULA

In this section, we present the main result for the quadrature formula (2.1).

Theorem 5.1. *The optimal coefficients of the quadrature formula (2.1), with nodes equidistantly spaced in the space $L_2^{(3)}(0, 2\pi)$, have the following form*

$$\begin{aligned}
C_0 = & \frac{5!}{h^5} \left[\frac{g_1h^4}{48} + \frac{g_2h^3}{24} + a_2^- h^2 - a_1^- h - 31 \cdot f_3(0) + f_3(1) + \sum_{n=1}^2 \frac{K_n F_n}{q_n} + \right. \\
& + \sum_{n=1}^2 \frac{K_n}{q_n} \left(\frac{D_{n4}g_1h^4}{48} + \frac{D_{n3}g_2h^3}{24} + D_{n2}a_2^- h^2 - D_{n1}a_1^- h + D_{n0}f_3(0) \right) + \\
& + \sum_{n=1}^2 K_n q_n^{N-1} \left(-\frac{D_{n4}g_1h^4}{48} + D_{n3}h^3 \left(\frac{g_2}{24} - \frac{g_1hN}{12} \right) + D_{n2}h^2 \left(\frac{g_2hN}{8} - \frac{g_1h^2N^2}{8} + a_2^+ \right) + \right. \\
& \left. + D_{n1}h \left(\frac{g_2h^2N^2}{8} - \frac{g_1h^3N^3}{12} + 2a_2^+ hN + a_1^+ \right) + D_{n0}f_3(N) \right) \Bigg];
\end{aligned}$$

$$\begin{aligned}
C_\beta &= \frac{5!}{h^5} \left[\sum_{n=1}^2 K_n q_n^{\beta-1} \left(\frac{D_{n4} g_1 h^4}{48} + \frac{D_{n3} g_2 h^3}{24} + D_{n2} a_2^- h^2 - D_{n1} a_1^- h + D_{n0} f_3(0) \right) + \right. \\
&+ \sum_{\gamma=0}^N \sum_{n=1}^2 K_n q_n^{|\beta-\gamma|-1} f_3(\gamma) + f_3(\beta-1) - 32 f_3(\beta) + f_3(\beta+1) + \\
&+ \sum_{n=1}^2 K_n q_n^{N-\beta-1} \left(-\frac{D_{n4} g_1 h^4}{48} + D_{n3} h^3 \left(\frac{g_2}{24} - \frac{g_1 h N}{12} \right) + D_{n2} h^2 \left(\frac{g_2 h N}{8} - \frac{g_1 h^2 N^2}{8} + a_2^+ \right) + \right. \\
&+ \left. D_{n1} h \left(\frac{g_2 h^2 N^2}{8} - \frac{g_1 h^3 N^3}{12} + 2a_2^+ h N + a_1^+ \right) + D_{n0} f_3(N) \right) \Big], \quad \beta = 1, 2, \dots, N-1; \\
C_N &= \frac{5!}{h^5} \left[-\frac{g_1 h^4}{48} + \frac{g_2 h^3}{24} - \frac{g_1 h^4 N}{12} + \frac{g_2 h^3 N}{8} - \frac{g_1 h^4 N^2}{8} + \frac{g_2 h^3 N^2}{8} - \right. \\
&- \frac{g_1 h^4 N^3}{12} + 2a_2^+ h^2 N + a_2^+ h^2 + a_1 h - 31 \cdot f_3(N) + f_3(N-1) + \\
&+ \sum_{n=1}^2 K_n q_n^{N-1} \left(\frac{D_{n4} g_1 h^4}{48} + \frac{D_{n3} g_2 h^3}{24} + D_{n2} a_2^- h^2 - D_{n1} a_1^- h + D_{n0} f_3(0) \right) + \\
&+ \sum_{n=1}^2 \sum_{\gamma=0}^N K_n q_n^{N-\gamma-1} f_3(\gamma) + \sum_{n=1}^2 \frac{K_n}{q_n} \left(-\frac{D_{n4} g_1 h^4}{48} + D_{n3} h^3 \left(\frac{g_2}{24} - \frac{g_1 h N}{12} \right) + \right. \\
&+ \left. D_{n2} h^2 \left(\frac{g_2 h N}{8} - \frac{g_1 h^2 N^2}{8} + a_2^+ \right) + D_{n1} h \left(\frac{g_2 h^2 N^2}{8} - \frac{g_1 h^3 N^3}{12} + 2a_2^+ h N + a_1^+ \right) + D_{n0} f_3(N) \right) \Big];
\end{aligned}$$

Proof: To prove the theorem, we use the expression (4.9). For this purpose, the discrete operator $D_3(h\beta)$ is defined by formula (4.5), while the function $u(h\beta)$ is determined using expression (4.11). Thus,

$$\begin{aligned}
C_\beta &= h D_3(h\beta) * u(h\beta) = h \sum_{\gamma=-\infty}^{\infty} D_3(h\beta - h\gamma) u(h\gamma) = \\
&= h \left(\sum_{\gamma=-\infty}^{-1} D_3(h\beta - h\gamma) u(h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma) u(h\gamma) + \sum_{\gamma=N+1}^{\infty} D_3(h\beta - h\gamma) u(h\gamma) \right) = \\
&= h \left(\sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma) u(-h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma) u(h\gamma) + \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N)) u(h(\gamma + N)) \right) = \\
&= h \left(\sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma) \left(\frac{g_1 h^4}{48} \gamma^4 + \frac{g_2 h^3}{24} \gamma^3 + a_2^- h^2 \gamma^2 - a_1^- h \gamma + f_3(0) \right) + \right. \\
&+ \sum_{\gamma=0}^N D_3(h\beta - h\gamma) f_3(\gamma) + \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N)) \left(-\frac{g_1 h^4}{48} \gamma^4 + \left(\frac{g_2 h^3}{24} - \frac{g_1 h^4 N}{12} \right) \gamma^3 + \right. \\
&+ \left. \left(\frac{g_2 h^3 N}{8} - \frac{g_1 h^4 N^2}{8} + a_2^+ h^2 \right) \gamma^2 + \left(\frac{g_2 h^3 N^2}{8} - \frac{g_1 h^4 N^3}{12} + 2a_2^+ h^2 N + a_1^+ h \right) \gamma + f_3(N) \right) \Big).
\end{aligned}$$

From the final expression, we obtain the coefficient C_0 when $\beta = 0$, the coefficients C_β when $\beta = 1, 2, \dots, N-1$, and the coefficient C_N when $\beta = N$.

The theorem is thus proven. $\square$

Problem 2 has been solved.

6. CONCLUSION

In this article, an optimal quadrature formula in the space $L_2^{(3)}(0, 2\pi)$ was developed using the Sobolev method for high-precision approximate calculation of singular integral equations with a Hilbert kernel, and an analytical expression for its optimal coefficients was derived.

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The Goursat Problem for a Higher-Order Hyperbolic Equation with a Spectral Parameter

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Abstract. In this article, the Goursat problem for higher-order hyperbolic equations with a spectral parameter has been formulated and studied. The main objective of the article is to find an exact analytical solution to this problem based on the Riemann function. The study considers a higher-order hyperbolic equation defined in a domain bounded by characteristic lines. The conditions of the Goursat problem imply that the function and its higher-order normal derivatives are given on these characteristic lines. To find the solution, the method involves transitioning to characteristic coordinates and determining the corresponding Riemann function. This function is expressed using a special function of the Mittag-Leffler. By applying Green's formula and taking the boundary conditions into account, the solution to the Goursat problem is found in an explicit form at any point and then transformed back to the original variables. The article proves that if the boundary conditions are given by sufficiently smooth functions, the problem has a unique solution. The main novelty lies in the fact that this type of problem is solved for the first time using the Riemann method, and the solution is explicitly expressed.

Keywords: Higher-order hyperbolic equation, Goursat problem, Riemann function, Green's formula, the Four-Parametric Mittag-Leffler function, spectral parameter

MSC (2020): 35L25

1. INTRODUCTION

Higher-order linear hyperbolic partial differential equations and problems for these equations are one of the fundamental problems of modern mathematical physics and are widely used in areas such as wave propagation, the theory of elasticity, electromagnetic dynamics, and quantum mechanics.

A. V. Bitsadze [1] pointed out the importance of studying higher-order equations of the following form:

$$L^m u = 0, m = 1, 2, \dots, \quad (1.1)$$

where, L is a second-order linear differential operator, and $L^m = L^{m-1}L$ is the m th composition of this operator.

In this direction, the work [2] can be mentioned, where equation (1.1) was studied when the Lavrentyev-Bitsadze operator is of type $m \geq 1$ and $L \equiv \partial^2/\partial x^2 + \operatorname{sign} x (\partial^2/\partial y^2)$. For $m = 2$, various problems for the equation (1.1) with the Lavrentyev-Bitsadze operator have been studied in the works [3] and [4].

An analogue of the Cauchy problem for a differential equation separable into wave multipliers was studied in [5]. In [6], the Cauchy problem for equation (1.1), where L is a wave operator, was solved using the Riesz method, with the solution obtained in the form of a sum of single- and double-layer multidimensional hyperbolic potentials.

In [7], the Cauchy problem for the higher-order equation (1.1) with one and several variables was reduced to a single-variable equation and solved using the method of spherical mean values, where L is a wave operator. Based on the exact solution obtained in this work, exact solutions of analogues of the Cauchy problem for multidimensional polywave equations involving the singular Bessel differential operator with respect to one or more variables were constructed in [8, 9, 10, 11] using the method of transmutation operators. In [12], the construction of the Riemann function for equations involving the singular Bessel differential operator with respect to one or more variables was presented. The monograph [13] provides a review of the literature on this problem.

In this article, we find the solution of the Goursat problem using the Riemann's method. This type of problem has not been previously studied by using the Riemann's method.

2. MAIN PART

Let us denote by Ω the domain of the plane (x, y) bounded by the lines $y = 0$, $x + y = 0$, $x - y = 1$, and consider the following equation in this domain:

$$L_1(u) \equiv \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} \right)^m u(x, y) - \lambda u(x, y) = f(x, y), \quad m \in N, \quad (2.1)$$

where, $u(x, y)$ is the unknown function, $\lambda \in R$ and $f(x, y)$ is a given continuous function.

Problem G. Find a function $u(x, y)$ that belongs to class $C^{2m}(\Omega) \cap C^{2m-1}(\bar{\Omega})$, satisfying the equation (2.1) and the following boundary conditions

$$\begin{aligned} u|_{OK} = \varphi_0(x), \quad \frac{\partial u}{\partial n_1} \Big|_{OK} = \varphi_1(x), \quad \frac{\partial^2 u}{\partial n_1^2} \Big|_{OK} = \varphi_2(x), \quad \dots, \quad \frac{\partial^{m-1} u}{\partial n_1^{m-1}} \Big|_{OK} = \varphi_{m-1}(x), \\ u|_{KN} = \psi_0(x), \quad \frac{\partial u}{\partial n_2} \Big|_{KN} = \psi_1(x), \quad \frac{\partial^2 u}{\partial n_2^2} \Big|_{KN} = \psi_2(x), \quad \dots, \quad \frac{\partial^{m-1} u}{\partial n_2^{m-1}} \Big|_{KN} = \psi_{m-1}(x), \end{aligned} \quad (2.2)$$

where

$$OK = \left\{ (x, y) : x + y = 0, 0 \leq x \leq \frac{1}{2} \right\}, KN = \left\{ (x, y) : x - y = 1, \frac{1}{2} \leq x \leq 1 \right\},$$

(Ω_0 is a domain contained within the interior of domain Ω , consisting of a rectangle $CEKF$, $C(x_0, y_0)$ is an arbitrary point of the inside of the domain Ω , and the segments CE and CF are located parallel to the characteristic lines.)

Here, $\varphi_0(x)$, $\psi_0(x)$, $\varphi_1(x)$, $\psi_1(x)$, $\varphi_2(x)$, $\psi_2(x)$, $\dots$, $\varphi_{m-1}(x)$, $\psi_{m-1}(x)$ are given sufficiently smooth functions such that

$$\begin{aligned} \varphi_0\left(\frac{1}{2}\right) = \psi_0\left(\frac{1}{2}\right), \quad -\varphi_1\left(\frac{1}{2}\right) = \psi_1\left(\frac{1}{2}\right), \quad \varphi_2\left(\frac{1}{2}\right) = \psi_2\left(\frac{1}{2}\right), \quad \dots, \\ (-1)^{m-1} \varphi_{m-1}\left(\frac{1}{2}\right) = \psi_{m-1}\left(\frac{1}{2}\right). \end{aligned} \quad (2.3)$$

n_1 and n_2 are the normals drawn to the characteristic lines, and

$$\begin{aligned} \frac{\partial u}{\partial n_1} \Big|_{OK} = \frac{\partial u}{\partial x} \cos \alpha_1 + \frac{\partial u}{\partial y} \cos \beta_1, \quad \frac{\partial^2 u}{\partial n_1^2} \Big|_{OK} = \frac{\partial^2 u}{\partial x^2} \cos^2 \alpha_1 + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \alpha_1 \cos \beta_1 + \\ + \frac{\partial^2 u}{\partial y^2} \cos^2 \beta_1, \quad \dots, \quad \frac{\partial^{m-1} u}{\partial n_1^{m-1}} \Big|_{OK} = \frac{\partial^{m-1} u}{\partial x^{m-1}} \cos^{m-1} \alpha_1 + 2 \frac{\partial^{m-1} u}{\partial x \partial y} \cos^{m-2} \alpha_1 \cos \beta_1 + \dots + \\ + \frac{\partial^{m-1} u}{\partial y^{m-1}} \cos^{m-1} \beta_1, \\ \frac{\partial u}{\partial n_2} \Big|_{NK} = \frac{\partial u}{\partial x} \cos \alpha_2 + \frac{\partial u}{\partial y} \cos \beta_2, \quad \frac{\partial^2 u}{\partial n_2^2} \Big|_{NK} = \frac{\partial^2 u}{\partial x^2} \cos^2 \alpha_2 + 2 \frac{\partial^2 u}{\partial x \partial y} \cos \alpha_2 \cos \beta_2 + \\ + \frac{\partial^2 u}{\partial y^2} \cos^2 \beta_2, \quad \dots, \quad \frac{\partial^{m-1} u}{\partial n_2^{m-1}} \Big|_{NK} = \frac{\partial^{m-1} u}{\partial x^{m-1}} \cos^{m-1} \alpha_2 + 2 \frac{\partial^{m-1} u}{\partial x \partial y} \cos^{m-2} \alpha_2 \cos \beta_2 + \dots + \\ + \frac{\partial^{m-1} u}{\partial y^{m-1}} \cos^{m-1} \beta_2, \end{aligned}$$

where $\alpha_1 = n_1 \wedge x = \frac{5\pi}{4}$, $\beta_1 = n_1 \wedge y = \frac{3\pi}{4}$, $\alpha_2 = n_2 \wedge x = \frac{7\pi}{4}$, $\beta_2 = n_2 \wedge y = \frac{5\pi}{4}$.

Let us switch to characteristic coordinates by $u(x, y) = w(\xi, \eta)$, $\xi = x + y$, $\eta = x - y$. Let us denote by Ω' the domain bounded by lines $\xi = \eta$, $\xi = 0$ and $\eta = 1$. Then equation (2.1), the boundary conditions (2.2), and the conditions (2.3) take the following forms, respectively.

$$\frac{\partial^{2m}}{\partial \xi^m \partial \eta^m} w(\xi, \eta) - \frac{1}{4^m} \lambda w(\xi, \eta) = \frac{1}{4^m} f\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right), \quad (2.4)$$

$$w|_{\xi=0} = \varphi_0\left(\frac{\eta}{2}\right), \quad \frac{\partial w}{\partial \xi}\bigg|_{\xi=0} = -\frac{\sqrt{2}}{2} \varphi_1\left(\frac{\eta}{2}\right), \quad \frac{\partial^2 w}{\partial \xi^2}\bigg|_{\xi=0} = \frac{1}{2} \varphi_2\left(\frac{\eta}{2}\right), \quad \dots,$$

$$\frac{\partial^{m-1} w}{\partial \xi^{m-1}}\bigg|_{\xi=0} = \left(-\frac{\sqrt{2}}{2}\right)^{m-1} \varphi_{m-1}\left(\frac{\eta}{2}\right), \quad w|_{\eta=1} = \psi_0\left(\frac{\xi+1}{2}\right), \quad \frac{\partial w}{\partial \eta}\bigg|_{\eta=1} = \frac{\sqrt{2}}{2} \psi_1\left(\frac{\xi+1}{2}\right),$$

$$\frac{\partial^2 w}{\partial \eta^2}\bigg|_{\eta=1} = \frac{1}{2} \psi_2\left(\frac{\xi+1}{2}\right), \quad \dots, \quad \frac{\partial^{m-1} w}{\partial \eta^{m-1}}\bigg|_{\eta=1} = \left(\frac{\sqrt{2}}{2}\right)^{m-1} \psi_{m-1}\left(\frac{\xi+1}{2}\right), \quad (2.5)$$

$$\varphi_0(1) = \psi_0(0), \quad -\varphi_1(1) = \psi_1(0), \quad \varphi_2(1) = \psi_2(0), \quad \dots, \quad (-1)^{m-1} \varphi_{m-1}(1) = \psi_{m-1}(0).$$

Let us denote the left-hand side of the equation (2.4) by $M(w)$, that is

$$M(w) = \frac{\partial^{2m}}{\partial \xi^m \partial \eta^m} w - \frac{1}{4^m} \lambda w.$$

Now, let us introduce the operator $M^*(v)$, referred to as the adjoint of this operator:

$$M^*(v) = \frac{\partial^{2m} v}{\partial \xi^m \partial \eta^m} - \frac{1}{4^m} \lambda v.$$

Accordingly, one can write the following equality

$$vM(w) - wM^*(v) = \frac{\partial P}{\partial \xi} + \frac{\partial Q}{\partial \eta}, \quad (2.6)$$

where, $P = \sum_{k=1}^m \frac{\partial^{2k-2} v}{\partial \xi^{k-1} \partial \eta^{k-1}} \times \frac{\partial^{2m-2k+1} w}{\partial \xi^{m-k} \partial \eta^{m+1-k}}, \quad Q = -\sum_{k=1}^m \frac{\partial^{2k-1} v}{\partial \xi^k \partial \eta^{k-1}} \times \frac{\partial^{2m-2k} w}{\partial \xi^{m-k} \partial \eta^{m-k}},$ $v = v(\xi_0, \eta_0; \xi, \eta)$ is the Riemann function and satisfies the following conditions:

$$\frac{\partial^{2m}}{\partial \xi^m \partial \eta^m} v(\xi, \eta; \xi_0, \eta_0) - \frac{1}{4^m} \lambda v(\xi, \eta; \xi_0, \eta_0) = 0, \quad (2.7)$$

$$v|_{\xi=\xi_0} = 0, \quad \frac{\partial v}{\partial \xi}\bigg|_{\xi=\xi_0} = 0, \quad \frac{\partial^2 v}{\partial \xi^2}\bigg|_{\xi=\xi_0} = 0, \dots, \quad \frac{\partial^{m-2} v}{\partial \xi^{m-2}}\bigg|_{\xi=\xi_0} = 0,$$

$$\frac{\partial^{m-1} v}{\partial \xi^{m-1}}\bigg|_{\xi=\xi_0} = \frac{(\eta - \eta_0)^{m-1}}{(m-1)!}, \quad (2.8)$$

$$v|_{\eta=\eta_0} = 0, \quad \frac{\partial v}{\partial \eta}\bigg|_{\eta=\eta_0} = 0, \quad \frac{\partial^2 v}{\partial \eta^2}\bigg|_{\eta=\eta_0} = 0, \dots, \quad \frac{\partial^{m-2} v}{\partial \eta^{m-2}}\bigg|_{\eta=\eta_0} = 0,$$

$$\frac{\partial^{m-1} v}{\partial \eta^{m-1}}\bigg|_{\eta=\eta_0} = \frac{(\xi - \xi_0)^{m-1}}{(m-1)!}. \quad (2.9)$$

Integrating (2.6) over the domain $\Omega'_0 = \{(\xi, \eta); 0 < \xi < \xi_0, \eta_0 < \eta < 1\}$, and applying Green's formula, we obtain

$$\iint_{\Omega'_0} [vM(w) - wM^*(v)] d\xi d\eta = \iint_{\Omega'_0} \left(\frac{\partial P}{\partial \xi} + \frac{\partial Q}{\partial \eta} \right) d\xi d\eta = \int_{\partial \Omega'_0} -Q d\xi + P d\eta =$$

$$= \int_{C' E' K' F'} -Q d\xi + P d\eta = 0. \quad (2.10)$$

In the equation (2.10), the boundary of the integral on the right-hand side consists of four segments; therefore, we evaluate it separately on each segment. Here, $C' E' K' F'$ represents a rectangle and is equal to the boundary of Ω'_0 .

a) On $C' F'$, $\xi = \xi_0$, $d\xi = 0$, $\eta_0 \leq \eta \leq 1$, therefore

$$\begin{aligned} \int_{C' F'} (-Q) d\xi + P d\eta &= \int_{C' F'} P d\eta = \int_{\eta_0}^1 \left[\sum_{k=1}^m \frac{\partial^{2k-2} v}{\partial \xi^{k-1} \partial \eta^{k-1}} \times \frac{\partial^{2m-2k+1} w}{\partial \xi^{m-k} \partial \eta^{m+1-k}} \right]_{\xi=\xi_0} d\eta = \\ &= \psi_0 \left(\frac{\xi_0 + 1}{2} \right) - w(\xi_0, \eta_0). \end{aligned}$$

b) On $F' K'$, $\eta = 1$, $d\eta = 0$, $0 \leq \xi \leq \xi_0$, so

$$\begin{aligned} \int_{F' K'} (-Q) d\xi + P d\eta &= \int_{F' K'} (-Q) d\xi = - \int_0^{\xi_0} \left[\sum_{k=1}^m \frac{\partial^{2k-1} v}{\partial \xi^k \partial \eta^{k-1}} \times \frac{\partial^{2m-2k} w}{\partial \xi^{m-k} \partial \eta^{m-k}} \right]_{\eta=1} d\xi = \\ &= - \int_0^{\xi_0} \left[\sum_{k=1}^m \left(\frac{\sqrt{2}}{2} \right)^{m-k} \left[\frac{\partial^{2k-1} v}{\partial \xi^k \partial \eta^{k-1}} \right]_{\eta=1} \psi_{m-k}^{(m-k)} \left(\frac{\xi+1}{2} \right) \right] d\xi. \end{aligned}$$

c) On $K' E'$, $\xi = 0$, $d\xi = 0$, $\eta_0 \leq \eta \leq 1$, therefore

$$\begin{aligned} \int_{K' E'} (-Q) d\xi + P d\eta &= \int_{K' E'} P d\eta = - \int_{\eta_0}^1 \left[\sum_{k=1}^m \frac{\partial^{2k-2} v}{\partial \xi^{k-1} \partial \eta^{k-1}} \times \frac{\partial^{2m-2k+1} w}{\partial \xi^{m-k} \partial \eta^{m+1-k}} \right]_{\xi=0} d\eta = \\ &= - \int_{\eta_0}^1 \left[\left(-\frac{\sqrt{2}}{2} \right)^{m-k} \left[\frac{\partial^{2k-2} v}{\partial \xi^{k-1} \partial \eta^{k-1}} \right]_{\xi=0} \varphi_{m-k}^{(m+1-k)} \left(\frac{\eta}{2} \right) \right] d\eta. \end{aligned}$$

d) On $E' C'$, $\eta = \eta_0$, $d\eta = 0$, $0 \leq \xi \leq \xi_0$, so

$$\int_{E' C'} (-Q) d\xi + P d\eta = \int_{E' C'} (-Q) d\xi = \int_0^{\xi_0} \left[\sum_{k=1}^m \frac{\partial^{2k-1} v}{\partial \xi^k \partial \eta^{k-1}} \times \frac{\partial^{2m-2k} w}{\partial \xi^{m-k} \partial \eta^{m-k}} \right]_{\eta=\eta_0} d\xi = 0.$$

By substituting the obtained expressions into (2.10), we obtain the solution to the Goursat problem for the equation (2.4), determined at an arbitrary point $C'(\xi_0, \eta_0) \in \Omega'$, as follows

$$\begin{aligned} w(\xi_0, \eta_0) &= \psi_0 \left(\frac{\xi_0 + 1}{2} \right) - \int_0^{\xi_0} \left[\sum_{k=1}^m \left(\frac{\sqrt{2}}{2} \right)^{m-k} \left[\frac{\partial^{2k-1} v(\xi, \eta; \xi_0, \eta_0)}{\partial \xi^k \partial \eta^{k-1}} \right]_{\eta=1} \psi_{m-k}^{(m-k)} \left(\frac{\xi+1}{2} \right) \right] d\xi - \\ &- \int_{\eta_0}^1 \left[\sum_{k=1}^m \left(-\frac{\sqrt{2}}{2} \right)^{m-k} \left[\frac{\partial^{2k-2} v(\xi, \eta; \xi_0, \eta_0)}{\partial \xi^{k-1} \partial \eta^{k-1}} \right]_{\xi=0} \varphi_{m-k}^{(m+1-k)} \left(\frac{\eta}{2} \right) \right] d\eta + \iint_{\Omega_0} \frac{1}{4^m} v f d\xi d\eta. \end{aligned} \quad (2.11)$$

According to equation (2.7), we perform integration with respect to variable ξ over the interval from ξ_0 to ξ , and with respect to variable η over the interval from η_0 to η . Using the conditions (2.8) and (2.9), and the following equality [9],

$$\int_a^{z_0} dz_1 \int_a^{z_1} dz_2 \dots \int_a^{z_{n-1}} \theta(\tau) d\tau = \frac{1}{(n-1)!} \int_a^{z_0} (z_0 - \tau)^{n-1} \theta(\tau) d\tau, \quad n \in N,$$

we derive the integral equation with respect to $\nu(\xi, \eta)$ from equation (2.12),

$$\begin{aligned} v(\xi, \eta) - \frac{1}{4^m} \times \frac{\lambda}{(m-1)!(m-1)!} \int_{\xi_0}^{\xi} \int_{\eta_0}^{\eta} (\xi - s)^{m-1} (\eta - t)^{m-1} v(s, t) dt ds = \\ = \frac{(\xi - \xi_0)^{m-1}}{(m-1)!} \times \frac{(\eta - \eta_0)^{m-1}}{(m-1)!}. \end{aligned} \quad (2.12)$$

Since the equation (2.12) is a Volterra-type integral equation, it has a unique solution. We solve it using the method of successive approximations.

Taking $v_0(\xi, \eta) = \frac{(\xi - \xi_0)^{m-1}}{(m-1)!} \times \frac{(\eta - \eta_0)^{m-1}}{(m-1)!}$ as the initial approximation, the subsequent approximations are constructed as follows:

$$\begin{aligned} v_k = \frac{(\xi - \xi_0)^{m-1}}{(m-1)!} \times \frac{(\eta - \eta_0)^{m-1}}{(m-1)!} - \frac{1}{4^m} \times \frac{\lambda}{(m-1)!(m-1)!} \times \\ \times \int_{\xi_0}^{\xi} \int_{\eta_0}^{\eta} (\xi - s)^{m-1} (\eta - t)^{m-1} v_{k-1}(s, t) dt ds. \end{aligned}$$

One can easily show that from the last equation the functions ν_k can be found as follows:

$$\begin{aligned} v_0 &= \frac{(\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1}}{\Gamma^2(m)}, \\ v_1 &= \frac{(\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1}}{\Gamma^2(m)} + \frac{\lambda}{4^m} \times \frac{(\xi - \xi_0)^{2m-1} (\eta - \eta_0)^{2m-1}}{\Gamma^2(2m)}, \\ v_2 &= \frac{(\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1}}{\Gamma^2(m)} + \frac{\lambda}{4^m} \times \frac{(\xi - \xi_0)^{2m-1} (\eta - \eta_0)^{2m-1}}{\Gamma^2(2m)} + \\ &+ \left(\frac{\lambda}{4^m} \right)^2 \times \frac{(\xi - \xi_0)^{3m-1} (\eta - \eta_0)^{3m-1}}{\Gamma^2(3m)}, \\ &\dots\dots\dots \end{aligned}$$

Continuing this process, for an arbitrary value of $n \in N$, we have

$$v_n = (\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1} \sum_{k=0}^n \left(\frac{\lambda}{4^m} \right)^k \frac{[(\xi - \xi_0) (\eta - \eta_0)]^{kn}}{\Gamma(mn + m) \Gamma(mn + m)}.$$

At this point, taking the limit as $n \rightarrow \infty$, and considering this equality [14],

$$E_{\alpha_1, \beta_1; \alpha_2, \beta_2}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha_1 n + \beta_1) \Gamma(\alpha_2 n + \beta_2)}$$

$\alpha_1, \alpha_2 \in R(\alpha_1^2 + \alpha_2^2 \neq 0)$ and $\beta_1, \beta_2 \in C$, we obtain

$$v = \lim_{n \rightarrow \infty} v_n = (\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1} E_{m, m; m, m} \left[\frac{\lambda}{4^m} ((\xi - \xi_0) (\eta - \eta_0))^m \right].$$

Thus, the Riemann function for equation (2.5) has the form

$$v(\xi, \eta; \xi_0, \eta_0) = (\xi - \xi_0)^{m-1} (\eta - \eta_0)^{m-1} E_{m,m;m,m} \left[\frac{\lambda}{4^m} ((\xi - \xi_0)(\eta - \eta_0))^m \right].$$

One can verify directly that this function indeed satisfies the conditions (2.8) and (2.9).

Returning to the variables x, y and introducing the notations $\xi_0 = x_0 + y_0$ and $\eta_0 = x_0 - y_0$, we find the Riemann function for the problem G :

$$v(x, y; x_0, y_0) = \left[(x - x_0)^2 - (y - y_0)^2 \right]^{m-1} E_{m,m;m,m} \left[\frac{\lambda}{4^m} \left((x - x_0)^2 - (y - y_0)^2 \right)^m \right]. \quad (2.13)$$

Considering the form (2.13) of the Riemann function the solution (2.11) can be written as follows

$$\begin{aligned} u(x_0, y_0) = & \psi_0 \left(\frac{x_0 + y_0 + 1}{2} \right) - \int_0^{x_0 + y_0} \left[\sum_{k=1}^m \left(\frac{\sqrt{2}}{2} \right)^{m-k} \frac{\partial}{\partial \xi} \left[(\xi - x_0 - y_0)^{m-k} \times \right. \right. \\ & \times (1 - x_0 + y_0)^{m-k} E_{m, m-k+1; m, m-k+1} \left. \left. \left[\frac{\lambda}{4^m} ((\xi - x_0 - y_0)(1 - x_0 + y_0))^m \right] \right] \psi_{m-k}^{(m-k)} \left(\frac{\xi + 1}{2} \right) \right] d\xi - \\ & - \int_{x_0 - y_0}^1 \left[\sum_{k=1}^m \left(-\frac{\sqrt{2}}{2} \right)^{m-k} (-x_0 - y_0)^{m-k} (\eta - x_0 + y_0)^{m-k} E_{m, m-k+1; m, m-k+1} \left[\frac{\lambda}{4^m} ((-x_0 - y_0) \times \right. \right. \\ & \times (\eta - x_0 + y_0))^m \right] \varphi_{m-k}^{(m+1-k)} \left(\frac{\eta}{2} \right) d\eta - \frac{1}{4^m} \iint_{\Omega_0} \left[(x - x_0)^2 - (y - y_0)^2 \right]^{m-1} \times \\ & \times E_{m,m;m,m} \left[\frac{\lambda}{4^m} \left[(x - x_0)^2 - (y - y_0)^2 \right]^m \right] f(x, y) dx dy, \end{aligned} \quad (2.14)$$

where Ω_0 is a domain located inside the domain Ω , consisting of a rectangle $CEKF$ bounded by the lines $x + y = 0$, $x + y = y_0 - x_0$, $x - y = 1$, and $x - y = x_0 - y_0$, and the point $C(x_0, y_0)$ is arbitrarily chosen from within the domain Ω .

Theorem. *Let the conditions $\psi_j \in C^{2m}(0, 1/2)$, $\varphi_j \in C^{2m}(1/2, 1)$ be satisfied for the functions ψ_j , φ_j , $j = \overline{0, m-1}$, $\lambda \in R$ and $f(x, y) \in C(\Omega)$, then the problem G has a unique solution given by the equality (2.14).*

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Nonlocal Problem for a Mixed Type Equation with Singular Coefficient in a Domain Whose Elliptic Part Is a Half-Strip

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Abstract. This paper investigates a problem for the Gellerstedt equation with a singular coefficient in a domain whose elliptic part is a vertical half-strip. The elliptic part uses the solution of the Problem N type. By using generalized fractional differentiation operators, we specify a linear combination that relates the value of the solution on characteristics of the equation with the value of the solution and its derivative on the parabolic degeneration line. The uniqueness of the solution of the problem is proved using the extremum principle and the energy integral method. The existence of a solution is proved using the method of integral equations.

Keywords: Mixed-type equation, singular coefficient, half-strip, extremum principle, energy integral method, integral equation, uniqueness of the solution, existence of the solution

MSC (2020): 35M10

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Many problems of plane gas flow [1], at transonic speeds, when the flow contains regions with subsonic and supersonic speeds, can be mathematically formulated as boundary value problems for a mixed elliptic-hyperbolic linear equation. For the mixed type equation $yu_{xx} + u_{yy} = 0$, the first fundamental studies were carried out by F. Tricomi [2]. In [3], the Tricomi problem was considered for the generalized Tricomi equation in an unbounded domain whose elliptic part is the upper half-plane and whose hyperbolic part is the characteristic triangle.

Several fundamental problems in transonic aerodynamics lead to the Tricomi boundary value problem, where the domain in the elliptic part of the plane represents a half-strip. The Tricomi problem for the equation $(\operatorname{sign} y)|y|^m u_{xx} + u_{yy} = 0$, $m > 0$ in a domain where the elliptical part is a half-strip has been studied in [4, 5, 6], and for the case of $m = 1$, in [7]. The paper [8] is devoted to the study of a problem with a Frankl-type condition on the degeneracy line for a mixed-type equation in a domain whose elliptic part is the first quadrant of the plane. A study of the boundary value problem in an infinite half-strip for the generalized biaxially symmetric Helmholtz equation was conducted in [9]. Boundary value problems for mixed-type equations with singular coefficient were explored in [10, 11, 12]. The degenerate elliptic-type equation with singular coefficient and a spectral parameter was studied in the vertical half-strip in [13, 14, 15, 16] and in the upper half-plane in [17].

Consider the equation

$$(\operatorname{sgn} y)|y|^m u_{xx} + u_{yy} + \frac{\beta_0}{y} u_y = 0, \quad (1.1)$$

where $m > 0$, $-\frac{m}{2} < \beta_0 < 1$ in the infinite domain D bounded by the half-lines $x = 0$ and $x = 1$ in the half-plane $y > 0$ and the characteristics

$$OC : \quad x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0, \quad BC : \quad x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1,$$

of Eq. (1.1) issuing from the points $O(0,0)$ and $B(1,0)$.

Denote by $D_1 = D \cap \{(x, y) : y > 0\}$, $D_2 = D \cap \{(x, y) : y < 0\}$ the elliptic and hyperbolic parts of the mixed domain D , respectively, and let $J = \{(x, y) : 0 < x < 1, y = 0\}$.

Problem. Find a solution of Eq. (1.1) in the domain $D = D_1 \cup D_2$, which satisfies the following conditions:

- 1) $u(x, y) \in C(\overline{D}) \cap C^2(D_1 \cup D_2)$ and satisfies Eq. (1.1) in $D_1 \cup D_2$;
- 2) $u(x, y)$ satisfies the boundary conditions

$$u(0, y) = \varphi_1(y), \quad u(1, y) = \varphi_2(y), \quad 0 \leq y < \infty, \quad (1.2)$$

$$\begin{aligned}
& a(x)(I_{0+}^{\mu_1, \rho_1, \eta_1} \delta(t) u[\Theta_0(t)])(x) + b(x)(I_{1-}^{\mu_2, \rho_2, \eta_2} \omega(t) u[\Theta_1(t)])(x) \\
& + c(x)u(x, 0) + d(x) \lim_{y \rightarrow -0} (-y)^{\beta_0} u_y(x, y) = g(x), \quad x \in J,
\end{aligned} \tag{1.3}$$

and the conjugation condition

$$\lim_{y \rightarrow -0} (-y)^{\beta_0} u_y = \lim_{y \rightarrow +0} y^{\beta_0} u_y, \quad x \in (0, 1). \tag{1.4}$$

Moreover, these limits at $x = 0$ can have a singularity of order lower than $1 - 2\beta$, and at $x = 1$ they are bounded, where $\beta = \frac{m+2\beta_0}{2(m+2)}$.

3)

$$\lim_{y \rightarrow +\infty} u(x, y) = 0 \quad \text{uniformly in } x \in [0, 1], \tag{1.5}$$

4)

$$y^{\beta_0} u_y(x, y), u_x(x, y) \text{ are bounded as } y \rightarrow +\infty. \tag{1.6}$$

Here μ_i , ρ_i , and η_i ($i = 1, 2$) are real parameters, $\Theta_0(x)$ and $\Theta_1(x)$ are the points of intersection of the characteristics of Eq. (1.1) issuing from the point $(x, 0) \in J$ with the characteristics OC and BC , respectively. The given functions $\varphi_i(y)$, $a(x)$, $b(x)$, $c(x)$, $d(x)$, $g(x)$, $\delta(x)$, and $\omega(x)$ satisfy the conditions

$$\varphi_i(y) \in C[0, +\infty), \quad y^{\frac{3m+2\beta_0}{4}} \varphi_i(y) \in L_1(0, +\infty), \quad \lim_{y \rightarrow +\infty} \varphi_i(y) = 0, \quad (i = 1, 2), \tag{1.7}$$

$$a(x), b(x), c(x), d(x), g(x) \in C^1(\bar{J}), \quad \delta(x) \in C^1(0, 1], \quad \omega(x) \in C^1[0, 1], \tag{1.8}$$

and moreover,

$$a^2(x) + b^2(x) + c^2(x) + d^2(x) \neq 0 \quad \forall x \in \bar{J}, \tag{1.9}$$

$(I_{0+}^{\mu, \rho, \eta} f)(x)$ and $(I_{1-}^{\mu, \rho, \eta} f)(x)$ are operators of generalized fractional integro-differentiation with the hypergeometric Gauss function, which were introduced in [18] (pp. 326–327) and, for real μ , ρ , η and $x > 0$, have the form

$$\begin{aligned}
(I_{0+}^{\mu, \rho, \eta} f)(x) &= \begin{cases} \frac{x^{-\mu-\rho}}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} F\left(\mu+\rho, -\eta; \mu; 1-\frac{t}{x}\right) f(t) dt, & \mu > 0, \\ \left(\frac{d}{dx}\right)^n (I_{0+}^{\mu+n, \rho-n, \eta-n} f)(x), & \mu \leq 0, \quad n = [-\mu] + 1, \end{cases} \\
(I_{1-}^{\mu, \rho, \eta} f)(x) &= \begin{cases} \frac{(1-x)^{-\mu-\rho}}{\Gamma(\mu)} \int_x^1 (t-x)^{\mu-1} F\left(\mu+\rho, -\eta; \mu; \frac{t-x}{1-x}\right) f(t) dt, & \mu > 0, \\ \left(-\frac{d}{dx}\right)^n (I_{1-}^{\mu+n, \rho-n, \eta-n} f)(x), & \mu \leq 0, \quad n = [-\mu] + 1. \end{cases}
\end{aligned} \tag{1.10}$$

The Problem posed above is a straightforward and essential generalization of the Tricomi problem, which was considered for Eq. (1.1) with $m = 1$, $\beta_0 = 0$ in the domain D in the monograph [19]. The Problem with the conditions (1.2)-(1.3) for the Eq. (1.1) with $\beta_0 = 0$ in the domain D is studied in [20].

In this paper, we study the Problem for Eq. (1.1) with singular coefficient.

2. UNIQUENESS OF THE SOLUTION

The solution of the Cauchy problem for Eq. (1.1) in the hyperbolic domain D_2 has the form [10]

$$u(x, y) = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)} \int_0^1 \tau \left(x + \frac{2}{m+2} (-y)^{\frac{m+2}{2}} (2t-1) \right) t^{\beta-1} (1-t)^{\beta-1} dt$$

$$-\frac{2}{m+2} \frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)} (-y)^{1-\beta_0} \int_0^1 \nu \left(x + \frac{2}{(m+2)} (-y)^{\frac{m+2}{2}} (2t-1) \right) t^{-\beta} (1-t)^{-\beta} dt, \quad (2.1)$$

where

$$\tau(x) = u(x, 0), \quad \forall x \in \bar{J}, \quad \nu(x) = \lim_{y \rightarrow -0} (-y)^{\beta_0} u_y(x, y), \quad \forall x \in J.$$

Using the representation (2.1) and taking into account the properties of the operator (1.10), we obtain the relations

$$\begin{aligned} u[\Theta_0(x)] &= \gamma_1(I_{0+}^{\beta, 0, \beta-1} \tau(t))(x) - \gamma_2(I_{0+}^{1-\beta, 2\beta-1, \beta-1} \nu(t))(x), \\ u[\Theta_1(x)] &= \gamma_1(I_{1-}^{\beta, 0, \beta-1} \tau(t))(x) - \gamma_2(I_{1-}^{1-\beta, 2\beta-1, \beta-1} \nu(t))(x), \end{aligned}$$

where

$$\gamma_1 = \frac{\Gamma(2\beta)}{\Gamma(\beta)}, \quad \gamma_2 = \frac{1}{2} \left(\frac{4}{m+2} \right)^{2\beta} \frac{\Gamma(1-2\beta)}{\Gamma(1-\beta)}.$$

Theorem 2.1. *In the domain D , there exists at most one solution of problem (1.1)–(1.3) provided that either*

$$\mu_1 = \mu_2 = -\beta, \quad \rho_1 = \rho_2 = 0, \quad \eta_1 = \eta_2 = 2\beta - 1, \quad \delta(x) = \omega(x) = 1, \quad (2.2)$$

and the condition (1.6),

$$E_1(x) = \gamma_1[a(x) + b(x)] + c(x) \neq 0, \quad x \in \bar{J}, \quad (2.3)$$

$$a(1) \geq 0, \quad b(0) \geq 0, \quad \left[\frac{a(x)}{E_1(x)} \right]' \leq 0, \quad \left[\frac{b(x)}{E_1(x)} \right]' \geq 0, \quad \frac{d(x)}{E_1(x)} \leq 0, \quad x \in \bar{J}, \quad (2.4)$$

are satisfied, or

$$\mu_1 = \mu_2 = \beta - 1, \quad \rho_1 = \rho_2 = 0, \quad \eta_1 = \eta_2 = 1 - 2\beta, \quad (2.5)$$

$$\delta(x) = x^{2\beta-1}, \quad \omega(x) = (1-x)^{2\beta-1}, \quad (2.6)$$

$$E_2(x) = \gamma_2[a(x) + b(x)] - d(x) \neq 0, \quad x \in \bar{J}. \quad (2.7)$$

Proof: Let $u(x, y)$ be a solution of the Problem satisfying the homogeneous conditions (1.2) and (1.3) ($\varphi_i(y) = 0$, $g(x) = 0$), then, by integrating the identity

$$\frac{\partial}{\partial x} (y^{m+\beta_0} u u_x) + \frac{\partial}{\partial y} (y^{\beta_0} u u_y) - y^{\beta_0} [y^m (u_x)^2 + (u_y)^2] = 0,$$

over the domain D_1 and taking into account the conditions imposed on the functions $u(x, y)$, $u_x(x, y)$ and $y^{\beta_0} u_y(x, y)$, we have

$$\iint_{D_1} y^{\beta_0} [y^m (u_x)^2 + (u_y)^2] dx dy + \int_0^1 \tau(x) \nu(x) dx = 0. \quad (2.8)$$

Let us show that if the assumptions of theorem 2.1 hold, then

$$I^* = \int_0^1 \tau(x) \nu(x) dx \geq 0$$

cannot be negative.

Indeed, substituting the functions $u[\Theta_0(x)]$ and $u[\Theta_1(x)]$ into condition (1.3), under conditions (2.2), taking into account the semigroup properties of generalized operators ([18], p. 327),

$$\left(I_{0+}^{\mu, \rho, \eta} \left(I_{0+}^{\gamma, \delta, \mu+\eta} f \right) (t) \right) (x) = \left(I_{0+}^{\mu+\gamma, \rho+\delta, \eta} f \right) (x), \quad \gamma > 0,$$

$$\left(I_{0+}^{\mu,\rho,\eta}\left(I_{1-}^{\gamma,\delta,\mu+\eta}f\right)(t)\right)(x)=\left(I_{1-}^{\mu+\gamma,\rho+\delta,\eta}f\right)(x), \quad \gamma>0,$$

and the well-known relations ([18], p. 327)

$$\left(I_{0+}^{\mu,-\mu,\eta}f\right)(x)=\left(I_{0+}^{\mu}f\right)(x), \quad \left(I_{1-}^{\mu,-\mu,\eta}f\right)(x)=\left(I_{1-}^{\mu}f\right)(x), \quad \mu>0,$$

where I_{0+}^{μ} and I_{1-}^{μ} are fractional Riemann–Liouville integrals, and by making simple transformations, we obtain the following relation between $\tau(x)$ and $\nu(x)$ extended to J from the hyperbolic part D_2 :

$$\tau(x)=a_1(x)\left(I_{0+}^{1-2\beta}\nu\right)(x)+b_1(x)\left(I_{1-}^{1-2\beta}\nu\right)(x)+c_1(x)\nu(x)+g_1(x), \quad (2.9)$$

where

$$a_1(x)=\gamma_2\frac{a(x)}{E_1(x)}, \quad b_1(x)=\gamma_2\frac{b(x)}{E_1(x)}, \quad c_1(x)=-\frac{d(x)}{E_1(x)}, \quad g_1(x)=\frac{g(x)}{E_1(x)}.$$

For $g_1(x)=0$, we have

$$I^*=\int_0^1a_1(x)\nu(x)\left(I_{0+}^{1-2\beta}\nu\right)(x)dx+\int_0^1b_1(x)\nu(x)\left(I_{1-}^{1-2\beta}\nu\right)(x)dx+\int_0^1c_1(x)\nu^2(x)dx.$$

By using the well-known formula for the gamma-function $\Gamma(z)$ [21],

$$\int_0^\infty t^{z-1}\cos(kt)dt=\frac{\Gamma(z)}{k^z}\cos\left(\frac{\pi z}{2}\right), \quad k>0, \quad 0<z<1,$$

and by setting $k=|x-\xi|$ and $z=2\beta$ in it, one can derive the relation

$$\frac{1}{|x-\xi|^{2\beta}}=\frac{1}{\Gamma(2\beta)\cos(\pi\beta)}\int_0^\infty t^{2\beta-1}\cos[t(x-\xi)]dt.$$

and hence

$$\begin{aligned} \frac{\pi}{2\sin(\pi\beta)}I^* &= \int_0^1a_1(x)\nu(x)dx\int_0^x\nu(\xi)d\xi\int_0^\infty t^{2\beta-1}\cos[t(x-\xi)]dt \\ &+ \int_0^1b_1(x)\nu(x)dx\int_x^1\nu(\xi)d\xi\int_0^\infty t^{2\beta-1}\cos[t(\xi-x)]dt + \frac{\pi}{2\sin(\pi\beta)}\int_0^1c_1(x)\nu^2(x)dx. \end{aligned}$$

By changing the order of integration and by integrating by parts, as a result, the following relation holds

$$\begin{aligned} \frac{\pi}{\sin(\pi\beta)}I^* &= a_1(1)\int_0^\infty t^{2\beta-1}dt\left[\left(\int_0^1\nu(\xi)\cos(t\xi)d\xi\right)^2+\left(\int_0^1\nu(\xi)\sin(t\xi)d\xi\right)^2\right] \\ &- \int_0^\infty t^{2\beta-1}dt\int_0^1a_1'(x)\left[\left(\int_0^x\nu(\xi)\cos(t\xi)d\xi\right)^2+\left(\int_0^x\nu(\xi)\sin(t\xi)d\xi\right)^2\right]dx \\ &+ b_1(0)\int_0^\infty t^{2\beta-1}dt\left[\left(\int_0^1\nu(\xi)\cos(t\xi)d\xi\right)^2+\left(\int_0^1\nu(\xi)\sin(t\xi)d\xi\right)^2\right] \\ &+ \int_0^\infty t^{2\beta-1}dt\int_0^1b_1'(x)\left[\left(\int_x^1\nu(\xi)\cos(t\xi)d\xi\right)^2+\left(\int_x^1\nu(\xi)\sin(t\xi)d\xi\right)^2\right]dx \end{aligned}$$

$$+ \frac{\pi}{\sin(\pi\beta)} \int_0^1 c_1(x) \nu^2(x) dx.$$

From conditions (2.3) and (2.4), we have $I^* \geq 0$. Then it follows from (2.8) that the solution of the Problem is unique.

Now, if the conditions (2.5)–(2.7) in the theorem 2.1 are satisfied, then the functional relation between $\tau(x)$ and $\nu(x)$ from the hyperbolic domain D_2 has the form

$$[\gamma_2(a(x) + b(x)) - d(x)]\nu(x) = \gamma_1 a(x)(D_{0+}^{1-2\beta}\tau)(x) + \gamma_1 b(x)(D_{1-}^{1-2\beta}\tau)(x) + c(x)\tau(x) - g(x), \quad (2.10)$$

where $D_{0+}^{1-2\beta}$ and $D_{1-}^{1-2\beta}$ are the Riemann–Liouville fractional differentiation operators ([18], p.43). We rewrite relation (2.10) in the form

$$\nu(x) = a_2(x)(D_{0+}^{1-2\beta}\tau)(x) + b_2(x)(D_{1-}^{1-2\beta}\tau)(x) + c_2(x)\tau(x) + g_2(x), \quad (2.11)$$

where

$$a_2(x) = \gamma_1 \frac{a(x)}{E_2(x)}, \quad b_2(x) = \gamma_1 \frac{b(x)}{E_2(x)}, \quad c_2(x) = \frac{c(x)}{E_2(x)}, \quad g_2(x) = -\frac{g(x)}{E_2(x)}.$$

Let D_h and D_{1h} be finite domains cut off from D and D_1 , respectively, by the line $y = h > 0$, where $O_h(0, h)$ and $B_h(1, h)$.

Lemma 2.1. *Let*

1) *the function $u(x, y) \in C(\overline{D_{1h}}) \cap C^2(D_{1h})$ satisfy the inequality $L(u) \geq 0$ ($L(u) \leq 0$) and take on a positive maximum (respectively, a negative minimum) value at some point $(x_0, 0)$, $x_0 \in (0, 1)$;*

2) *the value of $u(x, y)$ on $\sigma = \overline{OO_h} \cup O_h B_h \cup \overline{B_h B}$ is less (greater) than $u(x_0, 0)$. Then*

$$\nu_+(x) = \lim_{y \rightarrow +0} y^{\beta_0} u_y(x_0, 0) < 0 \quad (\nu_+(x) > 0)$$

provided that this limit exists.

Lemma 2.2. *Let $g(x) \equiv 0$, and let either $a(x), b(x), c(x) > 0$ and $d(x) \leq 0$ or $a(x), b(x), c(x) < 0$ and $d(x) \geq 0$. If the function $\tau_-(x)$ attains a positive maximum (respectively, a negative minimum) on the interval $[0, 1]$ at the point $x = x_0$, $0 < x_0 < 1$, then $\nu_-(x_0) > 0$ (respectively, $\nu_-(x_0) < 0$).*

The assertion of the lemma 2.2 readily follows from relation (2.10) and the maximum principle for fractional differentiation operators [19].

Let $u(x, y)$ be a solution of the homogeneous problem for Eq. (1.1) in the domain D_1 that satisfies the boundary conditions $u(0, y) = u(1, y) = 0$, $y \geq 0$. We show that $u(x, y) \equiv 0$ in D_1 .

Suppose that this is not true. Then there exists a domain D_{1h} such that, indeed, $\max_{\overline{D_{1h}}} |u(x, y)| > 0$. Without loss of generality, one can assume that $\max_{\overline{D_{1h}}} |u(x, y)| = \max_{\overline{D_{1h}}} u(x, y) > 0$.

Assume that $\max_{\overline{D_{1h}}} u(x, y)$ is attained on $O_h B_h$. Since $u(0, y) = u(1, y) = 0$ for $y \geq 0$, it should not be attained on $\sigma = \overline{OO_h} \cup O_h B_h \cup \overline{B_h B}$, and, by virtue of the well-known property of elliptic equations [22] (pp.24–25), it should be attained on the segment $\overline{OB}$ at some its interior point $(x_0, 0)$. But then, by the lemma 2.1 given above, it is $\nu_+(x_0) < 0$, which contradicts $\nu_-(x_0) > 0$.

Therefore, $\max_{D_{1h}} u(x, y)$ must be attained on $O_h B_h$. Take an arbitrary small number $\varepsilon > 0$ and, by (1.5), choose h large enough so that $|u(x, y)| < \varepsilon$ for $y = h$.

Let (x_0, y_0) be an arbitrary point in the domain D_1 . For a sufficiently large h , this point falls in the domain D_{1h} , and therefore $|u(x_0, y_0)| < \varepsilon$. Since ε is arbitrary, we have $u(x_0, y_0) = 0$. Then $u(x, y) \equiv 0$ in the domain $\overline{D_1}$, and, in particular, $u(x, +0) = \tau_+(x) \equiv 0$ on $[0, 1]$. But since $\tau_+(x) = \tau_-(x)$, it follows from (2.11) that $\nu_-(x) \equiv 0$. Then, from Darboux's formula (2.1), we obtain $u(x, y) \equiv 0$ in the domain D_2 .

Therefore, the Problem has at most one solution. $\square$

EXISTENCE OF A SOLUTION OF THE PROBLEM

To derive a functional relation between $\tau(x) = u(x, 0)$ and $\nu(x) = \lim_{y \rightarrow +0} y^{\beta_0} u_y(x, y)$ extended to J from the elliptic part D_1 of the mixed domain D , consider Problem N .

Problem N . In the domain D_1 , find a solution of Eq. (1.1) satisfying the inclusion $u(x, y) \in C(\overline{D}_1) \cap C^2(D_1)$ and conditions (1.2) and (1.5)–(1.7).

The solution of Problem N was found in [10] in the form

$$u(x, y) = u_1(x, y) + u_2(x, y),$$

where

$$u_1(x, y) = y^{\frac{1-\beta_0}{2}} \int_0^\infty [a_1(\lambda)e^{\lambda x} + a_2(\lambda)e^{-\lambda x}] J_{\frac{2\beta-1}{2}} \left(\frac{2\lambda}{m+2} y^{\frac{m+2}{2}} \right) d\lambda,$$

$$\begin{aligned} a_1(\lambda)e^{\lambda x} + a_2(\lambda)e^{-\lambda x} = & \frac{2\lambda}{(m+2)\sinh \lambda} \left[\sinh(\lambda(1-x)) \int_0^\infty y^{\frac{2m+1+\beta_0}{2}} J_{\frac{2\beta-1}{2}} \left(\frac{2\lambda}{m+2} y^{\frac{m+2}{2}} \right) \varphi_1(y) dy \right. \\ & \left. + \sinh(\lambda x) \int_0^\infty J_{\frac{2\beta-1}{2}} \left(\frac{2\lambda}{m+2} y^{\frac{m+2}{2}} \right) \varphi_2(y) dy \right], \end{aligned}$$

and

$$\begin{aligned} u_2(x, y) = & y^{\frac{1-\beta_0}{2}} \sum_{n=1}^\infty C_n K_{\frac{1-2\beta}{m+2}} \left(\frac{2\pi n}{m+2} y^{\frac{m+2}{2}} \right) \sin(\pi n x), \\ C_n = & \tilde{k} \int_0^1 \nu(t) \sin(\pi n t) dt, \quad \tilde{k} = -\frac{4}{\pi n} \left(\frac{\pi n}{m+2} \right)^{\frac{1+2\beta}{2}} \frac{1}{\Gamma\left(\frac{1+2\beta}{2}\right)}. \end{aligned}$$

Here, $J_\nu(z)$ is the Bessel function of the first kind and $K_\nu(z)$ is the Macdonald function [23].

The relation between $\tau(x)$ and $\nu(x)$ extended from D_1 to J has the form

$$\tau(x) = \gamma_0 \int_0^1 \nu(t) G(x, t) dt + \tau_1(x), \quad (2.12)$$

where

$$\begin{aligned} G(x, t) = & |x-t|^{-2\beta} - (x+t)^{-2\beta} \\ & + \sum_{n=1}^\infty [(2n-x+t)^{-2\beta} + (2n+x-t)^{-2\beta} - (2n-x-t)^{-2\beta} - (2n+x+t)^{-2\beta}], \\ \tau_1(x) = & u_1(x, 0) = k_0 \int_0^\infty t^{\frac{2m+1+\beta_0}{2}} \varphi_1(t) dt \int_0^\infty \frac{\sinh(\lambda(1-x))}{\sinh \lambda} \lambda^{\frac{1+2\beta}{2}} J_{\frac{2\beta-1}{2}} \left(\frac{2\lambda}{m+2} t^{\frac{m+2}{2}} \right) d\lambda \\ & + k_0 \int_0^\infty t^{\frac{2m+1+\beta_0}{2}} \varphi_2(t) dt \int_0^\infty \frac{\sinh(\lambda x)}{\sinh \lambda} \lambda^{\frac{1+2\beta}{2}} J_{\frac{2\beta-1}{2}} \left(\frac{2\lambda}{m+2} t^{\frac{m+2}{2}} \right) d\lambda, \\ \gamma_0 = & -\left(\frac{2}{m+2} \right)^{2\beta} \frac{\Gamma(\beta)}{2\sqrt{\pi}\Gamma\left(\frac{1+2\beta}{2}\right)}, \quad k_0 = \left(\frac{1}{m+2} \right)^{\frac{1+2\beta}{2}} \frac{2}{\Gamma\left(\frac{1+2\beta}{2}\right)}. \end{aligned}$$

By using the formula [23] (p. 191)

$$\int_0^\infty z^{\nu+1} \frac{\sinh(az)}{\sinh(bz)} I_\nu(cz) dz = \frac{2}{b} \left(\frac{\pi}{b} \right)^{\nu+1} \sum_{k=1}^\infty (-1)^{k-1} k^{\nu+1} \sin\left(\frac{ak\pi}{b}\right) K_\nu\left(\frac{ck\pi}{b}\right),$$

$$[\operatorname{Re} \nu > -1; \quad \operatorname{Re} b > |\operatorname{Re} a| + |\operatorname{Im} c|],$$

one can claim that the function $\tau_1(x)$ has derivatives of arbitrary order on the interval $(0, 1)$ and remains bounded as $x \rightarrow 0$ and $x \rightarrow 1$.

Let conditions (2.2) and (2.3) be satisfied. By substituting the function (2.12) into relation (2.9), we obtain, after some transformations,

$$d(x)\nu(x) + \int_0^1 \frac{K(x, t)\nu(t)dt}{|x - t|^{2\beta}} = f(x), \quad (2.13)$$

where

$$K(x, t) = \begin{cases} \gamma_0 E_1(x)(x - t)^{2\beta} G(x, t) - \frac{\gamma_2 a(x)}{\Gamma(1 - 2\beta)}, & t \leq x, \\ \gamma_0 E_1(x)(t - x)^{2\beta} G(x, t) - \frac{\gamma_2 b(x)}{\Gamma(1 - 2\beta)}, & t \geq x, \end{cases}$$

$$f(x) = g(x) - E_1(x)\tau_1(x).$$

By virtue of the above-stipulated assumptions on the smoothness of the given functions, one can obtain $f(x) \in C(\bar{J}) \cap C^1(J)$.

Therefore, relation (2.13) with $d(x) \neq 0$ is a Fredholm integral equation of the second kind with a weak singularity in the kernel, whose unconditional solvability follows from the uniqueness of the solution of the Problem.

By substituting the function (2.12) into relation (2.11) and by making simple transformations, we have

$$\begin{aligned} & [\gamma_2(a(x) + b(x)) - d(x)]\nu(x) - \gamma_0 c(x) \int_0^1 \nu(t)G(x, t)dt \\ & - \frac{\gamma_0 \gamma_1 a(x)}{\Gamma(2\beta)} \frac{d}{dx} \int_0^x \frac{d\xi}{(x - \xi)^{1-2\beta}} \int_0^1 \nu(t)G(\xi, t)dt \\ & + \frac{\gamma_0 \gamma_1 b(x)}{\Gamma(2\beta)} \frac{d}{dx} \int_x^1 \frac{d\xi}{(\xi - x)^{1-2\beta}} \int_0^1 \nu(t)G(\xi, t)dt = F(x), \end{aligned} \quad (2.14)$$

where

$$F(x) = g(x) + c(x)\tau_1(x) + \frac{\gamma_1}{\Gamma(2\beta)} \left[a(x) \int_0^x \frac{\tau_1'(t)dt}{(x - t)^{1-2\beta}} - b(x) \int_x^1 \frac{\tau_1'(t)dt}{(t - x)^{1-2\beta}} \right]. \quad (2.15)$$

Now, let us study the solvability of the integral equation (2.14). To this end, we transform the expression on the left-hand side in Eq. (2.14),

$$I_1(x) = \frac{d}{dx} \int_0^x (x - \xi)^{2\beta-1} d\xi \int_0^1 \nu(t)G(\xi, t)dt,$$

and then, by making the related transformations, the following is obtained

$$\begin{aligned} I_1(x) &= \pi \tan(\pi\beta)\nu(x) + \int_0^1 \left(\frac{t}{x}\right)^{1-2\beta} \frac{\nu(t)dt}{t - x} - \int_0^1 \left(\frac{t}{x}\right)^{1-2\beta} \frac{\nu(t)dt}{t + x} \\ &+ \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n+t}{x}\right)^{1-2\beta} \frac{\nu(t)dt}{2n - x + t} + \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n-t}{x}\right)^{1-2\beta} \frac{\nu(t)dt}{2n + x - t} \end{aligned}$$

$$-\sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n-t}{x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n-x-t} - \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n+t}{x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n+x+t}.$$

In a similar way, we have

$$\begin{aligned} I_2(x) &= \frac{d}{dx} \int_x^1 \frac{d\xi}{(\xi-x)^{1-2\beta}} \int_0^1 \nu(t)G(\xi, t)dt \\ &= -\pi \tan(\pi\beta)\nu(x) + \int_0^1 \left(\frac{1-t}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{t-x} + \int_0^1 \left(\frac{1+t}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{t+x} \\ &\quad - \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n+t-1}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n-x+t} - \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n-t+1}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n+x-t} \\ &\quad + \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n-t-1}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n-x-t} + \sum_{n=1}^{\infty} \int_0^1 \left(\frac{2n+t+1}{1-x} \right)^{1-2\beta} \frac{\nu(t)dt}{2n+x+t}. \end{aligned}$$

In what follows, we additionally assume that

$$a(x) = x^\varepsilon \tilde{a}_1(x), \quad b(x) = (1-x)^\varepsilon \tilde{b}_1(x), \quad d(x) = [x(1-x)]^\varepsilon \tilde{d}_1(x), \quad \varepsilon > 1-2\beta, \quad (2.16)$$

$$\tilde{a}_1(x), \tilde{b}_1(x), \tilde{d}_1(x) \in C^{(0,\lambda)}(\bar{J}) \cap C^1(J), \quad \lambda > 0.$$

By using conditions (2.16), we rewrite Eq. (2.14) in the form

$$A(x)\nu(x) + \frac{B(x)}{\pi i} \int_0^1 \frac{\nu(t)dt}{t-x} + \int_0^1 \frac{K(x, t)\nu(t)dt}{t-x} = F(x), \quad (2.17)$$

where

$$A(x) = \gamma_2 [x^\varepsilon \tilde{a}_1(x) + (1-x)^\varepsilon \tilde{b}_1(x)] (1 + \sin(\pi\beta)) - x^\varepsilon (1-x)^\varepsilon \tilde{d}_1(x),$$

$$B(x) = i\gamma_2 [x^\varepsilon \tilde{a}_1(x) + (1-x)^\varepsilon \tilde{b}_1(x)] \cos(\pi\beta),$$

and

$$\begin{aligned} K(x, t) &= -\gamma_0 c(x)(t-x) \left[\frac{1}{|x-t|^{2\beta}} - \frac{1}{(x+t)^{2\beta}} \right] \\ &\quad - \frac{\gamma_0 \gamma_1}{\Gamma(2\beta)} \left\{ x^\varepsilon \tilde{a}_1(x) \left[\left(\frac{t}{x} \right)^{1-2\beta} - 1 \right] - (1-x)^\varepsilon \tilde{b}_1(x) \left[\left(\frac{1-t}{1-x} \right)^{1-2\beta} + 1 \right] \right\} \\ &\quad + \frac{\gamma_0 \gamma_1}{\Gamma(2\beta)} \frac{t-x}{t+x} \left[x^\varepsilon \tilde{a}_1(x) \left(\frac{t}{x} \right)^{1-2\beta} + (1-x)^\varepsilon \tilde{b}_1(x) \left(\frac{1+t}{1-x} \right)^{1-2\beta} \right] \\ &\quad - \gamma_0 c(x)(t-x) \sum_{n=1}^{\infty} \left[(2n-x+t)^{-2\beta} + (2n+x-t)^{-2\beta} - (2n-x-t)^{-2\beta} - (2n+x+t)^{-2\beta} \right] \\ &\quad - \frac{\gamma_0 \gamma_1}{\Gamma(2\beta)} x^\varepsilon \tilde{a}_1(x)(t-x) \left[\sum_{n=1}^{\infty} \left(\frac{2n+t}{x} \right)^{1-2\beta} \frac{1}{2n-x+t} + \sum_{n=1}^{\infty} \left(\frac{2n-t}{x} \right)^{1-2\beta} \frac{1}{2n+x-t} \right. \\ &\quad \left. - \sum_{n=1}^{\infty} \left(\frac{2n-t}{x} \right)^{1-2\beta} \frac{1}{2n-x-t} - \sum_{n=1}^{\infty} \left(\frac{2n+t}{x} \right)^{1-2\beta} \frac{1}{2n+x+t} \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{\gamma_0\gamma_1}{\Gamma(2\beta)}(1-x)^{\varepsilon}\tilde{b}_1(x)(t-x)\left[\sum_{n=1}^{\infty}\left(\frac{2n+t-1}{1-x}\right)^{1-2\beta}\frac{1}{2n-x+t}+\sum_{n=1}^{\infty}\left(\frac{2n-t+1}{1-x}\right)^{1-2\beta}\frac{1}{2n+x-t}\right. \\
& \left.-\sum_{n=1}^{\infty}\left(\frac{2n-t-1}{1-x}\right)^{1-2\beta}\frac{1}{2n-x-t}-\sum_{n=1}^{\infty}\left(\frac{2n+t+1}{1-x}\right)^{1-2\beta}\frac{1}{2n+x+t}\right]
\end{aligned}$$

It follows from the form of the kernel $K(x, t)$, the smoothness of the given functions, and the conditions (2.16) that

$$K(x, t) \in C^{(0, \lambda)}(\bar{J} \times \bar{J}) \cap C^1(J \times J),$$

and the right-hand side $F(x) \in C(0, 1] \cap C^1(0, 1)$ of Eq. (2.17) from (2.15).

Thus, the singular integral equation (2.17) has a unique solution given by the following form

$$\nu(x) = A^*(x)P(x) - \frac{B^*(x)}{\pi i} \left(\frac{1-x}{x}\right)^{\frac{1}{4}(1-2\beta)} \int_0^1 \left(\frac{t}{1-t}\right)^{\frac{1}{4}(1-2\beta)} \frac{P(t)dt}{t-x},$$

where

$$A^*(x) = \frac{A(x)}{A^2(x) - B^2(x)}, \quad B^*(x) = \frac{B(x)}{A^2(x) - B^2(x)}, \quad P(x) = F(x) - \int_0^1 \frac{K(x, t)\nu(t)dt}{t-x}.$$

The condition $A^2(x) - B^2(x) \neq 0$ provides the existence of a regularizer [24] that reduces the singular integral equation (2.17) to a Fredholm integral equation of the second kind, whose unconditional solvability follows from the uniqueness of the solution of the problem.

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Regularization of the boundary value problem for an inhomogeneous high-order parabolic equation with one degeneracy line

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Abstract. In this paper, we study an ill-posed initial-boundary value problem for a high-order inhomogeneous parabolic equation with one degeneracy line. We obtain a priori estimate of the solution of the problem, determine the correctness set, and prove the uniqueness and conditional stability of the solution in this set. As a result, we construct a regularized approximate solution in the correctness set according to the theory of A.N. Tikhonov. In this case, an approximate solution is constructed depending on the evenness or oddness of the parameter n , which determines the order of the equation.

Keywords: equation with one line of degeneracy, ill-posed problem, a priori estimate, conditional stability, uniqueness, set of correctness, regularized solution

MSC (2020): 35R25, 65M30, 65N20

1. INTRODUCTION

In this research work, the initial-boundary value problem for the high-order inhomogeneous parabolic equation that changes direction is studied.

Let $Q_T = \{(x, y, t) : (x, y) \in \Omega, 0 < t < T\}$, $\Omega = \{(x, y) : |x| < l, 0 < y < d\}$. We consider the differential equation

$$u_t(x, y, t) + k(x)u_{xx}(x, y, t) + \partial_y^{2n}u(x, y, t) + cu(x, y, t) = f(x, y, t) \quad (1.1)$$

in the region $Q_T \cap \{x \neq 0\}$, where $k(x) = \begin{cases} \alpha, & x > 0, \\ -\beta, & x < 0, \end{cases}$ $\alpha > 0$, $\beta > 0$, c are given real numbers and $f(x, y, t) \in L_2(Q_T)$ is a sufficiently smooth, source function, $n \in N$.

Problem statement. Find a function $u(x, y, t) \in L_2(Q_T)$ in the region $Q_T \cap \{x \neq 0\}$ satisfying equation (1.1) and initial

$$u|_{t=0} = \varphi(x, y), (x, y) \in \bar{\Omega}, \quad (1.2)$$

boundary

$$u|_{x=-l} = u|_{x=l} = 0, 0 \leq y \leq d, 0 \leq t \leq T, \quad (1.3)$$

$$\partial_y^{2j}u|_{y=0} = \partial_y^{2j}u|_{y=d} = 0, j = 0, 1, \dots, (n-1), -l \leq x \leq l, 0 \leq t \leq T, \quad (1.4)$$

gluing

$$u(-0, y, t) = u(+0, y, t), \alpha u_x(-0, x, t) = -\beta u_x(+0, y, t), 0 \leq y \leq d, 0 \leq t \leq T \quad (1.5)$$

conditions, where $\varphi(x, y) \in L_2(\Omega)$ is sufficiently smooth function.

The theory of parabolic equations with changing direction plays an important role in both mathematics and applied sciences. Unlike the classical heat equation, these models describe processes where diffusion and conductivity vary depending on spatial regions or material properties, and in some cases, the direction of diffusion itself may change. Such equations arise naturally in the study of heat transfer in layered or composite materials, in groundwater filtration through soils with variable permeability, and in the modeling of phase transitions in physics. They are also applied in population dynamics, environmental sciences, and even in financial mathematics, where degenerate or direction-changing diffusion appears in certain pricing models. However, such equations lead to serious mathematical difficulties: the classical existence, uniqueness, and stability results for the equations no longer hold, and boundary value problems often become ill-posed in the Hadamard sense.

High-order parabolic equations are of particular interest because they describe more delicate physical processes than second-order models. Examples include higher-order diffusion, vibration of viscoelastic

plates, phase transition phenomena, and other systems where simple diffusion models are insufficient. The inhomogeneous case is especially important, as external forces or source terms inevitably appear in realistic applications. The presence of a degeneracy line in such equations substantially complicates the analysis, since the standard maximum principles, energy estimates, and semigroup methods cannot be applied directly.

The study of equations with changing direction began with the famous work of M. Gevrey[1]. This theory was developed by L. Cattabriga[2], M.S. Baouendi and P. Grisvard[3], C.D. Pagani and G. Talenti[4]. Parabolic equations whose direction changes with time have been extensively studied by C.D. Pagani[5], S.A. Tersenov[6], A.A. Kerefov[7], N.V. Kislov[8], S.G. Pyatkov[9], and other researchers. I.E. Egorov[10], D. Amanov[11], S.V. Popov and V.G. Markov[12] worked with parabolic equations of high-order with changing direction. Most of these works are devoted to the study of the existence and uniqueness of solutions to various boundary value problems.

Analytical calculation of the solution of initial-boundary value problems for high-order equations with changing direction create serious difficulties. Especially in cases where there is no classical solution to such problems, finding solutions based on the regularization approach is of great importance. Important scientific results in this regard were reflected in the scientific research of K.S. Fayazov, I.O. Khajiev, Y.K. Khudaiberganov and E.B. Shobdarov [13, 14, 15, 16, 17], [18]. In their works, they conducted research on higher-order parabolic equations with changing direction, focusing on the derivation of a priori estimates, the proof of uniqueness and stability theorems, and the construction of regularized solutions.

2. EIGENVALUE PROBLEM

Find such values of the parameter λ that the problem

$$k(x) \Phi_{xx}(x, y) + \frac{\partial^{2n} \Phi(x, y)}{\partial y^{2n}} + \lambda \Phi(x, y) = 0, (x, y) \in \Omega, n \in N, \quad (2.1)$$

$$\begin{cases} \Phi(-l, y) = \Phi(l, y) = 0, 0 \leq y \leq d, \\ \frac{\partial^{2i} \Phi(x, 0)}{\partial y^{2i}} = \frac{\partial^{2i} \Phi(x, d)}{\partial y^{2i}} = 0, -l \leq x \leq l, i = 0, 1, \dots, n-1, \\ \Phi(-0, y) = \Phi(+0, y), \quad \beta \Phi_x(-0, y) = -\alpha \Phi_x(+0, y), 0 \leq y \leq d. \end{cases} \quad (2.2)$$

has a nontrivial solution. In the problem (2.1)–(2.2), the differential operator is given by

$$L \equiv k(x) \partial_x^2 + \partial_y^{2n}.$$

This operator is self-adjoint under the boundary and gluing conditions with respect to the variables x and y . Since the coefficients of the operator L , as well as the boundary and gluing conditions (2.2), allow for separation with respect to the variables x and y , the solution can be sought in the form $\Phi(x, y) = X(x) Y(y)$, by applying the method of separation of variables.

Let $\lambda = \mu + \eta$. As a result, we obtain the eigenvalue problems

$$\begin{cases} k(x) X''(x) + \mu X(x) = 0, \\ X(-l) = X(l) = 0, \\ X(-0) = X(+0), \\ \beta X'(-0) = -\alpha X'(+0), \end{cases} \quad (2.3)$$

$$\begin{cases} \frac{\partial^{2n} Y(y)}{\partial y^{2n}} + \eta Y(y) = 0, \\ \frac{\partial^{2k} Y(0)}{\partial y^{2k}} = \frac{\partial^{2k} Y(d)}{\partial y^{2k}} = 0, k = 0, \dots, n-1, \end{cases} \quad (2.4)$$

with respect to functions $X(x)$, $Y(y)$, respectively. The eigenvalue problem 2.3 has positive eigenvalues μ_i^+ and negative eigenvalues μ_i^- , which are solutions of the transcendental equations

$$\sqrt{\beta} \tan \sqrt{\frac{\mu_i^+}{\alpha}} l - \sqrt{\alpha} \tanh \sqrt{\frac{\mu_i^+}{\beta}} l = 0,$$

$$\sqrt{\alpha} \tan \sqrt{\frac{-\mu_i^-}{\beta}} l - \sqrt{\beta} \tanh \sqrt{\frac{-\mu_i^-}{\alpha}} l = 0,$$

respectively, $i \in N$. We emphasize that the numbers μ_i^+ and $-\mu_i^-$ are non-decreasing sequences. The eigenfunctions corresponding to the eigenvalues are expressed in the form

$$X_i^+(x) = \begin{cases} \frac{1}{\sqrt{C(\mu_i^+)}} \frac{\sin \sqrt{\frac{\mu_i^+}{b}}(l+x)}{\cos \sqrt{\frac{\mu_i^+}{b}}l}, & 0 < x < -l, \\ \frac{1}{\sqrt{C(\mu_i^+)}} \sqrt{\frac{b}{a}} \frac{\sinh \left(\sqrt{\frac{\mu_i^+}{a}}(l-x) \right)}{\cosh \sqrt{\frac{\mu_i^+}{a}}l}, & 0 < x < l, \end{cases}$$

$$X_i^-(x) = \begin{cases} -\frac{1}{\sqrt{C(\mu_i^-)}} \sqrt{\frac{a}{b}} \frac{\sinh \sqrt{\frac{-\mu_i^-}{b}}(l+x)}{\cosh \sqrt{\frac{-\mu_i^-}{b}}l}, & -l < x < 0, \\ -\frac{1}{\sqrt{C(\mu_i^-)}} \frac{\sin \sqrt{\frac{-\mu_i^-}{a}}(l-x)}{\cos \sqrt{\frac{-\mu_i^-}{a}}l}, & 0 < x < l, \end{cases}$$

where $C(\mu_i^+) = \frac{l}{2} \left(1 + \tan^2 \sqrt{\frac{\mu_i^+}{\alpha}} l + \frac{\alpha}{\beta} \left(\tanh^2 \sqrt{\frac{\mu_i^+}{\beta}} l - 1 \right) \right)$, $C(\mu_i^-) = \frac{l}{2} \left(1 + \tan^2 \sqrt{\frac{-\mu_i^-}{\beta}} l + \frac{\beta}{\alpha} \left(\tanh^2 \sqrt{\frac{-\mu_i^-}{\alpha}} l - 1 \right) \right)$.

The eigenvalues of the problem (2.4) are determined by formula $\eta_j = (-1)^{n+1} \left(\frac{\pi j}{d} \right)^{2n}$, and the corresponding eigenfunctions

$$Y_j(y) = \sqrt{\frac{2}{d}} \sin \left(\frac{\pi j}{d} y \right), j \in N.$$

Therefore, the eigenvalues of the problem (2.1)-(2.2) are expressed by the formulas

$$\bar{\lambda}_{i,j} = \mu_i^+ + (-1)^{n+1} \left(\frac{\pi j}{d} \right)^{2n}, \quad \tilde{\lambda}_{i,j} = \mu_i^- + (-1)^{n+1} \left(\frac{\pi j}{d} \right)^{2n}$$

and the eigenfunctions are expressed by the formula $\Phi_{i,j}(x, y) = X_i(x) Y_j(y)$, $i, j \in N$, (see [15], p. 123).

Let the dot product be $(u, v) = \int_{-l}^l u v dx$ and the norm be $\|u\|^2 = (u, u)$ in the space $L_2[-l, l]$. Then the eigenfunctions have orthonormal properties

$$\int_{\Omega} \tilde{\Phi}_{i,j} \tilde{\Phi}_{p,q} dx dy = \begin{cases} 1, & i = p \wedge j = q, \\ 0, & i \neq p \vee j \neq q, \end{cases} \quad \int_{\Omega} \bar{\Phi}_{i,j} \bar{\Phi}_{p,q} dx dy = \begin{cases} 1, & i = p \wedge j = q, \\ 0, & i \neq p \vee j \neq q, \end{cases}$$

$$\int_{\Omega} \bar{\Phi}_{i,j} \tilde{\Phi}_{p,q} dx dy = 0.$$

Under the conditions (2.2), the operator L is self-adjoint, and since the domain Ω is bounded, its eigenvalues are discrete. According to the properties of the above eigenfunctions and by the Hilbert–Schmidt theorem, the system of eigenfunctions $\bar{\Phi}_{i,j}(x, y)$ and $\tilde{\Phi}_{i,j}(x, y)$ are complete and orthogonal in the space $L_2(\Omega)$ (see [18], p. 117).

Lemma 2.1. Suppose that a function $u(x, y, t)$ satisfies equation

$$u_t(x, y, t) + k(x) u_{xx}(x, y, t) + \partial_y^{2n} u(x, y, t) + cu(x, y, t) = 0 \quad (2.5)$$

and conditions $u(-l, y, t) = u(l, y, t) = 0$, $\partial_y^{2j} u(x, y, t)|_{y=0} = \partial_y^{2j} u(x, y, t)|_{y=d} = 0$, $j = 0, 1, \dots, (n-1)$, $u(-0, y, t) = u(+0, y, t)$, $\beta u_x(-0, y, t) = -\alpha u_x(+0, y, t)$, then the estimate

$$\|u(x, y, t)\| \leq \sqrt{2}(\|u(x, y, 0)\|)^{1-\frac{t}{T}} (\|u(x, y, T)\|)^{\frac{t}{T}} \quad (2.6)$$

is valid, $t \in (0, T)$.

Proof: According to the properties of the eigenfunctions $\bar{\Phi}_{i,j}(x, y)$, $\tilde{\Phi}_{i,j}(x, y)$, $(i, j = 1, 2, \dots)$ of the problem (2.1)-(2.2), the function $u(x, y, t)$ satisfying the conditions of Lemma 2.1, that is, the solution of equation (2.5), can be written in the form

$$u(x, y, t) = \sum_{i,j=1}^{\infty} \bar{u}_{i,j}(t) \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{u}_{i,j}(t) \tilde{\Phi}_{i,j}(x, y)$$

where the functions $\bar{u}_{i,j}(t)$, $\tilde{u}_{i,j}(t)$, $(i, j = 1, 2, \dots)$ are the solutions of problems

$$\begin{cases} \{\bar{u}_{i,j}(t)\}_t - (\bar{\lambda}_{i,j} - c) \bar{u}_{i,j}(t) = 0, & \{\tilde{u}_{i,j}(t)\}_t - (\tilde{\lambda}_{i,j} - c) \tilde{u}_{i,j}(t) = 0, \\ \bar{u}_{i,j}(0) = \bar{\varphi}_{i,j}(x, y), & \tilde{u}_{i,j}(0) = \tilde{\varphi}_{i,j}(x, y). \end{cases}$$

We know that for each function $\bar{u}_{i,j}(t)$, $\tilde{u}_{i,j}(t)$, the inequalities

$$\{\bar{u}_{i,j}(t)\}^2 \leq \left(\{\bar{u}_{i,j}(0)\}^2\right)^{1-\frac{t}{T}} \left(\{\bar{u}_{i,j}(T)\}^2\right)^{\frac{t}{T}}, \quad \{\tilde{u}_{i,j}(t)\}^2 \leq \left(\{\tilde{u}_{i,j}(0)\}^2\right)^{1-\frac{t}{T}} \left(\{\tilde{u}_{i,j}(T)\}^2\right)^{\frac{t}{T}}$$

hold, respectively (see [19], p. 820). By Hölder's inequality, we have either

$$\begin{aligned} \sum_{i,j=1}^{\infty} \{\bar{u}_{i,j}(t)\}^2 + \sum_{i,j=1}^{\infty} \{\tilde{u}_{i,j}(t)\}^2 &\leq 2 \left(\sum_{i,j=1}^{\infty} \{\bar{u}_{i,j}(0)\}^2 + \sum_{i,j=1}^{\infty} \{\tilde{u}_{i,j}(0)\}^2 \right)^{1-\frac{t}{T}} \times \\ &\times \left(\sum_{i,j=1}^{\infty} \{\bar{u}_{i,j}(T)\}^2 + \sum_{i,j=1}^{\infty} \{\tilde{u}_{i,j}(T)\}^2 \right)^{\frac{t}{T}} \end{aligned}$$

or

$$\|u(x, y, t)\| \leq \sqrt{2}(\|u(x, y, 0)\|)^{1-\frac{t}{T}} (\|u(x, y, T)\|)^{\frac{t}{T}}.$$

□

Lemma 2.2. For the solution $u(x, y, t)$ of the problem (1.1)-(1.5), the inequality

$$\|u(x, y, t)\| \leq \sqrt{2}(\|u(x, y, 0)\| + \gamma)^{1-\frac{t}{T}} (\|u(x, y, T)\| + \gamma)^{\frac{t}{T}} + \gamma \quad (2.7)$$

is valid, where $\gamma = \sqrt{T} e^{|c|(T+t)} \left(\int_0^T \|f(x, y, t)\|^2 dt \right)^{\frac{1}{2}}$, $t \in (0, T)$.

Proof: By introducing substitution $u(x, y, t) = w(x, y, t) e^{-ct}$ into equation (1.1), we can obtain equation

$$w_t(x, y, t) + k(x) w_{xx}(x, y, t) + \frac{\partial^{2n} w(x, y, t)}{\partial y^{2n}} = p(x, y, t),$$

where $p(x, y, t) = f(x, y, t) e^{ct}$, the function $w(x, y, t)$ satisfies conditions $w(-0, y, t) = w(+0, y, t)$, $\beta w_x(-0, y, t) = -\alpha w_x(+0, y, t)$. The solution of equation (2.3) can be expressed in the form

$$w(x, y, t) = \vartheta(x, y, t) + \omega(x, y, t).$$

Where, the functions $\vartheta(x, y, t)$ and $\omega(x, y, t)$ satisfy boundary conditions

$$\vartheta(-l, y, t) = \vartheta(l, y, t) = 0,$$

$$\omega(-l, y, t) = \omega(l, y, t) = 0,$$

and the gluing conditions

$$\begin{aligned} \vartheta(-0, y, t) &= \vartheta(+0, y, t), & \beta \vartheta_x(-0, y, t) &= -\alpha \vartheta_x(+0, y, t), \\ \omega(-0, y, t) &= \omega(+0, y, t), & \beta \omega_x(-0, y, t) &= -\alpha \omega_x(+0, y, t). \end{aligned}$$

Let $\vartheta(x, y, t)$ be the solution of the equation

$$\vartheta_t(x, y, t) + k(x) \vartheta_{xx}(x, y, t) + \frac{\partial^{2n} \vartheta(x, y, t)}{\partial y^{2n}} = 0$$

and $\omega(x, y, t)$ be the solution of the equation

$$\omega_t(x, y, t) + k(x) \omega_{xx}(x, y, t) + \frac{\partial^{2n} \omega(x, y, t)}{\partial y^{2n}} = p(x, y, t).$$

Suppose $\vartheta(x, y, t)$ and $\omega(x, y, t)$ exist. Then, according to the results obtained, the solutions $\vartheta(x, y, t)$ and $\omega(x, y, t)$ are expressed as:

$$\begin{aligned} \vartheta(x, y, t) &= \sum_{i,j=1}^{\infty} \bar{\vartheta}_{i,j}(t) \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{\vartheta}_{i,j}(t) \tilde{\Phi}_{i,j}(x, y), \\ \omega(x, y, t) &= \sum_{i,j=1}^{\infty} \bar{\omega}_{i,j}(t) \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{\omega}_{i,j}(t) \tilde{\Phi}_{i,j}(x, y). \end{aligned}$$

Here, each $\bar{\vartheta}_{i,j}(t)$, $\tilde{\vartheta}_{i,j}(t)$, $\bar{\omega}_{i,j}$, $\tilde{\omega}_{i,j}$ are solutions to the following problems, respectively:
 $\bar{\lambda}_{i,j} > 0$ and $\tilde{\lambda}_{i,j} > 0$

$$\begin{aligned} \left\{ \begin{array}{l} \{\bar{\vartheta}_{i,j}(t)\}_t - \bar{\lambda}_{i,j} \bar{\vartheta}_{i,j}(t) = 0, \\ \bar{\vartheta}_{i,j}(0) = \bar{\varphi}_{i,j} + \int_0^T e^{-\bar{\lambda}_{i,j}\tau} \bar{f}_{i,j}(\tau) d\tau, \end{array} \right. & \left\{ \begin{array}{l} \{\tilde{\vartheta}_{i,j}(t)\}_t - \tilde{\lambda}_{i,j} \tilde{\vartheta}_{i,j}(t) = 0, \\ \tilde{\vartheta}_{i,j}(0) = \tilde{\varphi}_{i,j} + \int_0^T e^{-\tilde{\lambda}_{i,j}\tau} \tilde{f}_{i,j}(\tau) d\tau, \end{array} \right. \\ \left\{ \begin{array}{l} \{\bar{\omega}_{i,j}(t)\}_t - \bar{\lambda}_{i,j} \bar{\omega}_{i,j}(t) = \bar{p}_{i,j}, \\ \bar{\omega}_{i,j}(0) = -\int_0^T e^{-\bar{\lambda}_{i,j}\tau} \bar{f}_{i,j}(\tau) d\tau, \end{array} \right. & \left\{ \begin{array}{l} \{\tilde{\omega}_{i,j}(t)\}_t - \tilde{\lambda}_{i,j} \tilde{\omega}_{i,j}(t) = \tilde{p}_{i,j}, \\ \tilde{\omega}_{i,j}(0) = -\int_0^T e^{-\tilde{\lambda}_{i,j}\tau} \tilde{f}_{i,j}(\tau) d\tau, \end{array} \right. \end{aligned}$$

$\bar{\lambda}_{i,j} < 0$ and $\tilde{\lambda}_{i,j} < 0$

$$\begin{aligned} \left\{ \begin{array}{l} \{\bar{\vartheta}_{i,j}(t)\}_t - \bar{\lambda}_{i,j} \bar{\vartheta}_{i,j}(t) = 0, \\ \bar{\vartheta}_{i,j}(0) = \bar{\varphi}_{i,j}, \end{array} \right. & \left\{ \begin{array}{l} \{\tilde{\vartheta}_{i,j}(t)\}_t - \tilde{\lambda}_{i,j} \tilde{\vartheta}_{i,j}(t) = 0, \\ \tilde{\vartheta}_{i,j}(0) = \tilde{\varphi}_{i,j}, \end{array} \right. \\ \left\{ \begin{array}{l} \{\bar{\omega}_{i,j}(t)\}_t - \bar{\lambda}_{i,j} \bar{\omega}_{i,j}(t) = \bar{p}_{i,j}, \\ \bar{\omega}_{i,j}(0) = 0, \end{array} \right. & \left\{ \begin{array}{l} \{\tilde{\omega}_{i,j}(t)\}_t - \tilde{\lambda}_{i,j} \tilde{\omega}_{i,j}(t) = \tilde{p}_{i,j}, \\ \tilde{\omega}_{i,j}(0) = 0, \end{array} \right. \end{aligned}$$

where $\bar{p}_{i,j}(t) = \int_{-l}^l \int_0^d p(x, y, t) \bar{\Phi}(x, y) dx dy$, $\tilde{p}_{i,j}(t) = \int_{-l}^l \int_0^d p(x, y, t) \tilde{\Phi}(x, y) dx dy$, ($i, j = 1, 2, \dots$). The solutions to these problems can be expressed as $\bar{\lambda}_{i,j} > 0$ is

$$\bar{\omega}_{i,j}(t) = - \int_t^T e^{\bar{\lambda}_{i,j}(t-\tau)} \bar{p}_{i,j}(\tau) d\tau,$$

$\bar{\lambda}_{i,j} < 0$ is

$$\bar{\omega}_{i,j}(t) = \int_0^t e^{\bar{\lambda}_{i,j}(t-\tau)} \bar{p}_{i,j}(\tau) d\tau,$$

$\tilde{\lambda}_{i,j} > 0$ is

$$\tilde{\omega}_{i,j}(t) = - \int_t^T e^{\tilde{\lambda}_{i,j}(t-\tau)} \tilde{p}_{i,j}(\tau) d\tau,$$

and $\tilde{\lambda}_{i,j} < 0$ is

$$\tilde{\omega}_{i,j}(t) = \int_0^t e^{\tilde{\lambda}_{i,j}(t-\tau)} \tilde{p}_{i,j}(\tau) d\tau.$$

These results lead to inequality

$$\begin{aligned} \|\omega(x, y, t)\|^2 &= \sum_{i,j=1}^{\infty} \bar{\omega}_{i,j}^2(t) + \sum_{i,j=1}^{\infty} \tilde{\omega}_{i,j}^2(t) = \\ &= \sum_{\bar{\lambda}_{i,j} > 0} \bar{\omega}_{i,j}^2(t) + \sum_{\bar{\lambda}_{i,j} < 0} \bar{\omega}_{i,j}^2(t) + \sum_{\tilde{\lambda}_{i,j} > 0} \tilde{\omega}_{i,j}^2(t) + \sum_{\tilde{\lambda}_{i,j} < 0} \tilde{\omega}_{i,j}^2(t) \leq \\ &\leq \sum_{\bar{\lambda}_{i,j} > 0} \left(\int_0^T e^{\bar{\lambda}_{i,j}(t-\tau)} \bar{p}_{i,j}(\tau) d\tau \right)^2 + \sum_{\bar{\lambda}_{i,j} < 0} \left(\int_0^t e^{\bar{\lambda}_{i,j}(t-\tau)} \bar{p}_{i,j}(\tau) d\tau \right)^2 + \\ &+ \sum_{\tilde{\lambda}_{i,j} > 0} \left(\int_t^T e^{\tilde{\lambda}_{i,j}(t-\tau)} \tilde{p}_{i,j}(\tau) d\tau \right)^2 + \sum_{\tilde{\lambda}_{i,j} < 0} \left(\int_0^t e^{\tilde{\lambda}_{i,j}(t-\tau)} \tilde{p}_{i,j}(\tau) d\tau \right)^2 \leq \\ &\leq T \int_0^T \|p(x, y, t)\|^2 dt \end{aligned} \quad (2.8)$$

We also express the solution of equation (1.1) in the form $u(x, y, t) = h(x, y, t) + g(x, y, t)$, where function $h(x, y, t)$ is the solution of the equation $h_t(x, y, t) + k(x) h_{xx}(x, y, t) + \frac{\partial^{2n} h(x, y, t)}{\partial y^{2n}} + c h(x, y, t) = 0$, and function $g(x, y, t)$ is the solution of the equation $g_t(x, y, t) + k(x) g_{xx}(x, y, t) + \frac{\partial^{2n} g(x, y, t)}{\partial y^{2n}} + c g(x, y, t) = f(x, y, t)$. Substituting $u(x, y, t) = w(x, y, t) e^{-ct}$ and from inequality 2.8 we get

$$\|g(x, y, t)\| \leq \sqrt{T} e^{|c|(T+t)} \left(\int_0^T \|f(x, y, t)\|^2 dt \right)^{\frac{1}{2}}.$$

By the result of Lemma 2.1, inequality (2.6) holds for function $h(x, y, t)$ and taking into account equality $h(x, y, t) = u(x, y, t) - g(x, y, t)$, we obtain inequality

$$\|u(x, y, t)\| \leq \sqrt{2} (\|u(x, y, 0)\| + \gamma)^{1-\frac{t}{T}} (\|u(x, y, T)\| + \gamma)^{\frac{t}{T}} + \gamma,$$

where $\gamma = \sqrt{T} e^{|c|(T+t)} \left(\int_0^T \|f(x, y, t)\|^2 dt \right)^{\frac{1}{2}}$. □

So, we introduce the correctness set as follows:

$$M = \{u(x, y, t) : \|u(x, y, T)\| \leq m, \quad m < \infty\}.$$

3. UNIQUENESS AND CONDITIONAL STABILITY THEOREMS

Theorem 3.1. *Let the solution of the problem (1.1)–(1.5) exists in the set M , $u(x, y, t) \in M$. Then it is unique.*

Proof: Let there are two solutions $u_1(x, y, t)$ and $u_2(x, y, t)$ of the problem (1.1) – (1.5). Let $U(x, y, t) = u_1(x, y, t) - u_2(x, y, t)$, then function $U(x, y, t)$ satisfies equation

$$U_t(x, y, t) + k(x) U_{xx}(x, y, t) + \frac{\partial^{2n} U(x, y, t)}{\partial y^{2n}} + cU(x, y, t) = 0$$

and $U(x, y, 0) = 0$ and conditions (1.3)–(1.5). From this, $\gamma = 0$ is deduced, and according to the inequality (2.7), it is established that either $\|U(x, y, t)\| \leq 0$ or $U(x, y, t) \equiv 0$ holds, which leads to $u_1(x, y, t) \equiv u_2(x, y, t)$, thereby proving the uniqueness of the solution of the problem. $\square$

Let $u(x, y, t)$ be the exact solution of the problem (1.1)–(1.5) corresponding to the precisely given initial data, and let $u_\varepsilon^N(x, y, t)$ be the approximate solution of the problem (1.1)–(1.5) corresponding to the approximately given initial data.

Theorem 3.2. *Let the solution of the problem (1.1)–(1.5) exists and $(u(x, y, t), u_\varepsilon(x, y, t)) \in M$. Furthermore, let $\|\varphi(x, y) - \varphi_\varepsilon(x, y)\| \leq \varepsilon$, $\|f(x, y, t) - f_\varepsilon(x, y, t)\| \leq \varepsilon$. Then the inequality*

$$\|u(x, y, t) - u_\varepsilon(x, y, t)\| \leq \sqrt{2}(\varepsilon + \gamma_\varepsilon)^{1 - \frac{t}{T}}(2m + \gamma_\varepsilon)^{\frac{t}{T}} + \gamma_\varepsilon$$

holds, where $\gamma_\varepsilon = Te^{|\varepsilon|(T+t)}\varepsilon$.

Proof: Let $U(x, y, t) = u(x, y, t) - u_\varepsilon(x, y, t)$. The function $U(x, y, t)$ satisfies equation

$$U_t(x, y, t) + k(x) U_{xx}(x, y, t) + \frac{\partial^{2n} U(x, y, t)}{\partial y^{2n}} + cU(x, y, t) = f(x, t) - f_\varepsilon(x, t)$$

in the region $Q_T \cap \{x \neq 0\}$ and initial

$$u(x, y, 0) = \varphi(x, y) - \varphi_\varepsilon(x, y), -l \leq x \leq l, 0 \leq y \leq d,$$

boundary

$$U(-l, y, t) = U(l, y, t) = 0, 0 \leq y \leq d, 0 \leq t \leq T,$$

$$\left. \frac{\partial^{2j} U(x, y, t)}{\partial y^{2j}} \right|_{y=0} = \left. \frac{\partial^{2j} U(x, y, t)}{\partial y^{2j}} \right|_{y=d} = 0, j = 0, 1, \dots, (n-1), -l \leq x \leq l, 0 \leq t \leq T,$$

and gluing

$$U(-0, y, t) = U(+0, y, t), \beta U_x(-0, x, t) = -\alpha U_x(+0, y, t), 0 \leq y \leq d, 0 \leq t \leq T,$$

conditions. Hence, function $U(x, y, t)$ satisfies inequality (2.7),

$$\|U(x, y, t)\| \leq \sqrt{2}(\|U(x, y, 0)\| + \gamma)^{1 - \frac{t}{T}}(\|U(x, y, T)\| + \gamma)^{\frac{t}{T}} + \gamma.$$

In this inequality,

$$\|U(x, y, 0)\| = \|\varphi(x, y) - \varphi_\varepsilon(x, y)\| \leq \varepsilon,$$

$$\|U(x, y, T)\| \leq \|u(x, y, T)\| + \|u_\varepsilon(x, y, T)\| \leq 2m,$$

$$\gamma_\varepsilon = Te^{|\varepsilon|(T+t)}\varepsilon.$$

Then we get

$$\|u(x, y, t) - u_\varepsilon(x, y, t)\| \leq \sqrt{2}(\varepsilon + \gamma_\varepsilon)^{1 - \frac{t}{T}}(2m + \gamma_\varepsilon)^{\frac{t}{T}} + \gamma_\varepsilon.$$

$\square$

4. REGULARIZATION

Let us assume that the solution of the problem (1.1)-(1.5) exists and it can be written in form

$$u(x, y, t) = \sum_{i,j=1}^{\infty} \bar{u}_{i,j}(t) \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{u}_{i,j}(t) \tilde{\Phi}_{i,j}(x, y).$$

We will consider the regularized solution of the problem (1.1)-(1.5) in two cases.

Case 1. Let $f(x, y, t) = 0$, $\|\varphi(x, y) - \varphi_{\varepsilon}(x, y)\| \leq \varepsilon$ and $u(x, y, t) \in M$. Then

$$\begin{cases} \bar{u}_{i,j}(t) = \bar{\varphi}_{i,j} e^{(\bar{\lambda}_{i,j}-c)t}, \\ \tilde{u}_{i,j}(t) = \tilde{\varphi}_{i,j} e^{(\tilde{\lambda}_{i,j}-c)t}. \end{cases}$$

From this equality, the solution of the problem is found in form

$$u(x, y, t) = \sum_{i,j=1}^{\infty} \bar{\varphi}_{i,j} e^{(\bar{\lambda}_{i,j}-c)t} \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{\varphi}_{i,j} e^{(\tilde{\lambda}_{i,j}-c)t} \tilde{\Phi}_{i,j}(x, y).$$

According to the eigenvalues of $\bar{\lambda}_{i,j}$ and $\tilde{\lambda}_{i,j}$, we consider separately when there are odd and even numbers of n :

n odd number. We obtain the approximate solution with respect to the initial data as

$$u^N(x, y, t) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \bar{\varphi}_{i,j} e^{(\bar{\lambda}_{i,j}-c)t} \bar{\Phi}_{i,j}(x, y) + \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} \tilde{\varphi}_{i,j} e^{(\tilde{\lambda}_{i,j}-c)t} \tilde{\Phi}_{i,j}(x, y),$$

and the approximate solution with respect to the initial approximate data as

$$u_{\varepsilon}^N(x, y, t) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \bar{\varphi}_{\varepsilon i,j} e^{(\bar{\lambda}_{i,j}-c)t} \bar{\Phi}_{i,j}(x, y) + \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} \tilde{\varphi}_{\varepsilon i,j} e^{(\tilde{\lambda}_{i,j}-c)t} \tilde{\Phi}_{i,j}(x, y),$$

where N_1, N_2 are regularization parameters.

Using inequality

$$\|u - u_{\varepsilon}^N\| \leq \|u - u^N\| + \|u^N - u_{\varepsilon}^N\|, \quad (4.1)$$

we calculate the norm of the difference between the exact solution and the approximate solution with respect to the initial approximate data.

We evaluate the second term $\|u^N - u_{\varepsilon}^N\|$ of the right-hand side of inequality (4.1) as follows:

$$\begin{aligned} \|u^N - u_{\varepsilon}^N\|^2 &= \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (\bar{\varphi}_{i,j} - \bar{\varphi}_{\varepsilon i,j})^2 e^{2(\bar{\lambda}_{i,j}-c)t} + \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} (\tilde{\varphi}_{i,j} - \tilde{\varphi}_{\varepsilon i,j})^2 e^{2(\tilde{\lambda}_{i,j}-c)t} \leq \\ &\leq \left(e^{2(\bar{\lambda}_{N_1, N_2}-c)t} + e^{2(\tilde{\lambda}_{1, N_2}-c)t} \right) \varepsilon^2. \end{aligned} \quad (4.2)$$

Now, we get

$$\begin{aligned} \|u - u^N\|^2 &= \sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t} + \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{N_2} \tilde{\varphi}_{i,j}^2 e^{2(\tilde{\lambda}_{i,j}-c)t} + \\ &+ \sum_{i=1}^{N_1} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t} + \sum_{i=1}^{\infty} \sum_{j=N_2+1}^{\infty} \tilde{\varphi}_{i,j}^2 e^{2(\tilde{\lambda}_{i,j}-c)t} \end{aligned} \quad (4.3)$$

for the first term. Under condition $\|u(x, y, T)\| \leq m$, the maximum value of each of the series

$$\sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t}, \quad \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{N_2} \tilde{\varphi}_{i,j}^2 e^{2(\tilde{\lambda}_{i,j}-c)t}, \quad \sum_{i=1}^{N_1} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t}, \quad \sum_{i=1}^{\infty} \sum_{j=N_2+1}^{\infty} \tilde{\varphi}_{i,j}^2 e^{2(\tilde{\lambda}_{i,j}-c)t} \text{ in}$$

equality (4.3) is found using the Lagrange multiplier method. First, we evaluate for the series

$\sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t}$. We have

$$\|u(x, y, T)\|^2 = \sum_{i,j=1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)T} + \sum_{i,j=1}^{\infty} \tilde{\varphi}_{i,j}^2 e^{2(\tilde{\lambda}_{i,j}-c)T} \leq m^2.$$

Based on the above information, we construct the Lagrangian function $F(\bar{\varphi}_{i,j}, \xi)$ as follows.

$$F(\bar{\varphi}_{i,j}, \xi) = \sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t} + \xi \left(\sum_{i,j=1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t} - m^2 \right),$$

where ξ is a parameter. Next, $\frac{\partial F(\bar{\varphi}_{i,j}, \xi)}{\partial \bar{\varphi}_{i,j}}$, $\frac{\partial F(\bar{\varphi}_{i,j}, \xi)}{\partial \xi}$ are calculated and equal to zero, and we find $\bar{\varphi}_{i,j}$ in the form

$$\bar{\varphi}_{i,j} = \begin{cases} m e^{-(\bar{\lambda}_{N_1+1, N_2+1}-c)T}, & i = N_1 + 1, j = N_2 + 1, \\ 0, & i \neq N_1 + 1, j \neq N_2 + 1. \end{cases}$$

By calculating the other series in the same way and putting the result in 4.3, we obtain

$$\begin{aligned} \|u - u^N\|^2 &\leq \left(e^{2(\bar{\lambda}_{N_1+1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{N_1+1, 1}-c)(t-T)} + \right. \\ &\quad \left. + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} \right) m^2. \end{aligned} \quad (4.4)$$

By putting the obtained values (4.2) and (4.4) into inequality (4.1), we get

$$\begin{aligned} \|u - u_\varepsilon^N\| &\leq \left(e^{2(\bar{\lambda}_{N_1+1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{N_1+1, 1}-c)(t-T)} + \right. \\ &\quad \left. + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} \right)^{1/2} m + \\ &\quad + \left(e^{2(\bar{\lambda}_{N_1, N_2}-c)t} + e^{2(\bar{\lambda}_{1, N_2}-c)t} \right)^{1/2} \varepsilon. \end{aligned} \quad (4.5)$$

n even number. The approximate solution with respect to the initial data is

$$u^N(x, y, t) = \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} \bar{\varphi}_{i,j} e^{(\bar{\lambda}_{i,j}-c)t} \bar{\Phi}_{i,j}(x, y) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tilde{\varphi}_{i,j} e^{(\tilde{\lambda}_{i,j}-c)t} \tilde{\Phi}_{i,j}(x, y),$$

the approximate solution with respect to the initial approximate data is

$$u_\varepsilon^N(x, y, t) = \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} \bar{\varphi}_{\varepsilon i,j} e^{(\bar{\lambda}_{i,j}-c)t} \bar{\Phi}_{i,j}(x, y) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tilde{\varphi}_{\varepsilon i,j} e^{(\tilde{\lambda}_{i,j}-c)t} \tilde{\Phi}_{i,j}(x, y),$$

where N_1 is regularization parameter. From the second term in inequality (4.1), we get the estimate

$$\begin{aligned} \|u^N - u_\varepsilon^N\|^2 &= \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} (\bar{\varphi}_{i,j} - \bar{\varphi}_{\varepsilon i,j})^2 e^{2(\bar{\lambda}_{i,j}-c)t} + \\ &+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (\tilde{\varphi}_{i,j} - \tilde{\varphi}_{\varepsilon i,j})^2 e^{2(\tilde{\lambda}_{i,j}-c)t} \leq \left(e^{2(\bar{\lambda}_{N_1, 1}-c)t} + e^{2(\bar{\lambda}_{1, 1}-c)t} \right) \varepsilon^2. \end{aligned} \quad (4.6)$$

We find the estimate

$$\|u - u^N\|^2 = \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{\infty} \bar{\varphi}_{i,j}^2 e^{2(\bar{\lambda}_{i,j}-c)t} \leq m^2 e^{2(\bar{\lambda}_{N_1+1, 1}-c)(t-T)} \quad (4.7)$$

for the first term as above, using the Lagrange multiplier method. Substituting the estimates (4.6) and (4.7) into inequality (4.1), we obtain the result. The estimate is

$$\|u - u_\varepsilon^N\| \leq m e^{(\bar{\lambda}_{N_1+1,1}-c)(t-T)} + \left(e^{2(\bar{\lambda}_{N_1,1}-c)t} + e^{2(\bar{\lambda}_{1,1}-c)t} \right)^{1/2} \varepsilon. \quad (4.8)$$

Case 2. Let $\varphi(x, y) = 0$, $\|f(x, y, t) - f_\varepsilon(x, y, t)\| \leq \varepsilon$ and $u(x, y, t) \in M$. In this case, we perform the substitution $u(x, y, t) = w(x, y, t) e^{-ct}$ and find the solution of the problem $w(x, y, t)$ in the form

$$w(x, y, t) = \sum_{i,j=1}^{\infty} \bar{w}_{i,j}(t) \bar{\Phi}_{i,j}(x, y) + \sum_{i,j=1}^{\infty} \tilde{w}_{i,j}(t) \tilde{\Phi}_{i,j}(x, y)$$

and from the equation

$$\begin{cases} \bar{w}_{i,j}(t) = \int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau, \\ \tilde{w}_{i,j}(t) = \int_0^t \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau, \end{cases}$$

we find the solution of the problem

$$\begin{aligned} w(x, y, t) &= \sum_{i,j=1}^{\infty} \int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \bar{\Phi}_{i,j}(x, y) + \\ &+ \sum_{i,j=1}^{\infty} \int_0^t \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \tilde{\Phi}_{i,j}(x, y), \end{aligned}$$

where $p(x, y, t) = f(x, y, t) e^{ct}$. In this case, we also consider separately the values of $\bar{\lambda}_{i,j}$ and $\tilde{\lambda}_{i,j}$ when n is an odd and even number.

n odd number. The approximate solution with respect to the initial data is

$$\begin{aligned} w^N(x, y, t) &= \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \bar{\Phi}_{i,j}(x, y) + \\ &+ \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} \int_0^t \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \tilde{\Phi}_{i,j}(x, y), \end{aligned}$$

and the approximate solution with respect to the initial approximate data is

$$\begin{aligned} w_\varepsilon^N(x, y, t) &= \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \int_0^t \bar{p}_{\varepsilon i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \bar{\Phi}_{i,j}(x, y) + \\ &+ \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} \int_0^t \tilde{p}_{\varepsilon i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \tilde{\Phi}_{i,j}(x, y). \end{aligned}$$

Using inequality (4.1), we calculate the norm of the difference between the exact solution and the approximate solution with respect to the initial approximations.

We evaluate the expression $\|w^N - w_\varepsilon^N\|$ on the right-hand side of inequality (4.1) as follows:

$$\|w^N - w_\varepsilon^N\|^2 = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \left(\int_0^t (\bar{p}_{i,j}(\tau) - \bar{p}_{\varepsilon i,j}(\tau)) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 +$$

$$+ \sum_{i=1}^{\infty} \sum_{j=1}^{N_2} \left(\int_0^t (\tilde{p}_{i,j}(\tau) - \tilde{p}_{\varepsilon i,j}(\tau)) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \leq \frac{1}{2} \left(\frac{e^{2\tilde{\lambda}_{N_1, N_2} t}}{\tilde{\lambda}_{1,1}} + \frac{e^{2\tilde{\lambda}_{1, N_2} t}}{\sigma_1} \right) \varepsilon^2, \quad (4.9)$$

where $\sigma_1 = \min_{j=1, \dots, N_2} \{|\tilde{\lambda}_{1,j}|\}$.

Now, under condition $\|w(x, y, T)\| \leq me^{cT}$ and using the Lagrange multiplier method, we obtain the estimate

$$\begin{aligned} \|w - w^N\|^2 &= \sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(\int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \\ &+ \sum_{i=1}^{N_1} \sum_{j=N_2+1}^{\infty} \left(\int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{N_2} \left(\int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \\ &+ \sum_{i=1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(\int_0^t \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \leq \\ &\leq \sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(2 \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \left(\int_t^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \right) + \\ &+ \sum_{i=1}^{N_1} \sum_{j=N_2+1}^{\infty} \left(2 \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \left(\int_t^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \right) + \\ &+ \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{N_2} \left(2 \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \left(\int_t^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \right) + \\ &+ \sum_{i=1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(2 \left(\int_0^T \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \left(\int_t^T \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \right) \leq \\ &\leq 2 \sum_{i=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \sum_{i=1}^{N_1} \sum_{j=N_2+1}^{\infty} \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \\ &+ 2 \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{N_2} \left(\int_0^T \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + 2 \sum_{i=1}^{\infty} \sum_{j=N_2+1}^{\infty} \left(\int_0^T \tilde{p}_{i,j}(\tau) e^{\tilde{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \\ &+ 2(T-t) \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_t^T \bar{p}_{i,j}^2(\tau) d\tau + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_t^T \tilde{p}_{i,j}^2(\tau) d\tau \right) \leq \\ &\leq 2m^2 \left(e^{2\tilde{\lambda}_{N_1+1, N_2+1}(t-T)+2cT} + e^{2\tilde{\lambda}_{1, N_2+1}(t-T)+2cT} + \right. \\ &\left. + e^{2\tilde{\lambda}_{N_1+1, 1}(t-T)+2cT} + e^{2\tilde{\lambda}_{1, N_2+1}(t-T)+2cT} \right) + 2(T-t) \frac{1}{(N_1 + N_2)^2}. \quad (4.10) \end{aligned}$$

Substituting the obtained estimates (4.9) and (4.10) into inequality (4.1), we get the estimate

$$\|u - u_{\varepsilon}^N\| \leq \left(\frac{e^{2(\tilde{\lambda}_{N_1, N_2} - c)t}}{2\tilde{\lambda}_{1,1}} + \frac{e^{2(\tilde{\lambda}_{1, N_2} - c)t}}{2\sigma_1} \right)^{1/2} \varepsilon +$$

$$\begin{aligned}
& + \sqrt{2}m \left(e^{2(\bar{\lambda}_{N_1+1, N_2+1}-)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-)(t-T)} + \right. \\
& \left. + e^{2(\bar{\lambda}_{N_1+1, 1}-)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-)(t-T)} \right)^{1/2} + \sqrt{2(T-t)} \frac{1}{(N_1 + N_2)}.
\end{aligned} \tag{4.11}$$

n even number. We obtain the approximate solution with respect to the initial data as

$$\begin{aligned}
w^N(x, y, t) &= \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} \int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \bar{\Phi}_{i,j}(x, y) + \\
&+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^t \tilde{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \tilde{\Phi}_{i,j}(x, y),
\end{aligned}$$

and the approximate solution with respect to the initial approximate data as

$$\begin{aligned}
w_{\varepsilon}^N(x, y, t) &= \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} \int_0^t \bar{p}_{\varepsilon i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \bar{\Phi}_{i,j}(x, y) + \\
&+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^t \tilde{p}_{\varepsilon i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \tilde{\Phi}_{i,j}(x, y).
\end{aligned}$$

From the second sum in inequality (4.1), we obtain an estimate of the form

$$\begin{aligned}
\|w^N - w_{\varepsilon}^N\|^2 &= \sum_{i=1}^{N_1} \sum_{j=1}^{\infty} \left(\int_0^t (\bar{p}_{i,j}(\tau) - \bar{p}_{\varepsilon i,j}(\tau)) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 + \\
&+ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\int_0^t (\tilde{p}_{i,j}(\tau) - \tilde{p}_{\varepsilon i,j}(\tau)) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \leq \\
&\leq \left(\frac{e^{2\bar{\lambda}_{N_1,1}t}}{2\sigma_2} + \frac{e^{2\bar{\lambda}_{1,1}t}}{2|\bar{\lambda}_{1,1}|} \right) \varepsilon^2.
\end{aligned} \tag{4.12}$$

where $\sigma_2 = \min_{i=1, \dots, N_1} \{|\bar{\lambda}_{i,1}|\}$.

We also find $\|w - w^N\|$ using the Lagrange multiplier method as above, and obtain the estimate

$$\begin{aligned}
\|w - w^N\|^2 &= \sum_{i=N_1+1}^{\infty} \sum_{j=1}^{\infty} \left(\int_0^t \bar{p}_{i,j}(\tau) e^{\bar{\lambda}_{i,j}(t-\tau)} d\tau \right)^2 \leq \\
&\leq m^2 e^{2\bar{\lambda}_{N_1+1,1}(t-T)+2cT}.
\end{aligned} \tag{4.13}$$

We substitute the estimates (4.12) and (4.13) with inequality (4.1) and obtain the result

$$\|u - u_{\varepsilon}^N\| \leq \left(\frac{e^{2(\bar{\lambda}_{N_1,1}-c)t}}{2\sigma_2} + \frac{e^{2(\bar{\lambda}_{1,1}-c)t}}{2|\bar{\lambda}_{1,1}|} \right)^{1/2} \varepsilon + m e^{(\bar{\lambda}_{N_1+1,1}-c)(t-T)}. \tag{4.14}$$

By minimizing the above results, which are the two cases we choose as $f(x, y, t) = 0$ and $\varphi(x, y) = 0$, and the right-hand side of the evaluations for the even and odd numbers n in them, the available regularization parameter N is found.

For the first case, we show that when $\|\varphi(x, y) - \varphi_\varepsilon(x, y)\| \leq \varepsilon$, it is possible to choose the regularization parameter N such that $\|u - u_\varepsilon^N\| \leq C\varepsilon$, where C is a constant. From estimate (4.5), as $\varepsilon \rightarrow 0$, there exists such an N that the following expression tends to zero:

$$D(t, \varepsilon) = \left(e^{2(\bar{\lambda}_{N_1+1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{N_1+1, 1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} \right)^{1/2} m + \\ + \left(e^{2(\bar{\lambda}_{N_1, N_2}-c)t} + e^{2(\bar{\lambda}_{1, N_2}-c)t} \right)^{1/2} \varepsilon.$$

Then one can show that

$$\lim_{\varepsilon \rightarrow 0} D(t, \varepsilon) = 0. \quad (4.15)$$

Assume that δ is a sufficiently small positive number. From the relation

$$\lim_{N \rightarrow \infty} J(t, m, N) = 0,$$

it follows that there exists $N(\delta)$ such that for all $N \geq N(\delta)$,

$$J(t, m, N) \leq \frac{\delta}{2},$$

where

$$J(t, m, N) = \left(e^{2(\bar{\lambda}_{N_1+1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{N_1+1, 1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} + e^{2(\bar{\lambda}_{1, N_2+1}-c)(t-T)} \right)^{1/2} m.$$

Now define $Z(\delta) = \inf_{N \geq N(\delta)} \left\{ \left(e^{2(\bar{\lambda}_{N_1, N_2}-c)t} + e^{2(\bar{\lambda}_{1, N_2}-c)t} \right)^{1/2} \right\}$. If

$$\varepsilon \leq \frac{1}{2} \cdot \frac{\delta}{Z(\delta)},$$

then the function $D(t, \varepsilon)$ satisfies the inequality $D(t, \varepsilon) \leq \delta$. This proves relation (4.15). Using the same approach, and relying on estimates (4.8), (4.11), and (4.14), one can similarly choose the parameter N depending on ε .

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Coupled coincidence point sets of α -dominated mappings and stability analysis

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Abstract. In this paper we utilize a rational metric inequality for the purpose of establishing a coupled coincidence point theorem for a coupled mapping and a single-valued mapping on a metric space. We define some regularity condition on the mappings which are extensions of ideas involved in the well known admissibility condition. In the second part of the paper we make a stability analysis of the set of coupled coincidence points obtained here. All the main results are supported with illustrative examples. The work is in the domain of set valued analysis.

Keywords: Metric space; dominated maps; Hausdorff distance; coupled coincidence point, stability
MSC (2020): 54H10, 54H25, 47H10

1. INTRODUCTION AND MATHEMATICAL BACKGROUND

The present paper is about multi-valued coupled coincidence points and stability associated with coupled coincidence point sets of certain multi-valued mappings and single-valued mapping defined on complete metric spaces. The idea of a coupled fixed point originated in the work of Guo et al [1] in 1987. It attracts much attention after the appearance of the work of Bhaskar et al [2] in 2006 where a coupled contraction mapping theorem was established in partially ordered metric spaces. Several other works followed in a large number exploring the coupled fixed point and related properties of different types of coupled mappings in various types of spaces. Some instances of works in the extensive literature created on this research area in the last decade are [3, 4, 5]. The concept of a coupled fixed point was extended to the domain of set-valued analysis by [6] after which several works have appeared in this direction like those noted in [7, 8, 9].

Fixed point theory in metric spaces equipped with a partial order has developed rapidly in recent years. Although an early result appeared in the work of Turinici [10] in 1986, the actual development of this line of research took place following the works of Ran et al. [11] and Nieto et al. [12]. Some recent works in this area are [13, 14, 15, 16]. The logical content of this study is that for the existence of fixed points, the various types of contractive inequality conditions to be satisfied by the operators for pairs of points collected from the metric space can be restricted. Several of these results are still valid in cases where the corresponding contractive conditions are assumed for those pairs of points related through the partial order. In fact, upon close examination of the proofs of several fixed point theorems of various types of contractive operators including the celebrated Banach's contraction mapping principle [17] one finds that the satisfaction of contraction inequality conditions are not used between every pair of points from the metric space. It is possible to restrict it to suitable pairs of points without disturbing the conclusions of the theorems. This is what is accomplished through the introduction of the partial order relation. In particular, in their work Bhaskar et al [2] considered partially ordered metric spaces and used the mixed-monotone property of the coupled maps corresponding to the partial ordering. Some works in this subject area are [3, 4, 5].

It has been shown by Samet et al [18] that the above purpose of introducing the partial order can be also served by a set of conditions called admissibility conditions. Originally α -admissibility was introduced in [18]. Later, several other types of admissibilities have been introduced and used in the domain of fixed point theory. These conditions, rather than introducing a new structure like partial order in metric spaces, are requirements on the operator under consideration. We note a few works from this recently originated branch of study in [19, 20, 21, 22, 23, 24].

Stability is a concept associated with the limiting behaviors of a system. It has been studied in various contexts of discrete and continuous dynamical systems [25, 26]. The study of the relationship between the convergence of a sequence of mappings and their fixed points, known as the stability of fixed points has also been widely studied in various settings [27, 28, 29, 30, 31]. The fixed point sets

of a sequence of mappings are said to be stable if they converge to the set of fixed points of the limit mapping in the Hausdorff metric. Multi-valued mappings often have more fixed points than their single-valued counterparts [29, 30, 31]. Therefore, the set of fixed points of multi-valued mappings becomes larger and hence more interesting for the study of stability.

In the present context, putting together some of the ideas mentioned above, we work out a fixed point result for coupled multi-valued mappings on complete metric spaces. In the last part of the paper we perform the stability analysis of the fixed point sets corresponding to a convergent sequence of multi-valued coupled mappings of the type considered here.

In the following, we discuss some concepts and definitions that we subsequently use in this paper.

Let (X, d) be a metric space. Let $N(X) :=$ the collection of all non-empty subsets of X ; $CL(X) :=$ the collection of all non-empty closed and bounded subsets of X ; and $K(X) :=$ the collection of all compact subset of X . Let $A, B \in CL(X)$, we use following notations and definitions

$$D(x, B) = \inf \{d(x, y) : y \in B\}, \text{ where } x \in X,$$

$$d(A, B) = D(A, B) = \inf \{d(a, b) : a \in A, b \in B\},$$

$$H(A, B) = \max \left\{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A) \right\},$$

H is known as the Hausdorff metric on $CL(X)$ [31]. Furthermore, if (X, d) is complete then $(CL(X), H)$ is also complete.

Lemma 1.1. Let (X, d) be a metric space and $B \in K(X)$. Then for every $x \in X$ there exists $y \in B$ such that $d(x, y) = D(x, B)$.

Proof. Let $B \in K(X)$ and $x \in X$. From the definition, $D(x, B) = \inf \{d(x, b) : b \in B\}$. Then there exists a sequence $\{y_n\}$ in B such that $d(x, y_n) \rightarrow D(x, B)$, as $n \rightarrow \infty$. Since B is compact, $\{y_n\}$ has a convergent subsequence $\{y_{n(k)}\}$. Hence, there exists $y \in X$ such that $y_{n(k)} \rightarrow y$ as $k \rightarrow \infty$. As B is compact, it is closed and hence $y \in B$. Now, $\lim_{n \rightarrow \infty} d(x, y_n) = D(x, B)$ implies that $\lim_{k \rightarrow \infty} d(x, y_{n(k)}) = D(x, B)$, that is, $d(x, y) = D(x, B)$.

Definition 1.1. [5] Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$. Then X is said to have α -regular property if $\{x_n\}$ is a sequence with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x$ then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

We propose the following three definitions for our purpose in the next section. These are conceptual extensions of the admissibility conditions[18].

Definition 1.2. Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping. A mapping $F : X \rightarrow X$ is said to be α -dominated if $\alpha(x, Fx) \geq 1$, for all $x \in X$.

Definition 1.3. Let (X, d) be a metric space and $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping. A multi-valued mapping $F : X \rightarrow N(X)$ is said to be α -dominated if $\alpha(x, y) \geq 1$, for $x \in X$ and $y \in Fx$.

Definition 1.4. Let X be a non-empty set and $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping. A multi-valued mapping $F : X \times X \rightarrow N(X)$ is said to be α -dominated with respect to a function $f : X \rightarrow X$ if for $x, y \in X$, $\alpha(fx, u) \geq 1$ and $\alpha(fy, v) \geq 1$ whenever $u \in F(x, y)$ and $v \in F(y, x)$.

Definition 1.5. [7] Let X be a non-empty set. An element $(x, y) \in X \times X$ is said to be a coupled coincidence point of a multi-valued mapping $F : X \times X \rightarrow N(X)$ and a single-valued $f : X \rightarrow X$ if $fx \in F(x, y)$ and $fy \in F(y, x)$.

We denote the set of all coupled coincidence points of F and f by the notation $C(F, f)$.

We propose the following definition of stability of coupled coincidence point sets. If (X, d) is a metric space, then $X \times X$ is also a metric space under the metric ρ defined by $\rho((x, y), (u, v)) = \max \{d(x, u), d(y, v)\}$, for all (x, y) and $(u, v) \in X \times X$. Let H_ρ be the Hausdorff metric induced by ρ .

Definition 1.6. Let (X, d) be a metric space, $F: X \times X \rightarrow CL(X)$ be a multi-valued mapping and $f: X \rightarrow X$. Let $\{F_n: X \times X \rightarrow CL(X)\}$ be a sequence of multi-valued mappings with $F = \lim_{n \rightarrow \infty} F_n$. Suppose that $\{C(F_n, f)\}$ is the sequence of coupled coincidence point sets of the sequence of pairs of mappings $\{(F_n, f) : n \in \mathbb{N}\}$ and $C(F, f)$ is the coupled coincidence point set of the pair of mappings (F, f) . We say that the coupled coincidence point sets of the sequence $\{(F_n, f) : n \in \mathbb{N}\}$ are stable if $\lim_{n \rightarrow \infty} H_\rho(C(F_n, f), C(F, f)) = 0$.

The above definition is proposed by incorporating the same ideas and in the same vein of Nadler on stability in his work [31] which is incidentally the first paper on set-valued fixed point theory.

2. MAIN RESULTS

Theorem 2.1. Let (X, d) be a complete metric space, $F: X \times X \rightarrow K(X)$ be a multi-valued mapping, $f: X \rightarrow X$ and $\alpha: X \times X \rightarrow [0, \infty)$. Suppose that (i) (X, d) has α -regular property, (ii) F is α -dominated with respect to f , (iii) $F(x, y) \subseteq f(X)$ for all $(x, y) \in X \times X$, (iv) $f(X)$ is a closed subset of X , (v) there exist nonnegative real numbers a, b, c, e with $(a + b + 2c + 2e) < 1$ such that for all $x, y, u, v \in X$ with $\alpha(fx, fu) \geq 1$ and $\alpha(fy, fv) \geq 1$,

$$\begin{aligned} H(F(x, y), F(u, v)) &\leq a \frac{D(fx, F(x, y)) D(fu, F(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\ &\quad + c [D(fx, F(x, y)) + D(fu, F(u, v))] \\ &\quad + e [D(fx, F(u, v)) + D(fu, F(x, y))]. \end{aligned}$$

Then the set of coupled coincidence point of F and f is non-empty, that is, $C(F, f) \neq \emptyset$.

Proof. Let $x_0, y_0 \in X$ be arbitrary. The assumption (iii) implies that for every $z \in F(x, y)$ there exists $p \in X$ such that $fp = z$. Suppose $x_1, y_1 \in X$ such that $fx_1 \in F(x_0, y_0)$ and $fy_1 \in F(y_0, x_0)$. By the assumption (ii), we have

$$\alpha(fx_0, fx_1) \geq 1 \quad \text{and} \quad \alpha(fy_0, fy_1) \geq 1.$$

By Lemma 1.1, there exist $x_2, y_2 \in X$ with $fx_2 \in F(x_1, y_1)$ and $fy_2 \in F(y_1, x_1)$ such that

$$d(fx_1, fx_2) = D(fx_1, F(x_1, y_1)) \quad \text{and} \quad d(fy_1, fy_2) = D(fy_1, F(y_1, x_1)).$$

By the assumption (ii), we have

$$\alpha(fx_1, fx_2) \geq 1 \quad \text{and} \quad \alpha(fy_1, fy_2) \geq 1.$$

Again, by Lemma 1.1, there exist $x_3, y_3 \in X$ with $fx_3 \in F(x_2, y_2)$ and $fy_3 \in F(y_2, x_2)$ such that

$$d(fx_2, fx_3) = D(fx_2, F(x_2, y_2)) \quad \text{and} \quad d(fy_2, fy_3) = D(fy_2, F(y_2, x_2)).$$

Using the assumption (ii), we have

$$\alpha(fx_2, fx_3) \geq 1 \quad \text{and} \quad \alpha(fy_2, fy_3) \geq 1.$$

Continuing this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $n \geq 0$,

$$fx_{n+1} \in F(x_n, y_n) \quad \text{and} \quad fy_{n+1} \in F(y_n, x_n), \quad (2.1)$$

$$d(fx_{n+1}, fx_{n+2}) = D(fx_{n+1}, F(x_{n+1}, y_{n+1})), \quad (2.2)$$

$$d(fy_{n+1}, fy_{n+2}) = D(fy_{n+1}, F(y_{n+1}, x_{n+1})) \quad (2.3)$$

and

$$\alpha(fx_n, fx_{n+1}) \geq 1 \quad \text{and} \quad \alpha(fy_n, fy_{n+1}) \geq 1. \quad (2.4)$$

We first prove that the sequences $\{fx_n\}$ and $\{fy_n\}$ are Cauchy sequences. By (2.3), (2.4) and the assumption (v), we have

$$\begin{aligned}
 d(fx_{n+1}, fx_{n+2}) &= D(fx_{n+1}, F(x_{n+1}, y_{n+1})) \leq H(F(x_n, y_n), F(x_{n+1}, y_{n+1})) \\
 &\leq a \frac{D(fx_n, F(x_n, y_n)) D(fx_{n+1}, F(x_{n+1}, y_{n+1}))}{1 + d(fx_n, fx_{n+1})} + b d(fx_n, fx_{n+1}) \\
 &\quad + c [D(fx_n, F(x_n, y_n)) + D(fx_{n+1}, F(x_{n+1}, y_{n+1}))] \\
 &\quad + e [D(fx_n, F(x_{n+1}, y_{n+1})) + D(fx_{n+1}, F(x_n, y_n))] \\
 &\leq a \frac{d(fx_n, fx_{n+1}) d(fx_{n+1}, fx_{n+2})}{1 + d(fx_n, fx_{n+1})} + b d(fx_n, fx_{n+1}) \\
 &\quad + c [d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2})] \\
 &\quad + e [d(fx_n, fx_{n+2}) + d(fx_{n+1}, fx_{n+1})] \\
 &\leq a d(fx_{n+1}, fx_{n+2}) + b d(fx_n, fx_{n+1}) \\
 &\quad + c [d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2})] \\
 &\quad + e [d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2})], \tag{2.5}
 \end{aligned}$$

which implies that

$$d(fx_{n+1}, fx_{n+2}) \leq \frac{[b + c + e]}{[1 - a - c - e]} d(fx_n, fx_{n+1}). \tag{2.6}$$

Similarly, we have

$$d(fy_{n+1}, fy_{n+2}) \leq \frac{[b + c + e]}{[1 - a - c - e]} d(fy_n, fy_{n+1}). \tag{2.7}$$

Let $k = \frac{(b+c+e)}{1-(a+c+e)}$, then $0 \leq k < 1$. By repeated application of (2.6) and (2.7), we respectively have

$$d(fx_{n+1}, fx_{n+2}) \leq k d(fx_n, fx_{n+1}) \leq k^2 d(fx_{n-1}, fx_n) \leq \dots \leq k^{n+1} d(fx_0, fx_1) \tag{2.8}$$

and

$$d(fy_{n+1}, fy_{n+2}) \leq k d(fy_n, fy_{n+1}) \leq k^2 d(fy_{n-1}, fy_n) \leq \dots \leq k^{n+1} d(fy_0, fy_1). \tag{2.9}$$

Using (2.8), we have for $m > n > 0$ that

$$\begin{aligned}
 d(fx_n, fx_m) &\leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \dots + d(fx_{m-1}, fx_m) \\
 &\leq [k^n + k^{n+1} \dots + k^{m-1}] d(fx_0, fx_1) \\
 &\leq \frac{k^n [1 - k^{n-m+1}]}{1 - k} d(fx_0, fx_1) \leq \frac{k^n}{1 - k} d(fx_0, fx_1). \tag{2.10}
 \end{aligned}$$

Since $0 < k < 1$, taking limit as $m, n \rightarrow \infty$ in (2.10), we have

$$\lim_{m, n \rightarrow \infty} d(fx_n, fx_m) = 0, \tag{2.11}$$

which implies that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Similarly, we can show that $\{fy_n\}$ is a Cauchy sequence in $f(X)$. Since X is complete, by the assumption (iv), $(f(X), d)$ is also complete. So there exist $x, y \in X$ such that

$$\lim_{n \rightarrow \infty} fx_n = fx \quad \text{and} \quad \lim_{n \rightarrow \infty} fy_n = fy. \tag{2.12}$$

We now prove that this (x, y) is a coupled coincidence point of F and f . Using (2.4), (2.12) and the assumption (i), we have $\alpha(fx_n, fx) \geq 1$ and $\alpha(fy_n, fy) \geq 1$ for all n . Now applying the assumption

(v), we have

$$\begin{aligned}
 D(fx_{n+1}, F(x, y)) &\leq H(F(x_n, y_n), F(x, y)) \\
 &\leq a \frac{D(fx_n, F(x_n, y_n)) D(fx, F(x, y))}{1 + d(fx_n, fx)} + b d(fx_n, fx) \\
 &\quad + c [D(fx_n, F(x_n, y_n)) + D(fx, F(x, y))] \\
 &\quad + e [D(fx_n, F(x, y)) + D(fx, F(x_n, y_n))] \\
 &\leq a \frac{d(fx_n, fx_{n+1}) D(fx, F(x, y))}{1 + d(fx_n, fx)} + b d(fx_n, fx) \\
 &\quad + c[d(fx_n, fx_{n+1}) + D(fx, F(x, y))] \\
 &\quad + e[D(fx_n, F(x, y)) + d(fx, fx_{n+1})].
 \end{aligned}$$

Now taking limit as $n \rightarrow \infty$ in the above inequality and using (2.12), we have

$$D(fx, F(x, y)) \leq c D(fx, F(x, y)) + e D(fx, F(x, y)),$$

that is,

$$(1 - c - e) D(fx, F(x, y)) \leq 0.$$

Since $1 - c - e > 0$, it follows that $D(fx, F(x, y)) = 0$. Since $F(x, y) \in K(X)$. So $fx \in F(x, y)$. Similarly we can prove that $fy \in F(y, x)$. Hence (x, y) is a coupled coincidence point of F and f , that is, $C(F, f) \neq \emptyset$.

Example 2.1. Let $X = [0, 1]$ and d be the usual metric on X . Define $F : X \times X \rightarrow K(X)$ by $F(x, y) = [0, \frac{x}{9}]$ for $(x, y) \in X \times X$ and $f : X \rightarrow X$ by $f(x) = \frac{x}{3}$ for $x \in X$. Let $\alpha(x, y) = 2$ for all $x, y \in X$ and $a = \frac{1}{10}$, $b = \frac{1}{2}$, $c = e = \frac{1}{12}$. All the conditions of Theorem 2.1 are satisfied and $(0, 0) \in C(F, f)$, that is, $C(F, f)$ is non-empty.

3. STABILITY OF COINCIDENCE POINT SETS

In this section, we investigate the stability of coupled coincidence point sets of the set valued contractions.

Theorem 3.1. Let (X, d) be a complete metric space, $F_j : X \times X \rightarrow K(X)$, $j = 1, 2$ be two multi-valued mappings, $f : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. Suppose that (i) (X, d) has α -regular property, (ii) each F_j ($j = 1, 2$) is α -dominated with respect to f , (iii) $F_j(x, y) \subseteq f(X)$ for all $(x, y) \in X \times X$ and $j = 1, 2$, (iv) $f(X)$ is a closed subset of X , (v) there exist nonnegative real numbers a, b, c, e with $(a + b + 2c + 2e) < 1$ such that for all $x, y, u, v \in X$ with $\alpha(fx, fu) \geq 1$ and $\alpha(fy, fv) \geq 1$,

$$\begin{aligned}
 H(F_j(x, y), F_j(u, v)) &\leq a \frac{D(fx, F_j(x, y)) D(fu, F_j(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\
 &\quad + c [D(fx, F_j(x, y)) + D(fu, F_j(u, v))] \\
 &\quad + e [D(fx, F_j(u, v)) + D(fu, F_j(x, y))].
 \end{aligned}$$

Then $C(F_j, f) \neq \emptyset$ for $j = 1, 2$. Suppose $M_j = \sup\{d(x, fx), d(y, fy) : (x, y) \in C(F_j, f)\}$, where $j = 1, 2$, and $\Delta = \sup_{x, y \in X} H(F_1(x, y), F_2(x, y))$ exist and $M = \max\{M_1, M_2\}$. Then

$$H_\rho(C(F_1, f), C(F_2, f)) \leq 2M + \frac{\Delta}{1 - k}, \text{ where } k = \frac{(b + c + e)}{1 - (a + c + e)}.$$

Proof. By Theorem 2.1, $C(F_j, f) \neq \emptyset$ for $j = 1, 2$. Let $(x_0, y_0) \in C(F_1, f)$; that is

$$fx_0 \in F_1(x_0, y_0) \text{ and } fy_0 \in F_1(y_0, x_0). \quad (3.1)$$

By the assumption (iii) and Lemma 1.1 there exist $x_1, y_1 \in X$ with $fx_1 \in F_2(x_0, y_0)$ and $fy_1 \in F_2(y_0, x_0)$ such that

$$d(fx_0, fx_1) = D(fx_0, F_2(x_0, y_0)) \leq H(F_1(x_0, y_0), F_2(x_0, y_0)) \leq \Delta \quad (3.2)$$

and

$$d(fy_0, fy_1) = D(fy_0, F_2(y_0, x_0)) \leq H(F_1(y_0, x_0), F_2(y_0, x_0)) \leq \Delta. \quad (3.3)$$

By the assumption (ii) on F_2 , we have

$$\alpha(fx_0, fx_1) \geq 1 \text{ and } \alpha(fy_0, fy_1) \geq 1.$$

By the assumption (iii) and Lemma 1.1 there exist $x_2, y_2 \in X$ with $fx_2 \in F_2(x_1, y_1)$ and $fy_2 \in F_2(y_1, x_1)$ such that

$$d(fx_1, fx_2) = D(fx_1, F_2(x_1, y_1)) \leq H(F_2(x_0, y_0), F_2(x_1, y_1))$$

and

$$d(fy_1, fy_2) = D(fy_1, F_2(y_1, x_1)) \leq H(F_2(y_0, x_0), F_2(y_1, x_1)).$$

By the assumption (ii) on F_2 , we have

$$\alpha(fx_1, fx_2) \geq 1 \text{ and } \alpha(fy_1, fy_2) \geq 1.$$

Then arguing as in the proof of Theorem 2.1 we construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$fx_{n+1} \in F_2(x_n, y_n) \text{ and } fy_{n+1} \in F_2(y_n, x_n), \quad (3.4)$$

$$d(fx_{n+1}, fx_{n+2}) = D(fx_{n+1}, F_2(x_{n+1}, y_{n+1})) \leq H(F_2(x_n, y_n), F_2(x_{n+1}, y_{n+1})), \quad (3.5)$$

$$d(fy_{n+1}, fy_{n+2}) = D(fy_{n+1}, F_2(y_{n+1}, x_{n+1})) \leq H(F_2(y_n, x_n), F_2(y_{n+1}, x_{n+1})) \quad (3.6)$$

and also (2.4) is satisfied. Arguing similarly as in the proof of Theorem 2.1 we prove that (1) (2.8) and (2.9) are satisfied; (2) $\{fx_n\}$ and $\{fy_n\}$ are Cauchy sequences in $f(X)$; (3) there exist $u, v \in X$ such that $\lim_{n \rightarrow \infty} fx_n = fu$ and $\lim_{n \rightarrow \infty} fy_n = fv$; and (4) (u, v) is a coupled coincidence of F_2 and f , that is, $(u, v) \in C(F_2, f)$, that is,

$$fu \in F_2(u, v) \text{ and } fv \in F_2(v, u). \quad (3.7)$$

Using (2.8), (3.2), we have

$$\begin{aligned} d(x_0, u) &\leq d(x_0, fx_0) + d(fx_0, u) \leq d(x_0, fx_0) + \sum_{i=0}^n d(fx_i, fx_{i+1}) + d(fx_{n+1}, u) \\ &\leq d(x_0, fx_0) + \sum_{i=0}^n k^i d(fx_0, fx_1) + d(fx_{n+1}, u) \\ &\leq M_1 + \frac{1}{1-k} d(fx_0, fx_1) + d(fx_{n+1}, u) \leq M + \frac{\Delta}{1-k} + d(fx_{n+1}, u), \end{aligned}$$

where $k = \frac{(b+c+e)}{1-(a+c+e)}$ is taken in the proof of Theorem 2.1. Taking the limit as $n \rightarrow \infty$ in the above inequality, we have

$$d(x_0, u) \leq M + \frac{\Delta}{1-k} + d(fu, u) \leq M + \frac{\Delta}{1-k} + M_2 \leq 2M + \frac{\Delta}{1-k}. \quad (3.8)$$

Similarly

$$d(y_0, v) \leq 2M + \frac{\Delta}{1-k}. \quad (3.9)$$

Combining (3.8) and (3.9), we have

$$\max \{d(x_0, u), d(y_0, v)\} \leq 2M + \frac{\Delta}{1-k}.$$

that is,

$$\rho(d(x_0, y_0), d(u, v)) \leq 2M + \frac{\Delta}{1-k}.$$

So we have for $(x_0, y_0) \in C(F_1, f)$ there exists $(u, v) \in C(F_2, f)$ such that $\rho(d(x_0, y_0), d(u, v)) \leq 2M + \frac{\Delta}{1-k}$. Similarly we can show that for $(s_0, t_0) \in C(F_2, f)$ there exists $(p, q) \in C(F_1, f)$ such that $\rho(d(s_0, t_0), d(p, q)) \leq 2M + \frac{\Delta}{1-k}$. Then it follows that $H_\rho(C(F_1, f), C(F_2, f)) \leq 2M + \frac{\Delta}{1-k}$.

Lemma 3.1. *Let (X, d) be a complete metric space, $\{F_n : X \times X \rightarrow K(X)\}$ be a sequence of multi-valued mappings, $f : X \rightarrow X$ with $f(X)$ is closed in X and $\alpha : X \times X \rightarrow [0, \infty)$. Suppose that the sequence $\{F_n\}$ converges uniformly to a multi-valued mapping $F : X \times X \rightarrow K(X)$. Suppose that (i) each F_n ($n \in \mathbb{N}$) is α -dominated with respect to f ; (ii) $F_n(x, y) \subseteq f(X)$ for all $(x, y) \in X \times X$ and $n \in \mathbb{N}$; (iii) there exist nonnegative real numbers a, b, c, e with $(a + b + 2c + 2e) < 1$ such that for every $n \in \mathbb{N}$ and for all $x, y, u, v \in X$ with $\alpha(fx, fu) \geq 1$ and $\alpha(fy, fv) \geq 1$,*

$$\begin{aligned} H(F_n(x, y), F_n(u, v)) &\leq a \frac{D(fx, F_n(x, y)) D(fu, F_n(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\ &\quad + c [D(fx, F_n(x, y)) + D(fu, F_n(u, v))] \\ &\quad + e [D(fx, F_n(u, v)) + D(fu, F_n(x, y))]. \end{aligned}$$

Also, suppose that for any $u \in X$ and for any sequence $\{x_n\}$ in X the following condition is satisfied.

$$\alpha(u, x_n) \geq 1 \text{ for every } n \in \mathbb{N} \Rightarrow \alpha(u, x) \geq 1, \text{ whenever } \{x_n\} \rightarrow x \text{ as } n \rightarrow \infty. \quad (3.10)$$

Then F also satisfies the assumptions (ii), (iii) and (v) of Theorem 2.1.

Proof. Let $(x, y) \in X \times X$ and $u \in F(x, y)$. Since $F_n \rightarrow F$ uniformly, there exists a sequence $\{x_n \in F_n(x, y)\}$ such that $x_n \rightarrow u$ and as $n \rightarrow \infty$. By the assumption (i), $\alpha(fx, x_n) \geq 1$ for every $n \in \mathbb{N}$. By (3.10), it follows that $\alpha(fx, u) \geq 1$. Thus we have that $\alpha(fx, u) \geq 1$ for all $u \in F(x, y)$. Similarly, we can show that $\alpha(fy, v) \geq 1$, for all $v \in F(y, x)$. Hence F satisfies the assumption (ii) of Theorem 2.1.

Let $(x, y) \in X \times X$ and $t \in F(x, y)$. Since $F_n \rightarrow F$ uniformly, there exists a sequence $\{x_n\}$ with $x_n \in F_n(x, y)$ such that $x_n \rightarrow t$, as $n \rightarrow \infty$. By the assumption (ii), $\{x_n\}$ is a sequence in $f(X)$. As $f(X)$ is closed, we have $t \in f(X)$. So $t \in F(x, y) \implies t \in f(X)$. Hence $F(x, y) \subseteq f(X)$, that is, F satisfies the assumption (iii) of Theorem 2.1.

Let $(x, y), (u, v) \in X \times X$ with $\alpha(fx, fu) \geq 1$ and $\alpha(fy, fv) \geq 1$. Then applying the assumption (iii), we have

$$\begin{aligned} H(F_n(x, y), F_n(u, v)) &\leq a \frac{D(fx, F_n(x, y)) D(fu, F_n(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\ &\quad + c [D(fx, F_n(x, y)) + D(fu, F_n(u, v))] \\ &\quad + e [D(fx, F_n(u, v)) + D(fu, F_n(x, y))]. \end{aligned}$$

Since F_n converges uniformly to F , taking limit $n \rightarrow \infty$, we get

$$\begin{aligned} H(F(x, y), F(u, v)) &\leq a \frac{D(fx, F(x, y)) D(fu, F(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\ &\quad + c [D(fx, F(x, y)) + D(fu, F(u, v))] \\ &\quad + e [D(fx, F(u, v)) + D(fu, F(x, y))]. \end{aligned}$$

Hence F satisfies the assumption (v) of Theorem 2.1.

Theorem 3.2. Let (X, d) be a complete metric space, $\{F_n : X \times X \rightarrow K(X)\}$ be a sequence of multi-valued mappings, $f : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. Suppose that the sequence $\{F_n\}$ converges uniformly to a multi-valued mapping $F : X \times X \rightarrow K(X)$. Suppose that (i) (X, d) has α -regular property; (ii) each F_n ($n \in \mathbb{N}$) is α -dominated with respect to f ; (iii) $F_n(x, y) \subseteq f(X)$ for all $(x, y) \in X \times X$ and $n \in \mathbb{N}$; (iv) $f(X)$ is closed in X ; (v) there exist nonnegative real numbers a, b, c, e with $(a + b + 2c + 2e) < 1$ such that for every $n \in \mathbb{N}$ and for all $x, y, u, v \in X$ with $\alpha(fx, fu) \geq 1$ and $\alpha(fy, fv) \geq 1$,

$$\begin{aligned} H(F_n(x, y), F_n(u, v)) &\leq a \frac{D(fx, F_n(x, y)) D(fu, F_n(u, v))}{1 + d(fx, fu)} + b d(fx, fu) \\ &\quad + c [D(fx, F_n(x, y)) + D(fu, F_n(u, v))] \\ &\quad + e [D(fx, F_n(u, v)) + D(fu, F_n(x, y))]. \end{aligned}$$

Also, suppose that (3.10) is satisfied. Then $C(F_n, f) \neq \emptyset$ for every $n \in \mathbb{N}$ and $C(F, f) \neq \emptyset$. Suppose that $M_n = \sup\{d(x, fx), d(y, fy) : (x, y) \in C(F_n, f) \cup C(F, f)\}$ and $\Delta_n = \sup_{x, y \in X} H(F_n(x, y), F(x, y))$ exist and $M_n \rightarrow 0$ as $n \rightarrow \infty$. Then $\lim_{n \rightarrow \infty} H_\rho(C(F_n, f), C(F, f)) = 0$, that is, the coincidence point sets of the sequence of pairs of mappings $\{(F_n, f)\}$ are stable.

Proof. By Lemma 3.1 and Theorem 2.1, $C(F_n, f) \neq \emptyset$ for every $n \in \mathbb{N}$ and $C(F, f) \neq \emptyset$. By Theorem 3.1, we have

$$H_\rho(C(F_n, f), C(F, f)) \leq 2M_n + \frac{\Delta_n}{1 - k}.$$

Since T_n converges to T uniformly, we get

$$\lim_{n \rightarrow \infty} H_\rho(C(F_n, f), C(F, f)) \leq 0,$$

which implies that $\lim_{n \rightarrow \infty} H_\rho(C(F_n, f), C(F, f)) = 0$, that is, the coincidence point sets of the sequence of pairs of mappings $\{(F_n, f)\}$ are stable.

Example 3.1. Let $X = [0, 1]$ be a set and d be the usual metric. Define $F : X \times X \rightarrow K(X)$ and $F_n : X \times X \rightarrow K(X)$ respectively by $F(x, y) = [0, \frac{x}{9}]$ and $F_n(x, y) = [0, \frac{x}{9} + \frac{2}{9n}]$ for $(x, y) \in X \times X$. Let $f(x) = \frac{x}{3}$ for all $x \in X$, $\alpha(x, y) = 2$ for all $x, y \in X$ and $a = \frac{1}{10}$, $b = \frac{1}{2}$, $c = e = \frac{1}{12}$. Then all the conditions of the Theorem 3.2 are satisfied. Here $C(F_n, f) = \{(x, y) : x, y \in [0, \frac{1}{n}]\}$ for every n , $C(F, f) = \{(0, 0)\}$ and the coincidence point sets of the sequence of pairs of mappings $\{(F_n, f)\}$ are stable.

Authors' contributions. The authors contributed equally to this work. All authors have read and approved the manuscript.

Acknowledgement: The authors gratefully acknowledge the suggestions made by the learned referee.

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The discrete spectrum of the three-particle Schrödinger operator on $\mathbb{Z}^2$

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Abstract. We study a three-particle lattice Hamiltonian describing two identical bosons and one fermion on $\mathbb{Z}^2$, interacting via attractive or repulsive zero-range pair potentials. We analyze the point spectrum of the associated three-particle lattice discrete Schrödinger operator and describe the existence and multiplicity of bound states. Particular attention is devoted to the case in which the bosons have infinite mass.

Keywords: Schrödinger operator, dispersion functions, zero-range pair potentials, discrete spectrum, essential spectrum

MSC (2020): Primary 81Q10; Secondary 35P20, 47N50

1. INTRODUCTION

The Efimov effect, first discovered by V. Efimov in 1970 [1], is one of the most remarkable and counterintuitive phenomena in quantum physics. It arises in three-body systems in three-dimensional space where the particles interact via short-range pairwise potentials. Notably, it is possible to tune the interaction strengths such that none of the two-particle subsystems supports a negative-energy bound state, yet at least two of the pairs exhibit a zero-energy resonance.

The existence of the Efimov effect for three-particle Schrödinger operators was rigorously established by D. R. Yafaev in the continuous case [2], and later extended to the discrete setting by several authors [3, 4, 5, 6, 7, 8]. The analysis of this phenomenon is closely linked to the asymptotic behavior of the eigenvalue counting function, which in turn is reduced to the study of compact integral operators of Faddeev type.

In recent studies [9, 10, 11, 12, 13, 14], systems of three quantum particles on the one- two- and three-lattices, interacting via pairwise attractive point potentials, were examined. These works focused on configurations where two of the three particles are assumed to have infinite mass. It was shown that, under such conditions, the associated three-particle Schrödinger operators admit an infinite sequence of negative eigenvalues for any nonzero strength of the interaction. Furthermore, the author of [11] derived precise asymptotic formulas describing the behavior of these eigenvalues in the strong coupling limit.

It should be noted that the operator investigated in the present paper has not been considered in these earlier works.

In physics, stable composite objects often result from attractive forces, enabling constituents the constituents to lower their energy by binding together. In contrast, repulsive forces tend to scatter particles when they are in free space. However, within structured environments such as a periodic potential, and in the absence of dissipation, stable composite objects can exist even with repulsive interactions arising from the lattice band structure [15].

The work referenced by [15] exemplifies the crucial link between the Bose-Hubbard model [16, 17] and atoms in optical lattices, and helps pave the way for many more interesting developments and applications, as discussed in [18]. In cold atom lattice physics, stable repulsively bound objects are expected to be common, which leading to potential composites with fermions [19] or Bose-Fermi mixtures [20], and can even be formed with more than two particles in a similar manner.

The main goal of the paper is to investigate the existence of infinite number of bound states of the three-particle discrete Schrödinger operator associated to a system of two identical bosons and a fermion, where the bosons have infinite mass and the fermion has a finite mass. This investigation is conducted on the two-dimensional lattice $\mathbb{Z}^2$ and involves repulsive or attractive zero-range pairwise interactions. The problem is reduced to the study the Fredholm determinant of a diagonal operator.

It should be noted that, unlike the last three articles [9, 10, 11], we study eigenvalues below and above the essential spectrum of the unperturbed operator for all repulsive or attractive zero-range pairwise interactions.

The paper is organized as follows. In Section 2, we introduce the three-particle discrete Schrödinger operator $H(K)$ and the two-particle discrete Schrödinger operators associated with subsystems of the system of two identical bosons and a fermion. In Section 3, we study the essential spectrum of the three-particle discrete Schrödinger operator $H(K)$. The eigenvalues of $H(K)$ below and above the spectrum of the non-perturbed operator $H_0(K)$ are investigated in Section 4. Section 5 is devoted to showing main result, Theorem 5.4.

2. THREE-PARTICLE DISCRETE SCHRÖDINGER OPERATOR ON THE LATTICE $\mathbb{Z}^2$

Let $L_s^2((\mathbb{T}^2)^2)$ be the linear subspace of the symmetric functions of the Hilbert space $L^2((\mathbb{T}^2)^2)$.

Let us consider the discrete Schrödinger operator $H(K)$, where $K \in \mathbb{T}^2$, associated with a system consisting of two identical bosons and a fermion moving on the two-dimensional lattice $\mathbb{Z}^2$ (see [9, 11]).

$$H(K) = H_0(K) - V \quad (2.1)$$

with zero-range attractive potentials

$$V = V_1 + V_2 + V_3,$$

where

$$\begin{aligned} (V_1 f)(p, q) &= \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} f(p, t) dt, & (V_2 f)(p, q) &= \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} f(t, q) dt \\ (V_3 f)(p, q) &= \frac{\mu}{(2\pi)^2} \int_{\mathbb{T}^2} f(t, p + q - t) dt, & f &\in L_s^2((\mathbb{T}^2)^2), \quad p, q \in \mathbb{T}^2, \end{aligned}$$

and numbers λ and μ serve as the parameters of boson-fermion interaction and boson-boson interaction, respectively.

In the physical literature the parameter K , $K \in \mathbb{T}^2$, is called the *total quasi-momentum* of the three-particle system ([21]).

Here the numbers λ and μ indicate repulsive pair-wise interaction when $\lambda < 0$ and $\mu < 0$, and attractive pair-wise interaction when $\lambda > 0$ and $\mu > 0$. The operator $H_0(K)$ is defined on the Hilbert space $L_s^2((\mathbb{T}^2)^2)$ by

$$(H_0(K)f)(p, q) = E(K; p, q)f(p, q), \quad f \in L_s^2((\mathbb{T}^2)^2),$$

and

$$E(K; p, q) = \varepsilon_b(p) + \varepsilon_b(q) + \varepsilon_f(K - p - q), \quad p, q \in \mathbb{T}^2. \quad (2.2)$$

Here, the real-valued continuous function $\varepsilon_b(\cdot)$ and $\varepsilon_f(\cdot)$, referred to as *the dispersion relation* associated with the free boson and fermion is defined as

$$\varepsilon_b(p) = \frac{1}{m} \varepsilon(p), \quad \varepsilon_f(p) = \frac{1}{\mathbf{m}} \varepsilon(p), \quad \varepsilon(p) = 2 - \cos p_1 - \cos p_2, \quad p = (p_1, p_2) \in \mathbb{T}^2, \quad (2.3)$$

respectively, and m and $\mathbf{m}$ represents the mass of the boson and fermion, respectively.

2.1. Statement of the main result in the case $m = \infty$ and $0 < \mathbf{m} < \infty$. With $m = \infty$ and $0 < \mathbf{m} < \infty$ and the equality (2.3), the three-particle dispersion function (2.2) can be written as

$$E(K; p, q) = \varepsilon_f(K - p - q), \quad p, q \in \mathbb{T}^2. \quad (2.4)$$

2.1.1. *Zeros of Birman–Schwinger determinants.* In this subsection we study the zeros of three functions that play the role of Birman–Schwinger determinants, i.e., determinants associated with Birman–Schwinger operators.

Firstly, for $n \in \mathbb{Z}^2$ we define the integral

$$d_n(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{e^{i(n,t)}}{\varepsilon_f(t) - z} dt, \quad z \in \mathbb{C} \setminus [0, 4/\mathfrak{m}]. \quad (2.5)$$

The first determinant is defined on $\mathbb{R} \setminus [0, 4/\mathfrak{m}]$ by

$$\Delta(\lambda; z) = 1 - \lambda d_0(z), \quad z \in \mathbb{R} \setminus [0, 4/\mathfrak{m}] \quad (2.6)$$

and proof of the next lemma is presented on Page 124

Lemma 2.1. (a) If $\lambda < 0$, then the function $\Delta(\lambda; z)$ has a unique simple zero z_1^0 in the interval $(4/\mathfrak{m}, \infty)$.

(b) If $\lambda > 0$, then $\Delta(\lambda; z)$ has a unique simple zero z_1^0 in the interval $(-\infty, 0)$.

Next, consider the second determinant

$$\tilde{\Delta}(\lambda, \mu; z) = \Delta(2\lambda; \mu + z) = 1 - 2\lambda d_0(\mu + z), \quad z \in \mathbb{R} \setminus [-\mu, 4/\mathfrak{m} - \mu] \quad (2.7)$$

and the proof of the next lemma is given on Page 126

Lemma 2.2. Assume $\mu \in \mathbb{R}$.

(a) If $\lambda < 0$, then $\tilde{\Delta}(\lambda, \mu; z)$ has a unique zero η in the interval $(4/\mathfrak{m} - \mu, \infty)$.

(b) If $\lambda = 0$, then $\tilde{\Delta}(\lambda, \mu; z)$ has no zeros.

(c) If $\lambda > 0$, then $\tilde{\Delta}(\lambda, \mu; z)$ has a unique zero η in the interval $(-\infty, -\mu)$.

Finally, for $n \in \mathbb{Z}^2$, define the third determinant

$$\Delta_n^+(\lambda; z) = 1 - \lambda(d_0(z) + d_n(z)), \quad z \in \mathbb{R} \setminus [0, 4/\mathfrak{m}], \quad (2.8)$$

and the proof of the following lemma can be found on Page 132.

Lemma 2.3. (a) Let $\lambda > 0$. Then $\Delta_n^+(\lambda; z)$ has a unique zero z_n in the interval $(-\infty, 0)$ such that $z_n < z_1^0 < 0$. Moreover,

$$\lim_{|n| \rightarrow \infty} z_n = z_1^0. \quad (2.9)$$

(b) Let $\lambda = 0$. Then $\Delta_n^+(\lambda; z)$ has no zeros.

(c) Let $\lambda < 0$. Then $\Delta_n^+(\lambda; z)$ has a unique zero z_n in the interval $(4/\mathfrak{m}, \infty)$ such that

$$4/\mathfrak{m} < z_1^0 < z_n.$$

Moreover, a similar limit to that in (2.9) holds.

2.1.2. *The essential spectrum of $H(K)$ for $m = \infty$ and $\mathfrak{m} < \infty$.*

Theorem 2.1. The essential spectrum of $H(K)$ is given by

$$\sigma_{\text{ess}}(H(K)) = \{z_1^0\} \cup \left([-\mu, 4/\mathfrak{m} - \mu] \cup [0, 4/\mathfrak{m}] \right).$$

Proof: The proof lies in Page 125. □

2.1.3. *Main result: the point spectrum of $H(K)$ for $m = \infty$ and $\mathfrak{m} < \infty$.* The following assertion states that three-particle Schrödinger operators admit an infinite sequence of eigenvalues for any nonzero strength of the interaction.

Theorem 2.2. *Assume that $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.*

(a) *Let $\lambda < 0$. Then*

$$\sigma_{pp}(H(K)) = \bigcup_{n \in \mathbb{Z}^2} \{z_n\} \cup \{\eta\},$$

where $4/\mathfrak{m} < z_1^0 < z_n$ and $4/\mathfrak{m} - \mu < \eta$.

(b) *If $\lambda = 0$, then*

$$\sigma_{pp}(H(K)) = \emptyset.$$

(c) *Let $\lambda > 0$. Then*

$$\sigma_{pp}(H(K)) = \bigcup_{n \in \mathbb{Z}^2} \{z_n\} \cup \{\eta\},$$

where $z_n < z_1^0 < 0$ and $\eta < -\mu$.

Proof: See Page 133. □

Remark 2.1. By Theorem 2.1, one branch of the essential spectrum, $[-\mu, 4/\mathfrak{m} - \mu]$, depends on μ . Theorem 2.2 then implies that some or all of the eigenvalues z_n and η may lie either in the essential spectrum $\sigma_{\text{ess}}(H(K))$, or in the gaps of this essential spectrum.

Remark 2.2. Let $U_K : L_s^2(\mathbb{T}^2 \times \mathbb{T}^2) \rightarrow L_s^2(\mathbb{T}^2 \times \mathbb{T}^2)$ be a unitary operator defined by

$$(U_K f)(p, q) = f(-K/2 + p, -K/2 + q). \quad (2.10)$$

When $m = \infty$ and $\mathfrak{m} < \infty$, the operator U_K establishes a unitary equivalence between $H(K)$ and $H(0)$. Therefore, it suffices to prove all subsequent results for the operator $H(0)$.

3. TWO-PARTICLE DISCRETE SCHRÖDINGER OPERATORS

In addition to the three-particle Schrödinger operator, we also examine the discrete Schrödinger operators associated with the two-particle boson–fermion and boson–boson subsystems, as these are necessary for analyzing the essential spectrum of $H(K)$.

Let $k \in \mathbb{T}^2$ and $L_k^2(\mathbb{T}^2)$ be a linear subspace of the Hilbert space $L^2(\mathbb{T}^2)$ defined by

$$L_k^2(\mathbb{T}^2) = \{f \in L^2(\mathbb{T}^2) \mid f(p) = f(k - p)\}.$$

The two-particle discrete Schrödinger operators corresponding to the boson–fermion and boson–boson subsystems of the three-particle system act on the Hilbert spaces $L^2(\mathbb{T}^2)$ and $L_k^2(\mathbb{T}^2)$ as

$$h_1(k) = h_1^0(k) - \lambda v_1, \quad \text{and} \quad h_2(k) = h_2^0(k) - \mu v_2, \quad k \in \mathbb{T}^2, \quad (3.1)$$

respectively.

In the physics literature, the parameter k , $k \in \mathbb{T}^2$, is known as the *total quasi-momentum* of the two-particle system ([21]).

Here, the operators $h_\alpha^0(k)$ are defined as

$$(h_1^0(k)f)(p) = E_k^{(1)}(p)f(p), \quad f \in L^2(\mathbb{T}^2),$$

and

$$(h_2^0(k)f)(p) = E_k^{(2)}(p)f(p), \quad f \in L_k^2(\mathbb{T}^2),$$

where

$$E_k^{(1)}(p) = \varepsilon_{\mathfrak{f}}(p) + \varepsilon_{\mathfrak{f}}(k - p), \quad E_k^{(2)}(p) = \varepsilon_{\mathfrak{b}}(p) + \varepsilon_{\mathfrak{b}}(k - p), \quad p \in \mathbb{T}^2. \quad (3.2)$$

The operators v_1 and v_2 are defined as

$$(v_1 f)(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(q) dq, \quad f \in L^2(\mathbb{T}^2), \quad p \in \mathbb{T}^2, \quad (3.3)$$

and

$$(v_2 f)(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(q) dq, \quad f \in L_k^2(\mathbb{T}^2), \quad p \in \mathbb{T}^2,$$

respectively. Observe that $v_1 f$ and $v_2 f$ are constant functions on $\mathbb{T}^2$.

3.1. Spectral properties of the two-particle discrete Schrödinger operators when $m = \infty$ and $0 < m < \infty$. With $m = \infty$ and $0 < m < \infty$ and the equality (2.3), the functions (3.2) can be written as

$$E_k^{(1)}(p) = \varepsilon_f(k - p) = \varepsilon(k - p)/m, \quad E_k^{(2)}(p) = 0, \quad p \in \mathbb{T}^2.$$

Consequently, since the potentials v_α , $\alpha = 1, 2$ have a convolution-type property, all three two-particle Schrödinger operators do not depend on the quasi-momentum $k \in \mathbb{T}^2$,

$$h_1 := h_1(k), \quad \text{and} \quad h_2 = h_2(k).$$

Then, the operators $h_1(k)$ and $h_2(k)$ act as

$$(h_1(k)f)(p) = \varepsilon_f(p)f(p) - \lambda(v_1 f)(p), \quad f \in L^2(\mathbb{T}^2) \quad (3.4)$$

and

$$(h_2(k)f)(p) = -\mu(v_2 f)(p), \quad f \in L_k^2(\mathbb{T}^2). \quad (3.5)$$

As v_1 is a finite rank operator, according to the Weyl theorem, the essential spectrum $\sigma_{ess}(h_1(k))$ of the operator $h_1(k)$ in (3.1) coincides with the spectrum $\sigma(h_1^0(k))$ of the non-perturbed operator $h_1^0(k)$. More specifically,

$$\sigma_{ess}(h_1(k)) = [E_{\min}^{(1)}(k), E_{\max}^{(1)}(k)],$$

where

$$E_{\min}^{(1)}(k) \equiv \min_{p \in \mathbb{T}^2} E_k^{(1)}(p), \quad E_{\max}^{(1)}(k) \equiv \max_{p \in \mathbb{T}^2} E_k^{(1)}(p).$$

Remark 3.1. Note that, under the given mass condition, statements concerning the properties of the operators $h_1(k)$ and $h_2(k)$ formally depend on the total quasi-momentum k , as discussed in Remark 2.2.

Therefore, in our case we have

$$\sigma_{ess}(h_1(k)) = [0, 4/m] \quad \text{and} \quad \sigma_{ess}(h_2(k)) = \{0\}.$$

The Fredholm determinant $\Delta(\lambda; z)$ defined in (2.6) is associated with the Birman–Schwinger operator

$$I - \lambda(h_1^0(0) - z)^{-1} v_1, \quad z \in \mathbb{C} \setminus [0, 4/m].$$

Here, $(h_1^0(0) - z)^{-1}$ is the resolvent of the unperturbed operator $h_1^0(0)$, that is, the operator of multiplication by the function $(\varepsilon_f(p) - z)^{-1}$.

3.1.1. Proof of Lemma 2.1. In the first place, we show that the number $z \in \mathbb{C} \setminus [0, 4/m]$ is an eigenvalue of $h_1(k)$ if and only if $\Delta(\lambda; z) = 0$.

The equation

$$h_1(0)f = zf \quad \text{i.e.,} \quad f = \lambda(h_1^0(0) - z)^{-1} v_1 f$$

has a non-trivial solution if and only if

$$\Delta(\lambda; z)C = 0, \quad C \in \mathbb{C},$$

has a non-trivial solution.

Therein, the solutions $C \in \mathbb{C}$ and $f \in L^2(\mathbb{T}^2)$ are connected by the following relations

$$C = v_1 f \quad \text{and} \quad f = \lambda(h_1^0(0) - z)^{-1} C.$$

(a) Let $\lambda > 0$. The function $\Delta(\lambda; z)$ is monotonic decreasing in $(-\infty, 0)$ and $\Delta(\lambda; z) > 1$ in $(4/\mathfrak{m}, \infty)$. Since

$$\lim_{z \rightarrow -\infty} \Delta(\lambda; z) = 1 \quad \text{and} \quad \lim_{z \rightarrow 0-} \Delta(\lambda; z) = -\infty,$$

the intermediate-value theorem implies the existence of a unique simple zero $z = z_1^0, z_1^0 \in (-\infty, 0) \cup (4/\mathfrak{m}, \infty)$ of the function $\Delta(\lambda; \cdot)$, and furthermore $z_1^0 \in (-\infty, 0)$.

(b) The case $\lambda < 0$ can be treated using the same method. $\square$

Since $h_2(k)$ defined in (3.5) is a rank-one integral operator with norm $|\mu|$, it follows that

$$\sigma_{\text{ess}}(h_2(k)) = \{0\} \quad \text{and} \quad \sigma_{\text{disc}}(h_2(k)) = \{-\mu\}.$$

Therefore, the results of this section can be summarized in the following lemma.

Lemma 3.1. *We have*

$$\begin{aligned} \sigma_{\text{disc}}(h_1(k)) &= \{z_1^0\}, \quad \text{if } \lambda \neq 0, \\ \sigma(h_1(k)) &= \{z_1^0\} \cup [0, 4/\mathfrak{m}], \quad \text{if } \lambda \neq 0 \end{aligned}$$

and

$$\sigma_{\text{disc}}(h_2(k)) = \{-\mu\}, \quad \sigma(h_2(k)) = \{-\mu\} \cup \{0\}.$$

4. PROOF OF THEOREM 2.1: ESSENTIAL SPECTRUM OF $H(K)$

One of the notable results in the spectral theory of multi-particle continuous Schrödinger operators is the characterization of the essential spectrum in terms of cluster operators (the HVZ theorem); see, for example, [22, 23, 24, 25, 26] for the discrete case and [27] for the pseudo-relativistic case.

Lemma 4.1. *The essential spectrum of $H(K)$ satisfies the relation*

$$\sigma_{\text{ess}}(H(K)) = \bigcup_{k \in \mathbb{T}^2} \left\{ \sigma(h_1(K - k)) + \varepsilon_{\mathfrak{b}}(k) \right\} \cup \bigcup_{k \in \mathbb{T}^2} \left\{ \sigma(h_2(K - k)) + \varepsilon_{\mathfrak{f}}(k) \right\}.$$

Proof: The proof can be found in [11, 24]. $\square$

4.0.1. *The essential spectrum of $H(K)$ with $m = \infty$, and $\mathfrak{m} < \infty$.* Due to Lemma 3.1 and the relations $\varepsilon_{\mathfrak{b}}(p) = 0$ and $\varepsilon_{\mathfrak{f}}(p) = \varepsilon(p)/\mathfrak{m}$, we obtain

$$\begin{aligned} \bigcup_{k \in \mathbb{T}^2} \left\{ \sigma(h_1(K - k)) + \varepsilon_{\mathfrak{f}}(k) \right\} &= \bigcup_{k \in \mathbb{T}^2} \left\{ \sigma(h_1(k)) + \varepsilon_{\mathfrak{f}}(k) \right\} = \{z_1^0\} \cup [0, 4/\mathfrak{m}], \\ \bigcup_{k \in \mathbb{T}^2} \left\{ \sigma(h_2(K - k)) + \varepsilon_{\mathfrak{f}}(k) \right\} &= \bigcup_{k \in \mathbb{T}^2} (\{-\mu\} + \varepsilon_{\mathfrak{f}}(k)) \cup \bigcup_{k \in \mathbb{T}^2} (\{0\} + \varepsilon_{\mathfrak{f}}(k)) \\ &= [-\mu, 4/\mathfrak{m} - \mu] \cup [0, 4/\mathfrak{m}]. \end{aligned}$$

According the last two relations and Lemma 4.1 we have the proof of Theorem 2.1

5. PROOF OF THEOREM 2.2: ANALYSIS OF THE POINT SPECTRUM OF $H(K)$ FOR $m = \infty$ AND $\mathfrak{m} < \infty$

Since the integral kernels of the partial integral operators V_1 and V_2 are constant functions, it is straightforward to verify that the subspace

$$\mathcal{A}_0 = \{f \in L_s^2(\mathbb{T}^2 \times \mathbb{T}^2) \mid f(p, q) = g(p + q), \ g \in L^2(\mathbb{T}^2)\}$$

is invariant under the action of the operator $H(K)$.

Therefore, we have

$$\sigma_{pp}(H(K)) = \sigma_{pp}(A_0(K)) \cup \sigma_{pp}(A_1(K)),$$

where $A_0(K)$ and $A_1(K)$ are restrictions of $H(K)$ on the linear subspaces $\mathcal{A}_0$ and $\mathcal{A}_0^\perp$, respectively.

Since $\mathcal{A}_0$ and $L^2(\mathbb{T}^2)$ are isomorphic, the operator $A_0(K)$ is unitarily equivalent to the operator B_0 on $L^2(\mathbb{T}^2)$, where

$$B_0 = E_0(K) - \mu I - 2\lambda v_1, \quad (5.1)$$

$E_0(K)$ denotes the multiplication by the function $\varepsilon_f(K - p)$, I is the identity operator, and v_1 is an integral operator defined by (3.3).

The operator $A_1(K)$ takes the form

$$A_1(K) = H_0(K) - V_1 - V_2.$$

5.1. The Spectrum of the Operator $A_0(K)$. Since the equivalence (5.1) and the Weyl theorem imply that

$$\sigma_{ess}(A_0(K)) = [-\mu, 4/\mathfrak{m} - \mu].$$

The second determinant $\tilde{\Delta}(\lambda, \mu; z)$ defined by (2.7) associated with the Birman-Schwinger operator

$$I - 2\lambda(h_1^0(0) - (\mu + z))^{-1}v_1, \quad z \in \mathbb{C} \setminus [-\mu, 4/\mathfrak{m} - \mu]$$

where $(h_1^0(0) - z)^{-1}$ is the resolvent of the unperturbed operator $h_1(k)$ in (3.4).

5.1.1. Proof of Lemma 2.2. Firstly, we show that $z \in \mathbb{C} \setminus [-\mu, 4/\mathfrak{m} - \mu]$ is an eigenvalue of $H(0)$ in $\mathcal{A}_0$ if and only if $\tilde{\Delta}(\lambda, \mu; z) = 0$.

The equation

$$H(0)f = zf, \quad f \in \mathcal{A}_0,$$

has a solution if and only if the equation

$$C(1 - 2\lambda d_0(\mu + z)) = 0, \quad (5.2)$$

has one, and their solutions are related by

$$f(p, q) := g(p + q) = \frac{2\lambda}{\varepsilon_f(-p - q) - \mu - z} C,$$

where

$$C = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} g(t) dt.$$

Equation (5.2) has a nontrivial solution C if and only if the equation

$$\Delta(2\lambda; \mu + z) := 1 - 2\lambda d_0(\mu + z) = 0.$$

has a zero.

The function $\Delta(2\lambda; \mu + z)$ is defined on

$$(-\infty, -\mu) \cup (4/\mathfrak{m} - \mu, \infty),$$

and, according to Lemma 2.1, the following assertions hold:

(a) If $\lambda > 0$, then $\Delta(2\lambda; \mu + z) = 1 - 2\lambda d_0(\mu + z)$ has a unique zero in $(-\infty, -\mu)$, but does not have one in $(4/\mathfrak{m} - \mu, \infty)$;

(b) If $\lambda < 0$, then $\Delta(2\lambda; \mu + z)$ has a unique zero in $(4/\mathfrak{m} - \mu, \infty)$, but does not have one in $(-\infty, -\mu)$;

(c) If $\lambda = 0$, then $B_0 = E_0(K) - \mu I$ is a multiplication operator, and so $A_0(K)$ has no eigenvalues for any $\mu \in \mathbb{R}$. $\square$

5.2. Spectrum of $A_1(K)$. Recall that

$$A_1(K) = H_0(K) - V_1 - V_2,$$

where $H_0(K)$ is the multiplication operator defined by the function $E(K; p, q) := \varepsilon_f(K - p - q)$ in (2.4):

$$(H_0(K)f)(p, q) = \varepsilon_f(K - p - q)f(p, q), \quad f \in \mathcal{A}_0^\perp.$$

The operators V_1 and V_2 are partially integral operators given in (2.1):

$$(V_1 f)(p, q) = \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} f(p, t) dt, \quad (V_2 f)(p, q) = \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} f(t, q) dt,$$

for $f \in \mathcal{A}_0^\perp$ and $p, q \in \mathbb{T}^2$.

Since the operator $A_1(K)$ does not depend on the parameter μ , Lemma 2.1 yields

$$\sigma_{\text{ess}}(A_1(K)) = \{z_1^0\} \cup [0, 4/\mathfrak{m}].$$

The function $d_n(z)$ defined in (2.5) plays a role in the investigation of the eigenvalues of $A_1(k)$. In this connection, we introduce the following lemma.

Lemma 5.1. (a) For any $z < 0$, the following asymptotics holds

$$\begin{aligned} d_n(z) &= \frac{\mathfrak{m}}{2^{|n|_1}} \frac{1}{|2 - \mathfrak{m}z|^{|n|_1+1}} \sum_{\xi=0}^{\infty} \frac{1}{(2 - \mathfrak{m}z)^{2\xi}} \frac{1}{2^{2\xi}} \\ &\quad \times \sum_{s_1+s_2=\xi} \frac{(|n|_1 + \xi)!}{s_1!s_2!(|n_1|! + s_1)!(|n_2|! + s_2)!}, \quad z < 0, \end{aligned} \quad (5.3)$$

where

$$|n|_1 = |n_1| + |n_2|.$$

(b) For any fixed $n \in \mathbb{Z}^2$, $d_n(z)$ is positive and monotonically increasing on $(-\infty, 0)$.

(c) The following asymptotics holds:

$$d_0(z) = -\frac{\sqrt{2}\mathfrak{m}}{(2\pi)^2} \ln(-\mathfrak{m}z) + C + O(1), \quad \text{as } z \rightarrow 0-. \quad (5.4)$$

Proof: We have the following representation:

$$\frac{e^{i(n,t)}}{\varepsilon_f(t) - z} = \mathfrak{m} \sum_{\eta=0}^{\infty} \frac{e^{i(n,t)}}{(2 - \mathfrak{m}z)^{\eta+1}} (e(t))^\eta, \quad (5.5)$$

valid for $z < 0$, since $2 - \mathfrak{m}z > e(t)$, where

$$e(t) = \cos t_1 + \cos t_2, \quad p = (p_1, p_2) \in \mathbb{T}^2.$$

We represent the function $d_n(z)$ as

$$d_n(z) = \sum_{\eta=0}^{\infty} \frac{1}{(2 - \mathfrak{m}z)^{\eta+1}} A_\eta, \quad \text{when } z < 0. \quad (5.6)$$

where

$$\begin{aligned} A_\eta &= \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{i(n,t)} (\cos t_1 + \cos t_2)^\eta dt \\ &= \frac{1}{2^\eta (2\pi)^2} \sum_{\substack{0 \leq k, k' \in \mathbb{N}_0^2, \\ k_1 + k_2 + k'_1 + k'_2 = \eta}} \frac{\eta!}{k_1! k_2! k'_1! k'_2!} \int_{\mathbb{T}^2} e^{i(n,t)} e^{i(k-k',t)} dt \end{aligned} \quad (5.7)$$

and

$$\mathbb{N}_0^2 = \mathbb{N}_0 \times \mathbb{N}_0$$

of the natural set

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}.$$

According to the identity

$$\frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{i(n,t)} e^{i(k-k',t)} dt = \begin{cases} 1, & \text{for } n + k - k' = 0 \\ 0, & \text{for } n + k - k' \neq 0, \end{cases} \quad k, k' \in \mathbb{N}_0^2, \quad (5.8)$$

the number A_η is non-negative for any $\eta \in \mathbb{N}$ and strictly positive if there exist $k, k' \in \mathbb{N}_0^2$ such that $n + k - k' = 0$ with $k_1 + k_2 + k'_1 + k'_2 = \eta$.

Let us take new expression for $k = (k_1, k_2)$, as

$$k_j = \begin{cases} -n_j + s_j, & \text{if } n_j < 0 \\ s_j, & \text{if } n_j \geq 0 \end{cases} \quad (5.9)$$

where $s_j \geq 0$ and for

$$\begin{aligned} k' &= (k'_1, k'_2), \\ k'_j &= \begin{cases} n_j + s_j, & \text{if } n_j > 0, \\ s_j, & \text{if } n_j \leq 0. \end{cases} \end{aligned} \quad (5.10)$$

Due to the conditions $n + k - k' = 0$ with $k_1 + k_2 + k'_1 + k'_2 = \eta$ we get

$$k_1 + k_2 + k'_1 + k'_2 = |n|_1 + 2(s_1 + s_2) = \eta$$

which implies that the number $\eta - |n|_1$ must be even, and moreover

$$\begin{aligned} A_\eta &= 0, \quad \text{for } \eta < |n|_1 \quad \text{or} \quad \eta - |n|_1 \text{ is odd,} \\ A_\eta &> 0, \quad \text{for } \eta \geq |n|_1 \quad \text{and} \quad \eta - |n|_1 \text{ is even} \end{aligned} \quad (5.11)$$

Taking into account the relations $\eta \geq |n|_1$, the evenness of $\eta - |n|_1$ and the equality $k_1 + k_2 + k'_1 + k'_2 = |n|_1 + 2(s_1 + s_2) = \eta$ we get

$$A_\eta = \frac{1}{2^\eta} \sum_{2(s_1 + s_2) = \eta - |n|_1} \frac{\eta!}{s_1! s_2! (|n|_1! + s_1)! (|n|_2! + s_2)!} \quad (5.12)$$

Introducing a new variable $\xi = (\eta - |n|_1)/2$ and with regard to equations (5.6) and (5.11) we receives

$$d_n(z) = \mathfrak{m} \sum_{\xi=0}^{\infty} \frac{1}{(2 - \mathfrak{m}z)^{2\xi + |n|_1 + 1}} \frac{1}{2^{2\xi + |n|_1}} A_{|n|_1 + 2\xi}, \quad \text{when } z < 0. \quad (5.13)$$

The relation (5.12), and the expansion (5.6) and (5.13) prove the first part of the lemma.

(b) This part is an immediate consequence of the part (a), as every term of (5.3) is positive and monotonically increasing in the interval $(-\infty, 0)$.

(c) The proof of the following asymptotics can be found in [28]:

$$d_0(z) = -\frac{\sqrt{2}\mathfrak{m}}{2\pi} \ln(-\mathfrak{m}z) + \mathfrak{m} \left(\frac{1}{2} - \frac{\sqrt{2}}{\pi} \right) + O(1), \quad \text{as } z \rightarrow 0- \quad (5.14)$$

As the inequality

$$\left| \frac{e^{ikx} - 1}{e^{ix} - 1} \right| \leq |k| \quad \text{for } x \neq 0,$$

we get

$$\left| \frac{e^{i(n_1 t_1 + n_2 t_2)} - 1}{\varepsilon(t)/\mathfrak{m} - z} \right| \leq \mathfrak{m}|n|_1 \quad \text{for } n \in \mathbb{Z}^2 \quad \text{and } t \in \mathbb{T}^2.$$

Thus, the integrand of the right hand side of the difference

$$d_n(z) - d_0(z)$$

is bounded, and $|d_n(z) - d_0(z)|$ is bounded and continuous at 0 from the left.

Consequently, the last fact, the expansion (5.14) and the relation

$$d_n(z) = d_0(z) + (d_n(z) - d_0(z))$$

complete the proof of (c). $\square$

The operators V_1 and V_2 can be expressed as $V_1 = \lambda \varphi_1^* \varphi_1$ and $V_2 = \lambda \varphi_2^* \varphi_2$, respectively, where $\varphi_1, \varphi_2 : L^2(\mathbb{T}^2 \times \mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$ are defined by

$$(\varphi_1 f)(p) = \frac{1}{2\pi} \int_{\mathbb{T}^2} f(p, t) dt, \quad (\varphi_2 f)(q) = \frac{1}{2\pi} \int_{\mathbb{T}^2} f(t, q) dt.$$

Their adjoint operators $\varphi_1^*, \varphi_2^* : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2 \times \mathbb{T}^2)$ are given by

$$(\varphi_1^* g)(p, q) = \frac{1}{2\pi} g(p), \quad (\varphi_2^* g)(p, q) = \frac{1}{2\pi} g(q), \quad g \in L^2(\mathbb{T}^2).$$

Lemma 5.2. *The number $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(A_1(K))$ is an eigenvalue of the operator $A_1(K)$ if and only if $D(z) = 0$, where*

$$D(z) = \frac{1}{\Delta^2(\lambda; z)} \prod_{n \in \mathbb{Z}^2} \Delta_n^+(\lambda; z), \quad (5.15)$$

with $\Delta_n^+(\lambda; z)$ defined in (2.8), i.e.

$$\Delta_n^+(\lambda; z) = 1 - \lambda(d_0(z) + d_n(z)).$$

If $z_n \in \mathbb{C} \setminus \sigma_{\text{ess}}(A_1(K))$ is an eigenvalue of $A_1(K)$, then $D(z_n) = 0$, $n \in \mathbb{Z}^2$ and the corresponding eigenfunction is of the form

$$f_n(p, q) = \frac{\lambda}{\varepsilon_f(K - p - q) - z_n} \left(\cos(n, (-K/2 + p)) + \cos(n, (-K/2 + q)) \right) \quad (5.16)$$

Proof: Given the unitary equivalence of $H(K)$ and $A_1(K)$ to $H(0)$ and $A_1(0)$, respectively, we first establish the claim for the latter operators.

Let $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(A_1(0))$ be an eigenvalue of $A_1(0)$, and let f be the corresponding eigenfunction, then

$$f = R_0(z)[V_1 + V_2]f, \quad (5.17)$$

where $R_0(z) = (H_0(0) - zI)^{-1}$ is a resolvent of $H_0(0)$.

This equation has a non-trivial solution if and only if the system of two linear equations

$$\tilde{\phi}_\alpha = \varphi_\alpha(R_0(z)[\lambda \varphi_1^* \tilde{\phi}_1 + \lambda \varphi_2^* \tilde{\phi}_2]), \quad \alpha = 1, 2, \quad \tilde{\phi}_1, \tilde{\phi}_2 \in L^2(\mathbb{T}^2) \quad (5.18)$$

on the space $L^2(\mathbb{T}^2) \oplus L^2(\mathbb{T}^2)$ has a non-zero solution.

Solutions of equations (5.17) and (5.18) are linked by

$$f(p, q) = R_0(z)[\lambda \varphi_1^* \tilde{\phi}_1 + \lambda \varphi_2^* \tilde{\phi}_2], \quad (5.19)$$

and

$$\tilde{\phi}_\alpha = \varphi_\alpha f, \quad \alpha = 1, 2.$$

Since f is symmetric, the functions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ coincide; only their arguments are denoted differently. In particular,

$$\tilde{\phi}_2(p) = \tilde{\phi}_1(p). \quad (5.20)$$

We note that the operators

$$\Delta_\alpha := I - \lambda \varphi_\alpha R_0(z) \varphi_\alpha^*, \quad z \in \mathbb{C} \setminus \sigma_{ess}(H_0(0)), \quad \alpha = 1, 2,$$

are invertible for any $z \in \mathbb{C} \setminus \sigma_{ess}(A_1(0))$. Therefore,

$$\Delta_\alpha^{-1} = (I - \lambda \varphi_\alpha R_0(z) \varphi_\alpha^*)^{-1}.$$

Then the functions $\tilde{\phi}_\alpha$, $\alpha = 1, 2$, satisfying equation (5.18), also satisfy the following system of integral equations:

$$\begin{cases} \tilde{\phi}_1 = \lambda \Delta_1^{-1} Q \tilde{\phi}_2, \\ \tilde{\phi}_2 = \lambda \Delta_2^{-1} Q^* \tilde{\phi}_1, \end{cases} \quad (5.21)$$

where

$$Q = \varphi_1 R_0(z) \varphi_2^*$$

is the integral operator on $L^2(\mathbb{T}^2)$ defined by

$$(Q\phi)(p) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{\phi(t) dt}{\varepsilon_f(-p-t) - z}, \quad \phi \in L^2(\mathbb{T}^2)$$

and Q^* is the adjoint operator of Q .

Using the substitution method, system (5.21) can be reduced to

$$\tilde{\phi}_1 = Q(z) \tilde{\phi}_1, \quad Q(z) = \lambda^2 \Delta_1^{-1} Q \Delta_2^{-1} Q^*. \quad (5.22)$$

Moreover, if

$$\Phi = (\tilde{\phi}_1, \tilde{\phi}_2)$$

is a solution of (5.21), then $\tilde{\phi}_1$ is an eigenfunction of the operator $Q(z)$ corresponding to the eigenvalue 1.

Then

$$\Phi = (\tilde{\phi}_1, \tilde{\phi}_2), \quad \tilde{\phi}_2 = \frac{\lambda}{\Delta(\lambda; z)} Q \tilde{\phi}_1,$$

is a solution of (5.21) (equivalently, of (5.18)).

Notice that the multiplicities of the linearly independent eigenvectors $\tilde{\phi}_1$ and Φ coincide.

We also note that the function f defined by (5.19) is an eigenfunction of $A_1(0)$ corresponding to the eigenvalue $z \in \mathbb{C} \setminus \sigma_{ess}(A_1(0))$.

Moreover, the multiplicity of the eigenvalue z of $A_1(0)$ coincides with the multiplicity of the eigenvalue 1 of the operator $Q(z)$.

The operator $Q(z)$ is a convolution-type trace-class integral operator. The standard Fourier transform

$$\mathcal{F}_1 : L^2(\mathbb{T}^2) \rightarrow \ell^2(\mathbb{Z}^2)$$

defined by

$$(\mathcal{F}_1 g)(n) = \frac{1}{2\pi} \int_{\mathbb{T}^2} e^{i(n,p)} g(p) dp, \quad g \in L^2(\mathbb{T}^2), \quad n \in \mathbb{Z}^2,$$

transforms $Q(z)$ into the multiplication operator

$$\hat{Q}(z) := \mathcal{F}_1 Q(z) \mathcal{F}_1^*$$

on $\ell^2(\mathbb{Z}^2)$.

Since the kernel of $Q(z)$ is of convolution type, a direct computation of its Fourier coefficients yields

$$\kappa_n(z) = \frac{\lambda^2}{(\Delta(\lambda; z))^2} d_n(z) d_{-n}(z), \quad n \in \mathbb{Z}^2. \quad (5.23)$$

Thus, the spectrum of $\hat{Q}(z)$ consists of the following union

$$\sigma(\hat{Q}(z)) = \{0\} \cup \bigcup_{n \in \mathbb{Z}^2} \{\kappa_n(z)\},$$

with the space of eigenfunctions

$$\hat{\varphi}_n(m) = \delta_{n,m}, \quad n, m \in \mathbb{Z}^2,$$

where $\delta_{\cdot, \cdot}$ is the Kronecker delta function on $\mathbb{Z}^2$.

Note that the compact operator $Q(z)$ has eigenvalues $\kappa_n(z)$, $n \in \mathbb{Z}^2$ with the corresponding eigenfunctions

$$\psi_n(p) = e^{i(n,p)}, \quad n \in \mathbb{Z}^2, p \in \mathbb{T}^2.$$

Therefore, the determinant of the operator $I - Q(z)$ can be written as the following product

$$\det(I - Q(z)) = \prod_{n \in \mathbb{Z}^2} (1 - \kappa_n(z)), \quad (5.24)$$

which takes the form (5.15), since $d_{-n}(z) = d_n(z)$ and (5.23).

Let $\kappa_n(z_n) = 1$ be an eigenvalue of $Q(z_n)$, then ψ_n is the first component of the solution $\varphi_n = (\psi_n, \tilde{\psi}_n)$ of equation (5.18), and the second component is defined as

$$\tilde{\psi}_n(q) := \tilde{\varphi}_2(q) = \lambda \Delta_2^{-1} Q^* \psi_n(q) = \frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} \tilde{\psi}_{-n}(q). \quad (5.25)$$

Using equality (5.20) in the last relation, we get

$$e^{i(n,q)} = \lambda \Delta_2^{-1} Q^* \psi_n(q) = \frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} e^{-i(n,q)},$$

which is contradiction.

However, if $\vartheta_n(p) = (e^{i(n,p)} + e^{-i(n,p)})/2 = \cos((n,p))$, it satisfies (5.20) and (5.25) if and only if

$$\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} = 1, \quad \text{i.e.} \quad \Delta_n^+(\lambda; z) = 1 - \lambda(d_0(z_n) + d_n(z_n)) = 0.$$

However, for the functions $\theta_n(p) = (e^{i(n,p)} - e^{-i(n,p)})/2i = \sin((n,p))$ the relation

$$\theta_n = \lambda \Delta_2^{-1} Q^* \theta_n = -\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} \theta_n,$$

holds, which contradicts with (5.20). It implies

$$-\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} = 1, \quad \text{i.e.} \quad \Delta_n^-(\lambda; z) = 1 - \lambda(d_0(z_n) - d_n(z_n)) = 0.$$

Since the system $\{\vartheta_n, \theta_n\}$, $n \in \mathbb{Z}^2$ is complete in $\ell^2(\mathbb{Z}^2)$, the last three relations and (5.23) allow us to use the Fredholm determinant (5.15) instead of (5.24).

Consequently, the number z_n is an eigenvalue of $A_1(0)$, and the corresponding eigenfunction is given by (5.19):

$$f_n^0 = R_0(z_n) [\lambda \varphi_1^* \psi_n + \lambda \varphi_2^* \tilde{\psi}_n].$$

i.e.,

$$f_n^0(p, q) = \frac{\lambda}{\varepsilon_f(-p - q) - z} (\cos(n, p) + \cos(n, q)).$$

According to the relation $H(0) = U_K^* H(K) U_K$, the number z_n is also an eigenvalue of $H(K)$ with the eigenfunction $f_n = U_K f_n^0$ in (5.16), where the unitary operator U_K is defined in (2.10). $\square$

5.3. Eigenvalues of $H(K)$ below and above the spectrum of $H_0(K)$. Let z_1^0 be a zero of $\Delta_\alpha(\cdot) = 0$ in Lemma 2.1.

Lemma 5.3. *For any fixed $n \in \mathbb{Z}^2$, the function $d_0(z) + d_n(z)$ is positive and monotonically increasing on the interval $(-\infty, 0)$.*

Additionally,

$$\lim_{z \rightarrow -\infty} (d_0(z) + d_n(z)) = 0, \quad (5.26)$$

$$\lim_{z \rightarrow 0-} (d_0(z) + d_n(z)) = \infty, \quad (5.27)$$

and for any $z \in (-\infty, 0)$

$$\lim_{|n| \rightarrow \infty} d_n(z) = 0 \quad (5.28)$$

hold.

Proof: The equalities

$$d_0(z) + d_n(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{1 + \cos(nt)}{\varepsilon_{\mathbf{f}}(t) - z} dt$$

and part (b) of Lemma 5.1 imply the positivity and monotonicity of $d_0(z) + d_n(z)$ in $(-\infty, 0)$.

Limits (5.26) and (5.28) follows from part (a) in Lemma 5.1.

Part (c) provides (5.27). □

5.3.1. Proof of Lemma 2.3. (a) Let z_1^0 be a zero of $\Delta(\lambda; \cdot) = 0$ (see Lemma 2.1).

According to Lemma 5.3, the function $\Delta_n^+(\lambda; z)$ is monotonically increasing in $(-\infty, 0)$ and

$$\begin{aligned} \lim_{z \rightarrow -\infty} \Delta_n^+(\lambda; z) &= 1, \\ \Delta_n^+(\lambda; z_1^0) &< \Delta(\lambda; z_1^0) = 0, \end{aligned}$$

holds.

Consequently, the function $\Delta_n^+(\lambda; z)$ has a unique zero z_n in the interval $(-\infty, z_1^0)$ if $\lambda > 0$.

The equalities

$$\begin{aligned} \Delta_n^+(\lambda; z_1^0) &= (\Delta_n^+(\lambda; \xi_n))' (z_1^0 - z_n), \\ \Delta_n^+(\lambda; z_1^0) &= -\lambda d_n(z_1^0), \\ (\Delta_n^+(\lambda; z))' &= -\lambda (d_0'(z) + d_n'(z)) \end{aligned}$$

where $z_n < \xi_n < z_1^0$ is a number obtained due to the intermediate value theorem, imply

$$z_n - z_1^0 = -\frac{d_n(z_1^0)}{d_0'(\xi_n) + d_n'(\xi_n)}$$

Applying the limit (5.28) in the last inequality, the last equality, we get the proof of the limit in (2.9), i.e.

$$\lim_{|n| \rightarrow \infty} z_n = z_1^0.$$

(b) If $\lambda = 0$, then $\Delta_n^+(\lambda; z) = 1$.

(c) The unitary operator $U_{\pi/2}$ in (2.10) is used to establish the equalities

$$U_{\pi/2} H_0(K) U_{\pi/2} = \frac{2}{\mathbf{m}} - H_0(K) \quad \text{and} \quad U_{\pi/2} V U_{\pi/2} = V$$

which implies the relation

$$U_{\pi/2} (H_0(K) - V) U_{\pi/2} = \frac{2}{\mathbf{m}} - (H_0(K) + V). \quad (5.29)$$

The final relation allows us to reduce the investigation of the eigenvalues of $H(K)$ above the interval $[0, 4/\mathbf{m}]$ to the study of those below it, and vice versa.

The proof of part (b) is a direct consequence of part (a) and the identity (5.29). □

5.4. **Proof of the main theorem: Theorem 2.2.** Lemmas 2.2 and 2.3 and

$$\sigma_{pp}(H(K)) = \sigma_{pp}(A_0(K)) \bigcup \sigma_{pp}(A_1(K)) \quad (5.30)$$

provide the proof.

For example, if $\lambda < 0$, part (a) of Lemma 2.2 provides that $A_0(K)$ has a simple eigenvalue η , $\eta \in (4/\mathfrak{m} - \mu, \infty)$ and part (c) of Lemma 2.3 implies that $A_1(K)$ has infinitely many eigenvalues $\{z_n\} \in (4/\mathfrak{m} - \mu, \infty)$.

The last facts together with (5.30) give the proof of the part (a) of the theorem. $\square$

CONCLUSION

We have considered the discrete Schrödinger operator associated with the Hamiltonian of a system of three quantum particles (two identical bosons and one fermion) on a two-dimensional lattice.

The bosons are assumed to have infinite mass, while the fermion has finite mass:

$$m = \infty, \quad \mathfrak{m} < \infty.$$

The system was studied for all nonzero interaction parameters. We proved that the corresponding three-particle discrete Schrödinger operator possesses infinitely many eigenvalues.

The appearance of infinitely many eigenvalues is related to the singular behavior of the associated Birman–Schwinger type operator near the threshold of the essential spectrum. In the case $m = \infty$, the kinetic energy contribution of the bosons disappears, which changes the threshold behavior of the corresponding resolvent and leads to an accumulation of eigenvalues near the edge of the essential spectrum.

ACKNOWLEDGEMENTS

The author acknowledge support from the Innovative Development Agency of the Republic of Uzbekistan (Grant No. FZ-20200929224).

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Uzbek Mathematical Journal
 2026, Volume 70, Issue 3, pp. 135–145
 DOI: 10.29229/uzmj.2026-3-14

A convex combination conjugate gradient algorithm with application in mode function

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Abstract. Conjugate gradient methods are a popular class of iterative methods for solving linear systems of equations and nonlinear optimization problems. In this paper, we suggest a new hybrid nonlinear conjugate gradient method, in which the conjugate gradient coefficient β_k is a convex combination of $\beta_k^{NVHS^*}$ and β_k^{DY} . With the strong Wolfe line search, the descent property and global convergence of the new hybrid method are proved. The numerical results also show that our method is robust and efficient. Furthermore, the proposed algorithm was extended to solve the problem of mode function.

Keywords: Hybrid conjugate gradient method, Line search, Sufficient descent condition, Global convergence, Numerical comparisons, Mode function, Kernel estimator.

MSC (2020): 90C30, 65K05, 62G05

1. INTRODUCTION

The optimization model is a needful mathematical problem since it has been connected to different fields such as economics, engineering and physics. In recent works, we have developed several efficient conjugate gradient methods with successful applications in image restoration, nonparametric estimation, and regression analysis [1, 2, 3, 4]. Today there are many optimization algorithms, such as Newton, quasi-Newton and bundle algorithms. Note that these algorithms fail to solve large-scale optimization problems because they need to store and calculate relevant matrices. In contrast, the conjugate gradient algorithm is successful due to its simplicity of iteration and low memory requirements. In this work, we study the nonlinear conjugate gradient method for the following unconstrained optimization problem:

$$\min \{f(x) : x \in \mathbb{R}^n\}, \quad (1.1)$$

where f is a smooth and nonlinear function. The conjugate gradient method generates a sequence $\{x_k\}_{k \geq 0}$ such that:

$$x_{k+1} = x_k + \alpha_k d_k, \quad (1.2)$$

where x_k is the current iteration point and $d_k \in \mathbb{R}^n$ is the search direction defined by the following formula:

$$d_{k+1} = -g_{k+1} + \beta_k d_k; \quad d_0 = -g_0, \quad (1.3)$$

where g_{k+1} the gradient of f at x_{k+1} and the parameter β_k is known as the conjugate gradient coefficient. The step length α_k is very important for the global convergence of conjugate gradient methods, one often requires the line search to satisfy the Wolfe conditions (WLS)

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k g_k^T d_k, \quad (1.4)$$

and

$$g_{k+1}^T d_k \geq \sigma g_k^T d_k. \quad (1.5)$$

Also, the strong Wolfe (SWLS) conditions consist of (1.4) and

$$|g_{k+1}^T d_k| \leq -\sigma g_k^T d_k, \quad (1.6)$$

TABLE 1. List of famous conjugate gradient coefficient formulas

Formula	Reference	Property
$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{y_k^T d_k}$	Hestenes-Stiefel [5]	Good numerical results
$\beta_k^{PRP} = \frac{g_{k+1}^T y_k}{\ g_k\ ^2}$	Polak-Ribière-Polyak [6, 7]	
$\beta_k^{LS} = \frac{g_{k+1}^T y_k}{-g_k^T d_k}$	Liu-Storey [8]	
$\beta_k^{FR} = \frac{\ g_{k+1}\ ^2}{\ g_k\ ^2}$	Fletcher-Reeves [9]	Strong convergence
$\beta_k^{CD} = \frac{\ g_{k+1}\ ^2}{-g_k^T d_k}$	Conjugate Descent [10]	
$\beta_k^{DY} = \frac{\ g_{k+1}\ ^2}{y_k^T d_k}$	Dai-Yuan [11]	

where $0 < \delta < \sigma < 1$. Now, we denote $y_k = g_{k+1} - g_k$, $\|\cdot\|$ the Euclidean norm. In Table 1, we will give some famous formulas of the parameter β_k :

Dai and Yuan [11] proved that the DY method always generate descent directions and converges globally with the standard Wolfe line conditions (1.4) and (1.5). On the other hand, the HS method is generally regarded to be one of the most efficient conjugate gradient methods, but their convergence property is not so good. In the latest years, based on the above six formulas and their hybridization, many works have devoted their time and effort to come up with new formulae in order to increase the efficiency and effectiveness of the HS method. Yao et al. [12] presented a nonlinear conjugate gradient algorithm, denoted by VHS, where the parameter β_k given by:

$$\beta_k^{VHS} = \frac{g_{k+1}^T \left(g_{k+1} - \frac{\|g_{k+1}\|}{\|g_k\|} g_k \right)}{y_k^T d_k}.$$

Yao et al. [12] proved that the VHS method also can produce sufficient descent direction and this method is globally convergent if strong Wolfe conditions (1.4) and (1.6) are used with $\sigma < \frac{1}{3}$. Zhang [13] gave the modified VHS method which we call NHS method. The β_k in NHS method is given by

$$\beta_k^{NHS} = \frac{\|g_{k+1}\|^2 - \frac{\|g_{k+1}\|}{\|g_k\|} |g_{k+1}^T g_k|}{d_k^T y_k}.$$

Zhang [13] proved that the NHS method satisfies the sufficient descent condition and is globally convergent under the strong Wolfe conditions (1.4) and (1.6) with $\sigma < \frac{1}{2}$. Du et al. [14] gave a new conjugate gradient method, called the NVHS* method. The β_k in the NVHS* method is defined as

$$\beta_k^{NVHS^*} = \frac{g_{k+1}^T \left(g_{k+1} - \frac{|g_{k+1}^T g_k|}{\|g_k\|^2} g_k \right)}{d_k^T y_k}.$$

The NVHS* method has sufficient descent conditions and is globally convergent if the strong Wolfe line search is used and the parameter $\sigma < \frac{1}{3}$. The hybrid conjugate gradient method considered is to combine the standard conjugate gradient methods in a two distinct ways. The first class is based on projection concept. Recently, Touati-Ahmed and Storey [15] introduced one of the first hybrid conjugate gradient algorithms, where the parameter β_k is computed as:

$$\beta_k^{TaS} = \min \{ \beta_k^{FR}, \beta_k^{PRP} \}.$$

The authors proved that β_k^{TaS} has good convergence properties and numerically outperforms both the β_k^{FR} and β_k^{PRP} algorithms. Soon afterward, Hu and Storey [16], Gilbert and Nocedal [17] further studied other hybrid schemes about PRP and FR methods. Dai and Yuan [18] combined DY method with HS method, such that the CG coefficient is computed by:

$$\beta_k^{hDYz} = \max \{0, \min \{\beta_k^{HS}, \beta_k^{DY}\}\}.$$

Also, this author [18] has given a another modified conjugate gradient method, in which β_k is defined by:

$$\beta_k^{hDY} = \max \{-c\beta_k^{DY}, \min \{\beta_k^{HS}, \beta_k^{DY}\}\},$$

where $c = \frac{1-\sigma}{1+\sigma}$. For the standard Wolfe conditions (1.4) and (1.5), under the Lipschitz continuity of the gradient, Dai and Yuan [19] established the global convergence of these hybrid computational schemes. The second class of hybrid conjugate gradient methods is based on the convex combination of the standard methods. Recently, Andrei [20] introduced a new hybrid conjugate gradient method based on HS and DY method (denoted as HYBRID method) for solving unconstrained optimization problems (1.1), calculating the parameter β_k^c as a convex combination of β_k^{HS} and β_k^{DY} i.e.

$$\beta_k^c = (1 - \theta_k) \beta_k^{HS} + \theta_k \beta_k^{DY},$$

where θ_k is a scalar parameter satisfying $0 \leq \theta_k \leq 1$. Convergence with the standard Wolfe condition was established and numerical results show that this hybrid computational scheme outperforms the Hestenes-Stiefel and the Dai-Yuan conjugate gradient algorithms. This author also studied the global convergence of the CCOMB method [21] under strong Wolfe line search, such that the parameter β_k as a convex combination of β_k^{PRP} and β_k^{DY} i.e.

$$\beta_k^N = (1 - \theta_k) \beta_k^{PRP} + \theta_k \beta_k^{DY}.$$

Liu and Li [2] proposed another hybrid conjugate gradient as a convex combination of LS and DY method, (denoted as HLSDY method), where β_k in this method given by:

$$\beta_k^{HLSDY} = (1 - \theta_k) \beta_k^{LS} + \theta_k \beta_k^{DY}.$$

The global convergence was established under a strong Wolfe line search. Numerical results show that the method is efficient for the standard unconstrained problems in a CUTE library [22].

The most important and new thing in this work is the application of our method in nonparametric statistics, where we are the first to use and access in this field.

This paper aims to propose new hybrid conjugate gradient as a convex combination of NVHS* and DY conjugate gradient algorithms. We establish, under a strong Wolfe line search, convergence properties of the proposed conjugate gradient method. Numerical results show that the hHD method is efficient and robust and outperforms as six conjugate gradient methods famous. Finally, an application of our method in nonparametric mode estimator is also considered.

Now, we will organize our work as follows. In the next section, we consist of a new hybrid method and determine the parameter θ_k . Also, we present the specific algorithm and we prove the sufficient descent condition. In section 3, we prove the global convergence of the proposed method with a strong Wolfe line search. The numerical results are contained in section 4. In section 5, we focus an application of the hHD method in statistics nonparametric. Finally, we make a summary of our paper.

2. CONVEX COMBINATION METHOD

The parameter β_k in the presented method, denoted as β_k^{hHD} , is computed as a convex combination of $\beta_k^{NVHS^*}$ and β_k^{DY} , i.e:

$$\beta_k^{hHD} = (1 - \theta_k) \beta_k^{NVHS^*} + \theta_k \beta_k^{DY}, \quad (2.1)$$

where θ_k is a scalar parameter satisfying $0 \leq \theta_k \leq 1$, which follows to be determined. It is obvious that if $\theta_k = 0$, then $\beta_k^{hHD} = \beta_k^{NVHS^*}$ and if $\theta_k = 1$, then $\beta_k^{hHD} = \beta_k^{DY}$. On the other hand, if $0 < \theta_k < 1$, then β_k^{hHD} is a convex combination of $\beta_k^{NVHS^*}$ and β_k^{DY} parameters. The search direction d_k of our algorithm is computed by

$$d_0 = -g_0, \quad d_{k+1} = -g_{k+1} + \beta_k^{hHD} d_k. \quad (2.2)$$

2.1. The conjugate condition. The traditional conjugacy condition $d_{k+1}^T y_k = 0$, plays an important role in the convergence analyses and numerical calculation. The select the parameter θ_k , we consider the following Lemma.

Lemma 2.1. If the conjugacy condition $d_{k+1}^T y_k = 0$, is satisfied at every iteration, we get

$$\theta_k = 1 - \frac{\|g_k\|^2}{|g_{k+1}^T g_k|}. \quad (2.3)$$

Proof. We multiply the two sides of the relation (2.2) by y_k^T , we obtain

$$-y_k^T g_{k+1} + \left((1 - \theta_k) \beta_k^{NVHS^*} + \theta_k \beta_k^{DY} \right) y_k^T d_k = 0.$$

After simplification, we find

$$\theta_k = \frac{y_k^T g_{k+1} - \beta_k^{NVHS^*} y_k^T d_k}{(\beta_k^{DY} - \beta_k^{NVHS^*}) y_k^T d_k} = \frac{y_k^T g_{k+1} - \|g_{k+1}\|^2 + \frac{|g_{k+1}^T g_k|}{\|g_k\|^2} g_{k+1}^T g_k}{\frac{|g_{k+1}^T g_k|}{\|g_k\|^2} g_{k+1}^T g_k} = \frac{-g_{k+1}^T g_k + \frac{|g_{k+1}^T g_k|}{\|g_k\|^2} g_{k+1}^T g_k}{\frac{|g_{k+1}^T g_k|}{\|g_k\|^2} g_{k+1}^T g_k}.$$

Then, we get

$$\theta_k = \frac{\|g_k\|^2}{|g_{k+1}^T g_k|} - 1.$$

The result can be achieved. $\square$

2.2. hHD Algorithm. The detailed framework of the proposed algorithm is presented in Algorithm 1

Algorithm 1 hHD Conjugate Gradient Algorithm

Require: Initial point $x_0 \in \mathbb{R}^n$, parameters $\delta = 10^{-3}$, $\sigma = 10^{-1}$, $\varepsilon = 10^{-6}$, $max_iter = 2000$, $max_time = 500$ s

Ensure: Approximate minimizer x_k

- 1: Compute $f_0 = f(x_0)$, $g_0 = \nabla f(x_0)$
 - 2: Set $d_0 = -g_0$, $k = 0$
 - 3: **while** $\|g_k\|_\infty > \varepsilon$ and $k < max_iter$ and elapsed time $< max_time$ **do**
 - 4: Compute α_k satisfying strong Wolfe conditions (1.4) and (1.6) using Hager-Zhang line search
 - 5: $x_{k+1} = x_k + \alpha_k d_k$
 - 6: Compute $g_{k+1} = \nabla f(x_{k+1})$
 - 7: $y_k = g_{k+1} - g_k$
 - 8: **if** $|g_{k+1}^T g_k| < 10^{-12}$ **then**
 - 9: $\theta_k = 0$
 - 10: **else**
 - 11: $\theta_k = \frac{\|g_k\|^2}{|g_{k+1}^T g_k|} - 1$
 - 12: $\theta_k = \max(0, \min(1, \theta_k))$ $\triangleright$ Clamp to $[0, 1]$
 - 13: **end if**
 - 14: Compute $\beta_k^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T y_k}$
 - 15: Compute $\beta_k^{NVHS^*} = \frac{g_{k+1}^T \left(g_{k+1} - \frac{|g_{k+1}^T g_k|}{\|g_k\|^2} g_k \right)}{d_k^T y_k}$
 - 16: $\beta_k^{hHD} = (1 - \theta_k) \beta_k^{NVHS^*} + \theta_k \beta_k^{DY}$
 - 17: $d_{k+1} = -g_{k+1} + \beta_k^{hHD} d_k$
 - 18: $k = k + 1$
 - 19: **end while**
 - 20: **return** x_k
-

2.3. Sufficient descent condition. Now, we prove that the search direction d_k obtained by hHD method satisfies the sufficient descent condition.

Theorem 2.1. Let the sequence $\{g_k\}_{k \geq 0}$ and $\{d_k\}_{k \geq 0}$ be generated by hHD algorithm, then for positive constant c , we have

$$g_k^T d_k \leq -c \|g_k\|^2, \forall k \geq 0. \quad (2.4)$$

Proof. The following proof is by induction. For $k = 0$, $g_0^T d_0 = -\|g_0\|^2$, we conclude that the sufficient descent condition holds for $k = 0$. Now, we assume (2.4) holds for k and prove that for $k + 1$. From (2.1) and (2.2), we get

$$d_{k+1} = -g_{k+1} + \left((1 - \theta_k) \beta_k^{NVHS*} + \theta_k \beta_k^{DY} \right) d_k.$$

After a simple calculation, we have

$$d_{k+1} = \theta_k d_{k+1}^{DY} + (1 - \theta_k) d_{k+1}^{NVHS*}. \quad (2.5)$$

Firstly, let $\theta_k = 0$, then

$$d_{k+1} = d_{k+1}^{NVHS*} = -g_{k+1} + \beta_k^{NVHS*} d_k. \quad (2.6)$$

Multiplying (2.6) by g_{k+1}^T from the left, we get

$$g_{k+1}^T d_{k+1}^{NVHS*} \leq -\|g_{k+1}\|^2 + \beta_k^{NVHS*} |g_{k+1}^T d_k|. \quad (2.7)$$

Thus from (1.6), (2.4), we have

$$d_k^T y_k \geq (1 - \sigma) (-g_k^T d_k) \geq 0. \quad (2.8)$$

We have from the definition of β_k^{NVHS*} and the Cauchy Schwarz inequality, that

$$0 \leq \beta_k^{NVHS*} \leq 2 \frac{\|g_{k+1}\|^2}{d_k^T y_k}. \quad (2.9)$$

Using (1.6), (2.7), (2.8) and (2.9), we concluded

$$g_{k+1}^T d_{k+1}^{NVHS*} \leq -k_1 \|g_{k+1}\|^2, \quad (2.10)$$

where $k_1 = \frac{1-3\sigma}{1-\sigma}$, so there is a constant $k_1 > 0$, with $0 < \sigma < \frac{1}{3}$.

Secondly, let $\theta_k = 1$, then

$$d_{k+1} = d_{k+1}^{DY} = -g_{k+1} + \beta_k^{DY} d_k. \quad (2.11)$$

Multiplying (2.11) by g_{k+1}^T from the left, we get

$$g_{k+1}^T d_{k+1}^{DY} \leq -\|g_{k+1}\|^2 + \|g_{k+1}\|^2 \frac{|g_{k+1}^T d_k|}{y_k^T d_k}. \quad (2.12)$$

Using the line search condition (1.6), (2.8) and (2.12), we obtain

$$g_{k+1}^T d_{k+1}^{DY} \leq -k_2 \|g_{k+1}\|^2, \quad (2.13)$$

where $k_2 = \frac{1-2\sigma}{1-\sigma}$. Since $\theta_k \in (0, 1)$ and the algorithm does not stagnate, we can define $a_1 = \inf_k \theta_k > 0$ and $a_2 = \sup_k \theta_k < 1$. Finally, we assume that $0 < \theta_k < 1$, i.e. there exists two positive constants a_1 and a_2 positives such that: $0 < a_1 \leq \theta_k \leq a_2 < 1$. From the relations (2.5), (2.10) and (2.13), we get

$$g_{k+1}^T d_{k+1} \leq -c \|g_{k+1}\|^2,$$

where $c = a_1 k_1 + (1 - a_2) k_2$. Hence, Theorem 2.1 is proved. $\square$

3. GLOBAL CONVERGENCE

- To establish the global convergence of our method, we need the following basic assumptions on the objective function.

Assumption 3.1. The level set

$$S = \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\},$$

is bounded.

Assumption 3.2. In some open convex neighborhood $\mathcal{N}$ of S , the function f is continuously differentiable and its gradient is Lipschitz continuous, namely, there exists a constant $L > 0$, such that:

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\| \quad \forall x, y \in \mathcal{N}. \quad (3.1)$$

From Assumption 3.2, we can deduce that for all $x \in \mathcal{N}$, there exists a positive constant $\Gamma \geq 0$, such that

$$\|\nabla f(x)\| \leq \Gamma, \text{ for all } x \in \mathcal{N}. \quad (3.2)$$

- It follows from Dai et al. [23] proved the sufficient condition for the convergence of conjugate gradient methods with strong Wolfe line search.

Lemma 3.1. Let Assumptions 3.1 and 3.2 hold. Consider the method (1.2) and (1.3), where d_k is a descent direction, and α_k is obtained by the strong Wolfe line search. If

$$\sum_{k \geq 0} \frac{1}{\|d_k\|^2} = \infty, \quad (3.3)$$

then

$$\liminf_{k \rightarrow \infty} \|g_k\| = 0.$$

Now, we need also this lemma to prove the convergence of our method.

Lemma 3.2. Let Assumptions 3.1 and 3.2 hold. If d_k is a descent direction and α_k satisfies the WLS (1.4) and (1.5). Then

$$\alpha_k \geq \frac{(1 - \sigma) |g_k^T d_k|}{L \|d_k\|^2}. \quad (3.4)$$

Proof. See the proof of Lemma 3.2 in Liu and Li [2]. $\square$

Remark 3.1. From (1.6), (2.4), the step size α_k obtained in the hHD algorithm satisfies (3.3). This indicates, the step size α_k obtained in our method is not equal to zero, i.e., there exists a constant $\lambda > 0$, such that

$$\alpha_k \geq \lambda, \forall k \geq 0. \quad (3.5)$$

Theorem 3.1. Suppose that Assumptions 3.1 and 3.2 hold. Consider any conjugate gradient method in the form (1.2) and (1.3), with the parameter $\beta_k = \beta_k^{hHD}$, in which the step length α_k is determined to satisfy the strong Wolfe line search condition (1.4) and (1.6), where d_k is a descent search direction. Then, this method converges in the sense that

$$\liminf_{k \rightarrow \infty} \|g_k\| = 0. \quad (3.6)$$

Proof. To prove Theorem 3.1, we use contradiction. That is, if equation (3.6) is not true. Then we can find a positive constant γ , such that

$$\|g_k\| \geq \gamma, \quad \text{for all } k. \quad (3.7)$$

From (2.1) and (2.9), we have

$$\beta_k^{hHD} \leq \beta_k^{DY} + \beta_k^{NVHS^*} \leq 3\beta_k^{DY}. \quad (3.8)$$

By using (2.8), (2.10) and (3.7), we obtain

$$y_k^T d_k \geq k_2(1 - \sigma)\gamma^2. \quad (3.9)$$

From (3.2), (3.8) and (3.9), we have

$$\beta_k^{hHD} \leq \frac{3\Gamma^2}{k_2(1 - \sigma)\gamma^2} = E. \quad (3.10)$$

Thus, it follows from (2.2), (3.2), (3.5) and (3.10), that

$$\|d_{k+1}\| \leq \|g_{k+1}\| + \beta_k^{hHD} \frac{\|x_{k+1} - x_k\|}{\alpha_k} \leq M, \quad (3.11)$$

where

$$M = \Gamma + E \frac{D}{\lambda}, \quad \text{and} \quad D = \max \{\|y - z\| : y, z \in \mathcal{N}\}.$$

By taking the summation $k \geq 0$,

$$\sum_{k \geq 0} \frac{1}{\|d_k\|^2} = \infty.$$

So, applying Lemma 3.1, we conclude that (3.6) is true. This is a contradiction with (3.7), so we have proved (3.6). $\square$

4. NUMERICAL EXPERIMENTS

In this section, we present some numerical experiments obtained with the new proposed conjugate gradient method. The test problems have been taken to the CUTE library [22, 24]. All the algorithms have been coded in MATLAB 2013 and compiler settings on the PC machine (2.5 GHz, 3.8 GB RAM) with Windows XP operating system. We compare the computational results of hHD method against the DY [11], NVHS* [14], CCOMB [21], hDYz [18], HYBRID [20] and HLSDY [2] methods. In this numerical result, all algorithms implement the SWLS condition with $\delta = 10^{-3}$ and $\sigma = 10^{-1}$. The iteration is terminated if one of the following conditions is satisfied (i) $\|g_k\|_\infty < 10^{-6}$, where $\|\cdot\|_\infty$ is the maximum absolute component of a vector, (ii) The number of iterations exceeded 2000, (iii) The computing time is more than 500 s. We show the performance difference clearly between our method and six conjugate gradient algorithms. We choose the performance profile introduced by Dolan and Moré [25] to compare the performance according to number iteration and CPU time to rule as follows. Let S is the set of methods and P is the set of the test problems with n_p , n_s is the number of the test problems and the number of the methods, respectively. For each problem $p \in P$ and solver $s \in S$, denote $\tau_{p,s}$ be the number of iterations or CPU time required to solve problems $p \in P$ by solver $s \in S$. Then comparison between different solvers based on the performance ratio is given by

$$r_{p,s} = \frac{\tau_{p,s}}{\min \{\tau_{p,i}, 1 \leq i \leq n_s\}}.$$

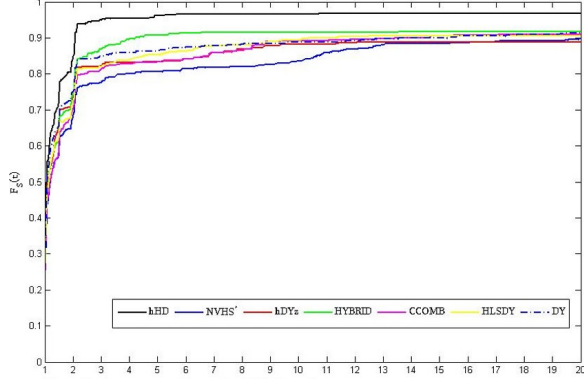
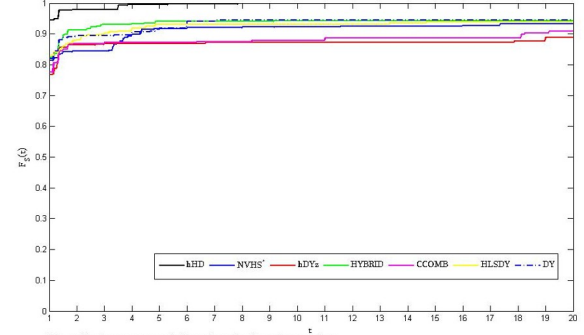
Suppose that a parameter $r_M \geq r_{p,s}$ for all problems and solvers chosen, and $r_M = r_{p,s}$ if and only if solver S does not solve problem p . The overall evaluation of the performance of the solvers is then given by the performance profile function given by

$$F_s(t) = \frac{\text{size} \{p : 1 \leq p \leq n_p, r_{p,s} \leq t\}}{n_p},$$

where $t \geq 1$ and $\text{size} \{p : 1 \leq p \leq n_p, r_{p,s} \leq t\}$ is the number of elements in the set

$\{p : 1 \leq p \leq n_p, r_{p,s} \leq t\}$. This function $F_s : [1, \infty[\rightarrow [0, 1]$ is the distribution function for the performance ratio. The value of $F_s(1)$ is the probability that the solver will win the rest of the solvers. In this numerical study, Dim denotes the dimension of the problem, ITR denotes the number of iterations, TIME denotes the CPU time and Inf denotes the algorithm failed to yield a solution for the problem.

Table 2, Fig. 1 and Fig. 2 give a performance comparison of the hHD method with those for the CPU time and the number of iterations. From these Figs. and Table 2, we can conclude that the hHD method is competitive and is superior to DY [11], NVHS* [14], CCOMB [21], hDYz [18], HYBRID [20] and HLSDY [2] conjugate gradient methods for the given test problems. These obtained preliminary results are indeed encouraging.

Fig. 1. Performance profile on the CPU time. t Fig. 2. Performance profile based on the iteration number. t

Functions	NVHS*		hHD		HLSDY		hDYz		HYBRID		CCOMB		DY		Dim
	ITR	TIME	ITR	TIME	ITR	TIME	ITR	TIME	ITR	TIME	ITR	TIME	ITR	TIME	
Hager	49	1.6870	7	0.1560	51	1.7970	8	0.2970	12	0.3070	50	1.6880	68	2.0160	1900
Griewank	43	5.3900	23	2.6280	23	2.7780	65	14.3720	20	2.6310	19	2.6200	22	2.7400	3000
Dixon	47	0.3310	42	0.3010	54	0.4820	56	0.4910	49	0.3640	48	0.3610	51	0.4520	800
Diagonal 4	23	3.7960	8	0.4580	7	2.1570	9	0.3910	7	0.4530	8	0.4530	7	0.4530	3000
Chung	5	2.1410	4	2.1200	5	2.1560	6	2.1410	5	2.1410	5	2.1410	5	2.1400	500
Branin	9	0.1620	5	0.0980	29	0.1720	8	0.1470	8	0.1470	29	0.1360	29	0.1560	2
Beale	21	0.0150	14	0.0150	21	0.0160	22	0.0160	22	0.0160	18	0.0320	21	0.0150	7
Tridiagonal	7	0.3980	5	0.3280	12	0.6250	9	0.4810	8	0.4530	8	0.4580	7	0.4030	1470
Alpine 2	9	2.9060	6	2.3450	9	2.8750	12	3.1010	8	2.3910	24	4.0840	10	2.9630	2600
Staircas S3	7	0.0310	7	0.0320	7	0.0320	7	0.0310	7	0.0310	7	0.0470	7	0.0310	400
Staircas S2	5	0.0610	3	0.0310	4	0.0470	4	0.0470	5	0.0630	4	0.0630	3	0.0320	470
Nonscomp	2	0.0160	2	0.0150	2	0.0310	2	0.0150	2	0.0150	2	0.0320	2	0.0310	1470
Perturbed	3	0.0310	3	0.0150	3	0.0310	3	0.0320	3	0.0160	3	0.0150	3	0.0310	500
Diagonal 1	12	0.3270	4	0.0460	6	0.0780	870	29.4060	5	0.0670	192	5.1710	12	0.3280	2800
Liarwhd	51	1.6810	42	1.3470	48	1.6280	48	1.6270	44	1.4840	46	1.6230	44	1.4810	1000
Schwefel223	10	0.1870	11	0.1970	10	0.1880	12	0.2040	10	0.1880	10	0.1880	12	0.2040	470
Cube	421	6.3280	224	3.3410	668	8.8470	Diverged	Diverged	298	3.8320	371	5.2350	267	3.8310	350
Rosenbrock	21	1.3440	8	0.5470	18	1.2480	119	5.8280	17	1.1090	79	3.1100	22	1.1930	2700
Power	1080	29.0750	700	25.1100	1190	27.1890	1297	31.9320	Diverged	Diverged	1385	32.8570	871	26.1480	1300
Perturbed	62	2.8320	57	2.6110	65	2.9220	104	3.5470	88	3.0880	65	2.8340	60	2.7440	1800
Lion	3	0.0310	3	0.0150	3	0.0310	3	0.0160	3	0.0150	3	0.0310	3	0.0160	2
Rayden1	7	0.4030	6	0.3210	9	0.4530	11	0.7030	10	0.6720	29	1.4770	15	0.8280	900
Zakharov	7	0.4030	9	0.3210	9	0.4530	11	0.7030	10	0.6720	29	1.4770	15	0.8280	1200
Rasting	30	1.9850	6	0.1020	Diverged	Diverged	8	0.1690	11	0.3720	53	2.1700	161	9.7480	1200
Quadratic	15	0.4130	22	0.5830	21	0.5190	17	0.4230	17	0.4350	18	0.4500	15	0.4180	3100
Staircas S1	56	2.9440	53	2.8800	102	4.9400	58	2.9460	55	2.9420	74	3.1250	78	3.1090	1470
Diagonal 2	Diverged	Diverged	7	0.0160	Diverged	Diverged	176	0.1250	Diverged	Diverged	180	0.1410	Diverged	Diverged	63
Almost Perturbed Quadratic	8	15.773	9	3.110	8	4.482	7	5.830	3	4.651	7	4.226	8	4.027	2000
England 1	7	58.871	2	0.390	8	0.346	8	0.418	7	0.416	6	0.631	8	0.442	3000
Almost Perturbed Quartic	23	128.456	5	0.411	2	2.246	4	1.859	9	1.437	4	1.542	8	2.628	1200
Nondia	Diverged	Diverged	8	0.043	Diverged	0.078	Diverged	Diverged	19	0.062	Diverged	Diverged	7	0.078	1800
Quartem	4	3.469	3	3.347	3	2.628	2	2.633	3	1.484	2	2.588	3	3.481	1800
Linear Perturbed	8	5.403	1	0.343	0.125	1.281	Diverged	Diverged	39	40.100	Diverged	Diverged	4	1.414	1000
England 8	Diverged	Diverged	5	13.746	Diverged	13.852	Diverged	Diverged	5	14.623	Diverged	Diverged	7	14.623	2000
Feletcher	28	42.191	2	3.845	1	1.875	2	7.162	2	2.391	2	2.086	2	2.921	2600
Dixon 3 dq	6	6.666	6	0.047	6	0.042	6	0.047	6	0.046	6	0.062	6	0.511	500
Tridia	12	8.470	0	5.118	11	6.189	3	5.932	Diverged	Diverged	85	2.857	6	5.148	12600
Double Border Arrow Up	8	5.667	8	0.416	2	0.284	0	0.381	8	0.375	0	0.382	2	0.402	4000
DiagonalDiag-Aup1	3	3.333	3	0.072	3	0.063	3	0.062	3	0.062	3	0.076	3	0.072	10000

TABLE 2. The simulation results of hHD, NVHS*, HLSDY, hDYz, HYBRID, CCOMB and DY methods

Note: In this table, “Diverged” indicates that the algorithm failed to converge. The possible failure modes are: (i) the number of iterations exceeded the maximum limit of 2000, (ii) the CPU time exceeded 500 seconds, or (iii) the gradient norm did not satisfy the stopping criterion $\|g_k\|_\infty < 10^{-6}$ within the allowed iterations.

5. APPLICATION IN MODE FUNCTION

Nonparametric estimation has received a great deal of attention in both theoretical and applied statistics literature. For the historical and mathematical survey, we refer to the reader to Sager [26]. In

statistics, it is always interesting to study the central tendency of the data, that is usually quantified using the location parameters (mean, mode, median). The problem of estimating the mode function of a probability density function (p.d.f.) has taken considerable attention in the past for both independent and dependent data, and a number of distinguished papers deal with this topic. For example, Parzen [27] and Eddy [28] for estimation of the unconditional mode in the independent and identically distributed (i.i.d.) case.

In this section, we consider the problem of estimating the mode of a multivariate unimodal probability density f with support in $\mathbb{R}^n$ from i.i.d. standard normal random variables $X_1, \dots, X_n$ with common probability density function f . This problem has been investigated in numerous papers. To quote a few of them, Konakov [19] and Samanta [15]. We assume that density f has a unique mode denoted by θ and defined by

$$f(\theta) = \max_{x \in \mathbb{R}^n} f(x). \quad (5.1)$$

A kernel estimator of the mode θ is defined as the random variable $\hat{\theta}$ which maximizes the kernel estimator $f_n(x)$ of $f(x)$, that is

$$f_n(\hat{\theta}) = \max_{x \in \mathbb{R}^n} f_n(x), \quad (5.2)$$

where

$$f_n(x) = \frac{1}{nh_n^n} \sum_{i=1}^n K\left(\frac{x - X_i}{h_n}\right). \quad (5.3)$$

The bandwidth (h_n) is a sequence of positive real numbers which goes to zero as n goes to infinity and the kernel K is a p.d.f. on $\mathbb{R}^n$. The kernel function K controls the smoothness of the estimated density. The Gaussian kernel produces infinitely smooth estimates, while the Epanechnikov kernel has compact support and is optimal in terms of asymptotic mean integrated squared error. The bandwidth h_n plays a crucial role in balancing bias and variance: a small bandwidth leads to undersmoothing (high variance), while a large bandwidth leads to oversmoothing (high bias). The cross-validation method selects the optimal bandwidth by minimizing an estimate of the prediction error, which provides a data-driven criterion for bandwidth selection.

In this simulation, we choose between two different types of kernel: the standard Gaussian kernel defined by

$$K(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{1}{2} \sum_{j=1}^n x_j^2\right),$$

and the Epanechnikov kernel given by

$$K(x) = \left(\frac{3}{4}\right)^n \prod_{j=1}^n (1 - x_j^2).$$

The selection of the bandwidth h is an important and basic problem in kernel smoothing techniques. In this simulation, we choose the optimal bandwidth by the cross-validation method.

In this context, we employ the hHD method to solve the problem (5.2) under the strong Wolfe line search technique and compare it with NVHS* [25] and DY [23] methods. We choose some initial points and obtain the results in Table 3. Additionally, we compare the hHD method with COMB method, and the results are shown in Table 4.

According to Table 3, it is clear that the hHD method is more efficient than NVHS* [25], COMB [21], and DY [23] methods based on the number of iterations and CPU time for solving the problem (5.2).

Conclusion

We have presented in this paper a new hybrid conjugate gradient algorithm by using the convex combination of NVHS* and DY methods. The global convergence properties and the sufficient descent condition of the proposed method have been established, under the strong Wolfe line search conditions.

TABLE 3. The simulation results of hHD, CCOMB, NVHS* and DY methods for solving problem (5.2)

Model	Initial Points	Dim	hHD		CCOMB		NVHS*		DY	
			ITR	TIME	ITR	TIME	ITR	TIME	ITR	TIME
Linear	0.2	26	80	6.5000	464	37.3410	975	67.5470	850	60.1230
		32	35	3.5000	59	6.7480	86	8.9830	75	7.8910
		37	90	10.5000	110	13.8640	57	7.7770	60	8.1020
		40	65	8.2000	84	12.3570	85	12.1750	82	11.9560
		44	160	28.0000	84	17.5780	84	17.6570	80	16.5280
		48	30	6.5000	54	12.1710	54	12.1710	50	11.0820
		58	6	2.1000	50	16.5830	43	15.0470	45	14.2000
		60	25	8.0000	14	4.3040	13	4.2660	15	5.4720
Nonlinear	0.5	160	70	180.00	22	36.8900	7	17.6860	10	20.8430
		170	26	78.0000	43	60.5780	63	80.5010	55	70.3570
		190	16	55.0000	15	45.3470	15	47.1470	18	50.0080
		180	15	48.0000	18	65.6030	18	65.5320	20	68.0140
		270	22	160.0000	47	341.400	48	350.620	45	330.3450
		290	10	85.0000	12	102.340	18	151.870	15	130.5320
		320	12	125.0000	16	157.340	28	289.170	25	250.0000
		340	8	90.0000	18	245.220	16	236.390	14	220.0230
Nonlinear	-0.3	54	32	7.5000	19	4.5742	16	3.9841	18	5.0000
		56	225	70.0000	41	12.6510	41	12.9020	45	15.0160
		66	34	13.5000	53	23.3400	53	23.6040	50	22.4560
		68	14	7.0000	35	14.6710	27	12.7020	30	13.9640
		78	390	185.0000	62	28.1250	124	51.6980	100	45.0030
		70	37	18.0000	56	27.8930	53	27.4250	55	26.0340
		72	22	11.5000	50	26.6290	52	27.3590	48	25.8730
		74	16	8.5000	20	10.3221	21	11.0090	22	12.2460

Numerical results also show that hHD method is robust and effective. The practical applicability of our method is also explored in the nonparametric estimation of the mode function.

Acknowledgments

The authors are very grateful to the anonymous referees for their valuable comments and useful suggestions, which improved the quality of this paper.

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On two problems with displacement at three points of boundary characteristics for a degenerate hyperbolic equation

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Abstract. This paper is devoted to proving the well-posedness of the boundary characteristic problem with three displacements for the equation $-(-y)^m u_{xx} + u_{yy} - \frac{m}{2y} u_y = 0$ in the characteristic triangle. The proof of the existence of a solution to the problem, based on integral representations of solutions to the modified Cauchy problem, is reduced to solving a functional equation with two shifts $p(x)$ and $q(x)$. A combined method of successive approximations and iterations is used to solve the functional equation. A counterexample is given indicating the importance of the class of functions where the solution to the functional equation is sought.

Keywords: singular coefficient, modified Cauchy problem, functional equation with two shifts, recurrence relation, combined method of iterations and successive approximations, functional series, majorant series, uniform convergence

MSC (2020): 11D04, 1141

1. INTRODUCTION

Over the past period of time, for degenerate hyperbolic equations, foreign and local scientists: G. Darboux, V.I. Zhegalov [1], A.M. Nakhushev [2] and M.S. Salakhitdinov, M. Mirsaburov [3],[4], A.K. Urinov, A.T. Abdukodirov [5], G.M. Mirsaburova [6], U.M. Mirsaburova [7] and others conducted scientific research on the study of boundary value and nonlocal problems and obtained significant scientific results.

The foundations of the theory of boundary value problems with displacement are laid in the works of V.I. Zhegalov [1], for the Lavrentiev-Bitsadze equation and A.M. Nakhushev [2], for the Tricomi equation. Unlike the Tricomi problem, where the boundary condition is specified only on one boundary characteristic, in problems with displacement [1], [2] both boundary characteristics are covered by a boundary condition where a nonlocal condition (displacement condition) is specified establishing pointwise connections between the function values on both characteristics [1] or pointwise connections are established using the fractional differentiation operator [2] in the Riemann-Liouville sense.

Problems with displacement were intensively developed in the works of S.K. Kumyкова [8], V.A. Eleev [9], O.A. Repin [10], and M.M. Smirnov [11]. In these problems, it was characteristic that the displacement condition was specified at two points on the boundary characteristics.

2. STATEMENT OF THE PROBLEM A

Let D_- be a finite simply connected domain of the half-plane $y < 0$, bounded by the segment AB of the axis $y = 0$, characteristics AC and BC of the equation

$$-(-y)^m u_{xx} + u_{yy} - \frac{m}{2y} u_y = 0, \quad (2.1)$$

where the constant $m > 0$, $A = A(-1, 0)$, and $B = B(1, 0)$.

In problems with displacement [1, 2, 3, 4], [8, 9, 10, 11], the displacement condition is specified at two points of the boundary characteristics. In this paper, in the characteristic triangle, for a degenerate hyperbolic equation with a singular coefficient, the correctness of the problem with the displacement condition at three points of the boundary characteristics is investigated.

On the segment AB , we consider the point $E = E(c, 0)$ where $c \in I = (-1, 1)$ – the interval of the axis $y = 0$, and introduce the linear functions $p(x) = ax - b$ and $q(x) = a - bx$ – mapping the segment $[-1, 1]$ onto the segments $[-1, c]$ and $[c, 1]$ respectively, where $p(-1) = -1$, $p(1) = c$, $q(-1) = 1$, $q(1) = c$, i.e. $a = (1 + c)/2$, $b = (1 - c)/2$.

A Problem . In the domain D find a regular solution $u(x, y) \in C(\bar{D}) \cap C^2(\bar{D})$ of equation (2.1), that satisfies the conditions:

$$u(x, 0) = \tau(x), \quad x \in [-1, 1], \quad (2.2)$$

$$u[\theta_0(x)] = \mu_2 u[\theta_1(q(x))] - \mu_1 u[\theta_1(p(x))] + \rho(x), \quad x \in [-1, 1], \quad (2.3)$$

where μ_1, μ_2 – are some positive constants, and $\tau(x), \rho(x) \in C^2(\bar{I})$ – are given functions,

$$\theta_0(x_0) = \frac{x_0 - 1}{2} - i \left[\frac{m+2}{4} (1 + x_0) \right]^{2/m+2}$$

– is the affix of the intersection point of the boundary characteristic AC with the characteristic emanating from the point $x_0 \in [-1, 1]$, and

$$\theta_1(p(x_0)) = \frac{1 + p(x_0)}{2} - i \left[\frac{m+2}{4} (1 - p(x_0)) \right]^{2/m+2},$$

$$\theta_1(q(x_0)) = \frac{1 + p(x_0)}{2} - i \left[\frac{m+2}{4} (1 - q(x_0)) \right]^{2/m+2}$$

– affixes of the intersection points of the boundary characteristic BC with the characteristics emanating from the points $(p(x_0), 0)$ and $(q(x_0), 0)$ respectively.

3. STUDY OF PROBLEM A.

Theorem 3.1. *Problem A is uniquely solvable under the condition*

$$\delta_1 + \delta_2 < 1, \quad (3.1)$$

where $\delta_1 = a\mu_1$, $\delta_2 = b\mu_2$

Proof. D'Alembert formula [12], which gives in the domain D for equation (2.1) a solution to the modified Cauchy problem with initial data

$$u(x, 0) = \tau(x), \quad x \in \bar{I}; \quad \lim_{y \rightarrow -0} (-y)^{-m/2} \frac{\partial u}{\partial y} = \nu(x), \quad x \in I;$$

has the form

$$\begin{aligned} u(x, y) = & \frac{1}{2} \left[\tau \left(x + \frac{2}{m+2} (-y)^{\frac{m+2}{2}} \right) + \tau \left(x - \frac{2}{m+2} (-y)^{\frac{m+2}{2}} \right) \right] - \\ & - \frac{(-y)^{\frac{m+2}{2}}}{m+2} \int_{-1}^1 \nu \left(x + \frac{2t}{m+2} (-y)^{\frac{m+2}{2}} \right) dt. \end{aligned} \quad (3.2)$$

Using the D'Alembert formula (3.2), it is easy to calculate the values

$$u[\theta_0(x)] = \frac{1}{2} [\tau(-1) + \tau(x)] - \frac{1}{2} \int_{-1}^x \nu(t) dt, \quad (3.3)$$

$$u[\theta_1(p(x))] = \frac{1}{2} [\tau(1) + \tau(p(x))] - \frac{1}{2} \int_{p(x)}^1 \nu(t) dt, \quad (3.4)$$

$$u[\theta_1(q(x))] = \frac{1}{2} [\tau(1) + \tau(q(x))] - \frac{1}{2} \int_{q(x)}^1 \nu(t) dt. \quad (3.5)$$

Now substituting (3.3), (3.4), and (3.5) into the boundary condition (2.3), and calculating the derivative with respect to x , taking into account equality (2.2), we have the functional equation:

$$\nu(x) = \delta_1 \nu(p(x)) + \delta_2 \nu(q(x)) + F(x), \quad (3.6)$$

where

$$F(x) = \tau'(x) - a\mu_1\tau'(p(x)) + b\mu_2\tau'(q(x)) - 2\rho(x),$$

is a known function. Equation (3.6) is a functional equation with two displacements, $p(x)$ and $q(x)$.

To solve equation (3.6), we apply a combined method of successive approximations and iterations [6].

To construct the terms of the sequence $\nu_0(x), \nu_1(x), \dots, \nu_n(x), \dots$ we use the recurrence relation

$$\nu_n(x) = \delta_1\nu_n(p(x)) + \delta_2\nu_{n-1}(q(x)) + F(x), x \in \bar{I}, n = 1, 2, \dots, \quad (3.7)$$

As the zero approximation of $\nu_0(x)$ we take the solution of the functional equation

$$\nu_0(x) = \delta_1\nu_0(p(x)) + F(x), x \in \bar{I}, \quad (3.8)$$

We will look for a solution to the functional equation (3.8) in the class of functions bounded at the point $x = -1$. If we abandon this requirement, then the corresponding homogeneous functional equation (3.8) (with $F(x) = 0$) may have a nontrivial solution.

Example. Let $F(x) = 0$, in equation (3.8), then it takes the form

$$\nu_0(x) = \delta_1\nu_0(p(x)), x \in \bar{I} \quad (3.9)$$

Consider the function $\nu_0(x) = (1+x)^\alpha$, choose α such that this function satisfies equation (3.9). It is easy to verify that

$$\nu_0(p(x)) = (1+p(x))^\alpha = a^\alpha(1+x)^\alpha \quad (3.10)$$

By virtue of (3.10), from the homogeneous functional equation (3.9), to find α we obtain the equation

$$\delta_1 a^\alpha - 1 = 0. \quad (3.11)$$

The solution of equation (3.11) has the form

$$\alpha = -\log_a \delta_1,$$

and according to (3.1), where ($\delta_1 < 1$) we conclude that the found $\alpha < 0$. Consequently, the solution $\nu(x) = (1+x)^\alpha$ of the homogeneous functional equation (3.9) is not bounded in the neighborhood of the point $x = -1$. Thus, the class of solutions where the solution of the functional equation (3.9) is sought is essential.

We will solve the functional equation (3.8) by the iteration method. In (11), replacing x with $p(x)$, we obtain

$$\nu_0(p(x)) = \delta_1\nu_0(p(p(x))) + F(p(x)), x \in \bar{I}. \quad (3.12)$$

Substituting now $\nu_0(p(x))$ from (3.12) into (3.8), we have

$$\nu_0(x) = \delta_1^2\nu_0(p_2(x)) + \delta_1 F(p_1(x)) + F(x), x \in \bar{I}, \quad (3.13)$$

here $p_1(x) = p(x), p_2(x) = p_1(p_1(x))$. Relation (3.13) is the first iteration of equation (3.8). Continuing this process, for the n -th iteration we have

$$\nu_0(x) = \delta_1^{n+1}\nu_0(p_{n+1}(x)) + \sum_{k=0}^n \delta_1^k F(p_k(x)), x \in \bar{I}, \quad (3.14)$$

here $p_0(x) = x, p_{n+1}(x) = p_n(p_1(x)) = p_1(p_n(x)), p_k(1) = 1, k \in \overline{1, n}$.

Note that, by (3.1) (where $\delta_1 < 1$),

$$\lim_{n \rightarrow \infty} \delta_1^n = 0. \quad (3.15)$$

Since $p(x) < x, p(-1) = -1$, for $x \in [-1, 1]$ and $p(-1) = -1$, then $p_n(x) = p_{n-1}(p_1(x)) = p_1(p_{n-1}(x)) < p_{n-1}(x)$ i.e., the functional sequence $\{p_n(x)\}$ monotonically decreases and is bounded

below by -1 , i.e. $p_n(x) \geq -1, \forall x \in \bar{I}$. Therefore, by the theorem on the limit of a monotonically decreasing and bounded sequence, the sequence $\{p_n(x)\}$ converges to some function $p^0(x)$

$$\lim_{n \rightarrow \infty} p_n(x) = p^0(x), \quad x \in \bar{I}. \quad (3.16)$$

Now we find $p^0(x)$. Passing to the limit as $n \rightarrow \infty$ in the equality $p_n(x) = p_1(p_{n-1}(x))$ we obtain $p^0(x) = p_1(p^0(x))$ and from this we conclude that $p^0(x) = -1, \forall x \in \bar{I}$, since $p(x) = p_1(x)$, it has a unique fixed point at $x = -1$. Thus,

$$\lim_{n \rightarrow \infty} p_n(x) = p^0(x) = -1, \quad \forall x \in \bar{I}. \quad (3.17)$$

In relation (3.14) passing to the limit as $n \rightarrow \infty$, taking into account (3.15)-(3.17) and the boundedness of the value $\nu_0(-1)$, we obtain

$$\nu_0(x) = \sum_{k=0}^{\infty} \delta_1^k F(p_k(x)) \in C(\bar{I}) \quad (3.18)$$

- the solution of the functional equation (3.8).

It is easy to verify that

$$|\nu_0(x)| \leq \sum_{k=0}^{\infty} \delta_1^k |F(p_k(x))| \leq \sum_{k=0}^{\infty} M \delta_1^k = \frac{M}{1 - \delta_1}, \quad x \in \bar{I}, \quad (3.19)$$

where $M = \max_{x \in \bar{I}} |F(x)|$. Consequently, by virtue of (3.19), the functional series on the right-hand side of (3.18) converges uniformly, and, by virtue of the continuity of $F(x)$ on $\bar{I}$, we conclude that $\nu_0(x) \in C(\bar{I})$. It is also easy to verify that $\nu_0(x) \in C^2(\bar{I})$

Now, in the recurrence relation (3.7) setting $n = 1$, we obtain a functional equation for determining $\nu_1(x)$:

$$\nu_1(x) = \delta_1 \nu_1(p(x)) + \delta_2 \nu_0(q(x)) + F(x) \quad x \in \bar{I}, \quad (3.20)$$

here $\nu_0(x)$ — is a known function defined by equality (3.18).

Applying the iteration method to this functional equation, as above, we obtain a solution to equation (3.20) in the form

$$\nu_1(x) = \sum_{k=0}^{\infty} \delta_1^k \delta_2 \nu_0(q(p_k(x))) + \sum_{k=0}^{\infty} \delta_1^k F(p_k(x)) \in C(\bar{I}). \quad (3.21)$$

where $\nu_0(q(p_k(x)))$ — is a known quantity determined by formula (3.18).

It is easy to verify that the functional series of the right-hand side of (3.21) converge uniformly.

Continuing this process successively, it is easy to find that

$$\nu_n(x) = \sum_{k=0}^{\infty} \delta_1^k \delta_2 \nu_{n-1}(q(p_k(x))) + \sum_{k=0}^{\infty} \delta_1^k F(p_k(x)) \in C(\bar{I}). \quad (3.22)$$

In equation (3.22), $\nu_{n-1}(q(p_k(x)))$ — is a known quantity.

Thus, we have constructed a functional sequence

$$\nu_0(x), \nu_1(x), \dots, \nu_n(x), \dots \quad (3.23)$$

Now, we will prove that the functional sequence (3.23) converges uniformly. To this consider the functional series

$$\nu_0(x) + [\nu_1(x) - \nu_0(x)] + [\nu_2(x) - \nu_1(x)] + \dots + [\nu_n(x) - \nu_{n-1}(x)] + \dots \quad (3.24)$$

Let us find the majorant series for the functional series (3.24) [6],[12]. Subtracting equality (3.18) from (3.21), we obtain

$$|\nu_1(x) - \nu_0(x)| \leq \sum_{k=0}^{\infty} \delta_1^k \delta_2 |\nu_0(q((p_k(x))))| \leq \frac{\delta_2}{1 - \delta_1} \frac{M}{1 - \delta_1} = \frac{M}{\delta_2} \left(\frac{\delta_2}{1 - \delta_1} \right)^2.$$

Continuing this process as in works [6],[12], and taking into account (3.22), it is easy to find an estimate for the members of the functional sequence

$$\begin{aligned} |\nu_n(x) - \nu_{n-1}(x)| &\leq \sum_{k=0}^{\infty} \delta_1^k \delta_2 |\nu_{n-1}(q((p_k(x)))) - \nu_{n-2}(q((p_k(x))))| \leq \\ &\leq \frac{\delta_2}{1 - \delta_1} \frac{M}{\delta_2} \left(\frac{\delta_2}{1 - \delta_1} \right)^n = \frac{M}{\delta_2} \left(\frac{\delta_2}{1 - \delta_1} \right)^{n+1}. \end{aligned} \quad (3.25)$$

By (3.25), the functional series (3.24) is majorized by the numerical series

$$\sum_{k=0}^{\infty} \frac{M}{\delta_2} \left(\frac{\delta_2}{1 - \delta_1} \right)^{n+1},$$

and by (3.1), this series is an infinitely decreasing geometric progression, therefore, the series (3.24) converges uniformly on the segment $\bar{I}$.

Consequently, the functional sequence $\{\nu_n(x)\}$ converges uniformly, i.e. there exists

$$\lim_{n \rightarrow \infty} \nu_n(x) = \nu(x), \quad \forall x \in \bar{I},$$

and $\nu(x) \in C(\bar{I})$.

Thus, we have found $\nu(x)$. Now, we find $\nu'(x)$, for this, differentiating equation (3.6) with respect to the variable x , we have

$$\nu'(x) = a\delta_1 \nu'(p(x)) - b\delta_2 \nu'(q(x)) + F'(x), \quad x \in \bar{I}, \quad (3.26)$$

i.e. we have obtained a functional equation with two displacements relative to the unknown function $\nu'(x)$. Since $0 < a < 1$, then, by (3.1), $a\delta_1 - b\delta_2 < 1$, therefore, equations (3.26) are investigated in the same way as equation (3.6), and we can find $\nu'(x) \in C(\bar{I})$. Similarly, $\nu''(x) \in C(\bar{I})$ is found. Thus, we have found $\nu(x) \in C^2(\bar{I})$. Next, the solution to problem A is reconstructed using d'Alembert's formula (3.2).

Theorem 1 is proven.

4. PROBLEM B.

Problem B. In the domain D find a regular solution $u(x, y) \in C(\bar{D}) \cap C^2(\bar{D})$ of equation (2.1), that satisfies the conditions:

$$\lim_{y \rightarrow -0} (-y)^{-m/2} \frac{\partial u}{\partial y} = \nu(x), \quad x \in (-1, 1), \quad (4.1)$$

$$u[\theta(x)] = \mu_2(x)u[\theta_1(q(x))] - \mu_1(x)u[\theta_1(p(x))] + \rho(x), \quad x \in [-1, 1], \quad (4.2)$$

where $\mu_1(x)$, $\mu_2(x)$, $\nu(x)$, $\rho(x) \in C^2(\bar{I})$ — are given functions.

Theorem 4.1. *Problem B under the condition*

$$\delta_1 + \delta_2 < 1, \quad (4.3)$$

is uniquely solvable, where $\delta_1 = \max_{x \in I} |\mu_1(x)|$, $\delta_2 = \max_{x \in I} |\mu_2(x)|$.

Substituting (3.3), (3.4), and (3.5) into the boundary condition (4.2), taking into account the equality (4.1), we obtain the functional equation:

$$\tau(x) = \mu_1(x)\tau(p(x)) + \mu_2(x)\tau(q(x)) + F_1(x), \quad (4.4)$$

where

$$F_1(x) = \int_{-1}^x \nu(t)dt - \mu_1(x) \int_{p(x)}^1 \nu(t)dt - \mu_2(x) \int_{q(x)}^1 \nu(t)dt + 2\rho(x),$$

is a known function. Equation (4.4) is a functional equation with two displacements $p(x)$ and $q(x)$ relative to the unknown function $\tau(x)$.

Equation (4.4) is investigated in the same way as equation (3.6).

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Optimal Control and Pursuit Game for Planar Rigid Segments with Inertial Dynamics

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Abstract. This paper studies time-optimal control and pursuit differential game problems for planar rigid segments with inertial dynamics, actuated by bounded forces applied perpendicularly at their endpoints. The objective of the control problem is to transfer a segment between prescribed configurations in minimum time. We derive closed-form expressions for the minimal transfer time, together with a complete analytical description of optimal motions. The obtained time-optimal steering results are then used to construct a capture strategy for the pursuit game, which guarantees finite-time capture.

Keywords: time-optimal control, pursuit differential game, Pontryagin's maximum principle, bang-bang control, inertial dynamics, rigid segment, finite-time capture

MSC (2020): 49J15, 49K15, 49N70

1. INTRODUCTION

Time-optimal control problems constitute an important class of problems in optimal control theory [1, 2, 3]. The main tool for their analysis is Pontryagin's Maximum Principle [4, 5, 6], which often leads to bang-bang optimal controls [7, 8, 9]. Time-optimal problems for planar motion systems and related models on $SE(2)$ were studied in [10, 11, 12], where optimal trajectories were described with alternating rotational and translational phases. Optimal control problems for rigid body rotation under bounded controls were investigated in [13, 14, 15].

Recent studies have also addressed time-optimal and controllability problems in more general settings. In particular, time-optimal control problems with integral constraints for heat transfer models were studied in [16], while the controllability and stability properties of infinite-dimensional linear systems were investigated in [17].

The theory of pursuit-evasion differential games originated in the pioneering works of Isaacs [18] and further developed in [19, 20]. Simple-motion models with one evader and multiple pursuers were studied in [21]. Group pursuit-evasion problems on manifolds were considered in [22], while infinite-dimensional formulations were analyzed in [23, 24, 25, 26].

In this paper, we consider a planar rigid segment actuated by bounded forces applied perpendicularly at its endpoints. The dynamics are described by a second-order control system with coupled translational and rotational motions. Using Pontryagin's Maximum Principle, we show that singular arcs do not occur and that every time-optimal control is of bang-bang type. Furthermore, the structure of time-optimal trajectories is characterized, and an explicit analytical formula for the minimal transfer time is derived. Finally, the obtained optimal steering results are applied to a pursuit differential game for two rigid segments, and a constructive pursuit strategy guaranteeing finite-time capture is established under the assumption that the pursuer possesses a larger control resource than the evader.

2. TIME-OPTIMAL CONTROL PROBLEM

2.1. Problem formulation. Consider a uniform rigid segment A_0B_0 of fixed length $l > 0$ and total mass m_0 moving in the Euclidean plane $\mathbb{R}^2$. The mass is continuously distributed along the segment. The initial positions of the endpoints are given by

$$A_0 = (a_0, b_0), \quad B_0 = (a_0 + l, b_0),$$

and the desired terminal positions are

$$C = (0, 0), \quad D = (l, 0).$$

During the motion, the length of the segment remains constant. Let $A(t), B(t)$ denote the positions of the endpoints at time $t \geq 0$. The segment is actuated by bounded external forces $F_A(t)$ and $F_B(t)$, applied perpendicularly at the endpoints $A(t)$ and $B(t)$, respectively:

$$|F_A(t)| \leq \rho, \quad |F_B(t)| \leq \rho,$$

where $\rho > 0$ is a given constant representing the maximal admissible force. The objective is to transfer the segment from the initial configuration A_0B_0 to the terminal configuration CD in minimal time.

2.2. Reduction to the equivalent problem. To reduce the problem to an equivalent formulation, we introduce a coordinate representation of the segment. Let $\theta(t) \in \mathbb{R}$ be the orientation angle, i.e., the angle between the segment and the x -axis (see Fig. 1). Define the unit vectors

$$e(\theta) = (\cos \theta, \sin \theta), \quad e^\perp(\theta) = (-\sin \theta, \cos \theta),$$

representing the directions along and orthogonal to the segment, respectively.

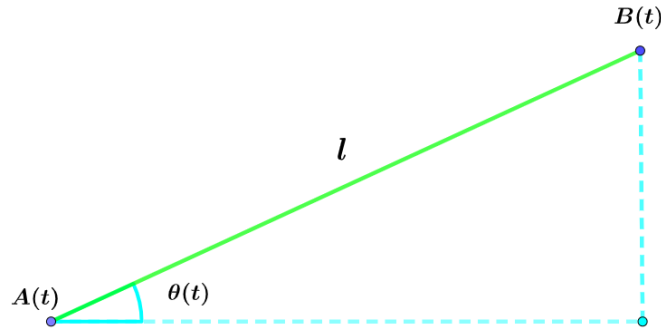


FIGURE 1. The orientation angle $\theta(t)$ of the rigid segment

The endpoints can be written as

$$A(t) = (x(t), y(t)), \quad B(t) = A(t) + l e(\theta(t)). \quad (2.1)$$

The midpoint (center of mass) of the segment is defined as

$$M(t) = \frac{1}{2}(A(t) + B(t)).$$

Using (2.1), this expression can be equivalently written as

$$M(t) = A(t) + \frac{l}{2} e(\theta(t)).$$

Hence, the initial and terminal positions of the midpoint are

$$M_0 = M(0) = \left(a_0 + \frac{l}{2}, b_0\right), \quad M_T = M(T) = \left(\frac{l}{2}, 0\right).$$

Since the forces are applied perpendicularly at the endpoints $A(t)$ and $B(t)$; therefore, the control inputs act along the direction $e^\perp(\theta(t))$ (see Fig. 2). Let $u_1(t)$ and $u_2(t)$ be scalar control functions associated with the endpoints $A(t)$ and $B(t)$, respectively. These controls satisfy

$$|u_1(t)| \leq \rho, \quad |u_2(t)| \leq \rho.$$

The corresponding control forces are given by

$$F_A(t) = u_1(t) e^\perp(\theta(t)), \quad F_B(t) = u_2(t) e^\perp(\theta(t)).$$

We now consider the following two components of motion.

Translational motion. By Newton's second law, the motion of the center of mass is governed by the resultant force:

$$m_0 \ddot{M}(t) = F_A(t) + F_B(t).$$

Substituting the expressions for the forces, we obtain

$$\ddot{M}(t) = \frac{u_1(t) + u_2(t)}{m_0} e^\perp(\theta(t)). \quad (2.2)$$

This equation describes the translational dynamics of the midpoint $M(t)$ of the segment.

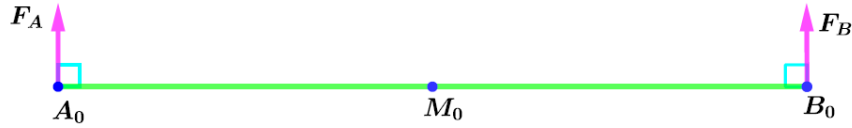


FIGURE 2. Control forces F_A and F_B acting on the segment endpoints

Rotational motion. The rotational dynamics are determined by the torque with respect to the midpoint $M(t)$. For planar motion, the torque is a scalar quantity given by

$$\tau = \overrightarrow{MA} \times F_A(t) + \overrightarrow{MB} \times F_B(t),$$

where

$$\overrightarrow{MA} = -\frac{l}{2} e(\theta(t)), \quad \overrightarrow{MB} = \frac{l}{2} e(\theta(t)),$$

and the symbol $\times$ denotes the scalar cross product in $\mathbb{R}^2$. Substituting the expressions for the forces and using the identity $e(\theta) \times e^\perp(\theta) = 1$, we obtain $\tau = \frac{l}{2}(u_2 - u_1)$. For a uniform segment of length l and mass m_0 , the moment of inertia about its midpoint is

$$J = \frac{m_0 l^2}{12}.$$

Hence, the rotational equation of motion takes the form

$$J \ddot{\theta}(t) = \tau,$$

which yields

$$\ddot{\theta}(t) = \frac{6}{m_0 l} (u_2(t) - u_1(t)). \quad (2.3)$$

Thus, the translational and rotational dynamics of the segment are completely determined by the scalar control inputs $u_1(t)$ and $u_2(t)$.

2.3. New Control Variables. The scalar controls $u_1(t)$ and $u_2(t)$ act along the same direction $e^\perp(\theta(t))$ and therefore produce only two independent effects: 1) a translation of the midpoint, 2) a rotation about the midpoint. To explicitly separate these effects, we introduce new control variables

$$s = \frac{1}{2}(u_1 + u_2), \quad a = \frac{1}{2}(u_2 - u_1). \quad (2.4)$$

The transformation (2.4) is linear and invertible, with its inverse given by $u_1 = s - a$, $u_2 = s + a$. Under this change of variables, the equations of motion (2.2), (2.3) take the form

$$\ddot{M}(t) = \frac{2}{m_0} s(t) e^\perp(\theta(t)), \quad \ddot{\theta}(t) = \frac{12}{m_0 l} a(t). \quad (2.5)$$

Thus, the control $s(t)$ governs the translational acceleration of the midpoint $M(t)$, while $a(t)$ governs the angular acceleration. This representation explicitly decouples the translational and rotational dynamics.

Since the original controls satisfy $|u_1(t)| \leq \rho$, $|u_2(t)| \leq \rho$, the admissible set in the (s, a) -coordinates is equivalently described by $|s - a| \leq \rho$, $|s + a| \leq \rho$. Using identities (2.4), we compute

$$|s| + |a| = \frac{1}{2}(|u_1 + u_2| + |u_2 - u_1|)$$

and obtain $|s| + |a| = \max\{|u_1|, |u_2|\} \leq \rho$. Therefore, the admissible control set can be written as

$$U = \{(s, a) \in \mathbb{R}^2 : |s| + |a| \leq \rho\}.$$

Geometrically, this set is a diamond-shaped region in the (s, a) -plane (see Fig. 3).

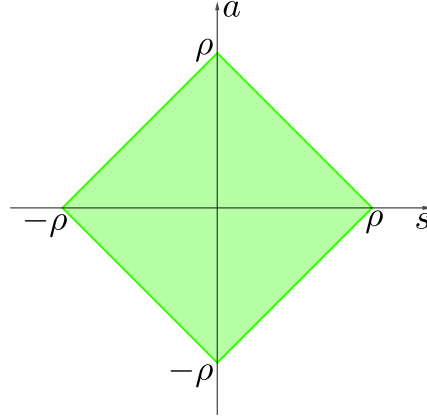


FIGURE 3. Admissible control set U in the transformed (s, a) plane

2.4. State-Space Formulation. Introduce the auxiliary variables

$$\dot{M} = m, \quad \dot{\theta} = \omega,$$

where m denotes the linear velocity of the midpoint, and ω denotes the angular velocity of the segment. The state vector is defined as

$$x = (M, m, \theta, \omega) \in \mathbb{R}^6, \quad M \in \mathbb{R}^2, \quad m \in \mathbb{R}^2.$$

The system can then be written in state-space form as

$$\begin{aligned} \dot{M} &= m, \\ \dot{m} &= \frac{2}{m_0} s e^\perp(\theta), \\ \dot{\theta} &= \omega, \\ \dot{\omega} &= \frac{12}{m_0 l} a, \end{aligned}$$

subject to the control constraint

$$|s| + |a| \leq \rho.$$

The initial conditions are

$$M(0) = M_0, \quad m(0) = 0, \quad \theta(0) = 0, \quad \omega(0) = 0,$$

and the terminal conditions are

$$M(T) = M_T, \quad m(T) = 0, \quad \theta(T) = 0, \quad \omega(T) = 0.$$

2.5. Optimal Control Problem. Based on the dynamic model, we formulate the time-optimal control problem. The control function

$$u(\cdot) = (s(\cdot), a(\cdot)) \in U$$

is to be chosen to minimize the transfer time T , subject to the system dynamics

$$\dot{x} = f(x, u) = \begin{pmatrix} m \\ \frac{2}{m_0} s e^\perp(\theta) \\ \omega \\ \frac{12}{m_0 l} a \end{pmatrix}$$

and the boundary conditions

$$x(0) = x_0 = (M_0, 0, 0, 0), \quad x(T) = x_T = (M_T, 0, 0, 0).$$

The following theorem describes the structure of time-optimal trajectories.

Theorem 2.1. *Every time-optimal trajectory consists of three phases of the form rotation $\rightarrow$ translation $\rightarrow$ rotation, and the minimal transfer time is given by*

$$T_{\min} = 2\sqrt{\frac{m_0 l |\theta^*|}{3\rho}} + \sqrt{\frac{2m_0 \sqrt{a_0^2 + b_0^2}}{\rho}},$$

where θ^* is the angle of the smallest absolute value such that

$$e^\perp(\theta^*) \parallel (M_T - M_0).$$

The proof of Theorem 2.1 is developed in Sections 2.6–2.8 using Pontryagin's Maximum Principle and a geometric analysis of optimal trajectories.

2.6. Pontryagin's Maximum Principle Analysis. Consider the control system

$$\dot{x} = f(x, u), \quad u(t) \in U, \quad x(0) = x_0, \quad x(T) = x_T.$$

To characterize optimal control and the corresponding trajectory, we apply Pontryagin's Maximum Principle (PMP), which provides necessary conditions for optimality in constrained control problems. If $u^*(\cdot)$ is an optimal control and $x^*(\cdot)$ is the corresponding optimal trajectory, then there exists a nontrivial absolutely continuous adjoint vector

$$p(\cdot) = (p_M(\cdot), p_m(\cdot), p_\theta(\cdot), p_\omega(\cdot)),$$

where $p_M, p_m \in \mathbb{R}^2$, $p_\theta, p_\omega \in \mathbb{R}$, and a constant $p_0 \leq 0$. For the normal minimum-time case, $p_0 = -1$. Moreover, $(p(\cdot), p_0) \neq 0$. The Hamiltonian is defined by

$$H(x, p, u) = p \cdot f(x, u) + p_0.$$

Necessary conditions of optimality:

1) Hamiltonian system:

$$\dot{p}_i = -\frac{\partial H}{\partial x_i},$$

or explicitly,

$$\begin{aligned} \dot{p}_M &= 0, \\ \dot{p}_m &= -p_M, \\ \dot{p}_\theta &= -s \frac{2}{m_0} \left\langle p_m, \frac{de^\perp(\theta)}{d\theta} \right\rangle = \frac{2}{m_0} s \langle p_m, e(\theta) \rangle, \\ \dot{p}_\omega &= -p_\theta. \end{aligned}$$

2) Maximum condition:

$$H(x^*(t), p(t), u^*(t)) = \max_{u \in U} H(x^*(t), p(t), u), \quad t \in [0, T].$$

3) Transversality condition:

$$H(x^*(t), p(t), u^*(t)) \equiv 0.$$

Define the switching functions:

$$\phi_1(t) = \frac{2}{m_0} \langle p_m(t), e^\perp(\theta(t)) \rangle, \quad \phi_2(t) = \frac{12}{m_0 l} p_\omega(t).$$

Then the Hamiltonian can be written as

$$H = -1 + p_M \cdot m + p_\theta \omega + s\phi_1 + a\phi_2.$$

The maximization problem

$$\max_{|s|+|a| \leq \rho} (s\phi_1 + a\phi_2)$$

leads to:

- If $|\phi_1| > |\phi_2|$, then $s = \rho \operatorname{sgn}(\phi_1)$, $a = 0$.
- If $|\phi_1| < |\phi_2|$, then $s = 0$, $a = \rho \operatorname{sgn}(\phi_2)$.
- If $|\phi_1| = |\phi_2|$, then $|s| + |a| = \rho$ (boundary control).

Thus, the optimal control is of bang–bang or boundary type.

2.7. Structure of Optimal Controls. We show that singular arcs do not occur for either control component.

Proposition 2.1. *No singular arc exists for either control component.*

Proof: **Absence of singular arcs for s .** Assume that $\phi_1(t) \equiv 0$ on a nontrivial interval I . Then

$$\langle p_m(t), e^\perp(\theta(t)) \rangle = 0,$$

which implies that $p_m(t)$ is collinear with $e(\theta(t))$, i.e.,

$$p_m(t) = \beta(t) e(\theta(t))$$

for some scalar function $\beta(t)$. From the adjoint system, we have $\dot{p}_m = -p_M$, $p_M = 0$, hence p_M is constant and $p_m(t) = -p_M t + C$, so that $p_m(t)$ is affine in time.

Thus, $p_m(t)$ is simultaneously an affine function of t and is aligned with the rotating vector $e(\theta(t))$. This is possible only if $p_m(t) \equiv 0$. Indeed, if $\theta(t)$ is nonconstant, the direction $e(\theta(t))$ rotates, which is incompatible with a nonzero affine function. If $\theta(t)$ is constant, then $a(t) = 0$, and the maximization condition implies a bang control unless $\phi_1 \equiv 0$, again leading to $p_m \equiv 0$.

From $\dot{p}_m = -p_M$, it follows that $p_M \equiv 0$, and consequently, all adjoint variables vanish. This contradicts the nontriviality condition of the adjoint vector. Hence, ϕ_1 cannot vanish on any nontrivial interval.

Absence of singular arcs for a . Assume that $\phi_2(t) \equiv 0$ on some interval I . Then $p_\omega(t) \equiv 0$. From $\dot{p}_\omega = -p_\theta$, it follows that $p_\theta(t) \equiv 0$. The adjoint equation for p_θ gives

$$\dot{p}_\theta = -\frac{2s}{m_0} \langle p_m, e(\theta) \rangle.$$

Hence, either $\langle p_m, e(\theta) \rangle \equiv 0$ or $s \equiv 0$. If $s \not\equiv 0$, then $p_m \equiv 0$, which again implies $p \equiv 0$, contradicting the nontriviality of the adjoint vector. If $s \equiv 0$, the system evolves under pure rotation. However, along a rotation arc, the switching function ϕ_2 cannot remain identically zero, since otherwise the maximum condition would be violated. This contradiction shows that ϕ_2 cannot vanish on any interval. Thus, no singular arcs exist for either control component, and every optimal control is of bang–bang type. $\square$

The Hamiltonian is affine with respect to the controls and the admissible control set

$$\{(s, a) : |s| + |a| \leq \rho\}$$

is a compact convex polygon with extreme points $(\pm\rho, 0)$ and $(0, \pm\rho)$. By the Pontryagin's Maximum Principle, optimal controls must attain their values at these extreme points. Therefore, optimal trajectories consist of a sequence of rotation and translation phases.

2.8. Optimal Trajectories and Time Computation. According to the maximum condition, the optimal control takes values in

$$(s, a) \in \{(\rho, 0), (-\rho, 0), (0, \rho), (0, -\rho)\}$$

almost everywhere. The selection is determined by the switching functions ϕ_1 and ϕ_2 . We consider the following cases separately.

Case A: Pure Rotation. If $|\phi_1(t)| < |\phi_2(t)|$, then the maximum condition $s(t) = 0$, $a(t) = \pm\rho$. The equations of motion (2.5) reduce to

$$\ddot{M}(t) = 0, \quad \ddot{\theta}(t) = \pm \frac{12\rho}{m_0 l}.$$

Under the initial conditions $M(0) = M_0$ and $\dot{M}(0) = 0$, it follows that $M(t) \equiv M_0$. Hence, the motion is a pure rotation about the midpoint M_0 . The angular acceleration is constant: $\alpha = \frac{12\rho}{m_0 l}$. The rotation angle θ^* is defined as the smallest angle in absolute value that satisfies

$$e^\perp(\theta^*) = \pm \frac{M_T - M_0}{\|M_T - M_0\|},$$

the direction of motion of the midpoint becomes aligned with the vector connecting the initial midpoint M_0 and the target midpoint M_T (see Fig. 4). By identifying components, we obtain

$$\sin \theta^* = \pm \frac{a_0}{\sqrt{a_0^2 + b_0^2}}, \quad \cos \theta^* = \mp \frac{b_0}{\sqrt{a_0^2 + b_0^2}}.$$

To achieve the rotation in minimal time with zero initial and final angular velocity, the optimal profile is symmetric bang–bang. Let $T_{\text{rot}} = 2t_1$ and

$$\ddot{\theta}(t) = \begin{cases} \alpha, & 0 \leq t \leq t_1, \\ -\alpha, & t_1 < t \leq 2t_1. \end{cases}$$

Integrating, we obtain $|\theta^*| = \alpha t_1^2$. Hence,

$$t_1 = \sqrt{\frac{m_0 l |\theta^*|}{12\rho}}, \quad T_{\text{rot}} = \sqrt{\frac{m_0 l |\theta^*|}{3\rho}}.$$

Case B: Pure Translation. If $|\phi_1(t)| > |\phi_2(t)|$, then the maximum condition yields $s(t) = \pm\rho$, $a(t) = 0$. The equations of motion (2.5) reduce to

$$\ddot{M}(t) = \pm \frac{2\rho}{m_0} e^\perp(\theta(t)), \quad \ddot{\theta}(t) = 0.$$

Hence, the angular velocity is constant. Since the translation phase starts after the rotation phase, we have $\dot{\theta}(T_{\text{rot}}) = 0$, $\theta(T_{\text{rot}}) = \theta^*$, and therefore $\theta(t) \equiv \theta^*$, $t \geq T_{\text{rot}}$. Thus, the motion reduces to a pure translation with fixed orientation θ^* along the direction $e^\perp(\theta^*)$. Let $d = |M_T - M_0| = \sqrt{a_0^2 + b_0^2}$. Introduce the scalar coordinate

$$x(t) = \langle M(t) - M_0, e^\perp(\theta^*) \rangle.$$

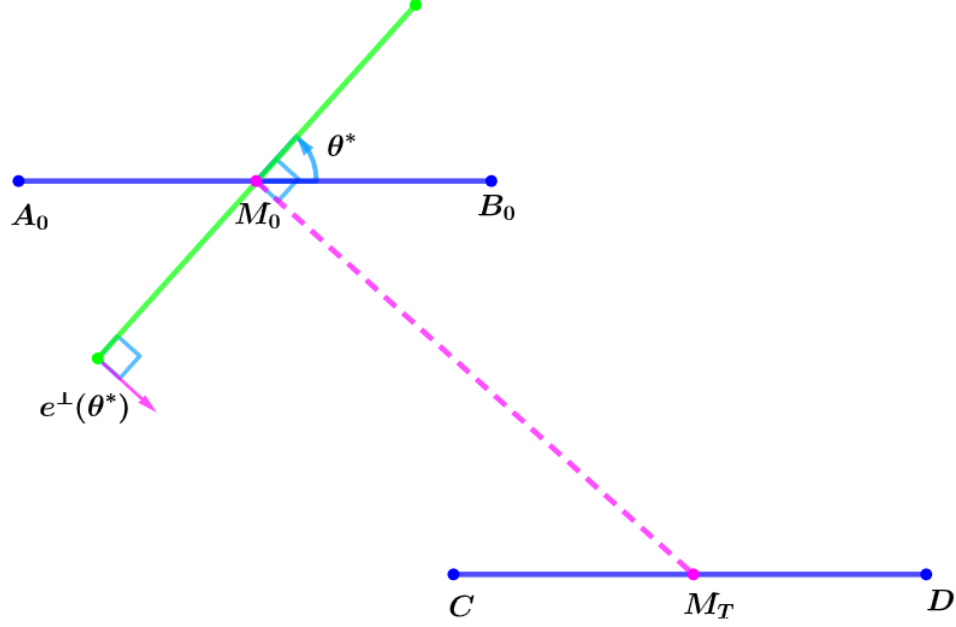


FIGURE 4. Bang-bang control structure for time-optimal rotation

Since $e^\perp(\theta^*)$ is a unit vector, we obtain

$$\ddot{x}(t) = \pm \frac{2\rho}{m_0},$$

with boundary conditions

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad x(T_1) = d, \quad \dot{x}(T_1) = 0.$$

This is exactly the time-optimal problem for the scalar double integrator already analyzed in Case A, with the substitutions

$$\theta \longrightarrow x, \quad \alpha \longrightarrow \frac{2\rho}{m_0}, \quad \theta^* \longrightarrow d.$$

By the same argument, the optimal control is bang-bang with one switching, and the minimal translation time is given by

$$T_{\text{trans}} = \sqrt{\frac{2m_0 d}{\rho}} = \sqrt{\frac{2m_0 \sqrt{a_0^2 + b_0^2}}{\rho}}.$$

The previous analysis implies that every time-optimal trajectory consists of exactly three phases: rotation $\rightarrow$ translation $\rightarrow$ rotation, namely (1) an initial rotation, (2) a pure translation, (3) a final rotation. Since the initial and final orientations coincide, the total rotation angle must vanish. Consequently, the trajectory necessarily contains two symmetric rotational phases of equal magnitude. From Case A, the minimal rotation time is

$$T_{\text{rot}} = \sqrt{\frac{m_0 l |\theta^*|}{3\rho}}.$$

Therefore, the minimal total transfer time is given by

$$T_{\text{min}} = 2T_{\text{rot}} + T_{\text{trans}} = 2\sqrt{\frac{m_0 l |\theta^*|}{3\rho}} + \sqrt{\frac{2m_0 \sqrt{a_0^2 + b_0^2}}{\rho}}.$$

This formula provides the minimal time required to steer the rigid segment from the initial configuration $A_0 B_0$ to the terminal configuration $C D$ under the bounded control constraint $|u_1|, |u_2| \leq \rho$. This completes the proof of Theorem 2.1.

Remark 2.1. The explicit formula for $T_{\min}$ holds under the assumption that the initial and terminal angular velocities vanish and that the initial and terminal orientations coincide. If nonzero boundary angular velocities are prescribed, additional switching patterns may arise, and the formula must be modified.

Example. For $m_0 = 1$, $l = 1$, $\rho = 1$, $a_0 = -5$, and $b_0 = 4$, the time-optimal motion of the rigid segment is illustrated in Fig. 5. The segment follows a rotation–translation–rotation strategy: it first undergoes a counterclockwise rotation with angle $\theta^* \approx 0.8961$ rad to align its direction with the target displacement; then moves along a straight translation path of length $d \approx 6.4031$; and finally performs a clockwise rotation with the identical angle, to reach the final configuration. The dashed curve represents the motion of the midpoint, while the solid lines show the movement of the endpoints. The minimal transfer time is $T_{\min} = 4.67$.

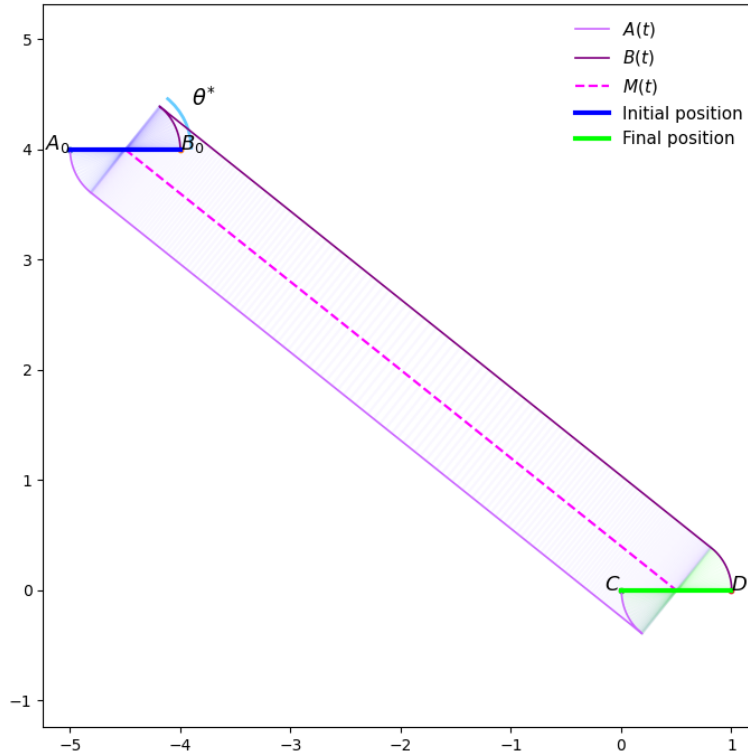


FIGURE 5. Time-optimal motion of the rigid segment from the initial configuration A_0B_0 to the terminal configuration CD

3. A PURSUIT DIFFERENTIAL GAME

Consider a planar pursuit differential game involving two rigid segments (the pursuer P and the evader E) with identical dynamics but different control resources. Let $M_P(t)$ and $M_E(t)$ denote the midpoints, and $\theta_P(t)$ and $\theta_E(t)$ denote the orientation angles of the pursuing and evading segments in $\mathbb{R}^2$, respectively.

The motions of the players are governed by the following system of differential equations:

$$\begin{aligned} \ddot{M}_P &= \frac{2}{m_0} s_P e^\perp(\theta_P), & \ddot{\theta}_P &= \frac{12}{m_0 l} a_P, \\ \ddot{M}_E &= \frac{2}{m_0} s_E e^\perp(\theta_E), & \ddot{\theta}_E &= \frac{12}{m_0 l} a_E, \end{aligned} \tag{3.1}$$

with the initial conditions

$$\begin{aligned} M_P(t_0) &= M_{P,1}, & \dot{M}_P(t_0) &= 0, & \theta_P(t_0) &= \theta_{P,0}, & \dot{\theta}_P(t_0) &= 0, \\ M_E(t_0) &= M_{E,1}, & \dot{M}_E(t_0) &= 0, & \theta_E(t_0) &= \theta_{E,0}, & \dot{\theta}_E(t_0) &= 0, \end{aligned} \quad (3.2)$$

where $(s_P(t), a_P(t))$ and $(s_E(t), a_E(t))$ are the translational and rotational control parameters of the pursuer P and the evader E , respectively.

Definition 3.1. The pairs of measurable functions $(s_P(t), a_P(t))$ and $(s_E(t), a_E(t))$ for $t \geq t_0$ are called the admissible translational and rotational controls of the pursuer P and the evader E , respectively, if they satisfy the following resource constraints:

$$|s_P(t)| + |a_P(t)| \leq \rho, \quad |s_E(t)| + |a_E(t)| \leq \sigma, \quad (3.3)$$

where ρ and σ are given numbers such that $\rho > \sigma > 0$ (i.e., the pursuer has a resource advantage).

Definition 3.2. We say that capture occurs in the game (3.1)-(3.3) if there exists a finite time $\tau > t_0$ such that the midpoints of the two rigid segments coincide, i.e., $M_P(\tau) = M_E(\tau)$.

The aim of the pursuer is to achieve capture, while the evader attempts to maintain the inequality $M_P(t) \neq M_E(t)$ for all $t > t_0$.

Problem 1. Construct a strategy for the pursuer that guarantees capture in the game (3.1)-(3.3) within a finite time.

A strategy for the pursuer. Motivated by the rotation-translation structure of time-optimal steering established in Theorem 2.1, we construct a stagewise pursuit strategy. Let $d_1 = |M_{P,1} - M_{E,1}|$.

First stage. The pursuer first performs a time-optimal rotation through the minimal angle $\theta_{P,1}$ such that $e^\perp(\theta_{P,0} + \theta_{P,1}) \parallel M_{E,1} - M_{P,1}$, and then executes a time-optimal translation toward the point $M_{E,1}$ (see Fig. 6). By Theorem 2.1, the duration of this maneuver is

$$\Delta t_1 = \sqrt{\frac{m_0 l |\theta_{P,1}|}{3\rho}} + \sqrt{\frac{2m_0 d_1}{\rho}}.$$

During the same interval Δt_1 , the evader applies an arbitrary admissible control and reaches a new position $M_{E,2}$. Denote $d_2 = |M_{E,2} - M_{E,1}|$. By Theorem 2.1, the minimal time required to transfer the evader from $M_{E,1}$ to $M_{E,2}$ under the control constraint σ is

$$\Delta \hat{t}_1 = \sqrt{\frac{m_0 l |\theta_{E,1}|}{3\sigma}} + \sqrt{\frac{2m_0 d_2}{\sigma}},$$

where $\theta_{E,1}$ denotes the time-optimal rotation angle of the evader. Since the evader performs this maneuver during the interval Δt_1 , we have $\Delta \hat{t}_1 \leq \Delta t_1$. At the end of the first stage, the pursuer reaches the previous position of the evader: $M_{P,2} = M_{E,1}$.

Second stage. The pursuer now repeats the same procedure. It performs a time-optimal rotation through the minimal angle $\theta_{P,2}$ such that

$$e^\perp(\theta_{P,0} + \theta_{P,1} + \theta_{P,2}) \parallel M_{E,2} - M_{P,2},$$

followed by a time-optimal translation toward the point $M_{E,2}$. The duration of the maneuver is

$$\Delta t_2 = \sqrt{\frac{m_0 l |\theta_{P,2}|}{3\rho}} + \sqrt{\frac{2m_0 d_2}{\rho}}.$$

During this interval, the evader reaches a new position $M_{E,3}$. Denoting $d_3 = |M_{E,3} - M_{E,2}|$, we obtain

$$\Delta \hat{t}_2 = \sqrt{\frac{m_0 l |\theta_{E,2}|}{3\sigma}} + \sqrt{\frac{2m_0 d_3}{\sigma}},$$

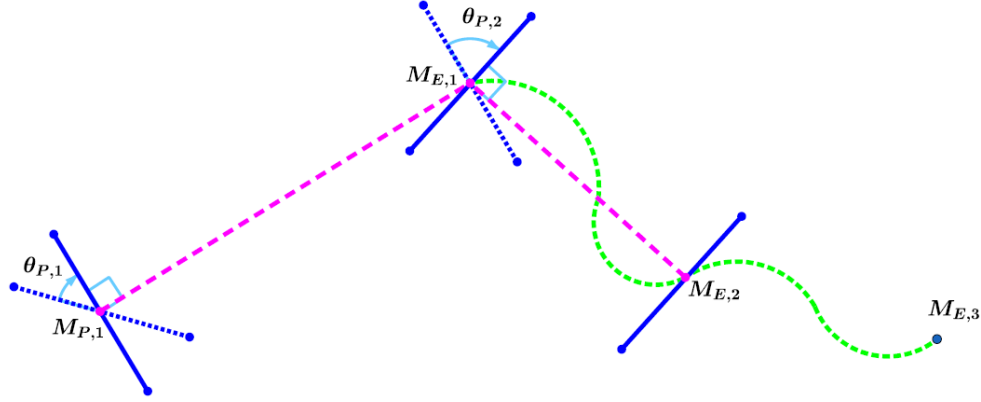


FIGURE 6. Pursuit strategy and trajectory convergence

with $\Delta \hat{t}_2 \leq \Delta t_2$. At the end of the second stage, $M_{P,3} = M_{E,2}$.

Proceeding inductively, let $M_{P,k}$ and $M_{E,k}$ denote the positions of the pursuer and the evader at the beginning of the k -th stage, and define $d_k = |M_{E,k} - M_{E,k-1}|$. At stage k , the pursuer rotates through the minimal angle $\theta_{P,k}$ such that

$$e^\perp \left(\sum_{j=0}^k \theta_{P,j} \right) \parallel M_{E,k} - M_{P,k},$$

and then performs a time-optimal translation toward $M_{E,k}$. The duration of the maneuver is

$$\Delta t_k = \sqrt{\frac{m_0 l |\theta_{P,k}|}{3\rho}} + \sqrt{\frac{2m_0 d_k}{\rho}}. \quad (3.4)$$

The next theorem shows that the above strategy is well-defined and leads to finite-time capture.

Theorem 3.1. *In game (3.1)–(3.3), the pursuer, under the constructed strategy, guarantees capture within the time*

$$T^* = \sqrt{\frac{m_0 l |\theta_{P,1}|}{3\rho}} + \sqrt{\frac{2m_0 d_1}{\rho}} + \frac{1}{\sqrt{\rho} - \sqrt{\sigma}} \left(\sqrt{\frac{m_0 l |\theta_{P,2}|}{3}} + \sqrt{2m_0 d_2} \right).$$

Proof: We prove the theorem by estimating the total capture time generated by the constructed stagewise strategy. At stage k , during the same interval Δt_k , the evader moves to a new position $M_{E,k+1}$. Denoting $d_{k+1} = |M_{E,k+1} - M_{E,k}|$, the minimal transfer time of the evader satisfies

$$\Delta \hat{t}_k = \sqrt{\frac{m_0 l |\theta_{E,k}|}{3\sigma}} + \sqrt{\frac{2m_0 d_{k+1}}{\sigma}}, \quad (3.5)$$

and consequently $\Delta \hat{t}_k \leq \Delta t_k$, where $\theta_{E,k}$ denotes the rotation angle appearing in the corresponding time-optimal steering maneuver. By construction, $M_{P,k+1} = M_{E,k}$, hence $d_k = |M_{E,k} - M_{P,k}|$, $k \geq 1$. Moreover, since at each stage the pursuer aligns its motion with the evader's previous displacement vector, and therefore

$$\theta_{P,k+1} = \theta_{E,k}, \quad k \geq 2. \quad (3.6)$$

Combining (3.4) and (3.5), and substituting (3.6), yields

$$\sqrt{\frac{m_0 l |\theta_{P,k}|}{3\rho}} + \sqrt{\frac{2m_0 d_k}{\rho}} \geq \sqrt{\frac{m_0 l |\theta_{P,k+1}|}{3\sigma}} + \sqrt{\frac{2m_0 d_{k+1}}{\sigma}}.$$

Introduce the quantity

$$S_k = \sqrt{\frac{m_0 l |\theta_{P,k}|}{3}} + \sqrt{2m_0 d_k}$$

and define $q = \sqrt{\frac{\sigma}{\rho}}$. Since $\rho > \sigma$, we have $0 < q < 1$, and the above inequality reduces to the recurrence relation $S_{k+1} \leq qS_k$, $k \geq 2$. Therefore, $S_k \leq q^{k-2}S_2$, $k \geq 3$. Thus, the sequence S_k decreases geometrically, which implies convergence of the total capture time. Using (3.4), we write

$$T_c = \sum_{k=1}^{\infty} \Delta t_k = \Delta t_1 + \Delta t_2 + \sum_{k=3}^{\infty} \Delta t_k.$$

From the definition of S_k , we obtain $\Delta t_k = \frac{S_k}{\sqrt{\rho}}$. Therefore,

$$T_c = \Delta t_1 + \Delta t_2 + \frac{1}{\sqrt{\rho}} \sum_{k=3}^{\infty} S_k \leq \Delta t_1 + \Delta t_2 \sum_{k=0}^{\infty} q^k.$$

Since

$$\sum_{k=0}^{\infty} q^k = \frac{\sqrt{\rho}}{\sqrt{\rho} - \sqrt{\sigma}},$$

we obtain

$$T_c \leq \Delta t_1 + \frac{\sqrt{\rho} \Delta t_2}{\sqrt{\rho} - \sqrt{\sigma}} = T^*.$$

Finally, since $S_k \rightarrow 0$ as $k \rightarrow \infty$, it follows that $d_k \rightarrow 0$. Consequently, the distance between the pursuer and the evader converges to zero, and the total capture time T_c is finite, which implies that capture occurs in a finite time T^* . This completes the proof of the theorem. $\square$

4. ACKNOWLEDGMENT

This research was carried out under Research Project No. FL-9524115148 of the Ministry of Higher Education, Science and Innovation of the Republic of Uzbekistan. The authors are sincerely grateful to Academician A. Azamov for formulating the problem and for valuable discussions of the results. The authors also thank Prof. G. Ibragimov for his valuable comments and suggestions, which have improved the presentation and clarity of the manuscript.

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Uzbek Mathematical Journal
2026, Volume 70, Issue 3, pp. 165–172
DOI: 10.29229/uzmj.2026-3-17

Stability and Convergence of a Nonlocal Explicit Finite-Difference Scheme for Numerical Solution of the Fractional Van der Pol–Airy Equation

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Abstract. This paper presents a study of a nonlocal explicit finite-difference scheme for the numerical solution of the Cauchy problem for the fractional Van der Pol–Airy oscillator, which describes forced oscillations with memory effects. The study focuses on the stability and convergence of the proposed numerical scheme. These properties are rigorously established using the properties of the scheme’s coefficients, the approximation of fractional derivative operators, and discrete Volterra inequalities. The conditional convergence and stability of the numerical scheme are proved. The scheme is shown to have first-order accuracy, provided that a condition on the time step of the computational grid is satisfied, which is also derived.

Keywords: fractional oscillator, Van der Pol–Airy equation, finite-difference scheme, stability, convergence, numerical modeling, fractional calculus, forced oscillations

MSC (2020): 34A08, 65L12, 65L20, 65L70, 34C15

1. INTRODUCTION

Currently, there is increased interest in the development of mathematical models of oscillatory systems with memory (fractional oscillators). This interest is driven by the presence of relevant applied problems and their applications in various fields of knowledge: from physical and engineering sciences [1] to economics [2, 3]. Indeed, oscillations arise in various systems, including those possessing heredity, and they can carry important information about the properties of the oscillatory process under consideration. Studying these properties will allow us to establish important relationships between various parameters of the system and describe a particular oscillatory regime. Fractional oscillators are studied within the framework of fractional dynamics theory [4], and the tool for studying fractional oscillators is typically the apparatus of fractional calculus [5, 6, 7].

This article continues the study of the nonlinear Van der Pol–Airy oscillatory system with heredity effects. The peculiarity of this oscillatory system lies in the presence in the model equation of nonlinear friction, characteristic of relaxation oscillations of the Van der Pol oscillator [8], a linear dependence of the natural oscillation frequency on time, characteristic of oscillations with a slowly decaying amplitude in the Airy oscillator [9], as well as the consideration of heredity effects, which are typical for viscoelastic and plastic media [10]. Heredity effects are described using fractional derivatives, understood in the sense of Gerasimov–Caputo [11, 12]. Note that in the authors’ works [13, 14], free oscillations of the fractional Van der Pol–Airy oscillator were studied, numerical methods were investigated, as well as qualitative properties of the numerical solution, such as the stability of limit cycles and the computational order of the scheme according to the Runge rule. A software package was developed in the Python programming language that allows visualizing the modeling results: constructing oscillograms and phase trajectories [15]. In the authors’ work [16], forced oscillations of the fractional Van der Pol–Airy oscillator were investigated, and a nonlocal explicit finite-difference scheme was proposed for an algorithm to calculate amplitude-frequency and phase-frequency characteristics. It was shown that the orders of fractional derivatives are related to the Q-factor of the oscillatory system.

Despite previous research, a rigorous study of the stability and convergence of the nonlocal explicit finite-difference scheme has not been conducted. Therefore, the present work is devoted to filling this gap. The paper investigates the stability and convergence of the nonlocal explicit finite-difference scheme. Through rigorous proofs of the corresponding theorems, it is established that the proposed scheme is conditionally stable and converges with first-order accuracy. A condition on the computational grid step is also obtained.

2. PROBLEM STATEMENT

Consider the Cauchy problem for the fractional Van der Pol–Airy equation:

$$\frac{1}{\theta^{2-\alpha}} \partial_{0t}^\alpha x(t) + \frac{\lambda (ax(t)^2 - b)}{\theta^{1-\beta}} \partial_{0t}^\beta x(t) + ctx(t) = \delta \cos(\varphi t), \quad x(0) = x_0, \quad \dot{x}(0) = y_0. \quad (2.1)$$

Here $x(t) \in C^2[0, T]$ is the solution function, $T > 0$ is the simulation time, $t \in [0, T]$ is the process time, θ is the time scale coefficient [17], $\lambda, a, b, c, \delta, \phi, x_0, y_0$ are given constants. The fractional differentiation operators in the model equation (2.1) are understood in the sense of Gerasimov–Caputo with fractional orders $1 < \alpha < 2$, $0 < \beta < 1$ and are defined as follows [11, 12]:

$$\begin{aligned} \partial_{0t}^\alpha x(t) &= \frac{1}{\Gamma(2-\alpha)} \int_0^t \ddot{x}(\tau) d\tau (t-\tau)^{\alpha-1}, \\ \partial_{0t}^\beta x(t) &= \frac{1}{\Gamma(1-\beta)} \int_0^t \dot{x}(\tau) d\tau (t-\tau)^\beta. \end{aligned} \quad (2.2)$$

Remark 2.1. The Van der Pol–Airy oscillator, which is described using fractional derivatives (2.2), will be called the fractional Van der Pol–Airy oscillator. The fractional derivatives here describe the heredity effect, where the current state of the oscillatory system depends on its previous states.

Remark 2.2. The Cauchy problem (2.1) describes forced oscillations with nonlinear friction and taking into account the increase in the natural frequency over time. Since nonlinear friction is characteristic of the Van der Pol oscillator, and the natural frequency increases linearly with time, which is characteristic of the Airy oscillator, we will call the Cauchy problem (2.1) the fractional Van der Pol–Airy oscillator.

Remark 2.3. Known special cases of the fractional Van der Pol–Airy oscillator (2.1) include:

- (1) For $c = \theta = 1$, $\delta = 0$, we obtain free oscillations of the Van der Pol–Airy oscillator, studied in [13, 14].
- (2) For $\alpha = 2$, $\beta = 1$, we arrive at the classical Van der Pol–Airy oscillator.
- (3) For $\lambda = 0$, we obtain the fractional Airy oscillator, studied in [18].

Remark 2.4. Note also that in the case where the natural oscillations in the model equation (2.1) are constant, we are dealing with the fractional Van der Pol oscillator, which has been fairly well studied [19].

For simplicity, we will henceforth use the notation $f(x) = \lambda(ax^2 - b)$. The function f satisfies the following Lipschitz condition.

Lemma 2.1. For any $M > 0$ the function $f(x) = \lambda(ax^2 - b)$ is Lipschitz continuous on $[-M, M]$ with constant $L_f = 2|\lambda a|M$. That is, for all $x, y \in [-M, M]$

$$|f(x) - f(y)| \leq L_f |x - y|. \quad (2.3)$$

3. APPROXIMATION OF FRACTIONAL DERIVATIVES

Because the model equation (2.1) is nonlinear, we solve it numerically. We assume that the exact solution $x(t)$ belongs to $C^3[0, T]$. Introduce a uniform grid $t_k = k\tau$, $k = 0, 1, \dots, N$, with step $\tau = T/N$, and let $x_k = x(t_k)$, $f_k = f(x_k)$.

We approximate the Gerasimov–Caputo derivatives using the well-known L1 discretization. For the second-order derivative inside the α -derivative we use a central difference, while for the first-order derivative inside the β -derivative we use a forward difference. This leads to

$$\begin{aligned} \partial_{0t}^\alpha x(t_k) &\approx A \sum_{j=0}^{k-1} \omega_j^\alpha \Delta^2 x_{k-j}, \\ \partial_{0t}^\beta x(t_k) &\approx B \sum_{j=1}^{k-1} \omega_j^\beta \Delta x_{k-j}, \end{aligned} \quad (3.1)$$

where $\Delta^2 x_m = x_{m+1} - 2x_m + x_{m-1}$, $\Delta x_m = x_{m+1} - x_m$,

$$A = \frac{\tau^{-\alpha}}{\theta^{2-\alpha}\Gamma(3-\alpha)}, \quad B = \frac{\tau^{-\beta}}{\theta^{1-\beta}\Gamma(2-\beta)},$$

and the weight coefficients are

$$\omega_j^\alpha = (j+1)^{2-\alpha} - j^{2-\alpha}, \quad \omega_j^\beta = (j+1)^{1-\beta} - j^{1-\beta}.$$

The coefficients possess the following elementary but important properties.

Lemma 3.1. *For all $1 < \alpha < 2$ and $0 < \beta < 1$ the following holds:*

- (1) $A > 0$, $B > 0$.
- (2) $\omega_j^\alpha > 0$, $\omega_j^\beta > 0$ for every $j \geq 0$ and the sequences are strictly decreasing: $\omega_{j+1}^\alpha < \omega_j^\alpha$, $\omega_{j+1}^\beta < \omega_j^\beta$.
- (3) $\sum_{j=0}^{\infty} |\Delta^2 \omega_j^\alpha| = \omega_1^\alpha$ and $\sum_{j=0}^{\infty} |\Delta \omega_j^\beta| = \omega_1^\beta$, where $\Delta \omega_j^\beta = \omega_{j+1}^\beta - \omega_j^\beta$ and $\Delta^2 \omega_j^\alpha = \omega_{j+1}^\alpha - 2\omega_j^\alpha + \omega_{j-1}^\alpha$ (with $\omega_{-1}^\alpha = 0$).

Proof: Property 1 follows directly from the positivity of the gamma function. For property 2, consider the function $f(x) = x^p$ with $p = 2 - \alpha \in (0, 1)$ or $p = 1 - \beta \in (0, 1)$. Then $\omega_j = f(j+1) - f(j)$. Since $f'(x) > 0$, the differences are positive. Concavity ($f''(x) < 0$) implies the differences decrease. Property 3 is verified by telescoping:

$$\sum_{j=0}^m \Delta^2 \omega_j^\alpha = \omega_{m+1}^\alpha - \omega_m^\alpha - \omega_1^\alpha \rightarrow -\omega_1^\alpha \quad (m \rightarrow \infty),$$

and because all $\Delta^2 \omega_j^\alpha$ are negative, their absolute values sum to ω_1^α . The proof for β is analogous. $\square$

Lemma 3.2. *Let $x \in C^3[0, T]$. Then the approximation errors satisfy*

$$\left| \partial_{0t}^\alpha x(t_k) - A \sum_{j=0}^{k-1} \omega_j^\alpha \Delta^2 x_{k-j} \right| \leq C_\alpha \tau, \quad \left| \partial_{0t}^\beta x(t_k) - B \sum_{j=1}^{k-1} \omega_j^\beta \Delta x_{k-j} \right| \leq C_\beta \tau,$$

with constants $C_\alpha = \frac{M_3 T^{2-\alpha}}{\theta^{2-\alpha}\Gamma(3-\alpha)}$, $C_\beta = \frac{M_2 T^{1-\beta}}{\theta^{1-\beta}\Gamma(2-\beta)}$, where $M_2 = \max_{[0, T]} |x''|$, $M_3 = \max_{[0, T]} |x'''|$.

Proof: We give the proof for the α -derivative; the β case is similar. Start from the definition

$$\partial_{0t}^\alpha x(t_k) = \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - s)^{1-\alpha} x''(s) ds.$$

On each subinterval $[t_j, t_{j+1}]$ we replace $x''(s)$ by its linear interpolant at t_j, t_{j+1} plus remainder:

$$x''(s) = \frac{t_{j+1} - s}{\tau} x''(t_j) + \frac{s - t_j}{\tau} x''(t_{j+1}) + R_j(s), \quad |R_j(s)| \leq \frac{M_3}{2} (s - t_j)(t_{j+1} - s).$$

Substituting and integrating, the principal part gives exactly the discrete formula with weights ω_j^α after summation. The remainder contributes at most

$$\frac{M_3}{2\Gamma(2-\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k - s)^{1-\alpha} (s - t_j)(t_{j+1} - s) ds \leq \frac{M_3 T^{2-\alpha}}{\Gamma(3-\alpha)} \tau.$$

Dividing by $\theta^{2-\alpha}$ yields the stated constant. The estimate for the β -derivative follows the same lines using $x'(s)$ and forward differences. $\square$

4. NONLOCAL EXPLICIT FINITE-DIFFERENCE SCHEME

Substituting the approximations (3.1) into equation (2.1) and solving for x_{k+1} we obtain the following explicit scheme

$$x_{k+1} = \frac{2A + P_k B - c\tau k}{D_k} x_k - \frac{A}{D_k} x_{k-1} - \frac{A}{D_k} \sum_{j=1}^{k-1} \omega_j^\alpha \Delta^2 x_{k-j} - \frac{P_k B}{D_k} \sum_{j=1}^{k-1} \omega_j^\beta \Delta x_{k-j} + \frac{\delta \cos(k\tau\varphi)}{D_k}, \quad (4.1)$$

where $P_k = f(x_k)$ and $D_k = A + P_k B$.

Our aim is to write the scheme in the form of a linear combination of previous values with coefficients that can be estimated. Using the identities

$$\sum_{j=1}^{k-1} \omega_j^\alpha \Delta^2 x_{k-j} = \sum_{m=0}^k c_{km}^\alpha x_m, \quad \sum_{j=1}^{k-1} \omega_j^\beta \Delta x_{k-j} = \sum_{m=0}^k c_{km}^\beta x_m,$$

with the convention $\omega_0^\alpha = \omega_0^\beta = 0$, we can rewrite (4.1) as

$$x_{k+1} = \sum_{m=0}^k a_{km} x_m + g_k, \quad k = 0, 1, \dots, N-1, \quad (4.2)$$

where $g_k = \delta \cos(k\tau\varphi)/D_k$ and the coefficients a_{km} are given explicitly by

$$\begin{aligned} a_{k,k} &= \frac{2A + P_k B - c\tau k}{D_k} - \frac{A\omega_1^\alpha + P_k B\omega_1^\beta}{D_k}, \\ a_{k,k-1} &= -\frac{A}{D_k} + \frac{A(2\omega_1^\alpha - \omega_2^\alpha) + P_k B(\omega_1^\beta - \omega_2^\beta)}{D_k}, \\ a_{k,m} &= -\frac{A(\omega_{k-m+1}^\alpha - 2\omega_{k-m}^\alpha + \omega_{k-m-1}^\alpha) + P_k B(\omega_{k-m+1}^\beta - \omega_{k-m}^\beta)}{D_k}, \quad 0 \leq m \leq k-2, \end{aligned}$$

with the understanding that $\omega_{-1}^\alpha = 0$ and $\omega_0^\beta = 0$ wherever needed.

For the subsequent analysis the following estimate of the total variation of the coefficients is crucial.

Lemma 4.1. *Assume that there exist constants $M > 0$ and $R_0 \in (0, 1)$ such that $|x_k| \leq M$ and*

$$\tau \leq \tau_0 = \left[\frac{R_0}{\lambda(|a|M^2 + |b|)C_0} \right]^{1/(\alpha-\beta)}, \quad C_0 = \theta^{1+\beta-\alpha} \frac{\Gamma(3-\alpha)}{\Gamma(2-\beta)}.$$

Then for all k

$$\sum_{m=0}^k |a_{km}| \leq \Gamma_0,$$

where Γ_0 is a constant independent of τ and k , given by

$$\Gamma_0 = \frac{1}{1-R_0} \left(2 + 2R_0 + \omega_1^\alpha + R_0\omega_1^\beta + \max_k \frac{c\tau k}{A} \right).$$

In particular, Γ_0 is uniformly bounded as $\tau \rightarrow 0$.

Proof: First, from the definition of A, B and the bound on P_k we have

$$|P_k| \leq \lambda(|a|M^2 + |b|), \quad \frac{|P_k|B}{A} \leq \lambda(|a|M^2 + |b|)\tau^{\alpha-\beta}C_0 \leq R_0.$$

Hence $D_k = A + P_k B \geq A - |P_k|B \geq A(1 - R_0) > 0$, and consequently

$$\frac{A}{D_k} \leq \frac{1}{1-R_0}, \quad \frac{|P_k|B}{D_k} \leq \frac{R_0}{1-R_0}.$$

Now estimate each a_{km} .

- For $m = k$:

$$|a_{k,k}| \leq \frac{2A + |P_k|B + c\tau k}{D_k} + \frac{A\omega_1^\alpha + |P_k|B\omega_1^\beta}{D_k} \leq \frac{2 + R_0 + c\tau k/A}{1 - R_0} + \frac{\omega_1^\alpha + R_0\omega_1^\beta}{1 - R_0}.$$

- For $m = k - 1$:

$$|a_{k,k-1}| \leq \frac{A}{D_k} + \frac{A|2\omega_1^\alpha - \omega_2^\alpha| + |P_k|B|\omega_1^\beta - \omega_2^\beta|}{D_k} \leq \frac{1 + (2\omega_1^\alpha + \omega_2^\alpha) + R_0(\omega_1^\beta + \omega_2^\beta)}{1 - R_0}.$$

- For $0 \leq m \leq k - 2$ we use the second differences. Because $\Delta^2\omega_j^\alpha < 0$ and $\Delta\omega_j^\beta < 0$, the absolute values are

$$|a_{k,m}| \leq \frac{A|\Delta^2\omega_{k-m-1}^\alpha| + |P_k|B|\Delta\omega_{k-m-1}^\beta|}{D_k} \leq \frac{|\Delta^2\omega_{k-m-1}^\alpha| + R_0|\Delta\omega_{k-m-1}^\beta|}{1 - R_0}.$$

Summing over m and using Lemma 3.1(3) we obtain

$$\sum_{m=0}^{k-2} |a_{k,m}| \leq \frac{1}{1 - R_0} \left(\sum_{j=1}^{\infty} |\Delta^2\omega_j^\alpha| + R_0 \sum_{j=1}^{\infty} |\Delta\omega_j^\beta| \right) = \frac{\omega_1^\alpha + R_0\omega_1^\beta}{1 - R_0}.$$

Adding the contributions from $m = k - 1, k$ yields the bound with the constant Γ_0 as stated. Note that $c\tau k/A \leq cT/A_0$ with $A_0 = \tau^\alpha A = 1/[\theta^{2-\alpha}\Gamma(3-\alpha)]$; this ratio tends to zero as $\tau \rightarrow 0$, so the term $\max_k c\tau k/A$ is bounded independently of τ . $\square$

5. STABILITY AND CONVERGENCE ANALYSIS

We now prove that the scheme is stable and convergent of order one. The proof is based on a discrete Gronwall inequality for Volterra sums, which we quote from [20].

Lemma 5.1 (Discrete Volterra Gronwall inequality). *Let $\{e_k\}_{k \geq 0}$ be a sequence of non-negative numbers, and let $\{g_n\}_{n \geq 0}$ be a non-negative sequence such that $G = \sum_{n=0}^{\infty} g_n < \infty$. If*

$$e_{k+1} \leq \sum_{j=0}^k g_{k-j} e_j + R, \quad k = 0, 1, \dots,$$

for some constant $R \geq 0$, then

$$e_k \leq (e_0 + R) \exp(G), \quad k \geq 0.$$

Theorem 5.1 (Stability). *Assume that the numerical solution remains bounded, i.e. $|x_k| \leq M$ for all k , and that the step τ satisfies the condition of Lemma 4.1. Then there exists a constant $C_{\text{stab}} > 0$ independent of τ such that*

$$\max_{0 \leq k \leq N} |x_k| \leq C_{\text{stab}} \left(|x_0| + \tau \sum_{j=0}^{N-1} |\delta \cos(j\tau\varphi)| \right).$$

In particular, the scheme is stable in the sense of continuous dependence on the data.

Proof: From (4.2) and Lemma 4.1 we have

$$|x_{k+1}| \leq \sum_{m=0}^k |a_{km}| |x_m| + |g_k| \leq \Gamma_0 \max_{0 \leq m \leq k} |x_m| + \frac{|\delta|}{A(1 - R_0)}.$$

Let $M_k = \max_{0 \leq i \leq k} |x_i|$. Then

$$M_{k+1} \leq \max(M_k, \Gamma_0 M_k + C_g),$$

where $C_g = |\delta|/[A(1 - R_0)]$. If $\Gamma_0 \leq 1$ we immediately obtain $M_{k+1} \leq M_k + C_g$, which yields linear growth. Even if $\Gamma_0 > 1$, the inequality implies

$$M_{k+1} \leq \Gamma_0 M_k + C_g.$$

Iterating this gives $M_k \leq \Gamma_0^k |x_0| + C_g(\Gamma_0^k - 1)/(\Gamma_0 - 1)$. Since $N = T/\tau$ and Γ_0 is independent of τ , the exponential factor $\Gamma_0^N = \exp(N \ln \Gamma_0) = \exp(\frac{T}{\tau} \ln \Gamma_0)$ can blow up as $\tau \rightarrow 0$. To avoid this, we must refine the estimate by exploiting the structure of the coefficients more carefully.

Instead, we write the inequality in the Volterra form. From (4.2) we bound

$$|x_{k+1}| \leq \sum_{j=0}^k \gamma_{k-j} |x_j| + \frac{|\delta|}{A(1 - R_0)},$$

where $\gamma_n = \max_k |a_{k,k-n}|$ for $n \geq 0$ (with the understanding $a_{k,m} = 0$ if $m < 0$). Using the explicit expressions and the bounds in the proof of Lemma 4.1 we obtain

$$\gamma_0 \leq C_0, \quad \gamma_1 \leq C_1, \quad \gamma_n \leq \frac{|\Delta^2 \omega_{n-1}^\alpha| + R_0 |\Delta \omega_{n-1}^\beta|}{1 - R_0} \quad (n \geq 2),$$

where C_0, C_1 are some constants. The important point is that $\sum_{n=0}^\infty \gamma_n < \infty$ (by Lemma 3.1(3) and the fact that the first two terms are finite). Denote $G = \sum_{n=0}^\infty \gamma_n$. Now we apply Lemma 5.1 with $g_n = \gamma_n$, $R = |\delta|/[A(1 - R_0)]$ and obtain

$$\max_{0 \leq k \leq N} |x_k| \leq (|x_0| + R)e^G.$$

Since $A^{-1} = \tau^\alpha \theta^{2-\alpha} \Gamma(3 - \alpha) \leq \tau^\alpha C_A$, the term R is $O(\tau^\alpha)$, but even without this simplification the right-hand side is bounded by a constant independent of τ (because R itself depends on τ through A , but we can absorb it by noting that $R \leq C\tau^\alpha$ and hence is uniformly bounded). Thus the scheme is stable. $\square$

Theorem 5.2 (Convergence). *Let $u(t) \in C^3[0, T]$ be the exact solution of (2.1) and let $\{x_k\}$ be the numerical solution obtained from (4.1) with τ satisfying the stability condition of Lemma 4.1 (with the same R_0 and the Lipschitz constant L_f in place of $\lambda(|a|M^2 + |b|)$). Then there exists a constant $C > 0$ independent of τ such that*

$$\max_{0 \leq k \leq N} |u(t_k) - x_k| \leq C\tau.$$

Proof: Set $e_k = u(t_k) - x_k$. Substituting the exact solution into the scheme (4.1) and subtracting the numerical scheme, we use Lemma 3.2 to obtain

$$e_{k+1} = \sum_{m=0}^k a_{km} e_m + \frac{1}{D_k} [f(u_k) - f(x_k)] B \left(\partial_{0t}^\beta u(t_k) - \sum_{j=1}^{k-1} \omega_j^\beta \Delta u_{k-j} \right) + \rho_k,$$

where $|\rho_k| \leq C_\rho \tau$ with C_ρ combining the approximation errors and the forcing term. The term involving $f(u_k) - f(x_k)$ can be bounded using the Lipschitz condition and the fact that the expression in parentheses is bounded by $C_\beta \tau$ (or, more directly, by the boundedness of the discrete derivative). Consequently,

$$\left| \frac{1}{D_k} [f(u_k) - f(x_k)] B \left(\partial_{0t}^\beta u(t_k) - \sum_{j=1}^{k-1} \omega_j^\beta \Delta u_{k-j} \right) \right| \leq L_f |e_k| \frac{BC_\beta \tau}{D_k}.$$

Since $B/D_k \leq B/[A(1 - R_0)] = \tau^{\alpha-\beta} \theta^{\beta-\alpha} \Gamma(3 - \alpha)/[\Gamma(2 - \beta)(1 - R_0)]$, this term is of the form $L_f \tau^{\alpha-\beta} C_1 |e_k|$.

Thus we arrive at an inequality

$$|e_{k+1}| \leq \sum_{m=0}^k \tilde{a}_{km} |e_m| + C_\rho \tau,$$

where $\tilde{a}_{km} = |a_{km}|$ for $m \neq k$, and $\tilde{a}_{kk} = |a_{kk}| + L_f C_1 \tau^{\alpha-\beta}$. Because the additional term is of higher order in τ , the sum $G' = \sum_{n=0}^{\infty} \tilde{\gamma}_n$ with $\tilde{\gamma}_n = \max_k \tilde{a}_{k,k-n}$ still remains finite and uniformly bounded for sufficiently small τ . Applying Lemma 5.1 with $g_n = \tilde{\gamma}_n$ and $R = C_\rho \tau$ yields

$$\max_{0 \leq k \leq N} |e_k| \leq (|e_0| + C_\rho \tau) e^{G'} \leq C\tau,$$

because $e_0 = 0$. This proves first-order convergence. $\square$

6. CONCLUSION

This paper has presented a rigorous study of the nonlocal explicit finite-difference scheme for the fractional Van der Pol–Airy oscillator. The stability and first-order convergence of the scheme have been proved using discrete Volterra inequalities, under a mild restriction on the time step. The proofs are completely constructive and provide explicit constants. Further work will involve numerical experiments to confirm the theoretical findings and a deeper qualitative analysis of the oscillator. **Funding.** The work was carried out within the framework of the framework agreement on cooperation between the Institute of Cosmophysical Research and Radio Wave Propagation, Far Eastern Branch of the Russian Academy of Sciences and the National University of Uzbekistan named after Mirzo Ulugbek (Tashkent, Uzbekistan) No. 1118 dated 04/28/2022 (01.0468/22 NTMI) (2022–2027) in physical and mathematical sciences.

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Hypo-EP elements in rings

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Abstract. In this paper, we introduce and investigate a natural extension of the notion of hypo-EP elements previously defined in the framework of C^* -algebras to the broader context of rings with involution. We provide several equivalent characterizations, factorization properties, and study the behavior of these elements under addition and multiplication. Examples and counterexamples are also given to illustrate the scope of the theory.

Keywords: EP element, hypo-EP element, Moore-Penrose inverse, group inverse, ring with involution
MSC (2020): 16B99, 16W10, 46L05.

1. INTRODUCTION

Let R be an associative ring with unity 1. An *involution* $a \mapsto a^*$ in R is an anti-isomorphism of degree 2, that is, $(a^*)^* = a$, $(a + b)^* = a^* + b^*$, $(ab)^* = b^*a^*$, for all $a, b \in R$.

The *Moore-Penrose inverse* $a^\dagger$ of $a \in R$ is defined as the unique solution of the following four equations:

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

There is at most one $a^\dagger$ such that the above conditions hold (see [1]). The set of all Moore-Penrose invertible elements of R will be denoted by $R^\dagger$.

Let $a \in R$. It can be shown that the set of elements $x \in R$ satisfying the conditions

$$axa = a, \quad xax = x, \quad \text{and} \quad ax = xa$$

is either empty or contains exactly one element. If the set contains a single element, this element is called the *group inverse* of a and is denoted by $a^\#$. The set of all group invertible elements will be denoted by $R^\#$.

An element $b \in R$ is **-cancellable* if for all $x \in R$,

$$b^*bx = 0 \implies bx = 0 \quad \text{and} \quad xbb^* = 0 \implies xb = 0. \quad (1.1)$$

Applying the involution to (1.1), we observe that b is *-cancellable if and only if b^* is *-cancellable. In C^* -algebras all elements are *-cancellable. It is demonstrated in [2] that $a \in R^\dagger$ if and only if a is *-cancellable and a^*a is group invertible.

We will also use the following notations: $aR = \{ax : x \in R\}$, $Ra = \{xa : x \in R\}$, $^\circ a = \{x \in R : xa = 0\}$ and $a^\circ = \{x \in R : ax = 0\}$. The usual notation for the commutator of a and b is used: $[a, b] = ab - ba$.

An element $a \in R$ is said to be an EP element if $a \in R^\dagger \cap R^\#$ and $a^\dagger = a^\#$ [2]. The set of all EP elements will be denoted by (EP). Mosić et al. [3, Theorem 2.1] gave several equivalent conditions for an element in R to be an EP element. Patrício and Puystjens in [4, Proposition 2] proved that for a Moore-Penrose invertible element $a \in R$, $a \in (\text{EP})$ if and only if $aR = a^*R$. As for a Moore-Penrose invertible element $a \in R$, $a \in (\text{EP})$ if and only if $[a, a^\dagger] = 0$, thus we deduce that $[a, a^\dagger] = 0$ if and only if $aR = a^*R$. In [5, Theorem 3.1], Rakić et al. investigated some other equivalent conditions. We observe that a is EP if and only if a^* is EP.

The concept of hypo-EP operators was originally introduced in the setting of bounded linear operators on Hilbert spaces (see [6, 7, 8, 9]), and was later extended to elements in C^* -algebras by S. Veeramani in [10] as follows: An element a in a C^* -algebra A is said to be a hypo-EP element if a is regular (i.e., there exists $x \in A$ such that $axa = a$) and $aA \subseteq a^*A$. It is therefore natural to explore a further generalization of this notion within the broader and purely algebraic framework of rings with involution. Such a generalization not only enables the transfer of several known results

into ring theory, but also opens up new directions for the study of generalized inverses in algebraic structures. Motivated by this perspective, the present work introduces and investigates the class of hypo-EP elements in rings with involution, provides several equivalent characterizations, and examines their behavior under algebraic operations such as sums and products. This study represents a first step toward a more complete theory of hypo-EP elements in rings.

2. HYPO-EP ELEMENTS

The class of hypo-EP operators in Hilbert spaces and hypo-EP elements in C^* -algebras in the presence of an involution is naturally extended to rings with involution. Furthermore, we can define hypo-EP elements in a ring with involution using the definition of hypo-EP in C^* -algebras as follows:

Definition 2.1. An element a of a ring R with involution $*$ is said to be **hypo-EP** if $a \in R^\dagger$ and $aR \subseteq a^*R$.

It is clear that if a and a^* are hypo-EP, then a is EP, and the same holds if a and $a^\dagger$ are hypo-EP. Moreover, it is easily seen that every left invertible element is hypo-EP. In the setting of complex matrices, If A is hypo-EP, then A is EP, because $R(A) \subseteq R(A^*)$ and $\text{rank}(A) = \text{rank}(A^*)$ imply $R(A) = R(A^*)$. Thus, in the matrix case, hypo-EP coincides with EP. The following example shows that in a general ring with involution, a hypo-EP element need not be EP.

Example 2.1. Let R denote the ring of all bi-finite real matrices (i.e., matrices that are both row-finite and column-finite) with the transpose acting as involution. Let $e_{i,j}$ be the matrix in R that has 1 in the (i,j) -position and 0 elsewhere. Define $a = \sum_{i=1}^{\infty} e_{i+1,i}$ and $b = a^*$. Then, we have:

$$ab = \sum_{i=2}^{\infty} e_{i,i}, \quad ba = 1.$$

Thus, $a^\dagger = b$, and it is straightforward to verify that b^2 is not left-invertible and that a is not EP (since $ab \neq ba$).

This example also demonstrates that the equality $aR = a^*R$ in the equivalence $(a \in (\text{EP})) \Leftrightarrow (a \in R^\dagger, aR = a^*R)$ cannot be substituted by the inclusions $aR \subseteq a^*R$ or $a^*R \subseteq aR$.

Since every self-adjoint element in $R^\dagger$ is hypo-EP, we then have the following result:

Proposition 2.1. Let $a \in R$ be a self-adjoint element and let $b \in R$ such that $b^*ab \in R^\dagger$; then b^*ab is hypo-EP.

Before characterizing hypo-EP elements, we recall some fundamental properties of the Moore-Penrose inverse in rings, which will be used throughout the paper.

Lemma 2.1. Let $a \in R^\dagger$. We have

- (1) $a^\dagger R = a^\dagger a R = a^* R = a^* a R$;
- (2) $aa^\dagger R = a^* a^\dagger a^* R = a^* a^\dagger R = a R$;
- (3) $(aa^*)^\circ = (a^*)^\circ = (a^\dagger)^\circ$;
- (4) $(1 - a^\dagger a)R = a^\circ$ and $(1 - aa^\dagger)R = (a^*)^\circ$.

Proof: (1) It is clear because $a^* = a^\dagger aa^* = a^* aa^\dagger$ and $a^\dagger = a^\dagger aa^\dagger$.

(2) We have $aa^\dagger R = (aa^\dagger)^* R = a^* a^\dagger a^* R$. From (1), we obtain $aa^\dagger R = a^* a^\dagger a^* R = a^* a^\dagger R = a R$.

(3) Since $a^* aa^\dagger = a^*$, it follows that if $a^\dagger x = 0$, then $a^* x = 0$. Hence, we obtain $(a^\dagger)^\circ \subseteq (a^*)^\circ$. Now, let $x \in (a^*)^\circ$, so that $a^* x = 0$. Multiplying on the left by $a^\dagger a^*$, we get

$$a^\dagger a^* a^\dagger a^* x = a^\dagger (aa^\dagger)^* x = a^\dagger x = 0.$$

Thus, $(a^*)^\circ = (a^\dagger)^\circ$.

Next, we show that $(aa^*)^\circ = (a^*)^\circ$. It is clear that $(a^*)^\circ \subseteq (aa^*)^\circ$. Now, let $x \in (aa^*)^\circ$, so that $a^* x = a^\dagger aa^* x = 0$. Thus, $(aa^*)^\circ \subseteq (a^*)^\circ$.

Therefore, we conclude that $(aa^*)^\circ = (a^*)^\circ = (a^\dagger)^\circ$.

(4) As $a((1 - a^\dagger a)y) = 0$, we have $(1 - a^\dagger a)y \in a^\circ$. Conversely, if $ax = 0$, then $(1 - a^\dagger a)x = x$ which implies $x \in (1 - a^\dagger a)R$. In the same manner, we show that $(1 - aa^\dagger)R = (a^*)^\circ$. $\square$

The following theorem provides several equivalent conditions for an element to be hypo-EP. These characterizations will serve as the foundation for the rest of our study.

Theorem 2.1. *Let $a \in R^\dagger$. The following conditions are equivalent:*

- (i) a is hypo-EP;
- (ii) $aR \subseteq a^\dagger R$;
- (iii) $[a, a^\dagger a] = 0$;
- (iv) $[a^*, a^\dagger a] = 0$;
- (v) $a^\circ \subseteq (a^\dagger)^\circ = (a^*)^\circ$;
- (vi) $[a^\dagger, a^\dagger a] = 0$;
- (vii) $aa^\dagger = a(a^\dagger)^2 a$;
- (viii) $(a^\dagger a)^\circ \subseteq (aa^\dagger)^\circ$;
- (ix) $[a^\dagger a, (a + a^\dagger)] = 0$;
- (x) $aa^\dagger = a^\dagger a^2 a^\dagger$.

Proof: (i) $\iff$ (ii) It is clear by $a^* R = a^\dagger R$.

(ii) $\implies$ (iii) Suppose $aR \subseteq a^\dagger R$; then $a = a^\dagger b$ for some $b \in R$. We get

$$a - a^\dagger a^2 = a^\dagger b - a^\dagger aa^\dagger b = 0.$$

Hence $a = a^\dagger a^2$.

(iii) $\implies$ (iv) Suppose $a = a^\dagger a^2$. By the definition of the Moore-Penrose invertibility of a , we have

$$a^* = ((a^\dagger a)a)^* = a^*(a^\dagger a)^* = a^* a^\dagger a.$$

(iv) $\implies$ (v) Assume $a^* = a^* a^\dagger a$. If $y \in a^\circ$, then $a^* y = a^* a^\dagger a y = 0$. Hence $y \in (a^*)^\circ = (a^\dagger)^\circ$. Thus $a^\circ \subseteq (a^\dagger)^\circ$.

(v) $\implies$ (vi) Suppose that $a^\circ \subseteq (a^*)^\circ$. By Lemma 2.1 (4), we have $(1 - a^\dagger a)R \subseteq (1 - aa^\dagger)R$. Thus $(1 - a^\dagger a) = (1 - aa^\dagger)b$ for some $b \in R$. Multiplying on the left by $a^\dagger$, we get

$$a^\dagger = (a^\dagger)^2 a.$$

(vi) $\implies$ (ii) Assume that $a^\dagger = (a^\dagger)^2 a$. We have $(a^\dagger)^* = a^\dagger a (a^\dagger)^*$ and by Lemma 2.1(2), $(a^\dagger)^* R = aR$, so we get $aR \subseteq a^\dagger R$.

(vi) $\iff$ (vii) Obvious.

(vii) $\implies$ (viii) Suppose $aa^\dagger = a(a^\dagger)^2 a$. Let $x \in (a^\dagger a)^\circ$; then $a^\dagger a x = 0$. Multiplying on the left by $aa^\dagger$, we get $aa^\dagger a^\dagger a x = 0$. Thus by hypothesis $x \in (aa^\dagger)^\circ$.

(viii) $\implies$ (v) Suppose $(a^\dagger a)^\circ \subseteq (aa^\dagger)^\circ$. Let $y \in a^\circ$; then $a^\dagger a y = 0$. By hypothesis $aa^\dagger y = 0$. Multiplying on the left by $a^\dagger$, we get $a^\dagger y = 0$. Hence $y \in (a^\dagger)^\circ$.

(ix) $\implies$ (vii) Assume $(a^\dagger a)(a + a^\dagger) = (a + a^\dagger)(a^\dagger a)$; then $a^\dagger a^2 + a^\dagger = a + (a^\dagger)^2 a$. Multiplying on the left by a , we get $a^2 + aa^\dagger = a^2 + a(a^\dagger)^2 a$. Therefore,

$$aa^\dagger = a(a^\dagger)^2 a.$$

(iii) $\implies$ (ix) By the equivalence of (iii) and (vi) we have

$$a^\dagger a(a + a^\dagger) = a^\dagger a^2 + a^\dagger = a + (a^\dagger)^2 a = (a + a^\dagger)a^\dagger a.$$

From the last two equations, we get the desired result.

(x) $\implies$ (iii) Right-multiply the assumption $aa^\dagger = a^\dagger a^2 a^\dagger$ by a to obtain $a = a^\dagger a^2$.

(iii) $\implies$ (x) Obvious. □

Using the characterizations above and the theorems in [11], we now relate the class of hypo-EP elements to the well-known class of EP elements.

Corollary 2.1. *Let $a \in R^\dagger$. Then the following are equivalent:*

- (i) a is EP;
- (ii) a is hypo-EP and $aR = a^2R$;
- (iii) a is hypo-EP and a is group invertible;
- (iv) $a^\dagger$ is hypo-EP and $Ra = Ra^2$;
- (v) $Ra \subseteq Ra^\dagger$ and $Ra = Ra^2$;
- (vi) $Ra \subseteq Ra^*$ and $Ra = Ra^2$.

In the following proposition we give some conditions for a hypo-EP element to be EP. This result is proved for operators in [12].

Proposition 2.2. *Let $a \in R$ be a hypo-EP element. If a satisfies one of the following statements,*

- (i) $[aa^\dagger, a + a^\dagger] = 0$;
- (ii) $[aa^\dagger, a + a^*] = 0$;
- (iii) $[a, aa^\dagger] = 0$,

then a is an EP element.

Proof: (i) It suffices to show that $a^\dagger$ is hypo-EP. We have

$$0 = [aa^\dagger, a + a^\dagger] = a + a(a^\dagger)^2 - a^2a^\dagger - a^\dagger.$$

Multiplying on the right by a , we get

$$a^\dagger a = a(a^\dagger)^2 a.$$

Thus, according to Theorem 2.1(x), $a^\dagger$ is hypo-EP, which implies that a^* is also hypo-EP. Therefore, a is EP.

(ii) As before,

$$0 = [aa^\dagger, a + a^*] = a + aa^\dagger a^* - a^2a^\dagger - a^*.$$

Multiplying on the left by $aa^\dagger$, we get $a = a^2a^\dagger$ and by Theorem 2.1(vi), $a^\dagger$ is hypo-EP. We conclude that a is EP.

(iii) Again we have $0 = [a, aa^\dagger] = a^2a^\dagger - a$, and by Theorem 2.1(vi) we derive that a is EP. $\square$

Remark 2.1. Proposition 2.2 can be seen as an analogue of the fact that for EP elements, the Moore-Penrose inverse and the group inverse coincide. Here, additional commutation conditions force a hypo-EP element to be EP.

For EP elements, the projection $p = 1 - aa^\dagger$ satisfies $pa = 0$ and $p + a$ is invertible (see [13]). The following proposition extends this characterization to hypo-EP elements.

Proposition 2.3. *Let $a \in R^\dagger$ be a hypo-EP element. Then the projection $p = 1 - aa^\dagger$ satisfies $pa = 0$ and $p + a$ is left-invertible.*

Proof: Let a be a hypo-EP element and set $p = 1 - aa^\dagger$. Then $p^2 = p$ and $pa = 0$. We prove that $p + a$ is left-invertible.

Compute

$$(a(a^\dagger)^2 + p)(p + a) = a(a^\dagger)^2 p + a(a^\dagger)^2 a + p.$$

Since

$$a(a^\dagger)^2 p = a(a^\dagger)^2 (1 - aa^\dagger) = a(a^\dagger)^2 - a(a^\dagger)^2 aa^\dagger = 0,$$

it follows that

$$(a(a^\dagger)^2 + p)(p + a) = a(a^\dagger)^2 a + p.$$

By Theorem 2.1(vii), $aa^\dagger = a(a^\dagger)^2 a$, and hence

$$(a(a^\dagger)^2 + p)(p + a) = aa^\dagger + (1 - aa^\dagger) = 1.$$

Therefore, $a(a^\dagger)^2 + p$ is a left inverse of $p + a$. $\square$

3. FACTORIZATIONS OF HYPO-EP ELEMENTS

Factorizations of hypo-EP elements provide useful structural insights and extend results already established in the setting of C^* -algebras (see [14, 15, 10]). They highlight connections with projection techniques and offer algebraic characterizations that remain valid in rings with involution.

Theorem 3.1. *Let $a \in R^\dagger$. The following statements are equivalent:*

- (i) a is hypo-EP;
- (ii) $\exists b_1 \in R$ such that $a = a^*b_1$;
- (iii) $\exists b_2 \in R$ such that $a^\dagger = b_2a$;
- (iv) $\exists c_1 \in R$ such that $aa^* = c_1a$;
- (v) $\exists b_3, u \in R$ such that $(b_3)^\circ \subseteq ((b_3)^*)^\circ$, $u^\circ = \{0\}$ and $a = ub_3u^*$;
- (vi) $\exists b_4, c_2, u, v, w \in R$ such that $(b_4)^\circ \subseteq (c_2)^\circ$, $u^\circ = v^\circ = \{0\}$ and $a = ub_4w = w^*c_2^*v^*$;
- (vii) $\exists b_5, c_3, u_1, v_1, w_1 \in R$ such that $(b_5)^\circ \subseteq (c_3)^\circ$, $(u_1)^\circ = (v_1)^\circ = \{0\}$, and $a^*a = u_1b_5w_1$, $aa^* = v_1c_3w_1$;
- (viii) $\exists b_6, c_4, u_2, v_2, w_2 \in R$ such that $(b_6)^\circ \subseteq (c_4)^\circ$, $(u_2)^\circ = (v_2)^\circ = \{0\}$, and $a = u_2b_6w_2$, $a^\dagger = v_2c_4w_2$.

Proof: (i) $\implies$ (ii) Since $aR \subseteq a^*R$, we have $a \in a^*R$. Thus we get the desired result.

(ii) $\implies$ (iii) Assume there exists $b_1 \in R$ such that $a = a^*b_1$, which implies $a^* = (b_1)^*a$. Consequently, we obtain

$$a^\dagger = a^\dagger(aa^\dagger)^* = a^\dagger(a^\dagger)^*a^* = b_2a,$$

where $b_2 = a^\dagger(a^\dagger)^*(b_1)^*$.

(iii) $\implies$ (i) Assume $a^\dagger = b_2a$. Let $y \in a^\circ$. Then

$$a^\dagger y = 0.$$

This implies $y \in (a^\dagger)^\circ$. Since $(a^\dagger)^\circ = (a^*)^\circ$, it follows that $a^\circ \subseteq (a^*)^\circ$. Therefore, by Theorem 2.1(v), a is hypo-EP.

(ii) $\implies$ (iv) Let $a = a^*b_1$. Thus $a^* = (b_1^*)a$. Now we have

$$aa^* = a(b_1^*)a = c_1a,$$

where $c_1 = ab_1^*$.

(iv) $\implies$ (i) Suppose $aa^* = c_1a$ for some $c_1 \in R$. Let $x \in a^\circ$; then

$$aa^*x = 0.$$

Hence $x \in (aa^*)^\circ$. By Lemma 2.1(3) and Theorem 2.1(v), a is hypo-EP.

(i) $\implies$ (v) By Theorem 2.1(v), it is enough to choose $b_3 = a$ and $u = 1$.

(v) $\implies$ (vi) Assume $a = ub_3u^*$ with $b_3, u \in R$ such that $(b_3)^\circ \subseteq ((b_3)^*)^\circ$ and $u^\circ = 0$. Let $w = u^*$ and $b_4 = b_3$, so $a = ub_4u$. We observe that

$$a = (u^*)^*((b_4)^*)^*u^*.$$

Now set $w = u^*$, $v = u$, and $c_2 = (b_4)^*$. Then $a = w^*c_2^*v^*$. Since $(b_4)^\circ \subseteq ((b_4)^*)^\circ$ and $u^\circ = 0$, it follows that $(b_4)^\circ \subseteq (c_2)^\circ$ and $v^\circ = 0$.

(vi) $\implies$ (i) By Theorem 2.1(v), it is sufficient to prove $a^\circ \subseteq (a^*)^\circ$. Let $x \in a^\circ$, meaning $ub_4wx = 0$. Since $u^\circ = 0$, we have $b_4wx = 0$, so $wx \in (b_4)^\circ$. Since $(b_4)^\circ \subseteq (c_2)^\circ$, we deduce $c_2wx = 0$, which further leads to $vc_2vx = 0$. Consequently $a^*x = 0$, so $x \in (a^*)^\circ$.

(vii) $\implies$ (i) Assume $a^*a = u_1b_5w_1$ and $aa^* = v_1c_3w_1$ for some $b_5, c_3, u_1, v_1 \in R$ such that $(b_5)^\circ \subseteq (c_3)^\circ$ and $(u_1)^\circ = (v_1)^\circ = 0$. We show $a^\circ \subseteq (a^*)^\circ$. Let $y \in a^\circ$; then $a^*ay = 0$. Substituting the given factorization,

$$u_1b_5wy = 0.$$

Thus $wy \in (b_5)^\circ$, and since $(b_5)^\circ \subseteq (c_3)^\circ$, we get $c_3w_1y = 0$. Then $w_1y \in (c_3)^\circ$, and $v_1c_3wy = 0$ implies $y \in (aa^*)^\circ$. By Lemma 2.1(3) and Theorem 2.1(v), the desired result holds.

(i) $\implies$ (vii) By Lemma 2.1(3) and Theorem 2.1(v), choose $w = 1$, $v_1 = 1$, $u_1 = 1$, $b_5 = a^*a$ and $c_3 = aa^*$.

(viii) $\implies$ (i) Assume $a = u_2b_6w_2$ and $a^\dagger = v_2c_4w_2$ for some $b_6, c_4, u_2, v_2, w_2 \in R$ such that $(b_6)^\circ \subseteq (c_4)^\circ$ and $(u_2)^\circ = (v_2)^\circ = 0$. Let $x \in a^\circ$. Then $u_2b_6w_2x = 0$, so $w_2x \in (b_6)^\circ$ and since $(b_6)^\circ \subseteq (c_4)^\circ$, we have $c_4w_2x = 0$. Hence $a^\dagger x = v_2c_4w_2x = 0$, so $x \in (a^*)^\circ$. By Theorem 2.1(v), the required result holds.

(i) $\implies$ (viii) Choose $u_2 = 1$, $v_2 = 1$, $w_2 = 1$, $b_6 = a$, and $c_4 = a^\dagger$. By Theorem 2.1(v), the desired result follows. $\square$

4. SUM OF HYPO-EP ELEMENTS

In general, the sum of two hypo-EP elements is not necessarily hypo-EP. The following example shows that $a + b$ is not necessarily hypo-EP even though a and b are hypo-EP.

Example 4.1. Let R be the ring of all complex 2×2 matrices. Consider

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a and b are invertible, hence they are hypo-EP. However, $a + b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not hypo-EP because $(a + b)^2 = 0$ and $a + b \neq 0$, which fails condition (iii) of Theorem 2.1.

It is important to understand whether the class of hypo-EP elements is closed under addition. The following result provides a sufficient condition for the sum to be hypo-EP.

Proposition 4.1. *Let a, b be hypo-EP elements such that $a + b \in R^\dagger$. If $(a + b)^\circ \subseteq a^\circ$, then $a + b$ is hypo-EP.*

Proof: By Theorem 2.1(v), it is enough to show $(a + b)^\circ \subseteq ((a + b)^*)^\circ$. Let $x \in (a + b)^\circ$; then $(a + b)x = 0$. By hypothesis, $ax = 0$, thus $bx = 0$. Since both a and b are hypo-EP, by Theorem 2.1 we have $a^*x = 0$ and $b^*x = 0$, so $(a + b)^*x = 0$. Therefore, $a + b$ is hypo-EP. $\square$

We now examine the converse of the previous result under additional assumptions. This helps to better understand the additive structure of hypo-EP elements. Suppose a and b are hypo-EP; then by (ii) of Theorem 3.1, $a^* = c_1a$ and $b^* = c_2b$, where $c_1, c_2 \in R$.

Theorem 4.1. *Let a, b be hypo-EP elements such that $a + b \in R^\dagger$ and $c_1 - c_2$ is invertible, where c_1, c_2 are as defined above. Then $a + b$ is hypo-EP if and only if $(a + b)^\circ \subseteq a^\circ$.*

Proof: Suppose $a + b$ is hypo-EP. Then we have

$$c_1a + c_2b = a^* + b^* = (a + b)^* = d(a + b)$$

for some $d \in R$. This implies

$$(c_1 - d)a = (d - c_2)b.$$

Set $e = c_1 - d$ and $f = d - c_2$; then $ea = fb$ and $(e + f)a = f(a + b)$, so

$$a = (e + f)^{-1}f(a + b).$$

Hence $(a + b)^\circ \subseteq a^\circ$. $\square$

Remark 4.1. The converse of Proposition 4.1 is not true in general if the condition that $c_1 - c_2$ is invertible is omitted. The following example illustrates this point. Let R be the ring of all complex 2×2 matrices. Consider

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then a , b and $a + b$ are hypo-EP (in fact, they are self-adjoint). However, $(a + b)^\circ \not\subseteq a^\circ$ because $a + b = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ has a larger nullspace. Notice that in this example $c_1 = c_2 = I$, hence $c_1 - c_2 = 0$, which is not invertible. This explains why the converse fails when the invertibility condition is dropped.

5. PRODUCT OF HYPO-EP ELEMENTS

This section deals with the product of two hypo-EP elements. We derive some equivalent conditions for the product of two hypo-EP elements to be hypo-EP. We first give an example to show that the product of two hypo-EP elements in a ring is not necessarily hypo-EP.

Example 5.1. Let $R = M_2(\mathbb{C})$ (complex 2×2 matrices) with the transpose as involution. Suppose

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then a is invertible (hence hypo-EP) and b is self-adjoint idempotent up to scalar (hence hypo-EP). Their product

$$ab = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

is not hypo-EP because one can verify that $(ab)^\dagger(ab)^2 \neq (ab)$.

It is natural to ask whether the product of two hypo-EP elements is again hypo-EP. The next theorem provides sufficient conditions for this to hold.

Theorem 5.1. *Let $a, b \in R$ such that $ab \in R^\dagger$. If a, b are hypo-EP, $bR \subseteq aR$ and ${}^\circ b \subseteq {}^\circ a$, then ab is hypo-EP.*

Proof: We begin by proving $aR \subseteq b^*R$. Since b is hypo-EP, it satisfies

$$b - b^\dagger b^2 = 0.$$

Rearranging, $(1 - b^\dagger b)b = 0$, so $1 - b^\dagger b \in {}^\circ b$. By assumption, $1 - b^\dagger b \in {}^\circ a$, leading to

$$a = b^\dagger ba.$$

Applying Lemma 2.1(1), which states $b^\dagger bR = b^*R$, we conclude $a \in b^*R$, thus $aR \subseteq b^*R$. Now, using Lemma 2.1(1) again,

$$\begin{aligned} abR &\subseteq aR \subseteq b^*R = b^*bR \\ &\subseteq b^*aR \subseteq b^*a^*R \\ &= (ab)^*R. \end{aligned}$$

This establishes that ab is hypo-EP. □

The following example demonstrates that the converse of Theorem 5.1 is not necessarily true.

Example 5.2. Let $R = M_2(\mathbb{C})$ and consider

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then a , b , and $ab = 0$ are all hypo-EP (they are self-adjoint projections). However, $aR \not\subseteq bR$ and ${}^\circ a \not\subseteq {}^\circ b$.

The following result connects the class of EP elements to hypo-EP elements via multiplication, giving a characterization in terms of range inclusion.

Theorem 5.2. *Let $a, b \in R$ be EP elements and $ab \in R^\dagger$. Then ab is hypo-EP if and only if $ab^*R \subseteq b^*aR$.*

Proof: By our assumption, we have

$$abR = a(bR) = ab^*R \quad \text{and} \quad b^*aR = b^*(aR) = b^*a^*R.$$

Thus $abR \subseteq (ab)^*R$ if and only if $ab^*R \subseteq b^*aR$. Therefore, ab is hypo-EP if and only if $ab^*R \subseteq b^*aR$. □

Corollary 5.1. *Let a, b be hypo-EP elements in R such that $ab \in R^\dagger$. If $ab^*R \subseteq b^*aR$, then ab is hypo-EP.*

We conclude this section with an alternative condition under which the product of two elements is hypo-EP, further illustrating the flexibility of the definition.

Theorem 5.3. *Let $a, b \in R$ such that $ab \in R^\dagger$. If a is hypo-EP and $(ab)^\circ \subseteq a^\circ$, then ab is hypo-EP.*

Proof: We have $(ab)^\circ \subseteq a^\circ \subseteq (a^*)^\circ \subseteq (b^*a^*)^\circ = [(ab)^*]^\circ$. Thus by Theorem 2.1(v), it follows that ab is hypo-EP. $\square$

Remark 5.1. The condition $(ab)^\circ \subseteq a^\circ$ is not necessary for ab to be hypo-EP. For

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

we have $(ab)^\circ \not\subseteq a^\circ$. But a, b and ab are all hypo-EP elements (over $\mathbb{C}$).

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Uzbek Mathematical Journal
 2026, Volume 70, Issue 3, pp. 181–189
 DOI: 10.29229/uzmj.2026-3-19

A Free Boundary Problem for a Diffusion-Convection Equation with Nonlocal Condition

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Abstract. In this paper, we consider a free boundary problem for a diffusion-convection equation with a nonlocal boundary condition. Methods for obtaining a priori estimates in the norms of Hölder spaces for the solution are proposed. On the basis of these estimates, the existence and uniqueness of the solution are proved.

Keywords: free boundary, diffusion-convection, parabolic equation, a priori bounds, existence and uniqueness

MSC (2020): 35K20, 35K59, 35R35

1. INTRODUCTION

Free boundary problems play a crucial role in physical, biological, and mathematical studies. They involve boundary value problems for partial differential equations, where determining an unknown moving boundary is essential [1, 2, 3, 4, 5, 6, 7, 8, 9]. In this paper, we formulate a free boundary problem for a nonlinear diffusion-convection equation, namely the Rosen-Fokas-Yorston equation [10, 11, 12]. This equation models fluid diffusion with convective effects in porous media and has numerous applications, including groundwater hydrology, oil reservoir engineering, and various biological contexts, such as drug propagation in arterial tissues.

In [10] a one-phase Stefan problem for the diffusion-convection equation with a particular temperature-dependent thermal conductivity and a particular unidirectional speed was studied. Further articles involving nonlinear convective terms in parabolic equations include [13, 14, 15, 16, 17].

We will study a one-dimensional free boundary problem for the diffusion-convection equation with a nonlocal boundary condition at the fixed face $x = 0$ and a Stefan condition on the free boundary which has a convective term:

$$u_t = u^2 (du_{xx} - u_x), \quad 0 < t \leq T, \quad 0 < x < s(t), \quad (1.1)$$

$$u(0, x) = u_0(x) > \beta, \quad 0 \leq x \leq s_0, \quad (1.2)$$

$$u(t, 0) = mu(t, x_0), \quad 0 \leq t \leq T, \quad (1.3)$$

$$u(t, s(t)) = \beta, \quad 0 \leq t \leq T, \quad (1.4)$$

$$s'(t) = -du_x(t, s(t)) + u(t, s(t)), \quad 0 \leq t \leq T, \quad (1.5)$$

$$s(0) = s_0, \quad 0 < m < 1, \quad 0 < x_0 < s_0, \quad \beta > 0, \quad (1.6)$$

where d is the diffusivity, u_0 is the initial concentration.

Problem (1.1)-(1.6) was posed and studied in [10, 12] for a variable Dirichlet and Neumann boundary conditions instead of the nonlocal condition (1.3). In [10, 12] authors through several transformations the problem reduced to a free boundary problem for a linear heat-diffusion equation. They obtained the existence and uniqueness of solution. Problems with nonlocal conditions can also be found in the works [18, 19, 20].

The goal of this study is to derive a priori estimates for solutions to the one-phase Stefan problem with a nonlocal boundary condition on a fixed boundary, ensuring solvability over any finite time interval and proving the uniqueness of the solution.

The present paper is organized as follows. In Section 2, we use several transformations to reformulate the free boundary problem for the Burgers equation. Section 3 focuses on deriving a priori bounds that will be used later in the analysis. Finally, in Section 4, we establish the local-in-time existence and uniqueness of the solution.

1.1. Reduction of the problem to the problem for the Burgers equation. As in [10], we will transform to problem (1.1)–(1.6) in the one which is governed by the Burgers equation. We have:

$$v(t, z) = u(t, x).$$

where

$$\begin{aligned} z(t, x) &= g_0 + \int_0^t (u(\tau, 0) - du_x(\tau, 0)) d\tau + \int_0^x \frac{1}{u(t, \xi)} d\xi, \\ z_0 = z(t, x_0) &= g_0 + \int_0^t \left(mv(\tau, z_0) - d \frac{v_z(\tau, z_0)}{mv(\tau, z_0)} \right) d\tau + \int_0^{x_0} \frac{1}{u(t, \xi)} d\xi, \\ g(t) = z(t, 0) &= g_0 + \int_0^t \left(mv(\tau, z_0) - d \frac{v_z(\tau, z_0)}{mv(\tau, z_0)} \right) d\tau, \\ h(t) = z(t, s(t)) &= h_0 + (\beta + 1)t - \frac{(\beta + 1)d}{\beta^2} \int_0^t v_z(h(\tau), \tau) d\tau, \end{aligned}$$

with g_0 and h_0 are arbitrary constants, is a solution to the problem given by the Burgers equation

$$v_t = dv_{zz} - 2vv_z, \quad (t, z) \in D, \quad (1.7)$$

$$v(0, z) = v_0(z), \quad g_0 \leq z \leq h_0, \quad (1.8)$$

$$v(t, g(t)) = mv(t, z_0), \quad 0 \leq t \leq T, \quad (1.9)$$

$$v(t, h(t)) = \beta, \quad 0 \leq t \leq T, \quad (1.10)$$

$$h'(t) = -\frac{(\beta + 1)d}{\beta^2} v_z(t, h(t)) + \beta + 1, \quad 0 \leq t \leq T, \quad (1.11)$$

$$g(0) = g_0, \quad h(0) = h_0, \quad (1.12)$$

where $D = \{(t, z) : 0 < t \leq T, g(t) < z < h(t)\}$, $h(t)$ is a free boundary,

$$v_0(z) = u_0(\mu^{-1}(z)), \quad \mu(x) = g_0 + \int_0^x \frac{1}{u_0(\eta)} d\eta, \quad h_0 = g_0 + \int_0^{s_0} \frac{1}{u_0(\eta)} d\eta,$$

and the constants s_0 , g_0 and h_0 satisfy the following relation $s_0 = \int_{g_0}^{h_0} v_0(z) dz$.

Throughout this we assume that:

$$v_0(z) \in C^2[g_0, h_0], \quad v_0(0) = mv(0, z_0), \quad v_0(h_0) = \beta. \quad (1.13)$$

2. A PRIORI ESTIMATES

In this section, we establish several a priori bounds that will be used later. First of all, let us establish the boundedness of the functions $v(t, z)$, $h'(t)$.

Theorem 2.1. *Let $v(t, z)$, $h(t)$ be a solution of (1.7)–(1.12). Then there hold estimates:*

$$0 < v(t, z) \leq M_1, \quad (t, z) \in \overline{D}, \quad (2.1)$$

$$0 < h'(t) \leq M_2, \quad 0 \leq t \leq T. \quad (2.2)$$

Proof: We now turn to prove (2.1). The nonnegativity of the trace of $v(t, 0)$ is established as follows: Assume that, at a certain boundary point $(t_0, g(t_0))$, the function $v(t, z)$ attains its negative minimum $v(t_0, g(t_0)) = -M$. Then, by virtue of condition (1.9), we get

$$-M = v(t_0, g(t_0)) = mv(t_0, z_0) > v(t_0, z_0).$$

Thus, we arrive at a contradiction. So, we show that the function $v(t, x)$ does not have negative values on the left boundary.

Next, applying the strong maximum principle to $v(t, z)$, we get

$$0 < v(t, z) \leq \max v_0(z) \equiv M_1, \quad (t, z) \in \overline{D}.$$

Since the conditions (1.8)-(1.10) and the positivity of the function $v(t, z)$ in the domain D , we find $v_z(t, h(t)) < 0$. It then follows from the free boundary condition in (1.11) that $h'(t) > 0$ for $t \in (0, T)$.

Next we prove an upper bound of $h'(t)$. To do so, we define

$$\Omega = \left\{ (t, z) : 0 < t < T, h(t) - \frac{1}{N} < z < h(t) \right\},$$

where N is a positive constant to be determined later. And we construct an auxiliary function on Ω as follows $w(t, z) = M_1 (2N(h(t) - z) - N^2(h(t) - z)^2)$.

We will choose N so that $w(t, z) \geq v(t, z)$ holds over Ω . Direct calculations show that, for $(t, z) \in \Omega$,

$$\begin{aligned} w_t &= 2M_1 N h'(t) (1 - N(h(t) - z)), \\ -w_z &= 2M_1 N (1 - N(h(t) - z)), \quad -w_{zz} = 2M_1 N^2. \end{aligned}$$

then $ww_z = M_1^2 N^2 (-4(h(t) - z) + 6N(h(t) - z)^2 - 2N^2(h(t) - z)^3)$.

Let $h(t) - z = p$, then $0 < p < \frac{1}{N}$ holds over Ω . By $ww_z = 0$, we obtain $p = 0$ or $p = \frac{-6M_1 \pm \sqrt{36N^2 - 32N^2}}{-4N^2}$, in other words, $p = \frac{2}{N}$ or $p = \frac{1}{N}$. Denote $ww_z = P(p)$, then

$$P'(p) = M_1^2 N^2 (-4 + 12Np - 6N^2 p^2).$$

By solving $P'(p) = 0$, we have $p = \left(1 \pm \frac{\sqrt{3}}{3}\right) \frac{1}{N}$. Since $P(p) = 0$ when $p = 0, \frac{1}{N}, \frac{2}{N}$. Using the properties of cubic function, we know that $P(p)$ reaches the minimum at $p = \left(1 - \frac{\sqrt{3}}{3}\right) \frac{1}{N}$ for $0 < p < \frac{1}{N}$, and $P\left(1 - \frac{\sqrt{3}}{3}\right) = -\frac{4}{9}\sqrt{3}M_1^2 N$.

It follows from $h'(t) > 0$ and (2.1) that

$$w_t - dw_{zz} + ww_z \geq \left(2dN^2 - \frac{4}{9}\sqrt{3}M_1 N\right) M_1 > 0 = v_t - dv_{zz} + vv_z.$$

on Ω , provided that $N \geq \frac{2\sqrt{3}}{9d}M_1$. Also note that

$$\begin{aligned} w\left(t, h(t) - \frac{1}{N}\right) &= M_1 \geq v\left(t, h(t) - \frac{1}{N}\right), \\ w(t, h(t)) &= \beta = v(t, h(t)). \end{aligned}$$

Thus, if we can choose N such that $v_0(z) \leq w(0, z)$ for $z \in [h_0 - \frac{1}{N}, h_0]$, then we can apply the maximum principle to $w - v$ over Ω to deduce that $v(t, z) \leq w(t, z)$ for $(t, z) \in \Omega$. It would then follow that

$$\begin{aligned} v_z(t, h(t)) &\geq w_z(t, h(t)) = -2NM_1, \\ h'(t) &= -\frac{(\beta + 1)d}{\beta^2} v_z(t, h(t)) + \beta + 1 \leq M_2 \equiv \frac{2(\beta + 1)NM_1 d}{\beta^2} + \beta + 1. \end{aligned}$$

To complete the proof, we need only find some N independent of T such that

$$v_0(z) \leq w(0, z) \quad \text{for } z \in \left[h_0 - \frac{1}{N}, h_0\right],$$

then we can apply the maximum principle to $w - v$.

By the definition of w , we have $w(0, z) = M_1 [2N(h_0 - z) - N^2(h_0 - z)^2]$ if $z \in [h_0 - \frac{1}{N}, h_0 - \frac{1}{2N}]$, then $w(0, z) \geq \frac{3}{4}M_1$; if $z \in [h_0 - \frac{1}{2N}, h_0]$, then $w_z(0, z) = -2M_1 N [1 - N(h_0 - z)] \leq -M_1 N$. Therefore upon choosing

$$N = \max \left\{ \frac{2\sqrt{3}}{9d}M_1, \frac{4\|v_0\|_{C^1([0, h_0])}}{3M_1} \right\}$$

we will have $w_z(0, z) \leq v'_0(z)$ for $z \in [h_0 - \frac{1}{2N}, h_0]$.

Since $w(0, h_0) = v_0(h_0) = 0$, the above inequality implies $w(0, z) \geq v_0(z)$ for $z \in [h_0 - \frac{1}{2N}, h_0]$.

Moreover, for $z \in [h_0 - \frac{1}{N}, h_0 - \frac{1}{2N}]$, we have $w(0, z) \geq \frac{3}{4}M_1$, $v_0(z) \leq \frac{\|v_0\|_{C^1([0, h_0])}}{N} \leq \frac{3}{4}M_1$. Therefore $v_0(z) \leq w(0, z)$ for $z \in [h_0 - \frac{1}{N}, h_0]$.

This completes the proof. $\square$

Initial estimates have been received. Now, using the results of the work [21], we establish a priori estimates for v_z and higher derivatives.

To transform the varying domain D into a fixed domain, we introduce the transformation:

$$t = t, \quad y = \frac{z - g(t)}{h(t) - g(t)}.$$

Then the problem (1.7)–(1.12) takes the following form in $Q = \{(t, y) : 0 < t \leq T, 0 < y < 1\}$:

$$V_t - A(t)V_{yy} = B(t, y, V, V_y), \quad (t, y) \in Q, \quad (2.3)$$

$$V(0, y) = V_0(y), \quad 0 \leq y \leq 1, \quad (2.4)$$

$$V(t, 0) = mV(t, y_0), \quad 0 \leq t \leq T, \quad (2.5)$$

$$V(t, 1) = \beta, \quad 0 \leq t \leq T, \quad (2.6)$$

where

$$\begin{aligned} A(t) &= \frac{d}{(h(t) - g(t))^2}, \\ B(t, y, V, V_y) &= \left(\frac{2}{h(t) - g(t)} V + \frac{1 - y(h'(t) - g'(t))}{h(t) - g(t)} \right) V_y, \\ h'(t) &= -\frac{1}{h(t) - g(t)} \frac{d^2(\beta + 1)}{\beta^2} V_y(t, h(t)) + \beta + 1. \end{aligned}$$

Without loss of generality, we may assume that $V(0, y) = 0$.

Theorem 2.2. Suppose that a function $V(t, y)$ continuous in $\bar{Q}$ satisfies the conditions of problem (2.3)–(2.6). Assume that, for $(t, y) \in \bar{Q}$, $|V| \leq M_1$ and any V_y , continuous functions A and B satisfy the condition

$$\begin{aligned} A &\geq A_0 > 0, \\ \frac{|B|}{A} &\leq K(V_y^2 + 1), \quad K > 0. \end{aligned} \quad (2.7)$$

Under the conditions (2.7), the following estimate is valid

$$|V_y(t, y)| \leq M_3(M_1, M_2, A_0, K, \delta), \quad (t, y) \in Q^\delta, \quad (2.8)$$

where $Q^\delta = \{(t, y) : 0 < \delta \leq y \leq 1 - \delta, 0 < \delta \leq t \leq T\}$.

In addition, in the domain $\{(t, y) \in \bar{Q}, |V| \leq M_1, |V_y| \leq M_3\}$ we have the estimate

$$|V|_{\frac{2}{3}}^{Q^{2\delta}} \leq M_4(M_1, K, \delta).$$

And if it is also known that $V(t, y)$ possesses in $\bar{Q}$ summable with a square generalized derivatives V_{ty} , V_{yy} , then

$$|V|_{1+\alpha}^{Q^{2\delta}} \leq M_5(M_1, A_0, M_3, K, \delta), \quad 0 < \alpha < 1, \quad (2.9)$$

$$|V|_{2+\alpha}^{Q^{4\delta}} \leq M_6(M_1, A_0, M_3, K, \delta). \quad (2.10)$$

If $0 < x_0 < 4\delta$ and $V(t, 1) = 0$, then the estimates (2.8)–(2.10) are also valid in $\bar{Q}$.

Proof: Since estimates $|v| \leq M_1$, $|h'(t)| \leq M_2$ are established, by virtue of Theorem 3 and Theorem 6 in [21], the inner estimates (2.8)-(2.10) are true. In order to establish estimates up to the left boundary, we act as follows: Choosing δ such that $4\delta \leq y_0$ and using the nonlocal condition (2.5) taking into account (2.10), we show that the bounded function $V_t(t, l)$ satisfies the Hölder condition. In the domain $Q_+ = \{(t, y) : 0 \leq t \leq T, 0 \leq y \leq 4\delta\}$ we now introduce new functions

$$W(t, y) = V(t, y) - g_1(t, y), \quad (2.11)$$

$$g_1(t, y) = \frac{4\delta - y}{4\delta} V(t, 0) + \frac{y}{4\delta} V(t, 4\delta).$$

From (2.11) we get

$$W_y(t, y) = V_y(t, y) - g_2(t), \quad W_{yy}(t, y) = V_{yy}(t, y), \quad W_t(t, y) = V_t(t, y) - g_3(t, y),$$

where

$$g_2(t) = \frac{1}{4\delta} (V(t, 4\delta) - V(t, 0)), \quad g_3(t) = g_{1t}(t, y).$$

Let us refine the smoothness of the functions g_i , $i = \overline{1, 3}$. By virtue of (2.10) and the fact that $V_t(t, l - 4\delta)$ and $V_t(t, l)$ are Hölder, we can assert that the functions $g_i(t, y)$, $i = 1, 2$, have a derivative with respect to t satisfying the Hölder condition, and with respect to the variable y they are linear. And the function $g_3(t, y)$ is bounded, satisfies the Hölder condition in t , and is linear in y .

For the function $W(t, y)$ in the domain Q_+ we get the problem

$$\begin{cases} W_t = dW_{yy} + B_2(W, W_y), & (t, y) \in Q_+, \\ W(0, y) = 0, & 0 \leq y \leq 4\delta, \\ W(t, 0) = 0, & W(t, 4\delta) = 0, \quad 0 \leq t \leq T. \end{cases}$$

Hence, applying Theorem 4 in [21], we find

$$|W|_{1+\alpha}^{Q_+} \leq M_6.$$

Therefore,

$$|V|_{1+\alpha}^{Q_+} \leq M_6. \quad (2.12)$$

Using the method of even continuation across the right boundary [21], we obtain estimates for $|V_y|$, $|V|_{1+\alpha}$, $|V|_{1+\alpha}$ in the domain $Q_- = \{(t, y) : 0 \leq t \leq T, 1 - 4\delta \leq y \leq 1\}$.

Estimates in the domains Q_- , Q_+ , $Q^{4\delta}$ give a general estimate in the closed domain $\bar{Q}$. $\square$

3. UNIQUENESS AND EXISTENCE OF A SOLUTION

Theorem 3.1. *Suppose that the conditions of Theorem 2.2 is satisfied. Then the solution of problem (1.7)–(1.12) is unique.*

Proof: To prove the uniqueness of the solution, we derive an integral representation equivalent to (1.11). Multiply equation (1.7) by z and for any $t \in (0, T)$ integrate over the domain D , and we have

$$\begin{aligned} h^2(t) = & h_0^2 + \frac{2(\beta+1)}{\beta} \int_{g_0}^{h_0} \xi v_0^2(\xi) d\xi - 2 \left(\frac{(\beta+1)^2 d}{\beta} + \beta + 1 \right) \int_0^t h(\tau) d\tau - 2(\beta+1)d \cdot t \\ & + \frac{2(\beta+1)dm}{\beta} \int_0^t v(\tau, z_0) d\tau + \frac{2(\beta+1)}{\beta} \int_{g(t)}^{h(t)} \xi v^2(t, \xi) d\xi + \frac{2(\beta+1)}{\beta} \int_0^t d\tau \int_0^{h(\tau)} v^2(\tau, \xi) d\xi. \end{aligned} \quad (3.1)$$

The theorem is proved first for small $t \in [0, t_0]$, and then it is established that it is valid for any $0 < t < \infty$. Let $h_1(t), v_1(t, z)$ and $h_2(t), v_2(t, z)$ be solution of (1.7)–(1.12) and, in addition, let $y(t) = \min(h_1(t), h_2(t))$ and $\chi(t) = \max(h_1(t), h_2(t))$. Due to (3.1) we get

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq k_1 \int_0^t |h_1(\tau) - h_2(\tau)| d\tau + k_2 \int_{g(t)}^{y(t)} \xi |v_1^2(t, \xi) - v_2^2(t, \xi)| d\xi \\ &\quad + dm k_2 \int_0^t |v_1(\tau, z_0) - v_2(\tau, z_0)| d\tau + k_2 \int_{y(t)}^{\chi(t)} \xi |v_i^2(t, \xi)| d\xi \\ &\quad + k_2 \int_0^t d\tau \int_{g(\tau)}^{y(\tau)} |v_1^2(\tau, \xi) - v_2^2(\tau, \xi)| d\xi + k_2 \int_0^t d\tau \int_{y(\tau)}^{\chi(\tau)} v_i^2(\tau, \xi) d\xi. \end{aligned} \quad (3.2)$$

where $k_1 = \frac{1}{h_0} \left(\frac{d(\beta+1)^2}{\beta} + \beta + 1 \right)$, $k_2 = \frac{\beta+1}{h_0\beta}$; v_i ($i=1,2$) is a solution between $y(t)$ and $\chi(t)$, i.e.,

$$v_i(t, z) = \begin{cases} v_1(t, z) & \text{if } h_2(t) < h_1(t), \\ v_2(t, z) & \text{if } h_2(t) > h_1(t). \end{cases}$$

By virtue of the established estimates for $h_i(t), v_i(t, z)$, we get

$$0 < v_i(t, z) \leq N(y(t) - z), \quad i = 1, 2,$$

$$|v_1(t, y(t)) - v_2(t, y(t))| \leq N|h_1(t) - h_2(t)|.$$

We now estimate the components of relation (3.2):

$$I_1 = k_2 \int_{g(t)}^{y(t)} \xi |v_1^2(t, \xi) - v_2^2(t, \xi)| d\xi \leq 2M_1 k_2 \int_{g(t)}^{y(t)} \xi |v_1(t, \xi) - v_2(t, \xi)| d\xi, \quad (3.3)$$

$$I_2 = k_2 \int_{y(t)}^{\chi(t)} \xi |v_i^2(t, \xi)| d\xi \leq \frac{M_1 k_2}{2} y(t) |h_1(t) - h_2(t)|^2 + \frac{M_1 k_2}{2} |h_1(t) - h_2(t)|^3, \quad (3.4)$$

$$I_3 = k_2 \int_0^t d\tau \int_{g(\tau)}^{y(\tau)} |v_1^2(\tau, \xi) - v_2^2(\tau, \xi)| d\xi \leq 2M_1 k_2 \int_0^t d\tau \int_{g(\tau)}^{y(\tau)} |v_1(\tau, \xi) - v_2(\tau, \xi)| d\xi, \quad (3.5)$$

$$I_4 = k_2 \int_0^t d\tau \int_{y(\tau)}^{\chi(\tau)} v_i^2(\tau, \xi) d\xi \leq k_2 M_1 t |h_1(t) - h_2(t)|. \quad (3.6)$$

Let $A(t_0) = \max_{0 \leq t \leq t_0} |h_1(t) - h_2(t)| > 0$. If $A(t_0) > 0$, taking into account the obtained inequalities from (3.3)–(3.6), we have:

$$\begin{aligned} (1 - k_1 t_0) A(t_0) &\leq 2M_1 k_2 \int_{g(t_0)}^{y(t_0)} \xi |v_1(t, \xi) - v_2(t, \xi)| d\xi + \frac{M_1 k_2}{2} y(t_0) A^2(t_0) + \frac{M_1 k_2}{2} A^3(t_0) \\ &\quad + 2M_1 k_2 \int_0^{t_0} d\tau \int_{g(\tau)}^{y(\tau)} |v_1(\tau, \xi) - v_2(\tau, \xi)| d\xi + k_2 M_1 t_0 A(t_0), \end{aligned} \quad (3.7)$$

where $t_0 < \frac{1}{k_1}$.

If (3.7) is divided by $A(t_0)$, we get:

$$\begin{aligned} (1 - k_1 t_0) \leq & 2M_1 k_2 \int_{g(t_0)}^{y(t_0)} \frac{\xi |v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi + \frac{M_1 k_2}{2} y(t_0) A(t_0) + \frac{M_1 k_2}{2} A^2(t_0) \\ & + 2M_1 k_2 \int_0^{t_0} d\tau \int_{g(\tau)}^{y(\tau)} \frac{|v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi + k_2 M_1 t_0. \end{aligned} \quad (3.8)$$

Let us rewrite (3.8) in the form

$$\begin{aligned} (1 - k_1 t_0) \leq & 2M_1 k_2 \int_{g(t_0)}^{y(0)} \frac{\xi |v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi + 2M_1 k_2 \int_{y(0)}^{y(t_0)} \frac{\xi |v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi \\ & + \frac{M_1 k_2}{2} y(t_0) A(t_0) + \frac{M_1 k_2}{2} A^2(t_0) + k_2 M_1 t_0 \\ & + 2M_1 k_2 \int_0^{t_0} d\tau \int_{g(\tau)}^{y(0)} \frac{|v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi + 2M_1 k_2 \int_0^{t_0} d\tau \int_{y(0)}^{y(\tau)} \frac{|v_1(t, \xi) - v_2(t, \xi)|}{A(t_0)} d\xi. \end{aligned} \quad (3.9)$$

and consider the function $V(t, x) = v_1(t, x) - v_2(t, x)$. Then we have the problem

$$\begin{cases} V_t = dV_{zz} + 2v_1 V_z + 2v_2 V, & 0 < t \leq t_0, \quad g_0 < z \leq y_0, \\ V(0, z) = 0, & g_0 \leq z \leq y_0, \\ V(t, g(t)) = mV(t, z_0), & 0 \leq t \leq t_0, \\ V(t, y(t)) \leq NA(t_0), & 0 \leq t \leq t_0. \end{cases}$$

Hence, according to the maximum principle $|V(z, t)| \leq NA(t_0)$.

Let us estimate the integral terms in formula (3.8):

$$\begin{aligned} 2M_1 k_2 \int_{g(t_0)}^{y(t_0)} \frac{\xi |V(t, \xi)|}{A(t_0)} d\xi & \leq 2M_1 k_2 \left(\int_{g(t_0)}^{y(0)} \frac{\xi |V(t, \xi)|}{A(t_0)} d\xi + \int_{y(0)}^{y(t_0)} \frac{\xi |V(t, \xi)|}{A(t_0)} d\xi \right) \\ & \leq 2M_1 k_2 \left(\int_{g(t_0)}^{y(0)} \frac{\xi |V(t, \xi)|}{A(t_0)} d\xi + 2M_1 k_2 \int_{y(0)}^{y(t_0)} \xi d\xi \right) \leq 2M_1 k_2 \int_{g(t_0)}^{y(0)} \frac{\xi |V(t, \xi)|}{A(t_0)} d\xi + M_7 t_0, \\ 2M_1 k_2 \int_0^{t_0} d\tau \int_{g(\tau)}^{y(\tau)} \frac{|V(t, \xi)|}{A(t_0)} d\xi & \leq 2M_1 k_2 \left(\int_0^{t_0} d\tau \int_{g(\tau)}^{y(0)} \frac{|V(t, \xi)|}{A(t_0)} d\xi + \int_0^{t_0} d\tau \int_{y(0)}^{y(\tau)} \frac{|V(t, \xi)|}{A(t_0)} d\xi \right) \\ & \leq 2M_1 k_2 \left(\int_0^{t_0} d\tau \int_{g(\tau)}^{y(0)} \frac{|V(t, \xi)|}{A(t_0)} d\xi + \int_0^{t_0} (y(\tau) - y(0)) d\tau \right) \leq 2M_1 k_2 \int_0^{t_0} d\tau \int_{g(\tau)}^{y(0)} \frac{|V(t, \xi)|}{A(t_0)} d\xi + M_8 t_0^2. \end{aligned}$$

Now let's consider the auxiliary problem:

$$\begin{cases} w_t = dw_{zz} + 2v_1 w_z + 2v_2 w, & 0 < t \leq t_0, \quad g_0 < z \leq y_0, \\ w(0, z) = 0, & g_0 \leq z \leq y_0, \\ w(t, g(t)) = mw(t, z_0), & 0 \leq t \leq t_0, \\ w(t, y_0) = 1, & 0 \leq t \leq t_0, \end{cases}$$

We introduce a function

$$W(t, z) = \frac{V(t, z)}{NA(t_0 \Delta t)} - w(t, z), \quad 0 < t \leq t_0, \quad g_0 < z \leq y_0.$$

This enables us to write

$$\begin{cases} W_t = dW_{zz} + 2v_1 W_z + 2v_2 W, & 0 < t \leq t_0, \quad g_0 < z \leq y_0, \\ W(0, z) = 0, & g_0 \leq z \leq y_0, \\ W(t, g(t)) = mW(t, z_0), & 0 \leq t \leq t_0, \\ W(t, y_0) = \frac{V(t, y_0)}{NA(t_0)} - w(t, y_0), & 0 \leq t \leq t_0. \end{cases}$$

By the maximum principle, we find

$$\frac{V(t, z)}{NA(t_0)} \leq w(t, z), \quad 0 \leq t \leq t_0.$$

Since $\lim_{t \rightarrow 0} W(t, z) = 0$, $g_0 \leq x \leq y_0$, we conclude that

$$\lim_{t \rightarrow 0} \int_{g_0}^{y(0)} W(t, z) dz = 0.$$

Therefore,

$$\lim_{t \rightarrow 0} \int_{g_0}^{y_0} \frac{|V(t, \xi)|}{A(t_0)} d\xi \rightarrow 0.$$

Since the right-hand side of (3.2) tends to zero as $t_0 \rightarrow 0$, then (3.2) does not hold for sufficiently small t_0 and we arrive at a contradiction. Consequently, $h_1(t) \equiv h_2(t)$ and further $v_1(t, z) = v_2(t, z)$, $0 \leq t \leq t_0$.

The uniqueness of the solution to the problem for any $0 < t < \infty$ is established as follows. Let $t_1 = \sup\{t : s_1(\eta) = s_2(\eta), 0 \leq \eta \leq t\}$. If $t_1 = \infty$, then the question will be solved. Otherwise, assuming that the parameter t_1 is bounded and repeating the above calculations in the interval $t_1 \leq t \leq t_1 + \Delta t$, we again arrive at a contradiction. $\square$

Theorem 3.2. *Suppose that the conditions of Theorem 2.2 are satisfied. Then, in Q there exists a solution $v(t, z) \in C^{2+\alpha}(\bar{Q})$, $h(t) \in C^{1+\alpha}(0 \leq t \leq T)$ of problem (1.7)–(1.12).*

Proof: To prove the solvability of a nonlinear problem, you can use various theorems from the theory of nonlinear equations, remembering that the uniqueness theorem of the classical solution is valid for it. Let us use the Leray-Schauder principle [22], established by a priori estimates $|\cdot|_{1+\alpha}$ for all possible solutions of nonlinear problems, and the solvability theorem in Hölder classes for linear problems. A more detailed exposition of the technique can be found, for example, in (Section VI, [5]; Section VII, [22]). $\square$

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Recovering the time dependent coefficient in a multidimensional multi-term fractional diffusion equation

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Abstract. We consider the inverse problem of determining the time-dependent coefficient for a diffusion equation involving a fractional Laplacian operator in space and Hilfer multi-term fractional derivatives in time with Dirichlet zero boundary conditions. The solution of the initial-boundary value problem was investigated using the Fourier method. First, we will study the spectral problem related to the fractional Laplacian in a spatial variable. In addition, the existence of a solution to the Cauchy problem for a fractional time differential equation is considered. Using the generalized Gronwall inequality, we obtain an estimate of the solution in terms of the unknown coefficient. The inverse problem is to recover a time-dependent coefficient with an integral type over-determination condition. We consider the analytical solution of the inverse problem and prove the existence and uniqueness of the analytical solution. The inverse problem reduces to an equivalent Volterra-type integral equation. The local existence and global uniqueness of the solution of the inverse problem are proven according to the Banach fixed-point theorem.

Keywords: Hilfer (generalized Riemann-Liouville) fractional derivative, fractional Laplacian, Mittag-Leffler function, inverse problem, integral equation, Fourier series, principle of contraction mapping

MSC (2020): 34A08 , 34K10 , 34K29 , 34K37 , 35R11 , 35R30

1. INTRODUCTION

In recent years, both linear and nonlinear fractional as well as integer order partial differential equations have gained significant interest in theoretical and applied mathematics. The increased attention to this field in recent years is primarily due to the fact that these equations play an important role in modeling many phenomena in various scientific and engineering sciences [1, 2, 3]. Fractional differential equations have been widely used in engineering, physics, chemistry, biology, and other fields. Many applications can be found in the work [3] and references therein.

Most of the research published to date is related to fractional differential equations. Initial and initial-boundary value problems for fractional-order time diffusion and diffusion-wave equations have been sufficiently studied, for example, see [3, 4, 5]. In the works [5, 6], the authors investigated the unique solvability of fractional initial-boundary value problems for diffusion-wave equations involving second-order elliptic differential operators. In these studies, various initial and boundary conditions are taken into account, and different solution approaches are applied.

In the midst of the inverse problems for the diffusion-wave equation involving fractional derivatives, the most common are inverse source problems with various overdetermination conditions [7, 8, 9, 10, 11, 12, 13]. In particular, in the article [14], the authors investigated direct and inverse problems for subdiffusion equations involving Hilfer fractional derivatives. An arbitrary positive self-adjoint operator A is taken as the elliptic part of the equation. In [15], the inverse problem of determining the coefficient in the lower term of the fractional-order diffusion equation is considered. A direct problem is an initial-boundary value problem for an equation with nonlocal initial and homogeneous Dirichlet conditions. The condition of overdetermination of the integral form with respect to the solution of the direct problem for determining the unknown coefficient is shown. In [16], the author proposed new formulas for representing the solution of the first (Dirichlet), second (Neumann) and third (Robin) initial-boundary value problems for one-dimensional fractional diffusion equations with the time-fractional Caputo derivative. These formulas are obtained by the continuation method used in the theory of differential equations with integer derivatives.

In [17], the correctness and long-term asymptotic behavior of the initial-boundary value problem for fractional-order diffusion equations were investigated, where the time differentiation consists of a finite summation of Caputo derivatives with decreasing orders in $(0, 1)$ and positive constant coefficients.

In [18], the well-posedness and the long-time behavior of the initial-boundary value problems for the multi-term Caputo time-fractional wave equations are considered. The multi-term time-fractional wave equation involves Caputo fractional derivatives in time of different orders between 0 and 2.

In [19], the inverse problem of determining the time-dependent source term along with the diffusion/temperature concentration for the space-time fractional-order diffusion equation is considered. The space-time fractional diffusion equation involves a Caputo fractional derivative in space and Hilfer fractional derivatives in time of different orders between 0 and 1. In [20], the inverse problem of determining the time-dependent source term along with the diffusion concentration for the multi-parameters space-time fractional diffusion equation with nonlocal boundary conditions is considered. The space-time fractional diffusion equation involves several Caputo fractional derivatives in time of different orders between 0 and 1. In [21], the inverse problem of determining the time-dependent source term along with the state variable for a non-homogeneous time-space fractional diffusion equation is considered. The time-space fractional diffusion equation involves a Caputo fractional derivative in time of order $\alpha \in (0, 1)$ and a fractional power of a symmetric uniformly elliptic operator in space.

In this article, we consider the inverse problem of determining the time-dependent coefficient with a state variable for a multidimensional multi-term time-fractional diffusion equation with zero Dirichlet boundary conditions. The multi-term time-fractional diffusion equation involves a multi-term Hilfer fractional derivative in time of different orders between 0 and 1, and a fractional Laplacian operator in space of order $1 < \sigma \leq 2$, where the unknown coefficient is recovered under an integral-type over-determination condition.

Next, we consider the full fractional diffusion-wave equation with the multi-term time-fractional Hilfer operator and a fractional Laplacian operator

$$\begin{aligned} & \left(D_{0+,t}^{\alpha_0,\beta_0} u \right) (x, t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j,\beta_j} u \right) (x, t) \\ & + (-\Delta)^{\frac{\sigma}{2}} u(x, t) + q(t)u(x, t) = f(x, t), \quad (x, t) \in Q_T, \end{aligned} \quad (1.1)$$

with initial and boundary conditions

$$I_{0+,t}^{(1-\alpha_j)(1-\beta_j)} u(x, t) \big|_{t=0} = \varphi_j(x), \quad j = 0, 1, \dots, m, \quad m \in \mathbb{N}, \quad x \in \Omega, \quad (1.2)$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \quad (1.3)$$

where $Q_T = \Omega \times (0, T)$, Ω is an open and bounded domain in $\mathbb{R}^d$, featuring a smooth boundary $\partial\Omega$, and T is a fixed final time ($T = \text{const} > 0$), $\mu_j > 0$, $j = 1, 2, \dots, m$, $(-\Delta)^{\frac{\sigma}{2}}$ is the fractional Laplacian operator of order $1 < \sigma \leq 2$, $D_{0+,t}^{\alpha_j,\beta_j}$, $j = 0, 1, \dots, m$, are Hilfer fractional derivatives in time variable of orders α_j , $j = 0, 1, \dots, m$, $0 < \alpha_n < \dots < \alpha_2 < \alpha_1 < \alpha_0 < 1$ and types $0 \leq \beta_j \leq 1$, $I_{0+,t}^{(1-\alpha_j)(1-\beta_j)}$ denotes the Riemann-Liouville fractional integrals in time variable of orders $(1 - \alpha_j)(1 - \beta_j)$, $j = 0, 1, \dots, m$, $m \in \mathbb{N}$ respectively.

Note that if σ tends to 2, the fractional Laplacian operator $(-\Delta)^{\frac{\sigma}{2}}$ tends to the Laplacian operator $-\Delta$, and thus model (1.1) reproduces the multi-term time-fractional diffusion-wave equation.

Given functions $q(t)$, $f(x, t)$, $\varphi_j(x)$, $j = 0, 1, \dots, m$ and parameters $0 < \alpha_n < \dots < \alpha_2 < \alpha_1 < \alpha_0 < 1$, $0 \leq \beta_j \leq 1$, $j = 0, 1, \dots, m$, $1 < \sigma \leq 2$, the problem of finding the solution to the initial boundary problem (1.1)-(1.3) is called the direct problem.

Inverse problem. For the given functions $f(x, t)$, $\varphi_j(x)$, $j = 0, 1, \dots, m$ and numbers $0 < \alpha_n < \dots < \alpha_2 < \alpha_1 < \alpha_0 < 1$, $0 \leq \beta_j \leq 1$, $j = 0, 1, \dots, m$, $1 < \sigma \leq 2$, it is required to determine the function $q(t)$, $t \in [0, T]$ in equation (1.1), using additional information about the solution to the direct problem (1.1)-(1.3)

$$\int_{\Omega} w(x)u(x, t)dx = h(t), \quad 0 \leq t \leq T, \quad (1.4)$$

where $w(x)$, $h(t)$ are given functions. In heat propagation in a thin rod in which the law of variation $h(t)$ of the total quantity of heat in the rod is given in [22]. This integral condition in parabolic problems is also called heat moments which are analyzed in [23].

Definition 1.1. A function $u(x, t)$ is called a classical solution to initial boundary value problem (1.1)-(1.3), for $q(t) \in C([0, T])$ if:

(i) $t^\gamma u(x, t) \in C(\overline{Q}_T)$, and $t^\gamma (-\Delta)^{\frac{\alpha}{2}} u(\cdot, t) \in C(\overline{Q}_T)$;

(ii) $t^\gamma \left(D_{0+,t}^{\alpha_0,\beta_0} + \sum_{i=1}^m \mu_i D_{0+,t}^{\alpha_i,\beta_i} \right) u(x, t) \in C(\overline{Q}_T)$;

(iii) $u(x, t)$ satisfies (1.1)-(1.3),

where $1 > \gamma > \max \left\{ (1 - \beta_0)(1 - \alpha_0), \max_{0 \leq j \leq m} [1 - \alpha_0 + \beta_j(1 - \alpha_j)] \right\}$.

2. PRELIMINARIES

In this section, we present some useful definitions and results, which will be used later.

Definition 2.1. (see [1]) Let $g \in L_{loc}^1(a, b)$, $-\infty < a < t < b < \infty$ be a locally integrable real-valued function. The left and right sided Riemann-Liouville fractional integral of order $\alpha > 0$ are defined as

$$I_{a+,t}^\alpha g(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} g(\tau) d\tau$$

and

$$I_{b-,t}^\alpha g(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} g(\tau) d\tau$$

respectively.

Definition 2.2. (see [2], pp. 113-116) Let $g_1 \in L^1(a, b)$, $-\infty < a < t < b < \infty$ and $g_1 * g_{(1-\alpha)(1-\beta)} \in AC([a, b])$. The Hilfer fractional derivative of order α , $0 < \alpha < 1$ and type β , $0 \leq \beta \leq 1$ is defined as

$$D_{a+,t}^{\alpha,\beta} g_1(t) := \left(I_{a+,t}^{\beta(1-\alpha)} \frac{d}{dt} I_{a+,t}^{(1-\alpha)(1-\beta)} g_1 \right) (t)$$

where $g_\alpha = t^{\alpha-1}/\Gamma(\alpha)$ and $"*"$ indicates integral convolution stated as

$$(g_1 * g)(t) = \int_0^t g_1(\tau) g(t - \tau) d\tau$$

The Hilfer fractional derivative interpolates both the Riemann-Liouville fractional derivative and the Caputo fractional derivative:

- For $\beta = 0$, the HFD becomes the RLFD, i.e.,

$$D_{a+,t}^{\alpha,0} g(z) = \frac{d}{dt} I_{a+,t}^{(1-\alpha)} g(t) := {}^{RL}D_{a+,t}^\alpha g(t).$$

- For $\beta = 1$, the HFD becomes the Caputo fractional derivative, i.e.,

$$D_{a+,t}^{\alpha,1} g(t) = I_{a+,t}^{1-\alpha} \frac{d}{dt} g(t) := {}^cD_{a+,t}^\alpha g(t).$$

Lemma 2.1. [1] Let g_i be a sequence of functions defined on $(a, b]$ for each $i \in \mathbb{N}$, such that

- $D_{a+,t}^{\alpha,\beta} g_i(t)$ exists $\forall i \in \mathbb{N}$, $t \in (a, b]$,
- both series $\sum_{i=1}^\infty g_i(t)$ and $\sum_{i=1}^\infty D_{a+,t}^{\alpha,\beta} g_i(t)$ are uniformly convergent on the interval $[a + \epsilon, b]$ for any $\epsilon > 0$.

Then,

$$D_{a+,t}^{\alpha,\beta} \sum_{i=1}^\infty g_i(t) = \sum_{i=1}^\infty D_{a+,t}^{\alpha,\beta} g_i(t), \quad 0 < \alpha \leq \beta < 1, \quad a < t < b.$$

We consider the weighted spaces of continuous functions

$$C_\gamma[a, b] := \{g : (a, b] \rightarrow \mathbb{R} : (t - a)^\gamma g(t) \in C[a, b], 0 \leq \gamma < 1, \},$$

$$C_\gamma^{\alpha, \beta, \sigma}(Q_T) = \left\{ u(x, t) \in C_\gamma(Q_T) : D_{0+, t}^{\alpha, \beta} u(x, t) \in C_\gamma(Q_T), 0 < \alpha \leq 1, 0 \leq \beta \leq 1, \right. \\ \left. (-\Delta)^{\frac{\sigma}{2}} u(x, t) \in C_\gamma(Q_T); x \in [0, l], 1 < \sigma \leq 2 \right\}.$$

2.1. Mittag-Leffler type functions. In this subsection, we describe the Mittag-Leffler and multinomial Mittag-Leffler functions with some important estimates.

Two parameter Mittag-Leffler (M-L) function. The two parameter M-L function $E_{\alpha, \beta}(z)$ is defined by the following series:

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\alpha)$ denotes the real part of the complex number α , $\Gamma(\cdot)$ is Euler's Gamma function. The Mittag-Leffler function has been studied by many authors who have proposed and studied various generalizations and applications. A very interesting work that meets many results about this function is due to Kilbas et al. (see [3], pp. 42-44).

For $\beta = 1$, β reduces to the Mittag-Leffler function of single parameter, i.e.

$$E_{\alpha, 1}(z) = E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Proposition 2.1. ([3], pp. 40-45) Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that

$$|E_{\alpha, \beta}(z)| \leq \frac{C}{1 + |z|}, \quad \kappa \leq |\arg(z)| \leq \pi.$$

Multinomial Mittag-Leffler functions

Definition 2.3. (see [24]) For $\beta > 0$, $\alpha_i > 0$, $z_i \in \mathbb{C}$, $i = \overline{1, n}$, $n \in \mathbb{N}$, the multinomial Mittag-Leffler functions is stated as

$$E_{(\alpha_1, \alpha_2, \dots, \alpha_n), \beta}(z_1, z_2, \dots, z_n) := \sum_{k=0}^{\infty} \sum_{\substack{l_1 + l_2 + \dots + l_n = k \\ l_1 \geq 0, \dots, l_n \geq 0}} \frac{(k; l_1, \dots, l_n) \prod_{i=1}^n z_i^{l_i}}{\Gamma\left(\beta + \sum_{i=1}^n \alpha_i l_i\right)},$$

where $(k; l_1, \dots, l_n) = \frac{k!}{l_1! \times \dots \times l_n!}$.

Remark 2.1. For $n = 2$, the multinomial Mittag-Leffler function takes the following form:

$$E_{(\alpha_1, \alpha_2), \beta}(z_1; z_2) = \sum_{k=0}^{\infty} \sum_{\substack{l_1 + l_2 = k \\ l_1 \geq 0, l_2 \geq 0}} \frac{k!}{l_1! l_2!} \frac{z_1^{l_1} z_2^{l_2}}{\Gamma(\beta + \alpha_1 l_1 + \alpha_2 l_2)} \\ = \sum_{k=0}^{\infty} \sum_{i=0}^k \frac{k!}{i!(k-i)!} \frac{z_1^i z_2^{k-i}}{\Gamma(\beta + \alpha_1 l_1 + \alpha_2 l_2)}.$$

Moreover, note that

$$E_{(\alpha_1, \alpha_2, \dots, \alpha_n), \beta}(z_1; z_2; \dots; z_n) = E_{(\alpha_n, \dots, \alpha_2, \alpha_1), \beta}(z_n; \dots; z_2; z_1)$$

Remark 2.2. For $z_1 \neq 0$ and $z_2 = z_3 = \dots = z_n = 0$, the multinomial Mittag-Leffler function takes the following form

$$E_{\alpha_1, \beta}(z_1) = \sum_{k=0}^{\infty} \frac{z_1^k}{\Gamma(\beta + \alpha_1 k)}.$$

Lemma 2.2. [19] For $\alpha_i, \beta, \tau, \sigma_i > 0, i = 1, 2, \dots, m, m \in \mathbb{N}$ the Laplace transform of the multinomial Mittag-Leffler function is given by

$$\mathcal{L}\{\tau^{\beta-1} E_{(\alpha_1, \alpha_2, \dots, \alpha_m), \beta}(-\sigma_1 \tau^{\alpha_1}, -\sigma_2 \tau^{\alpha_2}, \dots, -\sigma_m \tau^{\alpha_m})\} = \frac{s^{-\beta}}{1 + \sum_{i=1}^m \sigma_i s^{-\alpha_i}},$$

$$\text{if } \left| \sum_{i=1}^m \sigma_i s^{-\alpha_i} \right| < 1.$$

The Laplace transform for $m = 1$ is as follows

$$\mathcal{L}[t^{\beta-1} E_{\alpha, \beta}(\pm at^{\alpha})] = \int_0^{\infty} e^{-st} t^{\beta-1} E_{\alpha, \beta}(\pm at^{\alpha}) dt = \frac{s^{-\beta}}{1 \mp as^{-\alpha}}, \quad \Re(s) > |a|^{1/\alpha}.$$

Lemma 2.3. [25] For $0 < \beta < 1$ and $0 < \alpha_n < \dots < \alpha_2 < \alpha_1 < 1$ be given. Assume that $\frac{\alpha_1 \pi}{2} < \kappa < \alpha_1 \pi, \kappa \leq |\arg(\sigma_1 \tau^{\alpha_1})| \leq \pi$ and $\sigma_i \tau > 0$. Then there exists a constant C_0 depending only on $\kappa, \alpha_i, i = 1, 2$ such that

$$|E_{(\alpha_1 - \alpha_n, \dots, \alpha_1 - \alpha_2, \alpha_1), \beta}(-\sigma_n \tau^{\alpha_1 - \alpha_n}, \dots, -\sigma_2 \tau^{\alpha_1 - \alpha_2}, -\sigma_1 \tau^{\alpha_1})| \leq \frac{C_0}{1 + |\sigma_1 \tau^{\alpha_1}|}.$$

We give some definitions concerning the eigenvalues and eigenvectors of the fractional Laplacian.

Definition 2.4. [26] Suppose $\{\lambda_k, e_k\}$ be the eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in Ω with Dirichlet boundary condition on $\partial\Omega$:

$$\begin{aligned} -\Delta e_k(x) &= \lambda_k e_k(x), & \text{in } \Omega \\ e_k(x) &= 0, & \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Counting according to the multiplicities, we can assume

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = +\infty.$$

Let

$$H_0^{\sigma}(\Omega) := \left\{ u = \sum_{k=1}^{\infty} a_k e_k : \|u\|_{H_0^{\sigma}(\Omega)}^2 = \sum_{k=1}^{\infty} a_k^2 \lambda_k^{\sigma} < \infty \right\}$$

we define the operator $(-\Delta)^{\frac{\sigma}{2}}$ by

$$(-\Delta)^{\frac{\sigma}{2}} u = \sum_{k=1}^{\infty} a_k \lambda_k^{\frac{\sigma}{2}} e_k$$

which maps $H_0^{\frac{\sigma}{2}}(\Omega)$ onto $L^2(\Omega)$, with the following equivalence:

$$\|u\|_{H_0^{\sigma}} = \|(-\Delta)^{\frac{\sigma}{2}} u\|_{L^2(\Omega)}.$$

Let us present some assertions about eigenfunctions $v_k(x)$ and eigenvalues λ_k proved by V.A.Ilyin [27].

Lemma 2.4. The series $\sum_{k=1}^{\infty} \lambda_k^{-([\frac{d}{2}] + 1)} e_k^2(x)$ converges uniformly in a closed domain $\overline{\Omega}$.

Definition 2.5. Let κ be a fixed number. We will say that $g(x) \in D_{\kappa}(\Omega)$ if

- satisfy conditions
 - 1) $g(x) \in C^p(\overline{\Omega})$, $\frac{\partial^{p+1}g(x)}{\partial x_1^{p_1} \dots \partial x_n^{p_n}} \in L_2(\Omega)$, $p+1 = p_1 + p_2 + \dots + p_n$, $p \geq 1$
 - 2) $g(x)|_{\partial\Omega} = \Delta g(x)|_{\partial\Omega} = \dots = \Delta^{[\frac{p}{2}]}g(x)|_{\partial\Omega} = 0$
with the exponent $p = [\frac{d}{2}]$
- $\left\| A^{\frac{1}{2}([\frac{d}{2}]+1+\sigma)}g \right\|_{L_2(\Omega)} \leq \kappa$.

where $A := -\Delta$.

Lemma 2.5. ([28], p.189) Suppose $b \geq 0$, $\alpha > 0$, $\gamma > 0$, $\alpha + \gamma > 1$ and $a(t)$ nonnegative function locally integrable on $0 \leq t < T$ and suppose $t^{\gamma-1}u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\alpha-1} s^{\gamma-1} u(s) ds$$

a.e. in $(0, T)$; then

$$u(t) \leq a(t) Z_{\alpha, \gamma} \left((b\Gamma(\alpha))^{\frac{1}{\alpha+\gamma-1}} t \right),$$

where $Z_{\alpha, \gamma}(t) = \sum_{m=0}^{\infty} c_m t^{m(\alpha+\gamma-1)}$, $c_0 = 1$, $\frac{c_{m+1}}{c_m} = \frac{\Gamma(m(\alpha+\gamma-1)+\gamma)}{\Gamma(m(\alpha+\gamma-1)+\alpha+\gamma)}$ for $m \geq 0$. As $t \rightarrow +\infty$

$$Z_{\alpha, \gamma}(t) = O \left(t^{\frac{1}{2} \frac{\alpha+\gamma-1}{\alpha} - \gamma} \exp \left(\frac{\alpha + \gamma - 1}{\alpha} t^{\frac{\alpha+\gamma-1}{\alpha}} \right) \right).$$

Theorem 2.1. [29] Let (X, d) be a complete metric space, and assume $A : X \rightarrow X$. Furthermore, assume there exists an $a \in X$ and an $r > 0$ such that

(i) The ball $B(a, r) = \{x \in X : d(x, a) < r\}$ is an invariant set for A , i.e.

$$A : B(a, r) \rightarrow B(a, r)$$

(ii) the map A is a contraction on $B(a, r)$, t.e. there exists a constant $\gamma \in (0, 1)$ such that.

$$d(A(x), A(y)) \leq \gamma d(x, y)$$

for all $x, y \in B(a, r)$.

Then there exists a $x_0 \in B(a, r)$ such that $A(x_0) = x_0$ and, furthermore, this is the unique fixed point of A inside the ball $B(a, r)$.

3. INVESTIGATION OF DIRECT PROBLEM (1.1)–(1.3)

We investigate the solution of the direct problem (1.1)–(1.3) using the Fourier method. By applying the Fourier method, the solution $u(x, t)$ of the problem (1.1)–(1.3) can be expanded in a uniformly convergent series in term of eigenfunctions of the form

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) e_k(x), \quad (3.1)$$

where $u_k(t)$ are the unknowns and satisfy the following fractional differential equation

$$\left(D_{0+,t}^{\alpha_0, \beta_0} u_k \right) (t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} u_k \right) (t) + \lambda_k^{\frac{\sigma}{2}} u_k(t) + q(t) u_k(t) = f_k(t), \quad (3.2)$$

where λ_k and $\{e_n(x)\}_{n \geq 1}$ are eigenvalues and eigenfunctions of the classical Laplace operator $-\Delta$ defined in equation (2.1) and $(f(x, t), e_k(x))_{L_2(\Omega)} =: f_k(t)$.

From the initial conditions (1.2) and the equality (3.1) we obtain the following initial conditions:

$$I_{0+,t}^{(1-\alpha_j)(1-\beta_j)} u_k(t) \Big|_{t=0} = \varphi_{jk}, \quad j = 0, 1, \dots, m, \quad (3.3)$$

where $\varphi_{jk} = (\varphi_j, e_k)_{L_2(\Omega)}$.

Let us investigate the Cauchy problem (3.2) - (3.3). Applying the Laplace transform method to both sides of equation (3.2), we obtain:

$$\begin{aligned} & s^{\alpha_0} \mathcal{L}[u_k(t)](s) - s^{\beta_0(\alpha_0-1)} \left(I_{0+}^{(1-\beta_0)(1-\alpha_0)} u_k \right) (0+) + \sum_{j=1}^m \mu_j s^{\alpha_j} \mathcal{L}[u_k(t)](s) \\ & - \sum_{j=1}^m \mu_j s^{\beta_j(\alpha_j-1)} \left(I_{0+}^{(1-\beta_j)(1-\alpha_j)} u_k \right) (0+) + \lambda_k^{\frac{\sigma}{2}} \mathcal{L}[u_k(t)](s) = \mathcal{L}[f_k(t) - q(t)u_k(t)](s). \end{aligned}$$

Using the above equality and conditions (3.3), we find $\mathcal{L}[u_k(t)](s)$:

$$\begin{aligned} \mathcal{L}[u_k(t)](s) &= \frac{\varphi_{0k} s^{\beta_0(\alpha_0-1)}}{s^{\alpha_0} + \sum_{j=1}^m \mu_j s^{\alpha_j} + \lambda_k^{\frac{\sigma}{2}}} + \frac{\sum_{j=1}^m \varphi_{jk} \mu_j s^{\beta_j(\alpha_j-1)}}{s^{\alpha_0} + \sum_{j=1}^m \mu_j s^{\alpha_j} + \lambda_k^{\frac{\sigma}{2}}} \\ &+ \frac{1}{s^{\alpha_0} + \sum_{j=1}^m \mu_j s^{\alpha_j} + \lambda_k^{\frac{\sigma}{2}}} \mathcal{L}[f_k(t) - q(t)u_k(t)](s). \end{aligned}$$

According to Lemma 2.2 and the method of applying the inverse Laplace transform, we have:

$$\begin{aligned} u_k(t) &= \varphi_{0k} t^{(1-\alpha_0)(\beta_0-1)} E_{\alpha', 1-(1-\alpha_0)(1-\beta_0)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &+ \sum_{j=1}^m \mu_j \varphi_{jk} t^{\alpha_0-1+\beta_j(1-\alpha_j)} E_{\alpha', \alpha_0+\beta_j(1-\alpha_j)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &+ \int_0^t f_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau \\ &- \int_0^t q(\tau) u_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau, \quad (3.4) \end{aligned}$$

where $\alpha' = (\alpha_0, \alpha_0 - \alpha_1, \dots, \alpha_0 - \alpha_m)$.

Lemma 3.1. Let $q(t) \in C[0, T]$, $f_n(t) \in C[0, T]$. For any $n \in \mathbb{N}$ and $t \in [0, T]$, we have

$$\begin{aligned} t^\gamma |u_k(t)| &\leq \left(C |\varphi_{0k}| t^{\gamma+(1-\alpha_0)(\beta_0-1)} + C \sum_{j=1}^m \mu_j |\varphi_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \right. \\ &\quad \left. + C \|f_k\|_{C_\gamma[0, T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) \right) \\ &\quad \times Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0, T]} t^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} t \right), \quad t \in [0, T]; \quad (3.5) \end{aligned}$$

$$\begin{aligned} & \left| t^\gamma \left(D_{0+, t}^{\alpha_0, \beta_0} u_k \right) (t) \right| + \left| t^\gamma \sum_{j=1}^m \mu_j \left(D_{0+, t}^{\alpha_j, \beta_j} u_k \right) (t) \right| \leq \|f_k\|_{C_\gamma[0, T]} \\ &+ \left(\lambda_k^{\frac{\sigma}{2}} + \|q\|_{C[0, T]} \right) \left(C |\varphi_{0k}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \right. \\ &+ C \sum_{j=1}^m \mu_j |\varphi_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + C \|f_k\|_{C_\gamma[0, T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) \Big) \\ &\quad \times Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0, T]} t^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} t \right), \quad t \in [0, T], \quad (3.6) \end{aligned}$$

where C is a constant depending on α_j , β_j , $j = 0, 1, \dots, m$, σ .

Proof: Multiplying equation (3.4) by t^γ and using Lemma 2.3, then we can estimate $t^\gamma |u_k(t)|$ as follows:

$$\begin{aligned} t^\gamma |u_k(t)| &\leq C |\varphi_{0k}| t^{\gamma+(1-\alpha_0)(\beta_0-1)} + C \sum_{j=1}^m \mu_j |\varphi_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \\ &\quad + C \|f_k\|_{C_\gamma[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) + C \|q\|_{C[0,T]} \int_0^t (t-\tau)^{\alpha_0-1} |u_k(\tau)| d\tau. \end{aligned}$$

Then, according to lemma 2.5, we arrive at the first estimate of Lemma 3.1.

Next, let's take the second estimate of Lemma 3.1. For this, we obtain the following from equation (3.2):

$$\left(D_{0+,t}^{\alpha_0,\beta_0} u_k \right) (t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j,\beta_j} u_k \right) (t) = f_k(t) - \lambda_k^{\frac{\sigma}{2}} u_k(t) - q(t) u_k(t). \quad (3.7)$$

Multiply equality (13) by t^γ on both sides and we get estimate

$$\begin{aligned} &\left| t^\gamma \left(D_{0+,t}^{\alpha_0,\beta_0} u_k \right) (t) + t^\gamma \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j,\beta_j} u_k \right) (t) \right| \\ &\leq \|f_k\|_{C_\gamma[0,T]} + \left(\lambda_k^{\frac{\sigma}{2}} + \|q\|_{C[0,T]} \right) t^\gamma |u_k(t)|. \end{aligned}$$

According to the last inequality and (3.5) we obtain the second estimate of Lemma 3.1. Lemma 3.1 is proven. $\square$

The functions $\varphi_j(x)$, $j = 0, 1, \dots, m$, $w(x)$, $f(x, t)$ and $h(t)$ satisfy the following assumptions

A1) $\varphi_j(x) \in D_\kappa(\Omega)$, $j = 0, 1, \dots, m$;

A2) $f(x, t) \in C_\gamma(D_\kappa(\Omega), [0, T])$;

A3) $w(x) \in D_\kappa(\Omega)$;

A4) $D_{0+,t}^{\alpha,\beta} h(t) \in C_\gamma[0, T]$, $D_{0+,t}^{\alpha_j,\beta_j} h(t) \in C_\gamma[0, T]$ and $|t^\gamma h(t)| \geq h_{00} > 0$, h_{00} is a given number,

$$\int_\Omega w(x) \varphi_j(x) dx = I_{0+,t}^{(1-\alpha_j)(1-\beta_j)} h(t)|_{t=0}, \quad j = 0, 1, \dots, m, \quad m \in \mathbb{N}.$$

For the solution of the direct problem (1.1)–(1.3), the following theorem holds.

Theorem 3.1. *Let $q(t) \in C[0, T]$ and conditions A1) -A2) be satisfied. Then there exists a unique solution of the direct problem (1.1)–(1.3), $u(x, t) \in C_\gamma^{\alpha,\beta,\sigma}(\overline{Q}_T)$, where $\overline{Q}_T = \overline{\Omega} \times [0, T]$.*

Proof: First, we prove that the solution $u(x, t)$, given by formula (3.1), is a classical solution, that is $t^\gamma u(x, t) \in C(\overline{Q}_T)$, $t^\gamma (-\Delta)^{\frac{\sigma}{2}} u(\cdot, t) \in C(\overline{Q}_T)$ and $t^\gamma \left(D_{0+,t}^{\alpha_0,\beta_0} + \sum_{i=1}^m \mu_i D_{0+,t}^{\alpha_i,\beta_i} \right) u(x, t) \in C(\overline{Q}_T)$, $j = 0, 1, \dots, m$ represent continuous functions.

Formally, by term-by-term differentiation of the functional series (3.1), we construct the following series

$$(-\Delta)^{\frac{\sigma}{2}} u(x, t) = \sum_{k=1}^{\infty} \lambda_k^{\frac{\sigma}{2}} u_k(t) e_k(x), \quad (3.8)$$

$$\begin{aligned} &\left(D_{0+,t}^{\alpha_0,\beta_0} u \right) (x, t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j,\beta_j} u \right) (x, t) \\ &= \sum_{k=1}^{\infty} \left(D_{0+,t}^{\alpha_0,\beta_0} u_k \right) (t) e_k(x) + \sum_{j=1}^m \mu_j \sum_{k=1}^{\infty} \left(D_{0+,t}^{\alpha_j,\beta_j} u_k \right) (t) e_k(x). \end{aligned} \quad (3.9)$$

Let us prove the uniform convergence of the series (3.1), (3.8), and (3.9) in the domain $\overline{Q}_T$. To do this show the uniform convergence of series (3.1), (3.8), and (3.9) multiplied by t^γ in this domain. These series, for any $(x, t) \in \overline{Q}_T$, are respectively majorized by the following series

$$\begin{aligned} & \sum_{k=1}^{\infty} |e_k(x)| \left(|\varphi_{0k}| C_\epsilon \lambda_k^{-\frac{\sigma}{2} \frac{\gamma+(1-\alpha_0)(\beta_0-1)}{\alpha_0}} + C \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \right. \\ & \left. + C \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \right) Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right), \end{aligned} \quad (3.10)$$

$$\begin{aligned} & \sum_{k=1}^{\infty} |e_k(x)| \lambda_k^{\frac{\sigma}{2}} \left(|\varphi_{0k}| C_\epsilon \lambda_k^{-\frac{\sigma}{2} \frac{\gamma+(1-\alpha_0)(\beta_0-1)}{\alpha_0}} + C \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \right. \\ & \left. + C \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \right) Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right), \end{aligned} \quad (3.11)$$

$$\begin{aligned} & \sum_{k=1}^{\infty} |e_k(x)| \left[\|f_k\|_{C_\gamma[0,T]} + \left(\lambda_k^{\frac{\sigma}{2}} + \|q\|_{C[0,T]} \right) \left(|\varphi_{0k}| C_\epsilon \lambda_k^{-\frac{\sigma}{2} \frac{\gamma+(1-\alpha_0)(\beta_0-1)}{\alpha_0}} \right. \right. \\ & \left. \left. + C \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + C \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \right) \right] \\ & \times Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right). \end{aligned} \quad (3.12)$$

It is not difficult to notice that the sums of the series (3.10), (3.11), and (3.12) do not exceed the following expression:

$$\sum_{k=1}^{\infty} \lambda_k^{\frac{\sigma}{2}} \left(|\varphi_{0k}| + \sum_{j=1}^m \mu_j |\varphi_{jk}| + \|f_k\|_{C_\gamma[0,T]} \right) |e_k(x)|. \quad (3.13)$$

Applying the Cauchy-Bunyakovsky inequality we get

$$\begin{aligned} & \sum_{k=1}^{\infty} \lambda_k^{\frac{\sigma}{2}} \left(|\varphi_{0k}| + \sum_{j=1}^m \mu_j |\varphi_{jk}| + \|f_k\|_{C_\gamma[0,T]} \right) |e_k(x)| \\ & = \sum_{k=1}^{\infty} |\varphi_{0k}| \left(\sqrt{\lambda_k} \right)^{\left(\left[\frac{d}{2}\right]+1+\sigma\right)} \left(\sqrt{\lambda_k} \right)^{-\left(\left[\frac{d}{2}\right]+1\right)} |e_k(x)| \\ & + \sum_{j=1}^m \mu_j \sum_{k=1}^{\infty} |\varphi_{jk}| \left(\sqrt{\lambda_k} \right)^{\left(\left[\frac{d}{2}\right]+1+\sigma\right)} \left(\sqrt{\lambda_k} \right)^{-\left(\left[\frac{d}{2}\right]+1\right)} |e_k(x)| \\ & + \sum_{k=1}^{\infty} \|f_k\|_{C_\gamma[0,T]} \left(\sqrt{\lambda_k} \right)^{\left(\left[\frac{d}{2}\right]+1+\sigma\right)} \left(\sqrt{\lambda_k} \right)^{-\left(\left[\frac{d}{2}\right]+1\right)} |e_k(x)| \\ & \leq \left(\sum_{k=1}^{\infty} |\varphi_{0k}| \lambda_k^{\left(\left[\frac{d}{2}\right]+1+\sigma\right)} \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \lambda_k^{-\left(\left[\frac{d}{2}\right]+1\right)} |e_k^2(x)| \right)^{\frac{1}{2}} \\ & + \sum_{j=1}^m \mu_j \left(\sum_{k=1}^{\infty} |\varphi_{jk}| \lambda_k^{\left(\left[\frac{d}{2}\right]+1+\sigma\right)} \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \lambda_k^{-\left(\left[\frac{d}{2}\right]+1\right)} |e_k^2(x)| \right)^{\frac{1}{2}} \\ & + \left(\sum_{k=1}^{\infty} \|f_k\|_{C_\gamma[0,T]} \lambda_k^{\left(\left[\frac{d}{2}\right]+1+\sigma\right)} \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \lambda_k^{-\left(\left[\frac{d}{2}\right]+1\right)} |e_k^2(x)| \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq \Theta \left(\|\varphi_0\|_{D\left(A\left(\left[\frac{d}{2}\right]+1+\sigma\right)\right)} + \sum_{j=1}^m \mu_j \|\varphi_j\|_{D\left(A\left(\left[\frac{d}{2}\right]+1+\sigma\right)\right)} + \|f\|_{C_\gamma\left(D\left(A\left(\left[\frac{d}{2}\right]+1+\sigma\right)\right), [0, T]\right)} \right).$$

Furthermore, according to Lemma 2.4 the series $\Theta = \left(\sum_{k=1}^{\infty} \lambda_k^{-\left(\left[\frac{d}{2}\right]+1\right)} |e_k^2(x)| \right)^{\frac{1}{2}}$ converges uniformly in a closed domain $\overline{\Omega}$.

If the functions $\varphi_j(x)$, $j = 0, 1, \dots, m$ and $f(x, t)$ satisfy the conditions A1), A2), then, the series (3.1), (3.8), and (3.9) are absolutely and uniformly convergent in $\overline{Q}_T$. $\square$

INVESTIGATION OF THE INVERSE PROBLEM

Let us now investigate the inverse problem (1.1)-(1.4).

Multiplying equation (1.1) by the function $w(x)$ on both sides, we integrate in the domain Ω with respect to the variable x .

$$\begin{aligned} & \int_{\Omega} w(x) \left\{ \left(D_{0+,t}^{\alpha_0, \beta_0} u \right) (x, t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} u \right) (x, t) \right. \\ & \left. + (-\Delta)^{\frac{\sigma}{2}} u(x, t) + q(t) u(x, t) \right\} dx = \int_{\Omega} w(x) f(x, t) dx. \end{aligned}$$

Using condition (1.4), we get

$$\begin{aligned} & \left(D_{0+,t}^{\alpha_0, \beta_0} h \right) (t) + \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) + \int_{\Omega} w(x) (-\Delta)^{\frac{\sigma}{2}} u(x, t) dx \\ & + q(t) h(t) = \int_{\Omega} w(x) f(x, t) dx. \end{aligned}$$

From the above relation, we obtain the following equation for $q(t)$

$$\begin{aligned} q(t) = & \frac{1}{h(t)} \left(\int_{\Omega} w(x) f(x, t) dx - \left(D_{0+,t}^{\alpha_0, \beta_0} h \right) (t) - \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) \right) \\ & - \frac{1}{h(t)} \int_{\Omega} w(x) (-\Delta)^{\frac{\sigma}{2}} u(x, t) dx. \end{aligned} \quad (3.14)$$

According to definition 2.3, we can rewrite the integral $\int_{\Omega} w(x) (-\Delta)^{\frac{\sigma}{2}} u(x, t) dx$ as follows

$$\int_{\Omega} w(x) (-\Delta)^{\frac{\sigma}{2}} u(x, t) dx = \sum_{k=1}^{\infty} \lambda_k^{\frac{\sigma}{2}} u_k(t) \int_{\Omega} w(x) e_k(x) dx.$$

Using the above equality, we write (3.13) for $t \in (0, T]$ as follows

$$\begin{aligned} q(t) = & \frac{1}{h(t)} \left(\int_{\Omega} w(x) f(x, t) dx - \left(D_{0+,t}^{\alpha_0, \beta_0} h \right) (t) - \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) \right) \\ & - \frac{1}{h(t)} \sum_{k=1}^{\infty} \lambda_k^{\frac{\sigma}{2}} u_k(t) w_k. \end{aligned}$$

where $w_n = \int_{\Omega} w(x) e_n(x) dx$.

The functions $u_n(t)$ depend on $q(t)$, i.e. $u_n(t; q)$. After simple conversion, for $t \in [0, T]$ we get the following integral equation for determining $q(t)$:

$$q(t) = q_0(t) - \frac{1}{h_0(t)} \sum_{k=1}^{\infty} \lambda_k^{\frac{\alpha}{2}} t^{\gamma} u_k(t) w_k, \quad (3.15)$$

where $h_0(t) = t^{\gamma} h(t)$,

$$q_0(t) = \frac{1}{h_0(t)} \left(\int_{\Omega} w(x) t^{\gamma} f(x, t) dx - t^{\gamma} \left(D_{0+,t}^{\alpha_0, \beta_0} h \right) (t) - t^{\gamma} \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) \right).$$

We introduce an operator $\mathcal{G}$ defining it by the right hand side of (3.15):

$$\mathcal{G}[q](t) = q_0(t) + \frac{1}{h_0(t)} \sum_{n=1}^{\infty} w_n t^{\gamma} u_n(t; q). \quad (3.16)$$

Then, the equation (3.16) is written in more convenient form as

$$\mathcal{G}[q](t) = q(t). \quad (3.17)$$

Let

$$\begin{aligned} \|q_0\|_{C[0,T]} &:= \max_{t \in [0,T]} |q_0(t)| = \left\| \frac{1}{h_0(t)} \left(\int_{\Omega} w(x) t^{\gamma} f(x, t) dx \right. \right. \\ &\quad \left. \left. - t^{\gamma} \left(D_{0+,t}^{\alpha_0, \beta_0} h \right) (t) - t^{\gamma} \sum_{j=1}^m \mu_j \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) \right) \right\|_{C[0,T]} \\ &\leq \frac{1}{h_{00}} \left(w_0 \|f\|_{C_{\gamma}(D_{\kappa}(\Omega), [0,T])} + \|h\|_{C_{\gamma}^{\alpha, \beta}[0,T]} \sum_{j=1}^m \mu_j \right), \end{aligned}$$

where $w_0 = \max_{x \in [0,l]} |w(x)|$, $|h_0(t)| = |t^{\gamma} h(t)| \geq h_{00} > 0$, h_{00} is a given number,

$$\|h\|_{C_{\gamma}^{\alpha, \beta}[0,T]} := \max_{0 \leq j \leq m} \left\{ \max_{t \in [0,T]} \left| \left(D_{0+,t}^{\alpha_j, \beta_j} h \right) (t) \right| \right\}.$$

Fix a number $r > 0$ and consider the ball

$$B_r(q_0) := \{q(t) : q(t) \in C[0, T], \|q - q_0\|_{C[0,T]} \leq r\}.$$

Theorem 3.2. *Let A1)-A3) be satisfied. Then there exists a number $T^* \in (0, T)$, and $r > 0$, such that there exists a unique solution $q(t) \in C[0, T^*]$ of the inverse problem (1.1)-(1.4), $q(t) \in B_r(q_0)$.*

Proof: Let us first prove that for an enough small $T > 0$ the operator $\mathcal{G}$ maps the ball $B_r(q_0)$ implies that $\mathcal{G}[q](t) \in B_r(q_0)$. Indeed, for any continuous function $q(t)$, the function $\mathcal{G}[q](t)$ calculated using formula (3.15) will be continuous. Moreover, estimating the norm of the differences, we find that

$$\begin{aligned} \|\mathcal{G}[q](t) - q_0(t)\| &\leq \frac{w_0}{h_{00}} \sum_{k=1}^{\infty} \left(C |\varphi_{0k}| T^{\gamma + (1 - \alpha_0)(\beta_0 - 1)} \right. \\ &\quad \left. + C \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma + \alpha_0 - 1 + \beta_j(1 - \alpha_j)} + C \|f_k\|_{C_{\gamma}[0,T]} T^{\alpha_0} B(\alpha_0, 1 - \gamma) \right) \end{aligned}$$

$$\times Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0,T]} t^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} t \right).$$

Here we have used the estimate (3.5). In view of Lemmas 2.4 and conditions A1)-A3) last series is convergent series. Note that the function occurring on the right-hand side in this inequality is monotone increasing with T , and the fact that the function $q(t)$ belongs to the ball $B_r(q_0)$ implies the inequality $\|q\|_{C[0,T]} \leq \|q_0\|_{C[0,T]} + r$.

Therefore, we only strengthen the inequality if we replace $\|q\|_{C[0,T]}$ in this inequality with the expression $\|q_0\|_{C[0,T]} + r$. Performing these replacements, we obtain the estimate

$$\begin{aligned} \|\mathcal{G}[q](t) - q_0(t)\| &\leq \frac{w_0}{h_{00}} \sum_{k=1}^{\infty} \left(C |\varphi_{0k}| T^{\gamma+(1-\alpha_0)(\beta_0-1)} \right. \\ &\quad \left. + C \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + C \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \right) \\ &\quad \times Z_{\alpha, 1-\gamma} \left((C (\|q_0\|_{C[0,T]} + r) T^\gamma \Gamma(\alpha))^{\frac{1}{\alpha-\gamma}} T \right) =: m_1(T). \end{aligned} \quad (3.18)$$

Before applying the second condition of the Banach fixed-point theorem, we need the estimate for the difference between the solution of the original integral equation (3.4) and the solution of this equation with perturbed functions $\tilde{q}$, $\tilde{\varphi}_{jk}$, $j = 0, 1, 2, \dots, m$ and $\tilde{f}_k(t)$. Let $\tilde{u}_k(t)$ be solution of the integral equation (3.4) corresponding to the functions $\tilde{\varphi}_{jk}$, $j = 0, 1, 2, \dots, m$, $\tilde{f}_k(t)$ i.e.,

$$\begin{aligned} \tilde{u}_k(t) &= \tilde{\varphi}_{0k} t^{(1-\alpha_0)(\beta_0-1)} E_{\alpha', 1-(1-\alpha_0)(1-\beta_0)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &\quad + \sum_{j=1}^m \mu_j \tilde{\varphi}_{jk} t^{\alpha_0-1+\beta_j(1-\alpha_j)} E_{\alpha', \alpha_0+\beta_j(1-\alpha_j)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &\quad + \int_0^t \tilde{f}_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau \\ &\quad - \int_0^t \tilde{q}(\tau) \tilde{u}_k(\tau) (t-\tau)^{\alpha_0-1} \\ &\quad \times E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau. \end{aligned} \quad (3.19)$$

Composing the difference $u_n(t) - \tilde{u}_n(t)$ with the help of the equations (3.4), (3.19) and introducing the notations $\hat{u}_n(t) = u_n(t) - \tilde{u}_n(t)$, $\hat{q} = q(t) - \tilde{q}(t)$.

The following lemma holds for $|\hat{u}_n(t)|$:

$$\begin{aligned} \hat{u}_k(t) &= \hat{\varphi}_{0k} t^{(1-\alpha_0)(\beta_0-1)} E_{\alpha', 1-(1-\alpha_0)(1-\beta_0)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &\quad + \sum_{j=1}^m \mu_j \hat{\varphi}_{jk} t^{\alpha_0-1+\beta_j(1-\alpha_j)} E_{\alpha', \alpha_0+\beta_j(1-\alpha_j)} \left(-\lambda_k^{\frac{\sigma}{2}} t^{\alpha_0}, -\mu_1 t^{\alpha_0-\alpha_1}, \dots, -\mu_m t^{\alpha_0-\alpha_m} \right) \\ &\quad + \int_0^t \hat{f}_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau \\ &\quad - \int_0^t \hat{q}(\tau) u_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau \\ &\quad - \int_0^t \hat{q}(\tau) \hat{u}_k(\tau) (t-\tau)^{\alpha_0-1} E_{\alpha', \alpha_0} \left(-\lambda_k^{\frac{\sigma}{2}} (t-\tau)^{\alpha_0}, -\mu_1 (t-\tau)^{\alpha_0-\alpha_1}, \dots, -\mu_m (t-\tau)^{\alpha_0-\alpha_m} \right) d\tau. \end{aligned} \quad (3.20)$$

Lemma 3.2. Let $q(t), \tilde{q}(t) \in C[0, T]$, $f_n(t) \in C[0, T]$. For any $n \in \mathbb{N}$ and $t \in [0, T]$ we have the estimates:

$$\begin{aligned} t^\gamma |\hat{u}_k(t)| &\leq C \left(|\hat{\varphi}_{0k}| t^{\gamma+(1-\alpha_0)(\beta_0-1)} + \sum_{j=1}^m \mu_j |\hat{\varphi}_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \right. \\ &\quad + \|\hat{f}_k\|_{C_\gamma[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) + \|\hat{q}\|_{C[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) \left(|\varphi_{0k}| t^{\gamma+(1-\alpha_0)(\beta_0-1)} \right. \\ &\quad \left. \left. + \sum_{j=1}^m \mu_j |\varphi_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + \|f_k\|_{C_\gamma[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) \right) \right) \\ &\quad \times Z_{\alpha_0, 1-\gamma} \left((C\|q\|_{C[0,T]} t^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} t \right) Z_{\alpha_0, 1-\gamma} \left((C\|\tilde{q}\|_{C[0,T]} t^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} t \right). \end{aligned} \quad (3.21)$$

Proof: To prove this, we multiply the integral equation (3.20) by t^γ on both sides, then estimate

$$\begin{aligned} t^\gamma |\hat{u}_k(t)| &\leq C |\hat{\varphi}_{0k}| t^{\gamma+(1-\alpha_0)(\beta_0-1)} + C \sum_{j=1}^m \mu_j |\hat{\varphi}_{jk}| t^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} \\ &\quad + C \|\hat{f}_k\|_{C_\gamma[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) + C \|\hat{q}\|_{C[0,T]} \|u_k\|_{C_\gamma[0,T]} t^{\alpha_0} B(\alpha_0, 1-\gamma) \\ &\quad + C \|\tilde{q}\|_{C[0,T]} \int_0^t (t-\tau)^{\alpha_0-1} |\hat{u}_k(\tau)| d\tau. \end{aligned}$$

Using lemma 2.5 and estimate (3.5) in the above inequality, we get the estimate in lemma 3.1. Then according to lemma 2.5, we get the estimate in lemma 3.1. $\square$

Now let us take any functions $q(t), \tilde{q}(t) \in B_r(q_0)$ and estimate the distance between their images $\mathcal{G}[q](t)$ and $\mathcal{G}[\tilde{q}](t)$ in the space $C[0, T]$. Composing the difference $\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)$ with the help of equations (3.4), (3.20) and then estimating its norm, we obtain

$$\|\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)\| \leq \frac{w_0}{h_{00}} \sum_{k=1}^{\infty} \|(t^\gamma u_k(t, q) - t^\gamma \tilde{u}_k(t, \tilde{q}))\|.$$

In the above inequality, using estimate (3.21), we obtain the following inequality

$$\begin{aligned} \|\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)\| &\leq \frac{Cw_0}{h_{00}} \sum_{k=1}^{\infty} \left(|\hat{\varphi}_{0k}| T^{\gamma+(1-\alpha_0)(\beta_0-1)} \right. \\ &\quad + \sum_{j=1}^m \mu_j |\hat{\varphi}_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + \|\hat{f}_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \\ &\quad + \|\hat{q}\|_{C[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \left(|\varphi_{0k}| T^{\gamma+(1-\alpha_0)(\beta_0-1)} \right. \\ &\quad \left. \left. + \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \right) \right) \\ &\quad \times Z_{\alpha_0, 1-\gamma} \left((C\|q\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right) Z_{\alpha_0, 1-\gamma} \left((C\|\tilde{q}\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right). \end{aligned} \quad (3.22)$$

The function $\tilde{u}_n(t)$ corresponding to $\tilde{q}(t)$ satisfies the integral equation (3.20) with $\varphi_{jk} = \tilde{\varphi}_{jk}$, $j = 0, 1, 2, \dots, m$, $f_k(t) = \tilde{f}_k(t)$. Considering $\varphi_{jk} = \tilde{\varphi}_{jk}$, $j = 0, 1, 2, \dots, m$, $f_k(t) = \tilde{f}_k(t)$ in inequality (3.22), we obtain the following result.

$$\|\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)\| \leq \frac{CT^{\alpha_0} B(\alpha_0, 1-\gamma) w_0}{h_{00}} \sum_{k=1}^{\infty} \left(|\varphi_{0k}| T^{\gamma+(1-\alpha_0)(\beta_0-1)} \right.$$

$$\begin{aligned}
& + \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \Big) \\
& \quad \times Z_{\alpha_0, 1-\gamma} \left((C \|q\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right) \\
& \quad \times Z_{\alpha_0, 1-\gamma} \left((C \|\tilde{q}\|_{C[0,T]} T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right) \|\tilde{q}\|_{C[0,T]}. \tag{3.23}
\end{aligned}$$

The functions $q(t)$ and $\tilde{q}(t)$ belong to the ball $B_r(q_0)$, and hence for each of these functions one has inequality $\|q\|_{C[0,T]} \leq \|q_0\|_{C[0,T]} + r$. Note that the function on the right-hand side in inequality (3.23) at the factor $\|q - \tilde{q}\|_{C[0,T]}$ is monotone increasing with $\|q\|_{C[0,T]}$, $\|\tilde{q}\|_{C[0,T]}$ and T . Consequently, replacing $\|q\|_{C[0,T]}$ and $\|\tilde{q}\|_{C[0,T]}$ in inequality (3.23) with $\|q_0\|_{C[0,T]} + r$ will only strengthen the inequality. Thus, we have

$$\begin{aligned}
\|\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)\| & \leq \frac{CT^{\alpha_0} B(\alpha_0, 1-\gamma) w_0}{h_{00}} \sum_{k=1}^{\infty} \left(|\varphi_{0k}| T^{\gamma+(1-\alpha_0)(\beta_0-1)} \right. \\
& \quad + \sum_{j=1}^m \mu_j |\varphi_{jk}| T^{\gamma+\alpha_0-1+\beta_j(1-\alpha_j)} + \|f_k\|_{C_\gamma[0,T]} T^{\alpha_0} B(\alpha_0, 1-\gamma) \Big) \\
& \quad \times Z_{\alpha_0, 1-\gamma} \left((C (\|q_0\|_{C[0,T]} + r) T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right) \\
& \quad \times Z_{\alpha_0, 1-\gamma} \left((C (\|q_0\|_{C[0,T]} + r) T^\gamma \Gamma(\alpha_0))^{\frac{1}{\alpha_0-\gamma}} T \right) \|\tilde{q}\|_{C[0,T]}. \tag{3.24}
\end{aligned}$$

In view of these facts, we obtain

$$\begin{aligned}
\|\mathcal{G}[q](t) - q_0(t)\| & \leq \mu_1(T), \\
\|\mathcal{G}[q](t) - \mathcal{G}[\tilde{q}](t)\| & \leq \mu_2(T) \|\tilde{q}\|_{C[0,T]},
\end{aligned}$$

where $\mu_1(T)$ denotes the right-hand side of inequality (3.18) and $\mu_2(T)$ is the multiplier at $\|q - \tilde{q}\|_{C[0,T]} = \|\tilde{q}\|_{C[0,T]}$ on the right-hand side of (3.24) with the only difference that $q(t)$, $\tilde{q}(t)$ are replaced by $\|q_0\|_{C[0,T]} + r$ in both cases.

Note that $\mu_i(T)$, $i = 1, 2$, are positive monotonically increasing functions of T , and $\mu_1(0) = 0$, $\mu_2(0) = 0$. It follows that the equations $\mu_1(T) = r$ and $\mu_2(T) = 1$ have unique positive roots. Denote these roots by T_1 and T_2 , respectively. Then, it is clear that if we choose $T^* < \min(T_1, T_2)$, then the operator $\mathcal{G}$ is a contraction in the ball $B_r(q_0)$. However, in accordance with the Banach theorem 2.1, the operator $\mathcal{G}$ has a unique fixed point in the ball $B_r(q_0)$; i.e., there exists a unique solution of the equation (3.17). Theorem 2 is proven. $\square$

CONCLUSIONS

An inverse problem of determining the time-dependent coefficient for a space-time fractional diffusion equation, with multi-term Hilfer fractional derivatives in time of orders $0 < \alpha_n < \dots < \alpha_2 < \alpha_1 < \alpha_0 < 1$ and types $0 \leq \beta_j \leq 1$ and fractional Laplacian operator in space variable of order $1 < \sigma \leq 2$ with homogeneous boundary condition is investigated. Initially, the initial boundary value problem was investigated. At the same time, the spectral problem related to the fractional Laplacian in a spatial variable was studied. The Cauchy problem for the multi-term fractional order ordinary differential equations obtained by eigenfunction, was investigated using the Laplace transform method. A series representation involving multinomial Mittag-Leffler functions is obtained for $u(x, t)$. The functional series solution $u(x, t)$ is proved to be classical solution. The solution of the inverse problem is proved to be locally well-posed. For this, the inverse problem is reduced to an equivalent integral equation of Volterra type. The local existence and global uniqueness of the solution of the inverse problem were proven by the Banach theorem of fixed points.

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On the boundary value problem for the fourth-order partial differential equation

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Abstract. This paper investigates a boundary value problem for a fourth-order partial differential equation posed in a rectangular domain. The analysis focuses on determining the conditions on the domain's dimensions that guarantee the existence and uniqueness of the solution. The results provide constraints under which the problem is well-posed in the classical sense.

Keywords: boundary value problem, fourth-order partial differential equation, algebraic number, Fourier series

MSC (2020): 35G15, 11R04

1. INTRODUCTION

In recent years, considerable attention has been devoted to the study of ill-posed problems, including ill-posed boundary value problems for partial differential equations. This interest is driven both by practical applications and by purely theoretical considerations.

From the standpoint of boundary value problem formulation, partial differential equations of classical types and their direct generalizations are the most thoroughly studied ([1], [2]).

In the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : |x| < \pi, 0 < y < h\}$$

we consider the fourth-order partial differential equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = 0, \quad (1.1)$$

with boundary conditions

$$u(x, h) = \psi(x), \quad u_y(x, 0) = u_{yy}(x, 0) = u_{yyy}(x, 0) = 0, \quad (1.2)$$

where $\psi(x)$ is a 2π -periodic and sufficiently smooth function.

Set

$$\alpha = \frac{h}{\pi\sqrt{2}}. \quad (1.3)$$

Boundary value problems with data prescribed on the entire boundary of the domain for partial differential operators are, generally speaking, ill-posed. The question of their solvability is often related to the so-called small denominators problem. A clear example of this is the Dirichlet problem for the wave equation of a vibrating string, which has been extensively studied by both domestic and foreign researchers.

From a mathematical standpoint, the small denominators phenomenon is manifested in the fact that the solutions to the equations of motion, when represented as Fourier series, involve an infinite number of terms with coefficients whose denominators can become arbitrarily small, thereby leading to the divergence of the series.

As first noted by D. Borgin and R. Duffin [3] (see also [1]), the solvability of the Dirichlet problem for the vibrating string equation depends on whether the number (1.3) is rational or not.

It follows from Theorem 1 of [4] that if the number (1.3) is an irrational number, then the L_p -solution of the Dirichlet problem for the vibrating string equation is unique.

In [5] considered the question of the existence of an L_2 -solution of the Dirichlet problem for the wave equation in a rectangle.

In cases where the governing equation involves differences of arbitrary even-order derivatives, related problems have been investigated in [6], [7].

The following statement holds.

Theorem 1.1. *Let the number α , defined by equality (1.3), be an algebraic number of degree $n > 1$. Then, for any function $\psi(x)$ from the Sobolev class $W_2^{n+3}(|x| < \pi)$, the solution of the problem (1.1)-(1.2) exists and unique.*

The proof of Theorem 1.1 is based on finding the solution of the considered problem by the Fourier method.

Let us expand the boundary condition (1.2) into a Fourier series:

$$\psi(x) = \sum_{k=-\infty}^{\infty} \psi_k e^{ikx}.$$

We prove that the function

$$u(x, y) = \sum_{k=-\infty}^{\infty} \frac{\psi_k}{\cosh \frac{kh}{\sqrt{2}} \cos \frac{kh}{\sqrt{2}}} \cosh \frac{ky}{\sqrt{2}} \cos \frac{ky}{\sqrt{2}} e^{ikx} \quad (1.4)$$

is the desired solution. Prior to that, we obtain several estimates.

2. DEPENDENCE OF THE CONVERGENCE OF THE SERIES ON THE DEGREE OF THE ALGEBRAIC NUMBER

Consider the series

$$S = \sum_{k=-\infty}^{\infty} \frac{b_k}{\cos kz}, z \in \mathbb{R}. \quad (2.1)$$

Note that

$$\cos kz = 0 \quad \text{if} \quad kz = \pi m + \frac{\pi}{2},$$

where $m \in \mathbb{Z}$.

Set

$$\gamma_{mk} = \gamma_{mk}(z) = kz - \left(\pi m + \frac{\pi}{2}\right). \quad (2.2)$$

Then

$$kz = \pi \frac{2m+1}{2} + \gamma_{mk}. \quad (2.3)$$

It is admissible to assume that m can always be chosen in such a way that

$$|\gamma_{mk}| \leq \frac{\pi}{2}. \quad (2.4)$$

Set

$$\beta = \frac{z}{\pi}. \quad (2.5)$$

Then from (2.3) we obtain the representation of the variable (2.5)

$$\beta = \frac{2m+1}{2k} + \frac{\gamma_{mk}}{\pi k}. \quad (2.6)$$

Note that if $\beta = z/\pi$ is an algebraic number of degree $n > 1$, then it cannot be rational. Therefore, for any $m \in \mathbb{Z}$ and $k \in \mathbb{Z}$, the quantity

$$\gamma_{mk} \neq 0.$$

In the subsequent analysis, we employ the results of Liouville's theorem. Let us briefly recall the statement of this theorem

Theorem 2.1. (See [8] Chapter 1, §3) *Let β be an algebraic number of degree n . Then, for any fraction p/q , the following estimate*

$$\left| \beta - \frac{p}{q} \right| \geq \frac{c}{q^n}$$

is valid. The constant $c > 0$ does not depend on p and q .

From this theorem, we derive a corollary that is of significant importance for the problem under consideration.

Corollary 2.1. *Let β be an algebraic number of degree n and equality (2.6) holds. Then the following estimate*

$$|\gamma_{mk}| \geq \frac{c}{|k|^{n-1}} \quad (2.7)$$

is valid. The constant $c > 0$ does not depend on m and k .

Proof: Indeed, from (2.6) we get

$$\frac{\gamma_{mk}}{\pi k} = \beta - \frac{2m+1}{2k}.$$

Then, according to Liouville's theorem,

$$|\gamma_{mk}| = \pi k \left| \beta - \frac{2m+1}{2k} \right| \geq \pi k \frac{c}{|2k|^n} = \frac{c_0}{|k|^{n-1}}. \quad (2.8)$$

□

We now return to the estimation of the denominators in series (2.1). Taking into account (2.3), we can write the following

$$\cos kz = \cos \left[\left(\pi m + \frac{\pi}{2} \right) + \gamma_{mk} \right] = (-1)^{m+1} \sin \gamma_{mk}.$$

From estimates (2.4) and (2.8) it follows that, starting from some index k_0 , the inequality holds for all $|k| \geq k_0$

$$|\cos kz| = |\sin \gamma_{mk}| \geq \left| \sin \frac{c_0}{k^{n-1}} \right| \geq \frac{2}{\pi} \frac{c_0}{|k|^{n-1}}. \quad (2.9)$$

Since the condition $0 < |\gamma_{mk}| < \pi/2$ holds, inequality (2.9) is valid for all $k \in \mathbb{Z}$.

From this, we obtain an estimate for the series (2.1)

$$|S| \leq C \sum_{k=-\infty}^{\infty} |b_k| (1 + |k|^2)^{(n-1)/2}.$$

Then

$$|S|^2 \leq C \sum_{k=-\infty}^{\infty} |b_k|^2 (1 + |k|^2)^n \sum_{k=-\infty}^{\infty} \frac{1}{(1 + |k|^2)} \leq C \sum_{k=-\infty}^{\infty} |b_k|^2 (1 + |k|^2)^n. \quad (2.10)$$

Using the obtained estimates, we proceed to the proof of the main result of the paper.

3. ON THE EXISTENCE OF A SOLUTION TO THE PROBLEM

Let us prove the main result of the paper.

Proof: We seek the solution of problem (1.1)-(1.2) in the form

$$u(x, y) = \sum_{k=-\infty}^{\infty} v_k(y) e^{ikx}$$

Then we reduce the following equation

$$v_k^{(4)}(y) + k^4 v_k(y) = 0$$

Set

$$v_k(y) = e^{ay} [A \cos ay + B \sin ay] + e^{-ay} [C \cos ay + D \sin ay]$$

Then

$$v_k'(y) = ae^{ay} [A \cos ay + B \sin ay] + ae^{ay} [-A \sin ay + B \cos ay] -$$

$$\begin{aligned}
& -ae^{-ay}[C \cos ay + D \sin ay] + ae^{-ay}[-C \sin ay + D \cos ay] = \\
& = ae^{ay}[(A + B) \cos ay + (B - A) \sin ay] + ae^{-ay}[(D - C) \cos ay + (-C - D) \sin ay]
\end{aligned}$$

Hence,

$$\begin{aligned}
v_k''(y) &= a^2 e^{ay}[2B \cos ay - 2A \sin ay] + a^2 e^{-ay}[-2D \cos ay + 2C \sin ay] = \\
&= 2a^2 e^{ay}[B \cos ay - A \sin ay] + 2a^2 e^{-ay}[-D \cos ay + C \sin ay]
\end{aligned}$$

and

$$v_k'''(y) = 2a^3 e^{ay}[(B - A) \cos ay - (A + B) \sin ay] + 2a^3 e^{-ay}[(D + C) \cos ay + (D - C) \sin ay]$$

Therefore,

$$v_k^{(4)}(y) = 4a^4 e^{ay}[-A \cos ay - B \sin ay] + 4a^4 e^{-ay}[-C \cos ay - D \sin ay]$$

Consequently,

$$v_k''''(y) = -4a^4 v_k(x)$$

Then

$$u_{xxxx} + u_{yyyy} = k^4 u(x, y) - 4a^4 u(x, y) = 0$$

if

$$a = \frac{k}{\sqrt{2}}$$

According to the conditions (1.2), we get

$$e^{ah}[A \cos(ah) + B \sin(ah)] + e^{-ah}[C \cos(ah) + D \sin(ah)] = \psi_k,$$

$$A + B + D - C = 0,$$

$$B - D = 0,$$

$$A + B + D - C = 0.$$

Hence,

$$A = C = \frac{\psi_k}{2 \cosh(ah) \cos(ah)}, \quad D = B = 0.$$

Consequently, expression (1.4) is a formal solution of problem (1.1)-(1.2).

It remains to prove that the obtained solution and its fourth derivatives with respect to x and y converge uniformly. By direct differentiation of (1.4), we obtain

$$\frac{\partial^4 u(x, y)}{\partial y^4} = \sum_{k=-\infty}^{\infty} \frac{k^4 \psi_k}{\cosh \frac{kh}{\sqrt{2}} \cos \frac{kh}{\sqrt{2}}} \cosh \frac{ky}{\sqrt{2}} \cos \frac{ky}{\sqrt{2}} e^{ikx} \quad (3.1)$$

$$\frac{\partial^4 u(x, y)}{\partial x^4} = \sum_{k=-\infty}^{\infty} \frac{k^4 \psi_k}{\cosh \frac{kh}{\sqrt{2}} \cos \frac{kh}{\sqrt{2}}} \cosh \frac{ky}{\sqrt{2}} \cos \frac{ky}{\sqrt{2}} e^{ikx}.$$

Then the series majorizing series (3.1) has the form

$$\sum_{k=-\infty}^{\infty} \frac{k^4 |\psi_k|}{|\cos \frac{kh}{\sqrt{2}}|}. \quad (3.2)$$

Then, according to Corollary 2.2 starting from some k_0 , the denominator of series (3.2) satisfies estimate (2.9), where $z = \frac{h}{\sqrt{2}}$.

Now, for series (3.2), we can write the following estimate

$$\sum_{k=-\infty}^{\infty} \frac{k^4 |\psi_k|}{|\cos \frac{kh}{\sqrt{2}}|} \leq C \sum_{k=-\infty}^{\infty} |\psi_k| (1 + |k|^2)^{(n+3)/2}$$

The last term converges by virtue of the conditions of Theorem 1.1. Theorem is proved. $\square$

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Uzbek Mathematical Journal
 2026, Volume 70, Issue 3, pp. 211–229
 DOI: 10.29229/uzmj.2026-3-22

On dynamics of $(2, 2)$ -rational mapping over $\mathbb{Q}_2$

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Abstract. This paper is devoted to the study of the discrete dynamical systems associated with a $(2, 2)$ -rational mapping $f_{a,b}(x) = \frac{ax^2+bx+1}{x^2+bx+a}$ over the field of 2-adic numbers $\mathbb{Q}_2$, where $a, b \in \mathbb{Q}_2$ are parameters. We fully characterize the sets of fixed and 2-periodic points of the given mapping depending on the 2-adic norms of the parameters. It is shown that the system can exhibit various dynamical behaviors: in the case $|a|_2 = |b|_2 = 1$, the system is regular and all trajectories converge to a unique fixed point; whereas for $|a|_2 = |b|_2 < 1$, the fixed point becomes a repeller and an attracting 2-cycle emerges. Furthermore, for the case $|a|_2 = |b|_2 > 1$, we establish the existence of invariant spheres and provide necessary and sufficient conditions for the ergodicity of the mapping on these spheres with respect to the normalized Haar measure. These results contribute to the classification of p -adic rational dynamics for higher-degree mappings.

Keywords: 2-adic numbers, rational mapping, fixed point, periodic point, trajectory, invariant sphere, ergodicity

MSC (2020): 37P05, 37P20, 37P40, 11S82, 37A05

1. INTRODUCTION

The study of dynamical systems over non-Archimedean fields, especially over the field of p -adic numbers $\mathbb{Q}_p$, has become an active direction in modern dynamical systems theory. A systematic account of algebraic and p -adic dynamical systems was developed in the monograph of Anashin and Khrennikov, where ergodicity, measure-preserving transformations, and applications to mathematical physics and computer science are discussed in detail (see [1, 2, 3, 4, 5]). The ultrametric structure of $\mathbb{Q}_p$ makes the behavior of trajectories essentially different from the real and complex cases, since balls are simultaneously open and closed and the geometry of the phase space has a hierarchical nature [6].

A general theory of non-Archimedean dynamics has been extensively developed in the works of Benedetto [7, 8, 9] and others, including Rivera-Letelier [10] and Silverman [11]. These studies investigate periodic points, wandering domains, Fatou and Julia sets, and the role of good reduction for rational maps. In contrast to complex dynamics, non-Archimedean rational dynamics is strongly influenced by arithmetic properties of the coefficients and by the reduction of the map modulo powers of the prime.

Rational functions over $\mathbb{Q}_p$ form one of the central classes of p -adic dynamical systems [12]. Various families of rational maps have been studied from the point of view of fixed points, periodic orbits, Siegel disks, invariant spheres, attractors, and ergodic properties. In particular, Albeverio, Rozikov and Sattarov investigated $(2, 1)$ -rational p -adic dynamical systems, describing fixed points, invariant domains and attraction properties (see [13]). Later, Rozikov and Sattarov studied $(2, 2)$ -rational functions and obtained results on invariant spheres and ergodicity with respect to the Haar measure restricted to such spheres [14].

A substantial part of this theory is motivated by models of p -adic mathematical physics. In particular, rational mappings naturally arise from recurrence equations for Gibbs measures of p -adic Ising, Potts and related models on Cayley trees. Such mappings encode the structure of translation-invariant or periodic boundary laws, and their dynamical properties are closely connected with the existence, uniqueness and phase transition phenomena of corresponding p -adic Gibbs measures (see [15, 16, 17, 18, 19, 20, 21, 22, 23]).

Another important direction concerns ergodicity and minimality of p -adic transformations. For mappings on $\mathbb{Z}_p$ and on p -adic spheres, ergodicity is often studied with respect to the normalized Haar measure. In the non-Archimedean setting, minimality and ergodicity are frequently characterized by finite-level conditions modulo powers of p . For rational maps with good reduction, minimality and its relation to Haar ergodicity have also been studied (see [5, 24, 25, 26]).

Despite these developments, the asymptotic behavior of trajectories of higher-degree rational functions over $\mathbb{Q}_p$ is still far from being completely understood. In particular, for rational functions arising from statistical mechanics, it is important to describe not only fixed points and invariant spheres, but also the structure of ω -limit sets, the set of bad points, and the possible regularity of the system.

It is worth noting that the canonical form of a p -adic $(2, 2)$ -rational function is given by

$$f(x) = \frac{ax^2 + bx + c}{x^2 + dx + e},$$

where $a \neq 0$ and $|b - ad|_p + |c - ae|_p \neq 0$.

In [14, 27], the authors studied the dynamics of $(2, 2)$ -rational functions mainly over the field $\mathbb{C}_p$. According to the number of fixed points of such a function in $\mathbb{C}_p$, they reduced it, by means of topological conjugacy, to three types of rational dynamical systems. Using this classification of rational dynamics over $\mathbb{C}_p$, they then studied the corresponding rational dynamical systems over $\mathbb{Q}_p$. A delicate point is that the rational dynamical systems considered by the authors do not cover all $(2, 2)$ -rational dynamical systems over $\mathbb{Q}_p$. Nevertheless, it should be acknowledged that they obtained interesting results which enrich the theory of p -adic dynamical systems.

In this paper, we consider the following $(2, 2)$ -rational function, which arises in the theory of p -adic Gibbs measures:

$$f_{a,b}(x) = \frac{ax^2 + bx + 1}{x^2 + bx + a}. \quad (1.1)$$

Although functions of this form constitute only a subclass of $(2, 2)$ -rational functions, they are of practical importance in the study of the p -adic Vannimenus model. Moreover, under certain conditions on the parameters, this function may belong to a class which is not covered by the cases studied over $\mathbb{Q}_p$ in [14, 27]. The main object of the present paper is the function (1.1) over the field of 2-adic numbers.

Remark 1.1. We note that (1.1) represents a generalized form of the 2-adic Ising function, which has been thoroughly studied in [26, 28]. It is important to emphasize that (1.1) is not merely an object of abstract mathematical interest. Rather, this function naturally arises as a rational mapping in the construction of Gibbs measures associated with the Ising–Vannimenus model on a Cayley tree of order two. In this context, each periodic point of the map corresponds to a periodic Gibbs measure, thereby establishing a direct link between the dynamical properties of $f_{a,b}$ and the structure of Gibbs measures.

Depending on the parameters a and b , the map $f_{a,b}$ exhibits several distinct structural behaviors:

- it reduces to a $(1, 2)$ -rational function when $a = 0$;
- it becomes a $(1, 1)$ -rational function when $a \pm b + 1 = 0$;
- it degenerates to a constant function when $a = 1$;
- it coincides with an Ising-type function when $b^2 - 4a = 0$;
- it is conjugate to an Ising function in the case $b = 0$.

In the present paper, we focus exclusively on $(2, 2)$ -rational functions that are not of Ising type. Therefore, throughout the paper we impose the non-degeneracy condition

$$ab(a - 1)(a + b + 1)(a - b + 1)(b^2 - 4a) \neq 0. \quad (1.2)$$

2. PRELIMINARIES

Let $\mathbb{Q}_2$ denote the field of 2-adic numbers equipped with the standard non-Archimedean absolute value, called the 2-adic norm. It is well known that every nonzero element $x \in \mathbb{Q}_2$ admits a unique representation of the form

$$x = 2^{\text{ord}_2(x)} (1 + x_1 \cdot 2 + x_2 \cdot 2^2 + x_3 \cdot 2^3 + \cdots), \quad (2.1)$$

where $\text{ord}_2(x) \in \mathbb{Z}$ and the coefficients $x_j \in \{0, 1\}$ for all $j \in \mathbb{N}$. In this representation, the 2-adic norm of x is given by

$$|x|_2 = 2^{-\text{ord}_2(x)}.$$

For each $a \in \mathbb{Q}_2$, $r > 0$ we denote

$$B_r(a) = \{x \in \mathbb{Q}_2 : |x - a|_2 < r\}, \quad S_r(a) = \{x \in \mathbb{Q}_2 : |x - a|_2 = r\}.$$

We notice that $B_r(a)$ and $S_r(a)$ are clopen, compact subsets of $\mathbb{Q}_2$. Recall that $\mathbb{Z}_2 := B_2(0)$ and $\mathbb{Z}_2^* := S_1(0)$ are the set of all 2-adic integers and 2-adic units, respectively. One can check that for every $n \geq 1$ it holds:

$$S_{\frac{1}{2^n}}(1) = 1 + 2^n + 2^{n+1}\mathbb{Z}_2, \quad B_{\frac{1}{2^n}}(1) = 1 + 2^{n+1}\mathbb{Z}_2.$$

The proof of the following lemma is straightforward.

Lemma 2.1. *For any $x \in \mathbb{Z}_2^*$ one has $x^2 \in 1 + 8\mathbb{Z}_2$.*

Using Lemma 2.1, we can easily establish the following auxiliary result.

Lemma 2.2. *Let $a \in \mathbb{Q}_2$. Then $x^2 = a$ has a solution in $\mathbb{Q}_2$ iff $\text{ord}_2(a)$ is even and $\frac{a}{2^{\text{ord}_2(a)}} \in 1 + 8\mathbb{Z}_2$.*

A 2-adic dynamical system is a pair (X, f) , where $X \subset \mathbb{Q}_2$ and $f : X \rightarrow X$ is a continuous function with respect to the 2-adic topology. Throughout, we assume that the space X is either compact or equal to the entire field $\mathbb{Q}_2$. For a given point $x \in X$ the positive trajectory (or orbit) of x under iteration of f is defined as

$$\mathcal{O}^+(x, f) := \{x, f(x), f^2(x), \dots, f^n(x), \dots\}.$$

The central problem in the study of such systems is to describe the asymptotic behavior of this trajectory as $n \rightarrow \infty$. In other words, it is important to describe a set of all limiting point of the trajectory, which is denoted by $\omega_f(x)$ and is known as the omega-limit set of x .

A dynamical system (X, f) is said to be:

- *regular* if, for every $x \in X$, its omega-limit set is a singleton.
- *minimal* if the positive orbit of every point $x \in X$ is dense in X ; that is,

$$\overline{\mathcal{O}^+(x, f)} = X, \quad \forall x \in X.$$

- *topologically transitive* if for any pair of nonempty open sets $U, V \subset X$ there exists an integer $n \geq 0$ such that

$$f^n(U) \cap V \neq \emptyset.$$

It is obvious that every minimal dynamical system is topologically transitive. In general, a topologically transitive system need not be minimal. Transitivity only requires the existence of at least one point with a dense orbit, not that every point has this property.

3. FIXED POINTS AND PERIODIC POINTS OF $f_{a,b}$

Let us define the following sets:

$$\text{Fix}(f_{a,b}) = \{x \in \mathbb{Q}_2 : f_{a,b}(x) = x\},$$

$$\text{Per}_2(f_{a,b}) = \{x \in \mathbb{Q}_2 : f_{a,b}^2(x) = x\} \setminus \text{Fix}(f_{a,b}),$$

which are called the set of all fixed points and the set of all 2-periodic points of the function $f_{a,b}$, respectively.

In this section, we aim to describe these sets in detail.

The fixed point equation $f_{a,b}(x) = x$ reduces to

$$(x - 1)(x^2 + (b - a + 1)x + 1) = 0.$$

Hence, one solution is $x_0 = 1$. Denoting $D = (b - a + 1)^2 - 4$, we obtain that, whenever $\sqrt{D} \in \mathbb{Q}_2$, the remaining solutions are given by

$$x_{\pm} = \frac{a - b - 1 \pm \sqrt{D}}{2}. \quad (3.1)$$

We now analyze the condition $\sqrt{D} \in \mathbb{Q}_2$ in terms of the 2-adic norm of $b - a + 1$. To this end, we invoke Lemma 2.2.

Case $|b - a + 1|_2 \geq 1$. We observe that

$$\frac{D}{(b - a + 1)^2} \in \begin{cases} 1 + 16\mathbb{Z}_2, & \text{if } |b - a + 1|_2 > 1, \\ 1 + 4 + 8\mathbb{Z}_2, & \text{if } |b - a + 1|_2 = 1. \end{cases}$$

Consequently, $\sqrt{D} \in \mathbb{Q}_2$ whenever $|b - a + 1|_2 > 1$, whereas $\sqrt{D} \notin \mathbb{Q}_2$ in the case $|b - a + 1|_2 = 1$.

Case $|b - a + 1|_2 < 1$. First, assume that $|b - a + 1|_2 < \frac{1}{2}$. Then

$$\frac{D}{4} \in 1 + 2 + 4\mathbb{Z}_2,$$

which implies that $\sqrt{D} \notin \mathbb{Q}_2$.

Now consider the case $|b - a + 1|_2 = \frac{1}{2}$. Then there exists $\alpha \in \mathbb{Z}_2$ such that

$$b - a + 1 = 2 + 4\alpha.$$

Substituting this representation into the expression for D , we obtain

$$D = 16\alpha(1 + \alpha).$$

It follows that $\sqrt{D} \in \mathbb{Q}_2$ if and only if $\sqrt{\alpha(1 + \alpha)} \in \mathbb{Q}_2$. Since $\alpha \in \mathbb{Z}_2$, we have $|\alpha(1 + \alpha)|_2 < 1$. Therefore, by the characterization of quadratic residues in $\mathbb{Q}_2$, it follows that $\sqrt{\alpha(1 + \alpha)} \in \mathbb{Q}_2$ if and only if either $\alpha \in \{-1, 0\}$ or

$$\alpha(1 + \alpha) \in \bigsqcup_{n=1}^{\infty} 2^{2n} (1 + 8\mathbb{Z}_2). \quad (3.2)$$

Condition (3.2) follows directly from Lemma 2.2. We also note that, by Remark 1.1, one has $\alpha \neq 0$. Moreover, if $\alpha = -1$, then $x_- = x_+ = x_0$.

We now determine the canonical form of α under condition (3.2). To this end, we distinguish two cases: $\alpha \in \mathbb{Z}_2^*$ and $\alpha \in 2\mathbb{Z}_2$.

First, let $\alpha \in \mathbb{Z}_2^*$. Since $\alpha \neq -1$, there exist $k \geq 0$ and $\beta \in \mathbb{Z}_2$ such that

$$\alpha = \sum_{i=0}^k 2^i + 2^{k+2}\beta. \quad (3.3)$$

Then

$$\alpha(\alpha + 1) = 2^{k+1}(1 + 2\beta) \left(\sum_{i=0}^k 2^i + 2^{k+2}\beta \right). \quad (3.4)$$

By condition (3.2), the integer k must be odd. Hence, we write $k = 2n + 1$ for some $n \geq 0$. In this case, (3.4) becomes

$$\alpha(\alpha + 1) = 2^{2n+2} (-1 - 2\beta + 2^{2n+2} + 2^{2n+4}\beta(\beta + 1)). \quad (3.5)$$

Combining (3.2) with (3.5), we get

$$\beta \in \begin{cases} 1 + 4\mathbb{Z}_2, & \text{if } n = 0, \\ 1 + 2 + 4\mathbb{Z}_2, & \text{if } n > 0. \end{cases} \quad (3.6)$$

Substituting (3.6) into (3.3), we derive

$$\alpha \in \begin{cases} 1 + 2 + 8 + 32\mathbb{Z}_2, & \text{if } n = 0, \\ \sum_{i=0}^{2n+1} 2^i + 2^{2n+3} + 2^{2n+4} + 2^{2n+5}\mathbb{Z}_2, & \text{if } n > 0. \end{cases} \quad (3.7)$$

Next, assume that $\alpha \in 2\mathbb{Z}_2$. Since $\alpha \neq 0$, condition (3.2) implies that there exist $n \geq 1$ and $\beta \in \mathbb{Z}_2$ such that

$$\alpha = 2^{2n}(1 + 2\beta). \quad (3.8)$$

Consequently,

$$\alpha(\alpha + 1) = 2^{2n} (1 + 2\beta + 2^{2n} + 2^{2n+2}\beta(\beta + 1)). \quad (3.9)$$

Taking (3.2) into account, it follows from (3.9) that

$$\beta \in \begin{cases} 2 + 4\mathbb{Z}_2, & \text{if } n = 1, \\ 4\mathbb{Z}_2, & \text{if } n > 1. \end{cases} \quad (3.10)$$

Substituting (3.10) into (3.8), we obtain

$$\alpha \in \begin{cases} 4(1 + 4 + 8\mathbb{Z}_2), & \text{if } n = 1, \\ 2^{2n}(1 + 8\mathbb{Z}_2), & \text{if } n > 1. \end{cases} \quad (3.11)$$

Thus, by (3.7) and (3.11), condition (3.2) holds if and only if one of the following alternatives is satisfied:

- $\alpha \in 1 + 2 + 8 + 32\mathbb{Z}_2$;
- $\alpha \in \bigsqcup_{n=1}^{\infty} (1 + 2 + \dots + 2^{2n+1} + 2^{2n+3} + 2^{2n+4} + 2^{2n+5}\mathbb{Z}_2)$;
- $\alpha \in 4(1 + 4 + 8\mathbb{Z}_2)$;
- $\alpha \in \bigsqcup_{n=2}^{\infty} 2^{2n}(1 + 8\mathbb{Z}_2)$.

Keeping in mind that $b - a = 1 + 4\alpha$, we define the following sets:

$$\begin{cases} A_1 = 45 + 128\mathbb{Z}_2, \\ A_2 = \bigsqcup_{n=1}^{\infty} (1 + 2^2 + 2^3 + \dots + 2^{2n+3} + 2^{2n+5} + 2^{2n+6} + 2^{2n+7}\mathbb{Z}_2), \\ A_3 = 81 + 128\mathbb{Z}_2, \\ A_4 = \bigsqcup_{n=2}^{\infty} (1 + 2^{2n+2} + 2^{2n+5}\mathbb{Z}_2). \end{cases} \quad (3.12)$$

Thus, we obtain the following result:

Proposition 3.1. *Let $f_{a,b}$ be a $(2, 2)$ -rational function defined by (1.1). Then*

$$\text{Fix}(f_{a,b}) = \begin{cases} \{x_0, x_-, x_+\}, & \text{if } b - a \in \bigcup_{i=1}^5 A_i, \\ \{x_0\}, & \text{otherwise.} \end{cases}$$

Here A_1, A_2, A_3, A_4 are defined in (3.12), and $A_5 = \mathbb{Q}_2 \setminus \mathbb{Z}_2$.

Now we describe the set of all 2-periodic points of the function $f_{a,b}$. To this end, we consider the equation

$$\frac{f_{a,b}^2(x) - x}{f_{a,b}(x) - x} = 0,$$

which is equivalent to the quadratic equation

$$x^2 + \left(\frac{b-1}{a} + 1\right)x + 1 = 0. \quad (3.13)$$

The solutions of this equation are

$$x^{\pm} = \frac{-\frac{b-1}{a} - 1 \pm \sqrt{\left(\frac{b-1}{a} + 1\right)^2 - 4}}{2}, \quad (3.14)$$

whenever the square root exists in $\mathbb{Q}_2$.

By definition, $\text{Fix}(f_{a,b}) \cap \text{Per}_2(f_{a,b}) = \emptyset$. In particular, we exclude the case $\frac{b-1}{a} = -3$, since this would imply that the points x^+ and x^- coincide with the fixed point x_0 , i.e., $x^+ = x^- = x_0$.

Moreover, by condition (1.2), we have $\frac{b-1}{a} \neq 1$. Consequently, throughout this paper we impose the condition

$$\left(\frac{b-1}{a} + 1\right)^2 \neq 4.$$

Proposition 3.2. *Let $f_{a,b}$ be a $(2,2)$ -rational function defined by (1.1). Then*

$$\text{Per}_2(f_{a,b}) = \begin{cases} \{x^-, x^+\}, & \text{if } \frac{b-1}{a} \in \bigcup_{i=1}^5 A_i, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Here A_1, A_2, A_3, A_4 are defined in (3.12), and $A_5 = \mathbb{Q}_2 \setminus \mathbb{Z}_2$.

Remark 3.1. In this section, we have completely described the set of all fixed points and the set of all two-periodic points of the function $f_{a,b}$. It is worth emphasizing that this description is obtained for arbitrary values of the parameters a and b . Starting from the next section, we study the dynamical behaviour of $f_{a,b}$. Although we have explicit descriptions of the sets $\text{Fix}(f_{a,b})$ and $\text{Per}_2(f_{a,b})$ for general parameters, in what follows we restrict our attention to the case $|a|_2 = |b|_2$. The case $|a|_2 \neq |b|_2$ will be investigated in our future work. In general, the case $|a|_2 \neq |b|_2$ exhibits some similarities with the case $|a|_2 = |b|_2$. However, certain unexpected phenomena also arise. In particular, we have observed in some special cases that when $|a|_2 \neq |b|_2$, the dynamical system may display chaotic behaviour. Since a complete analysis of this case requires a separate and more detailed investigation, in the present paper we focus only on the case $|a|_2 = |b|_2$.

4. DYNAMICS OF THE FUNCTION $f_{a,b}$

In this section, we investigate the dynamical properties of the function $f_{a,b}$ in the case where $|a|_2 = |b|_2$.

We introduce the set

$$\mathcal{P}(f_{a,b}) = \left\{ x \in \mathbb{Q}_2 : \exists n \geq 0 \text{ such that } (f_{a,b}^n(x))^2 + b f_{a,b}^n(x) + a = 0 \right\},$$

which we call the set of preimages of the poles (or, equivalently, the full backward orbit of the poles) of the map $f_{a,b}$.

We note that $\mathcal{P}(f_{a,b}) \neq \emptyset$ if and only if $\sqrt{b^2 - 4a} \in \mathbb{Q}_2$. Whenever this condition holds, we define

$$\hat{x}_{\pm} := \frac{-b \pm \sqrt{b^2 - 4a}}{2}. \quad (4.1)$$

In the case $|a|_2 = |b|_2$ under consideration, there are the following three possibilities:

$$|a|_2 = |b|_2 = 1, \quad |a|_2 = |b|_2 < 1, \quad |a|_2 = |b|_2 > 1.$$

We investigate each case separately.

4.1. **Case** $|a|_2 = |b|_2 = 1$.

Remark 4.1. In this case, we have $|b - a|_2 < 1$ and $|b - 1|_2 < |a|_2$. Equivalently,

$$b - a \notin \bigcup_{i=1}^5 A_i, \quad \frac{b-1}{a} \notin \bigcup_{i=1}^5 A_i.$$

Therefore, by Propositions 3.1 and 3.2, we obtain

$$\text{Fix}(f_{a,b}) = \{x_0\}, \quad \text{Per}_2(f_{a,b}) = \emptyset.$$

Moreover, since $b^2 - 4a \in 5 + 8\mathbb{Z}_2$, it follows that $\sqrt{b^2 - 4a} \notin \mathbb{Q}_2$. Consequently, the function $f_{a,b}$ is well-defined on the entire field $\mathbb{Q}_2$.

Theorem 4.1. *Let $|a|_2 = |b|_2 = 1$. Then for any $x \in \mathbb{Q}_2$ it holds*

$$\omega_{f_{a,b}}(x) = \{x_0\}.$$

Proof: It is readily verified that

$$f_{a,b}(x) - 1 = \frac{(a-1)(x+1)(x-1)}{x^2 + bx + a}.$$

Using the estimates

$$|x+1|_2 \leq \begin{cases} |x|_2, & \text{if } |x|_2 > 1, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$|x^2 + bx + a|_2 = \begin{cases} |x|_2^2, & \text{if } |x|_2 > 1, \\ 1, & \text{otherwise,} \end{cases}$$

we obtain

$$\frac{|x+1|_2}{|x^2 + bx + a|_2} \leq 1.$$

Therefore,

$$|f_{a,b}(x) - 1|_2 \leq |(a-1)(x-1)|_2.$$

By iteration, it follows that

$$|f_{a,b}^n(x) - 1|_2 \leq |(a-1)^n(x-1)|_2.$$

Since $|a-1|_2 < 1$, we conclude that $f_{a,b}^n(x) \rightarrow 1$ as $n \rightarrow \infty$. □

4.2. **Case** $|a|_2 = |b|_2 = 2^m < 1$.

Remark 4.2. In this case, we have $|b - a|_2 < 1$ and $|b - 1|_2 > |a|_2$. Equivalently,

$$b - a \notin \bigcup_{i=1}^5 A_i, \quad \frac{b-1}{a} \in A_5.$$

Therefore, by Propositions 3.1 and 3.2, we obtain

$$\text{Fix}(f_{a,b}) = \{x_0\}, \quad \text{Per}_2(f_{a,b}) = \{x^-, x^+\}.$$

From (3.13) and Vieta's formula, we obtain

$$|x^- + x^+|_2 = \frac{1}{2^m}, \quad |x^- \cdot x^+|_2 = 1.$$

Hence the two roots have different 2-adic norms. More precisely, after relabelling the roots if necessary, we may assume that

$$x^- \in S_{2^m}(0), \quad x^+ \in S_{\frac{1}{2^m}}(0). \quad (4.2)$$

Theorem 4.2. Let $|a|_2 = |b|_2 = 2^m < 1$. Then for the initial point $x \in \mathbb{Q}_2$ it holds

$$\omega_{f_{a,b}}(x) = \begin{cases} \{x_0\} & \text{if } x \in \mathbb{Z}_2^*, \\ \{x^+, x^-\}, & \text{if } x \in \mathbb{Q}_2 \setminus (\mathbb{Z}_2^* \cup \mathcal{P}(f_{a,b})). \end{cases}$$

Moreover, $\mathcal{P}(f_{a,b})$ is finite.

Proof: A direct computation shows that

$$|f_{a,b}(x) - 1|_2 = |x^2 - 1|_2 < |x - 1|_2, \quad \forall x \in \mathbb{Z}_2^*.$$

Hence, $f_{a,b}(\mathbb{Z}_2^*) \subset \mathbb{Z}_2^*$ and

$$\lim_{n \rightarrow \infty} f_{a,b}^n(x) = 1, \quad \forall x \in \mathbb{Z}_2^*.$$

In order to analyse the dynamics of $f_{a,b}$ on $\mathbb{Q}_2 \setminus \mathbb{Z}_2^*$, we first establish the following preliminary facts:

$$(P_1) \quad S_{2^m}(0) \xrightarrow{f_{a,b}} S_{\frac{1}{2^m}}(0) \xrightarrow{f_{a,b}} S_{2^m}(0).$$

$$(P_2) \quad \text{for every } x \in 2\mathbb{Z}_2 \setminus \mathcal{P}(f_{a,b}) \text{ there exists } k \geq 1 \text{ such that } f_{a,b}^{2k}(x) \in S_{2^m}(0).$$

$$(P_3) \quad \text{for every } x \in \mathbb{Q}_2 \setminus (\mathbb{Z}_2 \cup \mathcal{P}(f_{a,b})) \text{ there exists } k \geq 1 \text{ such that } f_{a,b}^{2k}(x) \in S_{\frac{1}{2^m}}(0).$$

$$(P_4) \quad \mathcal{P}(f_{a,b}) \text{ is finite.}$$

Proof of (P_1) . Let $x \in S_{2^m}(0)$, i.e. $|x|_2 = 2^m$. Then by $|ax^2 + bx + 1|_2 = 1$ and $|x^2 + bx + a|_2 = 2^m$ we get $|f_{a,b}(x)|_2 = \frac{1}{2^m}$, i.e.

$$f_{a,b}(x) \in S_{\frac{1}{2^m}}(0).$$

Now pick $y \in S_{\frac{1}{2^m}}(0)$. Then by $|ay^2 + by + 1|_2 = \frac{1}{2^m}$ and $|y^2 + by + a|_2 = \frac{1}{4^m}$ we obtain $|f_{a,b}(y)|_2 = 2^m$, i.e.

$$f_{a,b}(y) \in S_{2^m}(0).$$

Proof of (P_2) . We note that, if $\hat{x}_-$ and $\hat{x}_+$ exist, then both of them lie on the sphere $S_{\sqrt{2^m}}(0)$, that is,

$$|\hat{x}_-|_2 = |\hat{x}_+|_2 = \sqrt{2^m}.$$

Let $x \in B_{\sqrt{2^m}}(0)$, i.e. $|x|_2 < 2^m$. Then we have

$$\begin{aligned} |ax^2 + bx + 1|_2 &= 1, \\ |x^2 + bx + a|_2 &= 2^m. \end{aligned}$$

Hence $f_{a,b}(x) \in S_{\frac{1}{2^m}}(0)$. By (P_1) we infer that $f_{a,b}^2(x) \in S_{2^m}(0)$. So, $k = 1$.

Take an arbitrary $y \in S_{\sqrt{2^m}}(0) \setminus \{\hat{x}_-, \hat{x}_+\}$. Then $|y^2|_2 = |a|_2 > |by|_2$. By strong triangle inequality,

$$|f_{a,b}(y)|_2 = \frac{1}{|y^2 + by + a|_2} > \frac{1}{2^m}.$$

Therefore, we have

$$\begin{aligned} |a(f_{a,b}(y))^2 + bf_{a,b}(y) + 1|_2 &= 2^m |f_{a,b}(y)|_2^2, \\ |(f_{a,b}(y))^2 + bf_{a,b}(y) + a|_2 &= |f_{a,b}(y)|_2^2, \end{aligned}$$

which implies that $f_{a,b}^2(y) \in S_{2^m}(0)$. So, $k = 1$.

Finally, pick $z \in 2\mathbb{Z}_2$ such that $|z|_2 > \sqrt{2^m}$. By strong triangle inequality we get $|f_{a,b}(z)|_2 = \frac{1}{|z^2|_2}$. Then using

$$\begin{aligned} |a(f_{a,b}(z))^2|_2 &= \frac{2^m}{|z^4|_2} < \frac{1}{|z^2|_2}, \\ |bf_{a,b}(z)|_2 &= \frac{2^m}{|z^2|_2} < 1, \end{aligned}$$

and $1 < \frac{1}{|z^2|_2}$ we get

$$|a(f_{a,b}(z))^2 + bf_{a,b}(z) + 1|_2 < \frac{1}{|z^2|_2}.$$

The last inequality together with $|(f_{a,b}(z))^2 + bf_{a,b}(z) + a|_2 = \frac{1}{|z^4|_2}$ implies that

$$|f_{a,b}^2(z)|_2 < |z^2|_2.$$

Thus, due to $|z|_2 < 1$ we have $|f_{a,b}^2(z)|_2 < |z|_2$. By the discreteness of the values of the 2-adic norm we conclude that there exists $i \geq 1$ such that $|f_{a,b}^{2i}(z)|_2 \leq \sqrt{2^m}$. Therefore, whenever $f_{a,b}^{2i}(z) \notin \{\hat{x}_-, \hat{x}_+\}$, choosing $k = i + 1$ gives

$$f_{a,b}^{2k}(z) \in S_{2^m}(0).$$

Proof of (P_3) . This follows directly by applying the identity

$$f_{a,b}\left(\frac{1}{x}\right) = \frac{1}{f_{a,b}(x)}$$

to properties (P_1) and (P_2) .

Proof of (P_4) . It follows from properties (P_1) – (P_3) that

$$\mathcal{P}(f_{a,b}) \subset \left\{x \in \mathbb{Q}_2 : \sqrt{2^m} \leq |x|_2 \leq \frac{1}{\sqrt{2^m}}\right\}.$$

Taking into account that

$$\omega_{f_{a,b}}(x) = \{x_0\}, \quad \forall x \in \mathbb{Z}_2^*,$$

and that, for every y with $1 < |y|_2 \leq \frac{1}{\sqrt{2^m}}$, one has

$$|f_{a,b}(y)|_2 \leq \frac{1}{|y^2|_2},$$

we conclude that

$$\mathcal{P}(f_{a,b}) \subset \left\{x \in \mathbb{Q}_2 : \sqrt{2^m} \leq |x|_2 \leq \frac{1}{\sqrt[4]{2^m}}\right\} \setminus \mathbb{Z}_2^*.$$

On the other hand, for every z satisfying $\sqrt{2^m} < |z|_2 < 1$, we have

$$|f_{a,b}^2(z)|_2 < |z|_2.$$

Therefore, by the symmetry relation

$$f_{a,b}\left(\frac{1}{x}\right) = \frac{1}{f_{a,b}(x)},$$

it follows that, for every t satisfying

$$1 < |t|_2 < \frac{1}{\sqrt[4]{2^m}},$$

one has

$$|f_{a,b}^2(t)|_2 > |t|_2.$$

Since the values of the 2-adic norm are discrete, the annulus

$$\left\{x \in \mathbb{Q}_2 : \sqrt{2^m} \leq |x|_2 \leq \frac{1}{\sqrt[4]{2^m}}\right\}$$

contains only finitely many possible values of $|x|_2$. Moreover, on the part $\sqrt{2^m} < |x|_2 < 1$ the norm of the even iterates strictly decreases, whereas on the part $1 < |x|_2 \leq \frac{1}{\sqrt[4]{2^m}}$ the norm of the even iterates strictly increases. Hence any orbit which does not hit a pole enters the invariant set

$$X = S_{2^m}(0) \sqcup S_{\frac{1}{2^m}}(0) \tag{4.3}$$

after a uniformly bounded number of iterates. Therefore a point may belong to $\mathcal{P}(f_{a,b})$ only if it is mapped to one of the poles within a bounded number of iterates. Since each fixed iterate of the rational map has only finitely many preimages of the finite set of poles, the set $\mathcal{P}(f_{a,b})$ is finite.

Thus properties $(P_1) - (P_4)$ have been established. Moreover, they imply that

$$X \cap \mathcal{P}(f_{a,b}) = \emptyset. \quad (4.4)$$

Thus, the dynamics of the map $f_{a,b}$ on $\mathbb{Q}_2$ can in fact be studied on the invariant set (4.3). Moreover, since both spheres are invariant under the second iterate, it suffices to analyse the dynamics of the map $g := f_{a,b}^2$ on these spheres.

We first show that g is a contraction on $S_{\frac{1}{2^m}}(0)$. Then, in view of the symmetry relation

$$g\left(\frac{1}{x}\right) = \frac{1}{g(x)},$$

the same contraction property follows for g on the sphere $S_{2^m}(0)$.

Let $y, z \in S_{\frac{1}{2^m}}(0)$. Then

$$\begin{aligned} |g(y) - g(z)|_2 &= \frac{|f_{a,b}(y) - f_{a,b}(z)|_2}{|a|_2} \\ &= \frac{|y - z|_2 |b(yz + 1) + (a + 1)(y + z)|_2}{|a|_2 |y^2 + by + a|_2 |z^2 + bz + a|_2} \\ &= \frac{|y - z|_2 |byz|_2}{|a|_2 |y^2|_2 |z^2|_2} = \frac{|y - z|_2}{|yz|_2} = |a|_2^2 |y - z|_2 < |y - z|_2. \end{aligned}$$

Thus, g is a contraction on $S_{\frac{1}{2^m}}(0)$. Hence, for every $x \in X$ we have

$$\lim_{n \rightarrow \infty} g^n(x) = \begin{cases} x_*, & \text{if } x \in S_{2^m}(0), \\ x^*, & \text{if } x \in S_{\frac{1}{2^m}}(0). \end{cases}$$

On the other hand, by (4.2) we have $g(x^-) = x^-$ and $g(x^+) = x^+$. By the uniqueness of the fixed point of a contraction on each sphere, we get $x_* = x^-$ and $x^* = x^+$. Since every point

$$x \in \mathbb{Q}_2 \setminus (\mathbb{Z}_2^* \cup \mathcal{P}(f_{a,b}))$$

eventually enters the invariant set X , the two subsequences of its orbit under $f_{a,b}$ converge to the two points of the two-cycle $\{x^-, x^+\}$. Therefore

$$\omega_{f_{a,b}}(x) = \{x^-, x^+\}, \quad \forall x \in \mathbb{Q}_2 \setminus (\mathbb{Z}_2^* \cup \mathcal{P}(f_{a,b})).$$

This completes the proof. □

4.3. Case $|a|_2 = |b|_2 > 1$.

Remark 4.3. A direct computation shows that, in this regime, neither fixed points nor 2-periodic points (whenever they exist) can be attracting. More precisely, any existing 2-periodic point is necessarily indifferent. Among the fixed points, only $x_0 = 1$ may be either indifferent or repelling, depending on the parameters. Moreover, in this case, $\mathcal{P}(f_{a,b}) \neq \emptyset$ and

$$\mathcal{P}(f_{a,b}) \subset \mathbb{Z}_2^* \cup (\mathbb{Q}_2 \setminus B_{2^m}(0)).$$

First, we try to determine the invariant sets of the function $f_{a,b}$.

Proposition 4.1. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements hold:*

$$(i) \quad S_{\frac{1}{2^m}}(0) \xrightarrow{f_{a,b}} B_{\frac{1}{2^m}}(0) \xrightarrow{f_{a,b}} S_{\frac{1}{2^m}}(0);$$

(ii) if $m > 1$ then $S_{\frac{1}{2^n}}(0) \xrightarrow{f_{a,b}} S_{\frac{1}{2^n}}(0)$ for any $n \in \{1, \dots, m-1\}$.

Proof: (i) Let us take an arbitrary point $x \in S_{\frac{1}{2^m}}(0)$. Then

$$|f_{a,b}(x)|_2 = \frac{|ax^2 + bx + 1|_2}{|a|_2} < \frac{1}{2^m}.$$

Now pick an arbitrary point $y \in B_{\frac{1}{2^m}}(0)$. Since

$$|ay^2|_2 < |by|_2 < 1,$$

we obtain

$$|f_{a,b}(y)|_2 = \frac{|ay^2 + by + 1|_2}{|a|_2} = \frac{1}{2^m}.$$

(ii) Assume that $m > 1$. Let $n \in \{1, \dots, m-1\}$. Then, for every $x \in S_{\frac{1}{2^n}}(0)$, we have

$$|bx|_2 > \max\{|ax^2|_2, 1\}.$$

Therefore,

$$|f_{a,b}(x)|_2 = \frac{|bx|_2}{|a|_2} = |x|_2.$$

Hence

$$f_{a,b}\left(S_{\frac{1}{2^n}}(0)\right) \subset S_{\frac{1}{2^n}}(0).$$

□

Proposition 4.2. Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements hold:

(i) The map $f_{a,b}$ is a bijective isometry on the closed ball $\overline{B}_{\frac{1}{2^m}}(0)$;

(ii) If $m > 1$, then $f_{a,b}$ is a bijective isometry on each sphere $S_{\frac{1}{2^n}}(0)$ for all $n \in \{1, \dots, m-1\}$.

Proof: (i) By Proposition 4.1, we have

$$f_{a,b}(\overline{B}_{\frac{1}{2^m}}(0)) \subset \overline{B}_{\frac{1}{2^m}}(0).$$

Let $x, y \in \overline{B}_{\frac{1}{2^m}}(0)$. Then

$$\begin{aligned} |f_{a,b}(x) - f_{a,b}(y)|_2 &= \frac{|(x-y)(a-1)(bxy + (a+1)(x+y) + b)|_2}{|(x^2 + bx + a)(y^2 + by + a)|_2} \\ &= \frac{|ab(x-y)|_2}{|a^2|_2} = |x-y|_2, \end{aligned}$$

which shows that $f_{a,b}$ is an isometry on $\overline{B}_{\frac{1}{2^m}}(0)$.

We now prove surjectivity. Fix an arbitrary $\gamma \in \overline{B}_{\frac{1}{2^m}}(0)$ and consider the equation

$$f_{a,b}(t) = \gamma,$$

which is equivalent to

$$(a - \gamma)t^2 + b(1 - \gamma)t + (1 - a\gamma) = 0. \quad (4.5)$$

Observe that

$$\begin{aligned} |a - \gamma|_2 &= 2^m, \\ |b(1 - \gamma)|_2 &= 2^m, \\ |1 - a\gamma|_2 &\leq 1. \end{aligned}$$

Hence,

$$\frac{|4(a - \gamma)(1 - a\gamma)|_2}{|b^2(1 - \gamma)^2|_2} \leq \frac{1}{2^{m+2}} \leq \frac{1}{8}.$$

Then, by Lemma 2.1, the discriminant

$$b^2(1 - \gamma)^2 \left(1 - \frac{4(a - \gamma)(1 - a\gamma)}{b^2(1 - \gamma)^2} \right)$$

admits a square root in $\mathbb{Q}_2$. Hence, equation (4.5) has two solutions in $\mathbb{Q}_2$, denoted by t_1 and t_2 .

By Vieta's formulas,

$$|t_1 + t_2|_2 = 1, \quad |t_1 t_2|_2 \leq \frac{1}{2^m}.$$

It follows from the non-Archimedean property that exactly one of the roots has unit norm. Without loss of generality, let $|t_1|_2 = 1$, i.e., $t_1 \in \mathbb{Z}_2^*$, while the other root satisfies

$$|t_2|_2 \leq \frac{1}{2^m},$$

that is, $t_2 \in \overline{B}_{\frac{1}{2^m}}(0)$.

Thus, for every $\gamma \in \overline{B}_{\frac{1}{2^m}}(0)$, there exists $t \in \overline{B}_{\frac{1}{2^m}}(0)$ such that $f_{a,b}(t) = \gamma$. Hence, $f_{a,b}$ is surjective on $\overline{B}_{\frac{1}{2^m}}(0)$, and therefore a bijective isometry.

(ii) Since the proof of the fact that $f_{a,b}$ is an isometry on the sphere $S_{\frac{1}{2^n}}(0)$ is exactly the same as in case (i), we only verify surjectivity. To this end, we show that, for any $\gamma \in S_{\frac{1}{2^n}}(0)$, the quadratic equation (4.5) has a solution belonging to the same sphere.

We have

$$\begin{aligned} |a - \gamma|_2 &= 2^m, \\ |b(1 - \gamma)|_2 &= 2^m, \\ |1 - a\gamma|_2 &= 2^{m-n}. \end{aligned}$$

Hence,

$$\frac{|4(a - \gamma)(1 - a\gamma)|_2}{|b^2(1 - \gamma)^2|_2} = \frac{1}{2^{n+2}} \leq \frac{1}{8}.$$

Then, by Lemma 2.1, the discriminant

$$b^2(1 - \gamma)^2 \left(1 - \frac{4(a - \gamma)(1 - a\gamma)}{b^2(1 - \gamma)^2} \right)$$

admits a square root in $\mathbb{Q}_2$. Hence, equation (4.5) has two solutions in $\mathbb{Q}_2$, denoted by t_1 and t_2 .

By Vieta's formulas,

$$|t_1 + t_2|_2 = 1, \quad |t_1 t_2|_2 \leq \frac{1}{2^m}.$$

It follows from the non-Archimedean property that exactly one of the roots has unit norm. Without loss of generality, let $|t_1|_2 = 1$, i.e., $t_1 \in \mathbb{Z}_2^*$, while the other root satisfies

$$|t_2|_2 \leq \frac{1}{2^m},$$

that is, $t_2 \in \overline{B}_{\frac{1}{2^m}}(0)$. □

Using the relation

$$f_{a,b}\left(\frac{1}{x}\right) = \frac{1}{f_{a,b}(x)},$$

one can prove the following results in a similar manner to Proposition 4.1 and Proposition 4.2.

Proposition 4.3. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements hold:*

- (i) $S_{2^m}(0) \xrightarrow{f_{a,b}} \mathbb{Q}_2 \setminus \overline{B}_{2^m}(0) \xrightarrow{f_{a,b}} S_{2^m}(0)$;
- (ii) if $m > 1$ then $S_{2^n}(0) \xrightarrow{f_{a,b}} S_{2^n}(0)$ for any $n \in \{1, \dots, m-1\}$.

Proposition 4.4. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements hold:*

- (i) The map $f_{a,b}$ is a bijective isometry on the set $\mathbb{Q}_2 \setminus (B_{2^m}(0) \cup \mathcal{P}(f_{a,b}))$;
- (ii) If $m > 1$, then $f_{a,b}$ is a bijective isometry on each sphere $S_{2^n}(0)$ for all $n \in \{1, \dots, m-1\}$.

We now analyse the behaviour of $f_{a,b}$ on the unit sphere. It is worth noting that the unit sphere is not invariant under this function.

Lemma 4.1. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then there exists a partition $\{C_0, C_1, C_2\}$ of $\mathbb{Z}_2^*$ such that*

- (i) $f_{a,b}(C_0) = 2\mathbb{Z}_2$,
- (ii) $f_{a,b}(C_1) \subset \mathbb{Z}_2^*$,
- (iii) $f_{a,b}(C_2) = \mathbb{Q}_2 \setminus \mathbb{Z}_2$.

Proof: In the proof of the Proposition 4.2, we actually proved that for every $\gamma \in 2\mathbb{Z}_2$ the equation $f_{a,b}(t) = \gamma$ has a solution on $\mathbb{Z}_2^*$. So, we set

$$C_0 := f_{a,b}^{-1}(2\mathbb{Z}_2) \cap \mathbb{Z}_2^*.$$

Analogously, let

$$C_2 := f_{a,b}^{-1}(\mathbb{Q}_2 \setminus \mathbb{Z}_2) \cap \mathbb{Z}_2^*.$$

By construction,

$$C_0 \cap C_2 = \emptyset.$$

On the other hand, the point $x_0 = 1 \in \mathbb{Z}_2^*$ is a fixed point of $f_{a,b}$, hence it does not belong to $C_0 \cup C_2$ under the above classification. Therefore, the family $\{C_0, C_2\}$ does not form a partition of $\mathbb{Z}_2^*$.

Consequently, defining

$$C_1 := \mathbb{Z}_2^* \setminus (C_0 \cup C_2),$$

we obtain the desired decomposition, which completes the proof. $\square$

As a conclusion of the case $|a|_2 = |b|_2 > 1$ we formulate the following theorem.

Theorem 4.3. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then for any $x \in \mathbb{Q}_2 \setminus (\mathbb{Z}_2^* \cup \mathcal{P}(f_{a,b}))$ it holds*

$$\mathcal{O}^+(x, f_{a,b}) \subset \omega_{f_{a,b}}(x).$$

Proof: It is enough to check only for $x \in \overline{B}_{\frac{1}{2^m}}(0)$. By Proposition 4.2 we have

$$|f_{a,b}^2(x) - f_{a,b}(x)|_2 = |f_{a,b}(x) - x|_2.$$

Due to the strong triangle inequality, we obtain

$$|f_{a,b}^2(x) - x|_2 \leq \frac{|f_{a,b}(x) - x|_2}{2}.$$

Then by induction, we can easily check that

$$|f_{a,b}^{2^n}(x) - x|_2 \leq \frac{|f_{a,b}(x) - x|_2}{2^n}.$$

This means that $x \in \omega_{f_{a,b}}(x)$. More generally, if $k \geq 0$ then $f^{2^n+k}(x) \rightarrow f^k(x)$ as $n \rightarrow \infty$. Hence,

$$\mathcal{O}^+(x, f_{a,b}) \subset \omega_{f_{a,b}}(x).$$

$\square$

5. ERGODICITY OF $f_{a,b}$

In the previous section, we established that the dynamical system $(\mathbb{Q}_2, f)$ is regular (or cyclic) in the case $|a|_2 = |b|_2 \leq 1$, which implies that the asymptotic behavior of trajectories is completely predictable.

In contrast, the regime $|a|_2 = |b|_2 > 1$ gives rise to more complex dynamical behavior. In order to study properties such as minimality and ergodicity, it is crucial to identify and analyze invariant spheres associated with the dynamics.

We now recall the notions of measure-preserving transformations and ergodicity, both of which are formulated with respect to a given measure.

Definition 5.1. Let $(X, \mathcal{B}, \mu)$ be a measure space. A measurable mapping $f : X \rightarrow X$ is called *measure-preserving* if

$$\mu(f^{-1}(A)) = \mu(A) \quad (5.1)$$

for every $A \in \mathcal{B}$. In this case, we also say that f preserves the measure μ .

Equivalently, if (5.1) holds, then the measure μ is said to be invariant under the mapping f . It is important to note that the condition (5.1) is stated in terms of the preimage $f^{-1}(A)$ rather than the image $f(A)$. This distinction is essential, especially when the mapping f is not invertible.

Definition 5.2. Let $(X, \mathcal{B}, \mu)$ be a probability space and let $f : X \rightarrow X$ be a measure-preserving mapping. The mapping f is called *ergodic* with respect to the measure μ if every invariant measurable set is trivial with respect to μ ; that is, whenever $A \in \mathcal{B}$ satisfies $f^{-1}(A) = A$, we have

$$\mu(A) = 0 \quad \text{or} \quad \mu(A) = 1.$$

In this case, the dynamical system (X, f) is said to be ergodic with respect to μ .

The space $\mathbb{Z}_2$ is endowed with its natural probability measure, namely the normalized Haar measure μ , for which

$$\mu(\mathbb{Z}_2) = 1.$$

The elementary μ -measurable sets are the 2-adic balls. Recall that, for $n \geq 1$, the 2-adic open ball of radius 2^{-n+1} with centre c is given by

$$B_{\frac{1}{2^{n-1}}}(c) = c + 2^n \mathbb{Z}_2 = \{x \in \mathbb{Z}_2 : x \equiv c \pmod{2^n}\}.$$

Since every residue class modulo 2^n has a unique representative in the set $\{0, 1, \dots, 2^n - 1\}$, we may, without loss of generality, choose the centre c of the ball $c + 2^n \mathbb{Z}_2$ so that

$$c \in \{0, 1, \dots, 2^n - 1\}.$$

The volume of this ball is defined as

$$\mu(c + 2^n \mathbb{Z}_2) = 2^{-n}.$$

Every ball in $\mathbb{Z}_2$ can be viewed as a copy of $\mathbb{Z}_2$ itself. Indeed, if $B_{\frac{1}{2^n}}(c) \subset \mathbb{Z}_2$, then the map

$$h(x) = \frac{x - c}{2^n}$$

defines a homeomorphism from $B_{\frac{1}{2^n}}(c)$ onto $\mathbb{Z}_2$. This identification allows us to define a *normalized Haar measure* on the ball by pulling back the standard Haar measure on $\mathbb{Z}_2$. Moreover, since the Haar measure is locally finite and translation-invariant, it can be naturally restricted to any open compact subset $X \subset \mathbb{Z}_2$.

In [24], the author builds upon the work of Anashin and Yurova to establish ergodicity criteria for 2-adic rational mappings.

Theorem 5.1. [24] Let $P, Q : \mathbb{Z}_2^* \rightarrow \mathbb{Z}_2^*$ be polynomials whose coefficients are 2-adic integers.

Set $P(x) = \sum_i p_i x^i$, $Q(x) = \sum_i q_i x^i$, $P_1 = \sum_{i \text{ odd}} p_i$, $P_2 = \sum_{i \text{ even}} p_i$, $Q_1 = \sum_{i \text{ odd}} q_i$ and $Q_2 = \sum_{i \text{ even}} q_i$.

The rational function $\frac{P}{Q}$ is ergodic if and only if one of the following situations occur:

- (1) $P_1 \equiv 1(\text{mod } 4)$, $P_2 \equiv 2(\text{mod } 4)$, $Q_1 \equiv 0(\text{mod } 4)$ and $Q_2 \equiv 1(\text{mod } 4)$.
- (2) $P_1 \equiv 3(\text{mod } 4)$, $P_2 \equiv 2(\text{mod } 4)$, $Q_1 \equiv 0(\text{mod } 4)$ and $Q_2 \equiv 3(\text{mod } 4)$.
- (3) $P_1 \equiv 1(\text{mod } 4)$, $P_2 \equiv 0(\text{mod } 4)$, $Q_1 \equiv 2(\text{mod } 4)$ and $Q_2 \equiv 1(\text{mod } 4)$.
- (4) $P_1 \equiv 3(\text{mod } 4)$, $P_2 \equiv 0(\text{mod } 4)$, $Q_1 \equiv 2(\text{mod } 4)$ and $Q_2 \equiv 3(\text{mod } 4)$.
- (5) One of the previous cases with P and Q interchanged.

Lemma 5.1. Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements equivalent:

- (i) $f_{a,b}^2$ is ergodic on $S_{\frac{1}{2^m}}(0)$;
- (ii) $|b + a - 1|_2 = |2a|_2$.

Proof: By Proposition 4.1 we have $f_{a,b}^2 : S_{\frac{1}{2^m}}(0) \rightarrow S_{\frac{1}{2^m}}(0)$. A direct computation yields

$$f_{a,b}^2(x) = \frac{Ax^4 + Bx^3 + Cx^2 + Dx + E}{Ex^4 + Dx^3 + Cx^2 + Bx + A},$$

where

$$\begin{aligned} A &= a^3 + ab + 1, \\ B &= 2a^2b + ab^2 + b^2 + 2b, \\ C &= a^2b + 2a^2 + 2a + ab^2 + b^3 + b^2 + b, \\ D &= ab^2 + b^2 + 4ab, \\ E &= a^2 + ab + a. \end{aligned}$$

Consider the homeomorphism $h(x) = ax$ from $S_{\frac{1}{2^m}}(0)$ onto $\mathbb{Z}_2^*$, and define the conjugate map $g := h \circ f_{a,b}^2 \circ h^{-1}$. Then

$$g(x) = \frac{a^{-6}Ax^4 + a^{-5}Bx^3 + a^{-4}Cx^2 + a^{-3}Dx + a^{-2}E}{a^{-7}Ex^4 + a^{-6}Dx^3 + a^{-5}Cx^2 + a^{-4}Bx + a^{-3}A}.$$

Set

$$\begin{aligned} P(x) &= a^{-6}Ax^4 + a^{-5}Bx^3 + a^{-4}Cx^2 + a^{-3}Dx + a^{-2}E, \\ Q(x) &= a^{-7}Ex^4 + a^{-6}Dx^3 + a^{-5}Cx^2 + a^{-4}Bx + a^{-3}A. \end{aligned}$$

Then $P, Q : \mathbb{Z}_2^* \rightarrow \mathbb{Z}_2^*$, and the ergodicity of $(\mathbb{Z}_2^*, g)$ can be verified using Theorem 5.1. Using the estimates

$$|A|_2 = |B|_2 = |C|_2 = |D|_2 = |a^3|_2, \quad |E|_2 < |a^2|_2,$$

we compute modulo 4:

$$\begin{aligned} P_1 &\equiv \frac{B}{a^5} + \frac{D}{a^3} \equiv \frac{D}{a^3} \equiv 1 + \frac{1}{a}(\text{mod } 4), \\ P_2 &\equiv \frac{A}{a^6} + \frac{C}{a^4} + \frac{E}{a^2} \equiv 1 + \frac{b}{a}(\text{mod } 4), \\ Q_1 &\equiv \frac{D}{a^6} + \frac{B}{a^4} \equiv \frac{b^2}{a^3} \equiv \frac{1}{a}(\text{mod } 4), \\ Q_2 &\equiv \frac{E}{a^7} + \frac{C}{a^5} + \frac{A}{a^3} \equiv 1 + \frac{b}{a^2} \equiv 1 + \frac{1}{a}(\text{mod } 4). \end{aligned}$$

Therefore, by Theorem 5.1, the ergodicity condition is equivalent to

$$1 + \frac{b}{a} - \frac{1}{a} \equiv 2(\text{mod } 4).$$

This congruence holds if and only if

$$|b + a - 1|_2 = |2a|_2.$$

□

Lemma 5.2. *Let $|a|_2 = |b|_2 = 2^m > 1$. Then the following statements are equivalent:*

- (i) $f_{a,b}$ is minimal on $\overline{B}_{\frac{1}{2^m}}(0)$;
- (ii) $f_{a,b}^2$ is minimal on $S_{\frac{1}{2^m}}(0)$;
- (iii) $f_{a,b}^2$ is minimal on $B_{\frac{1}{2^m}}(0)$.

Proof: By Proposition 4.1, the mapping $f_{a,b}$ sends $B_{\frac{1}{2^m}}(0)$ into $S_{\frac{1}{2^m}}(0)$ and $S_{\frac{1}{2^m}}(0)$ into $B_{\frac{1}{2^m}}(0)$. Therefore, both $B_{\frac{1}{2^m}}(0)$ and $S_{\frac{1}{2^m}}(0)$ are invariant under $f_{a,b}^2$.

We first prove (i) $\implies$ (ii). Let $x, y \in S_{\frac{1}{2^m}}(0)$ and $\varepsilon > 0$. Since $f_{a,b}$ is minimal on $\overline{B}_{\frac{1}{2^m}}(0)$, the positive orbit of x under $f_{a,b}$ is dense in $\overline{B}_{\frac{1}{2^m}}(0)$. Hence, for some $n \geq 0$, we have

$$|f_{a,b}^n(x) - y|_2 < \varepsilon.$$

Since $f_{a,b}$ maps $S_{\frac{1}{2^m}}(0)$ into $B_{\frac{1}{2^m}}(0)$ and $B_{\frac{1}{2^m}}(0)$ into $S_{\frac{1}{2^m}}(0)$, the point $f_{a,b}^n(x)$ belongs to $S_{\frac{1}{2^m}}(0)$ only when n is even. Thus $n = 2n_0$ for some $n_0 \geq 0$, and therefore

$$|f_{a,b}^{2n_0}(x) - y|_2 < \varepsilon.$$

This proves that $f_{a,b}^2$ is minimal on $S_{\frac{1}{2^m}}(0)$.

Now we prove (ii) $\implies$ (iii). Let $x, y \in B_{\frac{1}{2^m}}(0)$ and $\varepsilon > 0$. By Proposition 4.1,

$$f_{a,b}(x), f_{a,b}(y) \in S_{\frac{1}{2^m}}(0).$$

Since $f_{a,b}^2$ is minimal on $S_{\frac{1}{2^m}}(0)$, there exists $n_0 \geq 0$ such that

$$|f_{a,b}^{2n_0}(f_{a,b}(x)) - f_{a,b}(y)|_2 < \varepsilon. \quad (5.2)$$

Since

$$f_{a,b}^{2n_0}(f_{a,b}(x)) = f_{a,b}(f_{a,b}^{2n_0+1}(x)),$$

and since, by Proposition 4.2, $f_{a,b}$ is an isometry on $\overline{B}_{\frac{1}{2^m}}(0)$, from (5.2) we obtain

$$|f_{a,b}^{2n_0+1}(x) - y|_2 = |f_{a,b}(f_{a,b}^{2n_0}(x)) - f_{a,b}(y)|_2 < \varepsilon.$$

Thus the orbit of every point of $B_{\frac{1}{2^m}}(0)$ under $f_{a,b}^2$ is dense in $B_{\frac{1}{2^m}}(0)$. Hence $f_{a,b}^2$ is minimal on $B_{\frac{1}{2^m}}(0)$.

It remains to prove (iii) $\implies$ (i). Let $x, y \in \overline{B}_{\frac{1}{2^m}}(0)$ and let $\varepsilon > 0$ be arbitrary. We consider the following cases.

If $x, y \in B_{\frac{1}{2^m}}(0)$, then, by the minimality of $f_{a,b}^2$ on $B_{\frac{1}{2^m}}(0)$, there exists $n_0 \geq 0$ such that

$$|f_{a,b}^{2n_0}(x) - y|_2 < \varepsilon.$$

Let $x \in S_{\frac{1}{2^m}}(0)$ and $y \in B_{\frac{1}{2^m}}(0)$. Since $f_{a,b}(x) \in B_{\frac{1}{2^m}}(0)$, by the minimality of $f_{a,b}^2$ on $B_{\frac{1}{2^m}}(0)$, there exists $n_0 \geq 0$ such that

$$|f_{a,b}^{2n_0}(f_{a,b}(x)) - y|_2 < \varepsilon.$$

Equivalently,

$$|f_{a,b}^{2n_0+1}(x) - y|_2 < \varepsilon.$$

Now let $x \in B_{\frac{1}{2^m}}(0)$ and $y \in S_{\frac{1}{2^m}}(0)$. Since $f_{a,b}(y) \in B_{\frac{1}{2^m}}(0)$, there exists $n_0 \geq 1$ such that

$$|f_{a,b}^{2n_0}(x) - f_{a,b}(y)|_2 < \varepsilon.$$

Using the isometry of $f_{a,b}$ on $\overline{B}_{\frac{1}{2^m}}(0)$, we get

$$|f_{a,b}^{2n_0-1}(x) - y|_2 = |f_{a,b}(f_{a,b}^{2n_0-1}(x)) - f_{a,b}(y)|_2 < \varepsilon.$$

Finally, let $x, y \in S_{\frac{1}{2^m}}(0)$. Since $f_{a,b}(x), f_{a,b}(y) \in B_{\frac{1}{2^m}}(0)$, there exists $n_0 \geq 0$ such that

$$|f_{a,b}^{2n_0}(f_{a,b}(x)) - f_{a,b}(y)|_2 < \varepsilon.$$

That is,

$$|f_{a,b}(f_{a,b}^{2n_0}(x)) - f_{a,b}(y)|_2 < \varepsilon.$$

Again, by the isometry of $f_{a,b}$ on $\overline{B}_{\frac{1}{2^m}}(0)$, we obtain

$$|f_{a,b}^{2n_0}(x) - y|_2 < \varepsilon.$$

Thus, for arbitrary $x, y \in \overline{B}_{\frac{1}{2^m}}(0)$ and every $\varepsilon > 0$, the positive orbit of x under $f_{a,b}$ comes ε -close to y . Therefore,

$$\overline{\mathcal{O}^+(x, f_{a,b})} = \overline{B}_{\frac{1}{2^m}}(0)$$

for every $x \in \overline{B}_{\frac{1}{2^m}}(0)$. Hence $f_{a,b}$ is minimal on $\overline{B}_{\frac{1}{2^m}}(0)$. $\square$

Since ergodicity in the above sense is equivalent to minimality, the conclusions of Lemma 5.1 and Lemma 5.2 can be summarized in the following theorem.

Theorem 5.2. *Let $|a|_2 = |b|_2 = 2^m > 1$. Consider the probability space $(\overline{B}_{\frac{1}{2^m}}(0), \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ -algebra on the metric space $(\overline{B}_{\frac{1}{2^m}}(0), |\cdot|_2)$ and μ is the normalized Haar measure. Then the following statements equivalent:*

- (i) $f_{a,b}$ is ergodic on $\overline{B}_{\frac{1}{2^m}}(0)$;
- (ii) $|b + a - 1|_2 = |2a|_2$.

We now study the ergodicity of $f_{a,b}$ on each invariant sphere $S_{\frac{1}{2^n}}(0)$ under the assumption that case (ii) of Proposition 4.2 holds.

Theorem 5.3. *Let $|a|_2 = |b|_2 = 2^m$, $m > 1$. For $n \in \{1, \dots, m-1\}$ consider the probability space $(S_{\frac{1}{2^n}}(0), \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ -algebra on the metric space $(S_{\frac{1}{2^n}}(0), |\cdot|_2)$ and μ is the normalized Haar measure.*

Then the map $f_{a,b}$ is ergodic on $S_{\frac{1}{2^n}}(0)$ if and only if the following conditions hold:

- (a) $|b - a|_2 < |2a|_2$;
- (b) $n = m - 1$;
- (c) $n \neq -\text{ord}_2(b - a)$.

Proof: Let $n \in \{1, \dots, m-1\}$ and assume that $n \neq -\text{ord}_2(b - a)$. Define a homeomorphism $h : S_{\frac{1}{2^n}}(0) \rightarrow \mathbb{Z}_2^*$ by

$$h(x) = \frac{x}{2^n}.$$

Consider the conjugate map $g := h \circ f_{a,b} \circ h^{-1}$, that is,

$$g(x) = \frac{2^n x^2 + \frac{b}{a}x + \frac{1}{2^n a}}{\frac{4^n}{a}x^2 + \frac{2^n b}{a}x + 1}.$$

It is straightforward to verify that the set $\mathbb{Z}_2^*$ is invariant under both the numerator and denominator polynomials

$$P(x) = 2^n x^2 + \frac{b}{a}x + \frac{1}{2^n a}, \quad Q(x) = \frac{4^n}{a}x^2 + \frac{2^n b}{a}x + 1.$$

Moreover, one can check that

$$Q_1 \equiv \begin{cases} 0 \pmod{4}, & \text{if } n > 1, \\ 2 \pmod{4}, & \text{if } n = 1, \end{cases} \quad Q_2 \equiv 1 \pmod{4}.$$

Then, by Theorem 5.1, the dynamical system $(\mathbb{Z}_2^*, g)$ is ergodic if and only if

$$P_1 \equiv 1 \pmod{4},$$

and

$$P_2 \equiv \begin{cases} 0 \pmod{4}, & \text{if } n = 1, \\ 2 \pmod{4}, & \text{if } n > 1. \end{cases}$$

A direct computation shows that the condition $P_1 \equiv 1 \pmod{4}$ is equivalent to

$$|b - a|_2 < |2a|_2.$$

Furthermore, if $n = 1$, then $P_2 \equiv 0 \pmod{4}$ is equivalent to $|a|_2 = 4$, while for $n > 1$, the condition $P_2 \equiv 2 \pmod{4}$ is equivalent to

$$|a|_2 = 2^{n+1}.$$

Thus, we conclude that $(\mathbb{Z}_2^*, g)$ is ergodic if and only if the following conditions hold:

- $|b - a|_2 < |2a|_2$;
- $n = m - 1$.

Finally, suppose that $n = -\text{ord}_2(b - a)$. In this case, the sphere $S_{\frac{1}{2^n}}(0)$ contains a fixed point x_- . Therefore, the system cannot be ergodic on this sphere. $\square$

Remark 5.1. In general, using the identity

$$f_{a,b}\left(\frac{1}{x}\right) = \frac{1}{f_{a,b}(x)},$$

one can readily extend the results of Theorems 5.2 and 5.3 to characterize the ergodicity of the map $f_{a,b}$ on $\mathbb{Q}_2 \setminus \mathbb{Z}_2$.

6. ACKNOWLEDGEMENT

The authors are deeply grateful to the anonymous referees for their careful and thorough reading of the manuscript. Their insightful comments, constructive criticism, and helpful suggestions have played an important role in improving the presentation, mathematical clarity, and overall quality of the paper.

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Local and 2-local $\frac{1}{2}$ -derivations on filiform-type associative algebras

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Abstract. This paper is devoted to local and 2-local $\frac{1}{2}$ -derivations of null-filiform, filiform, and naturally graded quasi-filiform associative algebras. We obtain explicit descriptions of the corresponding $\frac{1}{2}$ -derivations and local $\frac{1}{2}$ -derivations and show that non-derivation local maps occur throughout these classes. For 2-local maps, however, a sharp dichotomy appears: every 2-local $\frac{1}{2}$ -derivation of a null-filiform associative algebra is a $\frac{1}{2}$ -derivation, whereas every filiform and naturally graded quasi-filiform family considered here admits a 2-local $\frac{1}{2}$ -derivation which is not a $\frac{1}{2}$ -derivation.

Keywords: Associative algebras, filiform associative algebras, quasi-filiform associative algebras, $\frac{1}{2}$ -derivation, local $\frac{1}{2}$ -derivation, 2-local $\frac{1}{2}$ -derivation

MSC (2020): 16W25, 16P10, 16W50

1. INTRODUCTION

The concept of δ -derivations was introduced by V. Filippov for Lie algebras in [1, 2]. This class of linear mappings contains several important special cases, including ordinary derivations when $\delta = 1$, antiderivations when $\delta = -1$, and mappings related to the centroid. In [2], it was shown that prime Lie algebras usually do not possess nonzero δ -derivations for $\delta \neq 1, -1, 0, \frac{1}{2}$. In the exceptional case $\delta = \frac{1}{2}$, Filippov described all $\frac{1}{2}$ -derivations of prime Lie algebras satisfying suitable restrictions. In particular, if A is a central simple Lie algebra over a field of characteristic $p \neq 2, 3$ and admits a non-degenerate symmetric invariant bilinear form, then every $\frac{1}{2}$ -derivation of A is a scalar multiplication. Further results on δ -derivations of prime alternative and non-Lie Mal'tsev algebras were obtained in [3].

Local derivations form another important generalization of derivations. They were introduced by R. V. Kadison and independently by D. R. Larson and A. R. Sourour in the study of operator algebras [4, 5]. Since then, local derivations have been investigated for many classes of associative, Lie, Leibniz, and related algebras; see, for instance, [6, 7, 8, 9, 10, 11]. One of the central problems in this direction is to determine whether every local derivation on a given algebra is an ordinary derivation. For example, in [8] it was proved that each local derivation on a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero is a derivation. On the other hand, solvable and nilpotent algebras may admit local derivations which are not derivations. Such phenomena were studied for solvable Lie algebras in [10, 11], for conformal Galilei and Schrödinger type Lie algebras in [7, 12, 13], and for several classes of Leibniz algebras in [9, 6].

The finite-dimensional theory has also been complemented by a number of results for infinite-dimensional Lie algebras. In particular, local derivations on locally simple Lie algebras, generalized Witt algebras, Witt algebras, and Witt algebras over fields of prime characteristic were considered in [14, 15, 16]. In these settings, it was shown that, for the classes under consideration, every local derivation is a derivation.

A further nonlinear version of this idea was introduced by P. Šemrl in [17] in the form of 2-local automorphisms and 2-local derivations. This notion has attracted considerable attention in the theory of Lie algebras and related structures [18, 14, 19, 20, 21]. For instance, [18] proves that every 2-local derivation on a finite-dimensional semisimple Lie algebra is a derivation, while every finite-dimensional nilpotent Lie algebra of dimension greater than two admits a 2-local derivation which is not a derivation. Similar rigidity results have been obtained for several finite- and infinite-dimensional Lie algebras, including semisimple Lie algebras, Witt algebras, Virasoro algebras, Schrödinger-Virasoro algebras, Jacobson-Witt algebras, and planar Galilean conformal algebras.

Derivations, local derivations, and 2-local derivations on filiform associative algebras were studied in [22]. In that work, local and 2-local derivations of filiform associative algebras were introduced

and described. The study of local and 2-local δ -derivations was developed in [23], where local and 2-local $\frac{1}{2}$ -derivations of finite-dimensional solvable Lie algebras with filiform, Heisenberg, and abelian nilradicals were investigated. Moreover, local $\frac{1}{2}$ -derivations of oscillator Lie algebras, conformal perfect Lie algebras, and Schrödinger algebras were described. Related results for naturally graded quasi-filiform Leibniz algebras of type I were obtained by B. Yusupov, N. Vaisova, and T. Madрахimov in [24], where it was shown that such algebras may admit local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations. Local and 2-local $\frac{1}{2}$ -derivations of solvable Leibniz algebras were further studied in [25]. More recently, explicit descriptions of $\frac{1}{2}$ -derivations and local $\frac{1}{2}$ -derivations on naturally graded filiform Lie and Leibniz algebras were obtained in [26]. Naturally graded non-Lie p -filiform Leibniz algebras were considered in [27], where it was shown that, in general, they admit local and 2-local $\frac{1}{2}$ -derivations which are not ordinary $\frac{1}{2}$ -derivations.

The aim of the present paper is to investigate $\frac{1}{2}$ -derivations, local $\frac{1}{2}$ -derivations, and 2-local $\frac{1}{2}$ -derivations on null-filiform, filiform, and naturally graded quasi-filiform associative algebras. We obtain explicit descriptions of $\frac{1}{2}$ -derivations and local $\frac{1}{2}$ -derivations for these algebras. It is shown that, in general, these algebras admit local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations. We also prove that every 2-local $\frac{1}{2}$ -derivation of a null-filiform associative algebra is a $\frac{1}{2}$ -derivation, whereas filiform and naturally graded quasi-filiform associative algebras admit 2-local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations.

Thus the null-filiform case is rigid with respect to 2-local $\frac{1}{2}$ -derivations, while the filiform and quasi-filiform cases exhibit genuinely nonlinear 2-local phenomena. This distinction is one of the principal structural conclusions of the paper.

The paper is organized as follows. In Section 2, we describe the $\frac{1}{2}$ -derivations of null-filiform, filiform, and naturally graded quasi-filiform associative algebras. Section 3 is devoted to local and 2-local $\frac{1}{2}$ -derivations. In Subsection 3.1, we give descriptions of local $\frac{1}{2}$ -derivations and show the existence of local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations. In Subsection 3.2, we prove the corresponding results for 2-local $\frac{1}{2}$ -derivations.

1.1. Preliminaries. For an algebra $\mathbf{A}$ from an arbitrary variety, we consider the series

$$\mathbf{A}^1 = \mathbf{A}, \quad \mathbf{A}^{i+1} = \sum_{k=1}^i \mathbf{A}^k \mathbf{A}^{i+1-k}, \quad i \geq 1.$$

We say that an algebra $\mathbf{A}$ is nilpotent if $\mathbf{A}^i = 0$ for some $i \in \mathbb{N}$. The smallest integer satisfying $\mathbf{A}^i = 0$ is called the nilpotency index, or nilindex, of $\mathbf{A}$.

Definition 1.1. An n -dimensional algebra $\mathbf{A}$ is called null-filiform if $\dim \mathbf{A}^i = (n+1)-i$, $1 \leq i \leq n+1$.

It is easy to see that an algebra has a maximum nilpotency index if and only if it is null-filiform. For a nilpotent algebra, the condition of null-filiformity is equivalent to the condition that the algebra is one-generated.

All null-filiform associative algebras were described in [28, 29]:

Theorem 1.1. *An arbitrary n -dimensional null-filiform associative algebra is isomorphic to the algebra:*

$$\mu_0^n: \quad e_i e_j = e_{i+j}, \quad 2 \leq i+j \leq n,$$

where $\{e_1, e_2, \dots, e_n\}$ is a basis of the algebra μ_0^n .

The following classes of nilpotent algebras are filiform and quasi-filiform algebras whose nilindex equals n and $n-1$, respectively.

Definition 1.2. An n -dimensional algebra is called filiform if $\dim(\mathbf{A}^i) = n-i$, $2 \leq i \leq n$.

Definition 1.3. An n -dimensional associative algebra $\mathbf{A}$ is called a quasi-filiform algebra if $\mathbf{A}^{n-2} \neq 0$ and $\mathbf{A}^{n-1} = 0$.

Definition 1.4. Given a nilpotent associative algebra $\mathbf{A}$, put $\mathbf{A}_i = \mathbf{A}^i / \mathbf{A}^{i+1}$, $1 \leq i \leq k-1$, and define

$$\text{gr}(\mathbf{A}) = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \cdots \oplus \mathbf{A}_k.$$

The product induced by the multiplication of $\mathbf{A}$ satisfies $\mathbf{A}_i \mathbf{A}_j \subseteq \mathbf{A}_{i+j}$, and therefore $\text{gr}(\mathbf{A})$ is a graded algebra. If $\text{gr}(\mathbf{A}) \cong \mathbf{A}$, then $\mathbf{A}$ is called a naturally graded algebra.

All filiform and naturally graded quasi-filiform associative algebras were classified in [30].

Theorem 1.2 ([30]). *Every n -dimensional ($n > 3$) complex filiform associative algebra is isomorphic to one of the following pairwise non-isomorphic algebras with basis $\{e_1, e_2, \dots, e_n\}$:*

$$\begin{array}{ll} \mu_{1,1}^n(\mu_0^{n-1} \oplus \mathbb{C}) & : e_i e_j = e_{i+j}, \\ \mu_{1,2}^n & : e_i e_j = e_{i+j}, \quad e_n e_n = e_{n-1}, \\ \mu_{1,3}^n & : e_i e_j = e_{i+j}, \quad e_1 e_n = e_{n-1}, \\ \mu_{1,4}^n & : e_i e_j = e_{i+j}, \quad e_1 e_n = e_{n-1}, \quad e_n e_n = e_{n-1} \end{array}$$

where $2 \leq i+j \leq n-1$.

Theorem 1.3 ([30]). *Let $\mathbf{A}$ be an n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebra. Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{ll} \mu_{2,1}^n & : e_i e_j = e_{i+j}, \quad e_{n-1} e_1 = e_n \\ \mu_{2,2}^n(\alpha) & : e_i e_j = e_{i+j}, \quad e_1 e_{n-1} = e_n, \quad e_{n-1} e_1 = \alpha e_n \\ \mu_{2,3}^n & : e_i e_j = e_{i+j}, \quad e_{n-1} e_{n-1} = e_n \\ \mu_{2,4}^n & : e_i e_j = e_{i+j}, \quad e_1 e_{n-1} = e_n, \quad e_{n-1} e_{n-1} = e_n \end{array}$$

where $\alpha \in \mathbb{C}$ and $2 \leq i+j \leq n-2$.

Definition 1.5. Let A be an algebra over a field $\mathbb{F}$, and let $D : A \rightarrow A$ be a linear map. The map D is called a δ -derivation if

$$D(xy) = \delta(D(x)y + xD(y))$$

for all $x, y \in A$.

Note that a 1-derivation is an ordinary derivation, and a (-1) -derivation is called an antiderivation. The set of all δ -derivations of an algebra A is denoted by $\text{Der}_\delta(A)$.

Definition 1.6. A linear operator $\Delta : A \rightarrow A$ is called a local δ -derivation if, for any $x \in A$, there exists a δ -derivation $D_x : A \rightarrow A$ depending on x such that $\Delta(x) = D_x(x)$. The set of all local δ -derivations on A is denoted by $\text{LocDer}_\delta(A)$.

Definition 1.7. A map $\nabla : A \rightarrow A$ (not necessarily linear) is called a 2-local δ -derivation if, for any $x, y \in A$, there exists a δ -derivation $D_{x,y} \in \text{Der}_\delta(A)$ such that

$$\nabla(x) = D_{x,y}(x), \quad \nabla(y) = D_{x,y}(y).$$

It should be noted that every 2-local δ -derivation is homogeneous. Indeed, for any $x \in A$ and any scalar λ , we have

$$\nabla(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \nabla(x).$$

In this work, we focus on local and 2-local $\frac{1}{2}$ -derivations. The main examples of $\frac{1}{2}$ -derivations are scalar multiplications, that is, maps of the form $D(x) = \lambda x$ for all $x \in A$. Such $\frac{1}{2}$ -derivations are called trivial $\frac{1}{2}$ -derivations.

2. $\frac{1}{2}$ -DERIVATIONS ON NULL-FILIFORM, FILIFORM AND QUASI-FILIFORM ASSOCIATIVE ALGEBRAS

Now we study $\frac{1}{2}$ -derivations on n -dimensional null-filiform, n -dimensional ($n > 3$) complex filiform and n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebras.

Proposition 2.1. $\frac{1}{2}$ -derivations of n -dimensional complex null-filiform associative algebras are given as follows:

$$D(e_i) = \sum_{k=i}^n \alpha_{k-i+1} e_k, \quad 1 \leq i \leq n,$$

Proof: Write

$$D(e_1) = \sum_{r=1}^n \alpha_r e_r.$$

Since $e_2 = e_1 e_1$, the $\frac{1}{2}$ -derivation identity gives

$$\begin{aligned} D(e_2) &= \frac{1}{2}(D(e_1)e_1 + e_1 D(e_1)) \\ &= \frac{1}{2} \left(\sum_{r=1}^{n-1} \alpha_r e_{r+1} + \sum_{r=1}^{n-1} \alpha_r e_{r+1} \right) = \sum_{k=2}^n \alpha_{k-1} e_k. \end{aligned}$$

Assume inductively that

$$D(e_i) = \sum_{k=i}^n \alpha_{k-i+1} e_k$$

for some $1 \leq i < n$. Using $e_{i+1} = e_i e_1$, we obtain

$$\begin{aligned} D(e_{i+1}) &= \frac{1}{2}(D(e_i)e_1 + e_i D(e_1)) \\ &= \frac{1}{2} \left(\sum_{k=i}^{n-1} \alpha_{k-i+1} e_{k+1} + \sum_{r=1}^{n-i} \alpha_r e_{i+r} \right) = \sum_{k=i+1}^n \alpha_{k-i} e_k. \end{aligned}$$

This proves the displayed formula for all i .

Conversely, let D have the displayed form. If $i + j \leq n$, then

$$\frac{1}{2}(D(e_i)e_j + e_i D(e_j)) = \sum_{k=i+j}^n \alpha_{k-i-j+1} e_k = D(e_{i+j}) = D(e_i e_j).$$

If $i + j > n$, both sides vanish. By bilinearity, the identity holds for all elements of μ_0^n , and hence D is a $\frac{1}{2}$ -derivation. $\square$

Proposition 2.2. $\frac{1}{2}$ -derivations of the n -dimensional ($n > 3$) complex filiform associative algebras are given as follows:

$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$	
$\mu_{1,1}^n : D(e_i) = \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k, \quad 2 \leq i \leq n-1,$	
$D(e_n) = \beta_{n-1} e_{n-1} + \beta_n e_n.$	
$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$	
$\mu_{1,2}^n : D(e_i) = \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k, \quad 2 \leq i \leq n-1,$	
$D(e_n) = \alpha_n e_{n-2} + \beta_{n-1} e_{n-1} + \alpha_1 e_n.$	
$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$	
$\mu_{1,3}^n : D(e_2) = \sum_{k=2}^{n-2} \alpha_{k-1} e_k + (\alpha_{n-2} + \frac{1}{2}\alpha_n) e_{n-1},$	
$D(e_i) = \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k, \quad 3 \leq i \leq n-1,$	
$D(e_n) = \beta_{n-1} e_{n-1} + \alpha_1 e_n.$	

$$\begin{aligned} D(e_1) &= \sum_{i=1}^n \alpha_i e_i, \\ \mu_{1,4}^n : \quad D(e_2) &= \sum_{k=2}^{n-2} \alpha_{k-1} e_k + (\alpha_{n-2} + \frac{1}{2} \alpha_n) e_{n-1}, \\ D(e_i) &= \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k, & 3 \leq i \leq n-1, \\ D(e_n) &= -\alpha_n e_{n-2} + \beta_{n-1} e_{n-1} + \alpha_1 e_n. \end{aligned}$$

Proof: Let us prove the proposition for the algebra $\mu_{1,1}^n$, and for the algebras $\mu_{1,2}^n$, $\mu_{1,3}^n$, $\mu_{1,4}^n$ the proofs are similar. Since the algebra $\mu_{1,1}^n$ has two generators $\{e_1, e_n\}$, any $\frac{1}{2}$ -derivation D on $\mu_{1,1}^n$ is completely determined by $D(e_1)$ and $D(e_n)$.

Let

$$D(e_1) = \sum_{i=1}^n \alpha_i e_i, \text{ and } D(e_n) = \sum_{j=1}^n \beta_j e_j.$$

Applying the $\frac{1}{2}$ -derivation identity, we have

$$\begin{aligned} D(e_2) &= D(e_1 e_1) = \frac{1}{2} (D(e_1) e_1 + e_1 D(e_1)) = \frac{1}{2} \left(\sum_{i=1}^n \alpha_i e_i e_1 + \sum_{i=1}^n \alpha_i e_1 e_i \right) \\ &= \frac{1}{2} \left(\sum_{i=2}^{n-1} \alpha_{i-1} e_i + \sum_{i=2}^{n-1} \alpha_{i-1} e_i \right) = \sum_{i=2}^{n-1} \alpha_{i-1} e_i. \end{aligned}$$

$$\begin{aligned} D(e_3) &= D(e_2 e_1) = \frac{1}{2} (D(e_2) e_1 + e_2 D(e_1)) = \frac{1}{2} \left(\sum_{i=2}^{n-1} \alpha_{i-1} e_i e_1 + \sum_{i=1}^n \alpha_i e_2 e_i \right) \\ &= \frac{1}{2} \left(\sum_{i=3}^{n-1} \alpha_{i-2} e_i + \sum_{i=3}^{n-1} \alpha_{i-2} e_i \right) = \sum_{i=3}^{n-1} \alpha_{i-2} e_i. \end{aligned}$$

Generally, one can prove by induction

$$D(e_i) = \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k, \quad 2 \leq i \leq n-1.$$

Suppose the above is true for some i . Now we can determine for $i+1$.

$$\begin{aligned} D(e_{i+1}) &= D(e_i e_1) = \frac{1}{2} (D(e_i) e_1 + e_i D(e_1)) = \frac{1}{2} \left(\sum_{k=i}^{n-1} \alpha_{k-i+1} e_k e_1 + \sum_{k=1}^n \alpha_k e_i e_k \right) \\ &= \frac{1}{2} \left(\sum_{k=i+1}^{n-1} \alpha_{k-i} e_k + \sum_{k=i+1}^{n-1} \alpha_{k-i} e_k \right) = \sum_{k=i+1}^{n-1} \alpha_{k-i} e_k. \end{aligned}$$

Using the zero product $e_1 e_n = 0$, we obtain

$$0 = D(e_1 e_n) = \frac{1}{2} (D(e_1) e_n + e_1 D(e_n)) = \frac{1}{2} \left(\sum_{i=1}^n \alpha_i e_i e_n + \sum_{j=1}^n \beta_j e_1 e_j \right) = \frac{1}{2} \sum_{j=2}^{n-1} \beta_{j-1} e_j.$$

Therefore we have $\beta_j = 0$, $1 \leq j \leq n-2$. We complete the proof of Proposition. $\square$

Proposition 2.3. *The $\frac{1}{2}$ -derivations of the n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebras are given as follows:*

$\mu_{2,1}^n :$	$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$ $D(e_2) = \sum_{k=2}^{n-2} \alpha_{k-1} e_k + \frac{\alpha_{n-1}}{2} e_n,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 3 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n,$ $D(e_n) = \frac{1}{2}(\alpha_1 + \beta_{n-1}) e_n.$
$\mu_{2,2}^n(\alpha = 1) :$	$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$ $D(e_2) = \sum_{k=2}^{n-2} \alpha_{k-1} e_k + \alpha_{n-1} e_n,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 3 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-3} e_{n-3} + \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n,$ $D(e_n) = \frac{1}{2} \beta_{n-3} e_{n-2} + \frac{1}{2}(\alpha_1 + \beta_{n-1}) e_n.$
$\mu_{2,2}^n(\alpha = -1) :$	$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$ $D(e_2) = \sum_{k=2}^{n-2} \alpha_{k-1} e_k,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 3 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n,$ $D(e_n) = \frac{1}{2}(\alpha_1 + \beta_{n-1}) e_n.$
$\mu_{2,2}^n(\alpha \neq -1, 1) :$	$D(e_1) = \sum_{i=1}^n \alpha_i e_i,$ $D(e_2) = \sum_{k=2}^{n-2} \alpha_{k-1} e_k + \frac{1}{2}(1 + \alpha) \alpha_{n-1} e_n,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 3 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n,$ $D(e_n) = \frac{1}{2}(\alpha_1 + \beta_{n-1}) e_n.$
$\mu_{2,3}^n :$	$D(e_1) = \sum_{i=1}^{n-2} \alpha_i e_i + \alpha_n e_n,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 2 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n,$ $D(e_n) = \beta_{n-1} e_n.$
$\mu_{2,4}^n :$	$D(e_1) = \sum_{i=1}^{n-2} \alpha_i e_i + \alpha_n e_n,$ $D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1} e_k, \quad 2 \leq i \leq n-2,$ $D(e_{n-1}) = \beta_{n-2} e_{n-2} + \alpha_1 e_{n-1} + \beta_n e_n,$ $D(e_n) = \alpha_1 e_n.$

Proof: We give the details for $\mu_{2,1}^n$; the remaining cases are obtained by the same computation with their additional defining products. Write

$$D(e_1) = \sum_{r=1}^n \alpha_r e_r, \quad D(e_{n-1}) = \sum_{r=1}^n \gamma_r e_r.$$

Since $e_2 = e_1e_1$ and $e_{n-1}e_1 = e_n$, we obtain

$$\begin{aligned} D(e_2) &= \frac{1}{2}(D(e_1)e_1 + e_1D(e_1)) \\ &= \sum_{k=2}^{n-2} \alpha_{k-1}e_k + \frac{1}{2}\alpha_{n-1}e_n, \end{aligned}$$

and induction based on $e_i = e_{i-1}e_1$ gives

$$D(e_i) = \sum_{k=i}^{n-2} \alpha_{k-i+1}e_k, \quad 3 \leq i \leq n-2.$$

The zero product $e_1e_{n-1} = 0$ yields

$$0 = D(e_1e_{n-1}) = \frac{1}{2}(D(e_1)e_{n-1} + e_1D(e_{n-1})),$$

and hence $\gamma_1 = \dots = \gamma_{n-3} = 0$. Therefore

$$D(e_{n-1}) = \beta_{n-2}e_{n-2} + \beta_{n-1}e_{n-1} + \beta_n e_n.$$

Finally,

$$D(e_n) = D(e_{n-1}e_1) = \frac{1}{2}(D(e_{n-1})e_1 + e_{n-1}D(e_1)) = \frac{1}{2}(\alpha_1 + \beta_{n-1})e_n.$$

This is the first form in the statement.

For $\mu_{2,2}^n(\alpha)$, applying the identity to $e_1e_{n-1} = e_n$ and $e_{n-1}e_1 = \alpha e_n$ shows that

$$(\alpha - 1)\gamma_j = 0, \quad 1 \leq j \leq n-3.$$

If $\alpha = 1$, the zero products involving $e_{n-1}e_i$ for $i \geq 2$ leave only the possible extra coefficient γ_{n-3} ; if $\alpha \neq 1$, all these coefficients vanish. This gives the three $\mu_{2,2}^n(\alpha)$ rows displayed above. The defining products $e_{n-1}^2 = e_n$ and, respectively, $e_1e_{n-1} = e_n = e_{n-1}^2$ give the stated forms for $\mu_{2,3}^n$ and $\mu_{2,4}^n$.

Conversely, substitution into every nonzero defining product and into the zero products involving the generators verifies the $\frac{1}{2}$ -derivation identity. Bilinearity then completes the verification for arbitrary elements. $\square$

3. LOCAL AND 2-LOCAL $\frac{1}{2}$ -DERIVATIONS ON NULL-FILIFORM, FILIFORM AND QUASI-FILIFORM ASSOCIATIVE ALGEBRAS

In this section, we consider local and 2-local $\frac{1}{2}$ -derivations of null-filiform ($n > 2$), filiform ($n > 3$), and quasi-filiform ($n > 5$) algebras. All these algebras admit a local $\frac{1}{2}$ -derivation which is not a $\frac{1}{2}$ -derivation. At the same time we show that every 2-local $\frac{1}{2}$ -derivation of a null-filiform algebra is a $\frac{1}{2}$ -derivation. An example of a 2-local $\frac{1}{2}$ -derivation which is not a $\frac{1}{2}$ -derivation is given for filiform and quasi-filiform algebras.

3.1. Local $\frac{1}{2}$ -derivations on null-filiform, filiform and quasi-filiform associative algebras.

Now we study local $\frac{1}{2}$ -derivations on n -dimensional null-filiform, n -dimensional ($n > 3$) complex filiform and n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebras.

Theorem 3.1. *Let Δ be a linear map from a null-filiform associative algebra into itself. Then Δ is a local $\frac{1}{2}$ -derivation if and only if*

$$\Delta(e_i) = \sum_{j=i}^n c_{j,i}e_j, \quad 1 \leq i \leq n.$$

Proof: Let $\mathfrak{C}$ be the matrix of a local $\frac{1}{2}$ -derivation Δ on μ_0^n :

$$\mathfrak{C} = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n-1} & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n-1} & c_{2,n} \\ c_{3,1} & c_{3,2} & \cdots & c_{3,n-1} & c_{3,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-1,1} & c_{n-1,2} & \cdots & c_{n-1,n-1} & c_{n-1,n} \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n-1} & c_{n,n} \end{pmatrix}.$$

By the definition for every $x = \sum_{i=1}^n x_i e_i \in \mu_0^n$ there exists a $\frac{1}{2}$ -derivation D_x on μ_0^n such that

$$\Delta(x) = D_x(x).$$

By Proposition 2.1, the $\frac{1}{2}$ -derivation D_x has the following matrix form:

$$\mathfrak{C}_x = \begin{pmatrix} \alpha_1^x & 0 & 0 & \cdots & 0 & 0 \\ \alpha_2^x & \alpha_1^x & 0 & \cdots & 0 & 0 \\ \alpha_3^x & \alpha_2^x & \alpha_1^x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-2}^x & \alpha_{n-3}^x & \alpha_{n-4}^x & \cdots & 0 & 0 \\ \alpha_{n-1}^x & \alpha_{n-2}^x & \alpha_{n-3}^x & \cdots & \alpha_1^x & 0 \\ \alpha_n^x & \alpha_{n-1}^x & \alpha_{n-2}^x & \cdots & \alpha_2^x & \alpha_1^x \end{pmatrix}.$$

For the matrix $\mathfrak{C}$ of Δ by choosing subsequently $x = e_1, \dots, x = e_n$ and using $\Delta(x) = D_x(x)$, i.e. $\mathfrak{C}\bar{x} = D_x(\bar{x})$, where $\bar{x}$ is the vector corresponding to x , we obtain

$$\mathfrak{C} = \begin{pmatrix} c_{1,1} & 0 & \cdots & 0 & 0 \\ c_{2,1} & c_{2,2} & \cdots & 0 & 0 \\ c_{3,1} & c_{3,2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-2,1} & c_{n-2,2} & \cdots & 0 & 0 \\ c_{n-1,1} & c_{n-1,2} & \cdots & c_{n-1,n-1} & 0 \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n-1} & c_{n,n} \end{pmatrix}.$$

Conversely, suppose that the matrix of Δ has the above form and let us show that Δ is a local $\frac{1}{2}$ -derivation. For each x we must find a $\frac{1}{2}$ -derivation D_x such that $\Delta(x) = D_x(x)$. We have $\mathfrak{C}\bar{x} = \mathfrak{C}_x(\bar{x})$, where $\bar{x}$ is the coordinate vector of $x = \sum_{i=1}^n x_i e_i$. This implies the following system of equalities

$$\begin{aligned} c_{1,1}x_1 &= \alpha_1^x x_1, \\ c_{2,1}x_1 + c_{2,2}x_2 &= \alpha_2^x x_1 + \alpha_1^x x_2, \\ \cdots &\cdots \\ c_{n-1,1}x_1 + c_{n-1,2}x_2 + \cdots + c_{n-1,n-1}x_{n-1} &= \alpha_{n-1}^x x_1 + \alpha_{n-2}^x x_2 + \cdots + \alpha_1^x x_{n-1}, \\ c_{n,1}x_1 + c_{n,2}x_2 + \cdots + c_{n,n}x_n &= \alpha_n^x x_1 + \alpha_{n-1}^x x_2 + \cdots + \alpha_1^x x_n. \end{aligned}$$

Let us consider two cases separately:

Case 1: If $x_1 \neq 0$, then $\alpha_1^x = c_{1,1}$,

$$\alpha_i^x = c_{i,1} + \frac{1}{x_1} \left(\sum_{j=2}^i (c_{i,j} - \alpha_{i-j+1}^x) x_j \right), \quad 2 \leq i \leq n.$$

Case 2: If $x_1 = x_2 = \dots = x_{t-1} = 0$ and $x_t \neq 0$, then $\alpha_1^x = c_{t,t}$, $2 \leq t \leq n$

$$\alpha_{i-t+1}^x = c_{i,t} + \frac{1}{x_t} \left(\sum_{j=t+1}^i (c_{i,j} - \alpha_{i-j+1}^x) x_j \right) \quad t+1 \leq i \leq n.$$

Thus, the parameters α_i^x , $1 \leq i \leq n$, can be chosen so that the corresponding $\frac{1}{2}$ -derivation D_x satisfies $\Delta(x) = D_x(x)$. The proof is complete. $\square$

Theorem 3.2. *Let Δ be a linear map on an n -dimensional ($n > 3$) complex filiform associative algebra. Then Δ is a local $\frac{1}{2}$ -derivation if and only if*

- for the algebra $\mu_{1,1}^n$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) = \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_i) = \sum_{j=i}^{n-1} c_{j,i} e_j, \quad 2 \leq i \leq n-1 \\ \Delta(e_n) = c_{n-1,n} e_{n-1} + c_{n,n} e_n. \end{cases} \quad (3.1)$$

- for the algebra $\mu_{1,3}^n$, the same support conditions hold and

$$c_{i,i} = c_{n,n}, \quad 2 \leq i \leq n-2. \quad (3.2)$$

- for the algebras $\mu_{1,2}^n$, $\mu_{1,4}^n$

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) = \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_i) = \sum_{j=i}^{n-1} c_{j,i} e_j, \quad 2 \leq i \leq n-1 \\ \Delta(e_n) = c_{n-2,n} e_{n-2} + c_{n-1,n} e_{n-1} + c_{n,n} e_n. \\ c_{i,i} = c_{n,n}, \quad 2 \leq i \leq n-3. \end{cases} \quad (3.3)$$

Proof: Applying the local condition first to the basis vectors and using Proposition 2.2 gives the displayed support restrictions. Some additional relations are forced by sums of basis vectors. For example, in $\mu_{1,2}^n$, apply the local condition to $e_i + e_n$ with $2 \leq i \leq n-3$. In every $\frac{1}{2}$ -derivation, the coefficients of e_i in $D(e_i)$ and of e_n in $D(e_n)$ are both α_1 . Consequently,

$$c_{i,i} = c_{n,n}, \quad 2 \leq i \leq n-3.$$

The same argument applies to $\mu_{1,4}^n$. For $\mu_{1,3}^n$ there is no e_{n-2} component in $D(e_n)$ that can absorb the corresponding coefficient, so the relation extends to $i = n-2$. This proves necessity.

We now give the sufficiency argument in the two cases singled out above. For $\mu_{1,2}^n$, Proposition 2.2 gives

$$D(x) = x_1 \sum_{r=1}^n \alpha_r e_r + \sum_{i=2}^{n-1} x_i \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k + x_n (\alpha_n e_{n-2} + \beta_{n-1} e_{n-1} + \alpha_1 e_n).$$

If $x_1 \neq 0$, comparison with $\Delta(x)$ determines successively $\alpha_1, \alpha_2, \dots, \alpha_n$ and then β_{n-1} . Suppose $x_1 = 0$, and let t be the first index in $\{2, \dots, n-1\}$ for which $x_t \neq 0$. If $t \leq n-3$, set $\alpha_1 = c_{t,t} = c_{n,n}$ and solve successively for the remaining α 's; the free parameters α_n and β_{n-1} resolve the e_{n-2} and e_{n-1} equations when $x_n \neq 0$. If $t \in \{n-2, n-1\}$, first use the e_n equation when $x_n \neq 0$ and then the two preceding equations. If all $x_1, \dots, x_{n-1}$ vanish, choose $\alpha_1 = c_{n,n}$, $\alpha_n = c_{n-2,n}$, and $\beta_{n-1} = c_{n-1,n}$. Thus a suitable D_x exists in every case.

For $\mu_{1,4}^n$, the corresponding expression is

$$\begin{aligned} D(x) = & x_1 \sum_{r=1}^n \alpha_r e_r + x_2 \left(\sum_{k=2}^{n-2} \alpha_{k-1} e_k + \left(\alpha_{n-2} + \frac{1}{2} \alpha_n \right) e_{n-1} \right) \\ & + \sum_{i=3}^{n-1} x_i \sum_{k=i}^{n-1} \alpha_{k-i+1} e_k + x_n (-\alpha_n e_{n-2} + \beta_{n-1} e_{n-1} + \alpha_1 e_n). \end{aligned}$$

The same successive solution works: α_n solves the last equation when $x_1 \neq 0$ and the e_{n-2} equation when $x_1 = 0$ and $x_n \neq 0$; β_{n-1} then solves the e_{n-1} equation. The relations $c_{i,i} = c_{n,n}$ are exactly the compatibility conditions in the remaining cases. The arguments for $\mu_{1,1}^n$ and $\mu_{1,3}^n$ are identical, with the latter using its stated extended diagonal relation. Hence every map in the displayed classes is local. $\square$

Theorem 3.3. *A linear map Δ on an n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebra is a local $\frac{1}{2}$ -derivation if and only if it has the corresponding form below:*

- for the algebra $\mu_{2,1}^n$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_2) &= \sum_{j=2}^{n-2} c_{j,2} e_j + c_{n,2} e_n, \\ \Delta(e_i) &= \sum_{j=i}^{n-2} c_{j,i} e_j, \quad 3 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-2,n-1} e_{n-2} + c_{n-1,n-1} e_{n-1} + c_{n,n-1} e_n, \\ \Delta(e_n) &= c_{n,n} e_n. \end{cases}$$

- for the algebra $\mu_{2,2}^n(\alpha = 1)$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_2) &= \sum_{j=2}^{n-2} c_{j,2} e_j + c_{n,2} e_n, \\ \Delta(e_i) &= \sum_{j=i}^{n-2} c_{j,i} e_j, \quad 3 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-3,n-1} e_{n-3} + c_{n-2,n-1} e_{n-2} \\ &\quad + c_{n-1,n-1} e_{n-1} + c_{n,n-1} e_n, \\ \Delta(e_n) &= c_{n-2,n} e_{n-2} + c_{n,n} e_n. \end{cases}$$

- for the algebra $\mu_{2,2}^n(\alpha = -1)$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_2) &= \sum_{j=2}^{n-2} c_{j,2} e_j, \\ \Delta(e_i) &= \sum_{j=i}^{n-2} c_{j,i} e_j, \quad 3 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-2,n-1} e_{n-2} + c_{n-1,n-1} e_{n-1} + c_{n,n-1} e_n, \\ \Delta(e_n) &= c_{n,n} e_n. \end{cases}$$

- for the algebra $\mu_{2,2}^n(\alpha \neq -1, 1)$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= \sum_{j=1}^n c_{j,1} e_j, \\ \Delta(e_2) &= \sum_{j=2}^{n-2} c_{j,2} e_j + c_{n,2} e_n, \\ \Delta(e_i) &= \sum_{j=i}^{n-2} c_{j,i} e_j, \quad 3 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-2,n-1} e_{n-2} + c_{n-1,n-1} e_{n-1} + c_{n,n-1} e_n, \\ \Delta(e_n) &= c_{n,n} e_n. \end{cases}$$

- for the algebra $\mu_{2,3}^n$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= \sum_{j=1}^{n-2} c_{j,1} e_j + c_{n,1} e_n, \\ \Delta(e_i) &= \sum_{j=i}^{n-2} c_{j,i} e_j, \quad 2 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-2,n-1} e_{n-2} + c_{n-1,n-1} e_{n-1} + c_{n,n-1} e_n, \\ \Delta(e_n) &= c_{n,n} e_n. \end{cases}$$

- for the algebra $\mu_{2,4}^n$,

$$\text{LocDer}_{\frac{1}{2}} : \begin{cases} \Delta(e_1) &= c_{n,n} e_1 + \sum_{j=2}^{n-2} c_{j,1} e_j + c_{n,1} e_n, \\ \Delta(e_i) &= c_{n,n} e_i + \sum_{j=i+1}^{n-2} c_{j,i} e_j, \quad 2 \leq i \leq n-2, \\ \Delta(e_{n-1}) &= c_{n-2,n-1} e_{n-2} + c_{n,n} e_{n-1} + c_{n,n-1} e_n, \\ \Delta(e_n) &= c_{n,n} e_n. \end{cases}$$

Proof: For every basis vector e_i , choose a $\frac{1}{2}$ -derivation D_{e_i} such that $\Delta(e_i) = D_{e_i}(e_i)$. Proposition 2.3 then gives all the displayed support restrictions. In the case $\mu_{2,4}^n$, apply the local condition also to $e_i + e_n$ for $1 \leq i \leq n-2$ and to $e_{n-1} + e_n$. The diagonal coefficients of the two summands are governed by the same parameter α_1 ; hence

$$c_{1,1} = c_{2,2} = \cdots = c_{n-2,n-2} = c_{n-1,n-1} = c_{n,n}.$$

This proves necessity in every case.

Conversely, let Δ have one of the stated forms and fix $x = \sum_i x_i e_i$. Substituting the corresponding form from Proposition 2.3 into $D_x(x) = \Delta(x)$ gives a triangular system. Starting with the first nonzero coordinate among $x_1, \dots, x_{n-2}$ determines the chain parameters successively. The remaining equations involve only the free coefficients in $D_x(e_{n-1})$ and $D_x(e_n)$. For $\mu_{2,1}^n$ these are $\beta_{n-2}, \beta_{n-1}, \beta_n$; for $\mu_{2,2}^n(1)$ they are $\beta_{n-3}, \beta_{n-2}, \beta_{n-1}, \beta_n$; and in the other cases they are the three terminal coefficients displayed in Proposition 2.3. They solve the final two or three equations. If all chain coordinates vanish, those same terminal coefficients solve the system directly. In $\mu_{2,4}^n$, the common-diagonal relations above are precisely the compatibility conditions. Thus a suitable D_x exists for every x , proving sufficiency. $\square$

The dimensions of the spaces of $\frac{1}{2}$ -derivations and local $\frac{1}{2}$ -derivations are summarized in Table 1.

Corollary 3.1. *In each of the considered classes of null-filiform, filiform and naturally graded quasi-filiform non-split associative algebras, there exist local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations whenever the space of local $\frac{1}{2}$ -derivations is strictly larger than the space of $\frac{1}{2}$ -derivations.*

Remark 3.1. We also note that local $\frac{1}{2}$ -derivations of an arbitrary low-dimensional algebra can be similarly described using the fact that the $\frac{1}{2}$ -derivations have a common matrix form. The technique developed in this paper for constructing local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations can be applied to any low-dimensional algebra for which the $\frac{1}{2}$ -derivations have a common matrix form.

3.2. 2-local $\frac{1}{2}$ -derivations on null-filiform, filiform and quasi-filiform associative algebras. Now we study 2-local $\frac{1}{2}$ -derivations on n -dimensional null-filiform, n -dimensional ($n > 3$) complex filiform and n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebras.

Theorem 3.4. *Any 2-local $\frac{1}{2}$ -derivation on the null-filiform associative algebra μ_0^n is a $\frac{1}{2}$ -derivation.*

Proof: Consider a 2-local $\frac{1}{2}$ -derivation ∇ on μ_0^n such that $\nabla(e_1) = 0$.

TABLE 1. Dimensions of the spaces of $\frac{1}{2}$ -derivations and local $\frac{1}{2}$ -derivations.

Algebra	Dimension of the space of $\frac{1}{2}$ -derivations	Dimension of the space of local $\frac{1}{2}$ -derivations
μ_0^n	n	$\frac{n^2+n}{2}$
$\mu_{1,1}^n$	$n+2$	$\frac{n^2-n+6}{2}$
$\mu_{1,2}^n$	$n+1$	$\frac{n^2-3n+16}{2}$
$\mu_{1,3}^n$	$n+1$	$\frac{n^2-3n+12}{2}$
$\mu_{1,4}^n$	$n+1$	$\frac{n^2-3n+16}{2}$
$\mu_{2,1}^n$	$n+3$	$\frac{n^2-3n+16}{2}$
$\mu_{2,2}^n, \alpha = 1$	$n+4$	$\frac{n^2-3n+20}{2}$
$\mu_{2,2}^n, \alpha = -1$	$n+3$	$\frac{n^2-3n+14}{2}$
$\mu_{2,2}^n, \alpha \neq -1, 1$	$n+3$	$\frac{n^2-3n+16}{2}$
$\mu_{2,3}^n$	$n+2$	$\frac{n^2-3n+12}{2}$
$\mu_{2,4}^n$	$n+1$	$\frac{n^2-5n+14}{2}$

For the element $x = t_1 e_1 + t_2 e_2 + \cdots + t_n e_n \in \mu_0^n$, there is a $\frac{1}{2}$ -derivation $D_{e_1, x}$ such that

$$\nabla(e_1) = D_{e_1, x}(e_1), \quad \nabla(x) = D_{e_1, x}(x).$$

Then we have

$$0 = \nabla(e_1) = D_{e_1, x}(e_1) = \sum_{i=1}^n \alpha_i e_i.$$

Hence $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ and therefore $D_{e_1, x} = 0$. Since x is arbitrary, $\nabla(x) = 0$ for all $x \in \mu_0^n$, that is, $\nabla = 0$.

Now let ∇ be an arbitrary 2-local $\frac{1}{2}$ -derivation of the algebra μ_0^n . Then there is a $\frac{1}{2}$ -derivation D such that $\nabla(e_1) = D(e_1)$. The map $\nabla - D$ is also a 2-local $\frac{1}{2}$ -derivation and $(\nabla - D)(e_1) = 0$, which implies $\nabla \equiv D$. Hence ∇ is a $\frac{1}{2}$ -derivation. $\square$

Theorem 3.5. *The n -dimensional ($n > 3$) complex filiform associative algebras and the n -dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebras admit 2-local $\frac{1}{2}$ -derivations which are not $\frac{1}{2}$ -derivations.*

Proof: We first prove the theorem for the algebra $\mu_{1,1}^n$. Let us define a homogeneous non-additive function f on $\mathbb{C}^2$ by

$$f(z_1, z_n) = \begin{cases} \frac{z_1^2}{z_n}, & \text{if } z_n \neq 0, \\ 0, & \text{if } z_n = 0. \end{cases}$$

where $(z_1, z_n) \in \mathbb{C}^2$. Consider the map $\nabla : \mu_{1,1}^n \rightarrow \mu_{1,1}^n$ defined by the rule

$$\nabla(x) = f(x_1, x_n)e_n, \quad \text{where } x = \sum_{i=1}^n x_i e_i \in \mu_{1,1}^n.$$

Since f is not additive, ∇ is not a $\frac{1}{2}$ -derivation.

Let us show that ∇ is a 2-local $\frac{1}{2}$ -derivation. For the elements

$$x = \sum_{i=1}^n x_i e_i, \quad y = \sum_{i=1}^n y_i e_i,$$

we look for a $\frac{1}{2}$ -derivation D in the form:

$$D(e_1) = \alpha_n e_n, \quad D(e_i) = 0, \quad 2 \leq i \leq n-1, \quad D(e_n) = \beta_n e_n.$$

Assume that $\nabla(x) = D(x)$ and $\nabla(y) = D(y)$. Then we obtain the following system of equations for α_n and β_n :

$$\begin{cases} x_1 \alpha_n + x_n \beta_n = f(x_1, x_n), \\ y_1 \alpha_n + y_n \beta_n = f(y_1, y_n). \end{cases} \quad (3.4)$$

Case 1. Let $x_1 y_n - x_n y_1 = 0$. Since f is homogeneous, the right-hand side of the system is compatible, and therefore the system has a solution.

Case 2. Let $x_1 y_n - x_n y_1 \neq 0$. Then the coefficient matrix of the system is nonsingular, and hence the system has a unique solution.

For the remaining algebras, the same construction works with the two coordinates and the common output vector shown below:

Algebra	coordinates	output vector
$\mu_{1,2}^n, \mu_{1,3}^n, \mu_{1,4}^n$	(x_1, x_n)	e_{n-1}
$\mu_{2,1}^n, \mu_{2,2}^n(\alpha), \mu_{2,3}^n, \mu_{2,4}^n$	(x_1, x_{n-1})	e_n

Indeed, Proposition 2.2 contains, for each filiform algebra in the first row, all maps

$$D(x) = (ax_1 + bx_n)e_{n-1},$$

and Proposition 2.3 contains, for each quasi-filiform algebra in the second row, all maps

$$D(x) = (ax_1 + bx_{n-1})e_n.$$

Replacing (x_1, x_n) in the definition of f by the indicated coordinate pair therefore gives a 2-local $\frac{1}{2}$ -derivation in every case. Since f is not additive, none of these maps is a $\frac{1}{2}$ -derivation. The proof is complete. $\square$

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Computer imposition: *A.F. Aliyev*

The journal was registered by the Press and Information Agency of the Republic of Uzbekistan on December 22, 2006. Registration No. 0044.

Handed over to the set on 06/10/2026. Signed for printing on 16/10/2026.
Format 60×84 1/16. Literary typeface. Offset printing.

V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences
University str., 9, Tashkent 100174, Uzbekistan

Printed in a printing house “MERIT-PRINT”
Tashkent city, Yakkasaray district, Sh. Rustaveli street, 91