UZBEKISTAN ACADEMY OF SCIENCES
V.I.ROMANOVSKIY INSTITUTE OF MATHEMATICS

UZBEK MATHEMATICAL JOURNAL

Journal was founded in 1957. Until 1991 it was named by "Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk". Since 1991 it is known as "Uzbek Mathematical Journal". It has 4 issues annually.

Volume 69   Issue 3   2025

Uzbek Mathematical Journal is abstracting and indexing by

SCOPUS   EBSCO   CrossRef

MathSciNet   Zentralblatt   Math VINITI

TASHKENT - 2025

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.5-14
DOI: 10.29229/uzmj.2025-3-1

p-adic quasi-Gibbs measures for the three-state SOS model on the binary tree

Akhmedov O.U.

Abstract. In this paper, we study p-adic quasi-Gibbs measures for the three-state SOS model on a Cayley tree of order two. The existence of new translation-invariant p-adic quasi-Gibbs measures have been found. Moreover, the boundedness of the found translation-invariant p-adic quasi-Gibbs measures is proven. Furthermore, we show if $\left(\frac{\theta_s}{p}\right) = 1$, s is even and $p \equiv 1 (\bmod 6)$, then a phase transition occurs.

Keywords: Cayley tree, configuration, p-adic numbers, p-adic SOS model, p-adic quasi-Gibbs measure, periodic measure.

MSC (2020): 46S10, 12J12, 11S99, 30D05, 54H20.

1. Introduction

p-adic numbers, the first introduced by the German mathematician K. Hensel, have garnered significant attention within the mathematical community. Initially, regarded as objects of pure mathematics, these numbers have since found diverse applications in theoretical physics, including quantum mechanics, p-adic-valued physical observables [1, 2, 3], and many other areas [4]-[9].

One of the main motivations for studying statistical mechanics models on lattice systems (see, for example, [10]-[14]) is to explore the phenomenon of phase transitions. In ultrametric spaces, this phenomenon is marked by the existence of at least two distinct p-adic quasi-Gibbs measures one is bounded and the other is unbounded, making the analysis of the boundedness of these measures a natural focus of research.

The solid-on-solid (SOS) model serves as an extension of the Ising model ([15]-[23]) or as a less symmetric variant of the Potts model ([24]-[28]). For a comprehensive overview of SOS models on trees, see ([29]-[35]). Rigorous studies of the p-adic SOS model on the Cayley tree have largely concentrated on the cases $m = 2$ and $p = 3$. In particular, in [36] O. Khakimov derived a functional equation for the model using the p-adic version of the Kolmogorov extension theorem [37]. It was also rigorously demonstrated that under certain conditions, the model does not exhibit a phase transition. Furthermore, the conditions for the occurrence of a phase transition were identified through the analysis of the functional equation.

In [38], the one-dimensional p-adic SOS model with countable set of spin values was investigated, and it was shown that the set of all p-adic Gibbs measures has the cardinality of the continuum. This work revealed the existence of a quasi-phase transition in the one-dimensional p-adic SOS model.

In this paper, we enlarge the set of Gibbs measures, which is mostly studied in [36], for the three-state SOS model on the Cayley tree of order two. Namely, we show that if $\left(\frac{\theta_s}{p}\right) = 1$, s is even, $p \equiv 1 (\bmod 6)$, then for the three-state p-adic SOS model on the Cayley tree of order two there are four translation-invariant quasi-Gibbs measures (TIQGMs), otherwise there is no TIQGM. Moreover, we show the existence of the phase transition relying on the fact that one of quasi-Gibbs measures is bounded and four quasi-Gibbs measures are unbounded.

The primary focus of this paper is to extend these findings by studying translation-invariant quasi-Gibbs measures for the SOS model on a Cayley tree of order two. The paper is structured as follows: Section 2 provides essential definitions and established results. In Section 3, we introduce concepts related to the construction of p-adic quasi-Gibbs measures for the p-adic SOS model. In Section 4, we analyse the p-adic TIQGM for the SOS model. Finally, in Section 5, we study the boundedness of the obtained p-adic quasi-Gibbs measures for the SOS model.

2. Preliminaries

2.1. **p-adic numbers.** Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime number p, any rational number $x \neq 0$ can be represented as:

$$x = p^r \frac{n}{m}, \quad r, n, m \in \mathbb{Z},$$

where m and n are coprime with p (the greatest common divisors $(p, n) = 1$ and $(p, m) = 1$). The p-adic norm of a number $x \in \mathbb{Q}$ is defined by the formula (see [39]):

$$| x |_p = \begin{cases} p^{-r}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

The set of p-adic integers and p-adic unit denoted by $\mathbb{Z}_p$ and $\mathbb{Z}_p^*$, respectively, as follows

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\},$$

$$\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}.$$

Any p-adic number $x \neq 0$ can be represented by $x = \frac{x^*}{|x|_p}$, where $x^* \in \mathbb{Z}_p^*$. The canonical expansion for the p-adic number x and p-adic unit x^* is given by (2.1) and (2.2) below,

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \dots), \tag{2.1}$$

$$x^* = x_0 + x_1 p + x_2 p^2 + \dots, \tag{2.2}$$

where $1 \leq x_0 \leq p - 1$ and $0 \leq x_i \leq p - 1$ for $i \in \mathbb{N}$ ([39]-[40]).

Due to [40], the equation $x^2 = a$, for

$$a = p^{\gamma(a)}(a_0 + a_1 p + a_2 p^2 + \dots) \neq 0, \quad a_0 \neq 0, \quad a_j \in \{0, 1, \dots, p-1\},\ j \in \mathbb{N},$$

has a solution $x \in \mathbb{Q}_p$ if and only if the following conditions are satisfied:

(1) The number $\gamma(a)$ is even;

(2) If $p \neq 2$, the congruence $y^2 \equiv a_0 (\operatorname{mod} p)$ is solvable; if $p = 2$, the equality $a_1 = a_2 = 0$ holds.

A more general type of the equation was studied in [41, 42].

In [41], new symbols "O" and "o" were introduced, which allowed simplifying some calculations. These symbols replace the notation "$\equiv \pmod{p^s}$" without paying attention to the degree s. A given p-adic number x the symbol $O[x]$ means a p-adic number with norm $p^{-\gamma(x)}$, i.e., $|x|_p = |O(x)|_p$. The symbol $o[x]$ means a p-adic number with norm strictly less than $p^{-\gamma(x)}$, i.e., $|o(x)|_p < |x|_p$. It is easy to see that $y = O[x]$ if and only if $x = O[y]$.

Let $a \in \mathbb{Q}_p$ and $r > 0$, we introduce the notation

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}.$$

The *p-adic exponential* is defined by the series

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Note that the set

$$\mathcal{E}_p = \{x \in \mathbb{Q}_p : |x - 1|_p < p^{-1/(p-1)}\},$$

is the range of the p-adic exponential (see [40]).

2.2. **p-adic Measure and Cayley Tree.** Let X be any nonempty set, $\mathcal{B}$ be an algebra of subsets in X, and $(X,\mathcal{B})$ be a measurable space. A function $\mu:\mathcal{B}\to\mathbb{Q}_p$ is called a *p-adic measure* if for any collection of sets $A_1,\dots,A_n\in\mathcal{B}$ such that $A_i\cap A_j=\varnothing$ for $i\neq j$, the equality

$$\mu\left(\bigcup_{j=1}^{n}A_j\right)=\sum_{j=1}^{n}\mu(A_j)$$

holds; a p-adic measure is called *probabilistic* if $\mu(X)=1$; a p-adic measure is called *bounded* (see [3]), if it satisfies the following condition

$$\sup_{A\in\mathcal{B}}|\mu(A)|_p<\infty.$$

The Cayley tree Γ^k of order $k\geq 1$ is an infinite tree i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Denote by V the set of vertices, and by L the set of edges of the Cayley tree Γ^k. Two vertices x and y are called *nearest neighbours* if there exist an edge $l\in L$ connecting them and denote by $l=\langle x,y\rangle$.

Fix $x_0\in\Gamma^k$ and given vertex x, denote by $|x|$ the number of edges in the shortest path connecting x_0 and x. For $x,y\in\Gamma^k$, denote by $d(x,y)$ the number of edges in the shortest path connecting x and y. For $x,y\in\Gamma^k$,we write $x\leq y$ if x belongs to the shortest path connecting x_0 with y, and we write $x<y$ if $x\leq y$ and $x\neq y$. If $x\leq y$ and $|y|=|x|+1$, then we write $x\to y$.

We set

$$W_n=\{x\in V:|x|=n\},\quad V_n=\{x\in V:|x|\leq n\},\quad L_n=\{l=\langle x,y\rangle\in L:x,y\in V_n\},$$

$$S(x)=\{y\in V:x\to y\},\quad S_1(x)=\{y\in V:d(x,y)=1\}.$$

The set $S(x)$ is called the set of direct successors of the vertex x.

3. Construction of p-adic Gibbs measures for the p-adic SOS model

We consider p-adic SOS model on the Cayley tree. Let $\mathbb{Q}_p$ be a field of p-adic numbers and $\Phi=\{0,1,..,m\}$. A configuration σ on V is defined by the function $x\in V\to\sigma(x)\in\Phi$. Similarly, one can define the configuration σ_n and $\sigma^{(n)}$ on V_n and W_n, respectively. The set of all configurations on V (resp. V_n, W_n) is denoted by $\Omega=\Phi^V$ (resp. $\Omega_{V_n}=\Phi^{V_n}$, $\Omega_{W_n}=\Phi^{W_n}$).

For given configurations $\sigma_{n-1}\in\Omega_{V_{n-1}}$ and $\varphi^{(n)}\in\Omega_{W_n}$ we define a configuration in Ω_{V_n} as follows

$$(\sigma_{n-1}\vee\varphi^{(n)})(x)=\begin{cases}\sigma_{n-1}(x), & \text{if } x\in V_{n-1},\\ \varphi^{(n)}(x), & \text{if } x\in W_n.\end{cases}$$

A formal p-adic Hamiltonian $H:\Omega\to\mathbb{Q}_p$ of the SOS model is defined by

$$H_n(\sigma)=J\sum_{\langle x,y\rangle\in L_n}|\sigma(x)-\sigma(y)|_\infty,\quad \sigma\in\Omega_{V_n},\tag{3.1}$$

where $J\in B(0,p^{-1/(p-1)})$ is the coupling constant, L_n is the set of edges in V_n, and $|\cdot|_\infty$ denotes the usual absolute value.

We define a function $z:x\to z_x$, $\forall x\in V\setminus\{x_0\}$, $z_x\in\mathbb{Q}_p$ and consider p-adic probability distribution $\mu_z^{(n)}$ on Ω_{V_n} defined by

$$\mu_z^{(n)}(\sigma_n)=\frac{1}{Z_n^{(z)}}\exp_p\{H_n(\sigma_n)\}\prod_{x\in W_n}z_{\sigma(x),x}\quad n=1,2,\dots,\tag{3.2}$$

where $Z_n^{(z)}$ is the normalizing constant

$$Z_n^{(z)}=\sum_{\varphi\in\Omega_{V_n}}\exp_p\{H_n(\varphi)\}\prod_{x\in W_n}z_{\sigma(x),x}.\tag{3.3}$$

A p-adic probability distribution $\mu_z^{(n)}$ is said to be consistent if for all $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, we have

$$\sum_{\varphi\in\Omega_{W_n}} \mu_z^{(n)}(\sigma_{n-1}\vee\varphi) = \mu_z^{(n-1)}(\sigma_{n-1}). \tag{3.4}$$

In this case, by the p-adic analogue of the Kolmogorov theorem [4], there exists a unique measure μ_z on the set Ω such that $\mu_z\left(\{\sigma|_{V_n} \equiv \sigma_n\}\right) = \mu_z^{(n)}\left(\sigma_n\right)$ for all n and $\sigma_n \in \Omega_{V_n}$.

If for some function z the measures $\mu_z^{(n)}$ satisfy the consistency condition, then there exists a unique p-adic probability measure, denoted by μ_z, since it depends on z. Such a measure μ_z is called *p-adic quasi-Gibbs measure*, corresponding to the p-adic SOS model. By $\mathcal{QG}(H)$ we denote the set of all p-adic quasi-Gibbs measures associated with functions $z = \{\tilde{z}_x, x \in V\}$. If there exist two different functions s and z defined on V, such that there exist corresponding measures μ_s, μ_z, one is bounded, and the other is unbounded, then it is said that there exists a *phase transition* (see [45]).

It is known [36] that the p-adic distributions $\mu_{\tilde{z}_x}^{(n)}(\sigma), n = 1, 2, ...,$ defined in (3.2), are consistent for the p-adic SOS model if and only if for every $x \in V \setminus \{x^0\}$, the following system of equations holds:

$$z_{i,x} = \prod_{y\in S(x)} \frac{\sum_{j=0}^{m-1}\theta^{|i-j|_\infty} z_{j,y} + \theta^{m-i}}{\sum_{j=0}^{m-1}\theta^{m-j} z_{j,y} + 1}, \qquad i = 0, 1, ..., m-1, \tag{3.5}$$

here, $\theta = \exp_p\{J\}$ and $z_{i,x} = \tilde{z}_{i,x}/\tilde{z}_{m,x}$ for $i = 0, 1, ..., m-1$.

It follows that for any function $z = \{z_x, x \in V\}$ satisfying condition (3.5), there exists a unique p-adic quasi-Gibbs measure μ.

4. Existence of the p-adic TIQGM

Let G_k be the free product of $k+1$ cyclic groups of order two with corresponding generators $a_1, a_2, \ldots, a_{k+1}$. It is known that there is a one-to-one correspondence between the set V of vertices of the Cayley tree Γ^k and the group G_k (see [31, 47]).

Let G_k^* be a normal subgroup of the group G_k. A function z_x defined for $x \in G_k$ is called *G_k^*-periodic* if $z_{yx} = z_x$ for all $x \in G_k$ and $y \in G_k^*$. A G_k-periodic function is termed *translation-invariant* (TI). Consider the set of all p-adic TIQGM for the model (3.1). Note that this set is contained in $\mathcal{QG}(H)$, but its description for arbitrary m poses a challenging task.

For the system (3.5), the translation-invariant solutions take the form $z_x = z = (z_0, z_1, z_2, ..., z_m) \in \mathbb{Q}_p^{m+1}$ for all $x \in V$. Let $m = 2$, i.e., the spin takes values 0, 1, and 2. In this case, from (3.5), we obtain

$$\begin{cases} z_0 = \left(\frac{z_0+\theta z_1+\theta^2}{\theta^2 z_0+\theta z_1+1}\right)^k, \\ z_1 = \left(\frac{\theta z_0+z_1+\theta}{\theta^2 z_0+\theta z_1+1}\right)^k. \end{cases} \tag{4.1}$$

In this subsection we study (4.1) in the case $k = 2$. Introducing the notations $x = \sqrt{z_0}$, $y = \sqrt{z_1}$ from (4.1) we have

$$\begin{cases} x = \frac{x^2+\theta\, y^2+\theta^2}{\theta^2 x^2+\theta\, y^2+1}, \\ y = \frac{\theta x^2+y^2+\theta}{\theta^2 x^2+\theta y^2+1}. \end{cases} \tag{4.2}$$

We remark that in [36], the author studied the solutions belong to $\mathcal{E}_p$ of the system of equations (4.1) and it was shown that for $m = 2$ and $k = 2$, $p \neq 3$ there does not exists any solution in the set $\mathcal{E}_p$. In this work, we study the system of equations (4.1) in the $\mathbb{Q}_p$.

We rewrite the system (4.2) as

$$\begin{cases} (x-1)\,(\theta^2x^2+\theta^2x+\theta\, y^2+\theta^2-x) = 0, \\ \theta\, y^3-y^2+(\theta^2x^2+1)\, y-(x^2+\theta) = 0. \end{cases} \tag{4.3}$$

The system (4.3) can be separated as follows:

$$\begin{cases} x-1=0, \\ \theta\, y^3 - y^2 + (\theta^2 x^2+1)\, y - (\theta\, x^2+\theta) = 0, \end{cases} \tag{4.4}$$

or

$$\begin{cases} \theta^2 x^2 + \theta^2 x + \theta\, y^2 + \theta^2 - x = 0, \\ \theta\, y^3 - y^2 + (\theta^2 x^2+1)\, y - (\theta\, x^2+\theta) = 0. \end{cases} \tag{4.5}$$

For the case $x = 1$, we have the following result:

Proposition 4.1. [44] *Let $p > 3$. For p-adic SOS model on the Cayley tree of order two, there exist three TIQGMs corresponding to the solutions of equation* (4.4) *if and only if $p \equiv 1 (\bmod 8)$ or $p \equiv 3 (\bmod 8)$.*

Now we study the case $x \neq 1$. Substituting y^2 from the first equation of (4.5) we obtain

$$y^2 = \frac{x - \theta^2(x^2+x+1)}{\theta}. \tag{4.6}$$

Putting the values of equation (4.6) into the second equation (4.5) we have the following

$$y = \frac{x}{\theta(x+1)}. \tag{4.7}$$

Using equation (4.7), we can reformulate equation (4.6) in the following form:

$$\theta^3 x^4 + \theta(3\theta^2-1)x^3 + (4\theta^3+1-2\theta)x^2 + \theta(3\theta^2-1)x + \theta^3 = 0. \tag{4.8}$$

Introducing a new variable $\xi = x + \frac{1}{x}$, from (4.8) we have

$$\theta^3 \xi^2 + \theta(3\theta^2-1)\xi + 2\theta^3 - 2\theta + 1 = 0. \tag{4.9}$$

Note that equation (4.9) is a quadratic equation with respect to ξ. We calculate its discriminant:

$$\Delta(\theta) := \theta^2(\theta^4 + 2\theta^2 + 1 - 4\theta). \tag{4.10}$$

Equation (4.9) has a solution in $\mathbb{Q}_p$ if and only if $\sqrt{\Delta(\theta)}$ exists in $\mathbb{Q}_p$.

Letting $\theta = 1 + \theta_s p^s + \theta_{s+1} p^{s+1} + ...$, $\theta_s \neq 0$ and $s \in \mathbb{N}$, then we have the following lemma

Lemma 4.2. *If $\left(\frac{\theta_s}{p}\right) = 1$ and s is even, then equation* (4.9) *has two solutions; otherwise, equation* (4.9) *does not have any solution.*

Proof. Expanding (4.10) in series with respect to θ, we obtain

$$\theta^4 + 2\theta^2 + 1 - 4\theta = (\theta-1)^4 + 4(\theta-1)^3 + 8(\theta-1)^2 + 4(\theta-1). \tag{4.11}$$

Since $\theta = 1 + \theta_s p^s + \theta_{s+1} p^{s+1} + \ldots$, we can rewrite equation (4.11) as follows

$$\begin{aligned} \theta^4 + 2\theta^2 + 1 - 4\theta = (\theta_s p^s + \theta_{s+1} p^{s+1} + ...)((\theta_s p^s + \theta_{s+1} p^{s+1} + ...)^3 + \\ + 4((\theta_s p^s + \theta_{s+1} p^{s+1} + ...)^2) + 8(\theta_s p^s + \theta_{s+1} p^{s+1} + ...) + 4). \end{aligned} \tag{4.12}$$

Using the representation (4.12) we see that the first term of the discriminant (4.11) is equal to $4\theta_s p^s$. Thus, under the condition that s is even and $\left(\frac{\theta_s}{p}\right) = 1$, equation (4.9) has two solutions

$$\xi_{1,2} = \frac{1 - 3\theta^2 \pm \sqrt{\theta^4 + 2\theta^2 - 4\theta + 1}}{2\theta^2}. \tag{4.13}$$

□

Taking into account $\xi = x + \frac{1}{x}$, we obtain $x = \frac{\xi \pm \sqrt{\xi^2 - 4}}{2}$. Therefore, we have

$$x_{1,2} = \frac{\xi_1 \pm \sqrt{\xi_1^2 - 4}}{2}, \quad x_{3,4} = \frac{\xi_2 \pm \sqrt{\xi_2^2 - 4}}{2}. \tag{4.14}$$

Using (4.13), we find

$$\sqrt{\xi^2 - 4} = \frac{\sqrt{2u(\theta) \pm 2v(\theta)\sqrt{\Delta(\theta)}}}{2\theta^2},$$

where

$$u(\theta) = -3\theta^4 - 2\theta^2 - 2\theta + 1,$$

$$v(\theta) = 3\theta^2 - 1.$$

Denote

$$\Delta_1(\theta) = 2u(\theta) \pm 2v(\theta)\sqrt{\Delta(\theta)}.$$

The existence of x in $\mathbb{Q}_p$ is equivalent to the existence of $\sqrt{\Delta_1(\theta)}$ in $\mathbb{Q}_p$.

Lemma 4.3. *The system of equations* (4.5) *has four solutions if and only if* $\sqrt{\Delta_1(\theta)}$ *exists in* $\mathbb{Q}_p$.

Proof. Let $\left(\frac{\theta_s}{p}\right) = 1$ and s be even. We rewrite $u(\theta)$ in the following manner:

$$u(\theta) = -6 - 18(\theta - 1) - 20(\theta - 1)^2 - o[(\theta - 1)^3],$$

$$\sqrt{\Delta(\theta)} = o[1].$$

In what follows

$$\Delta_1(\theta) = 2(-6 - 18(\theta - 1) - 20(\theta - 1)^2 - o[(\theta - 1)^3]) = -12 + o[1].$$

It is clear that equation (4.8) has four solutions in $\mathbb{Q}_p$ if and only if $\sqrt{\Delta_1(\theta)}$ exists in $\mathbb{Q}_p$. Moreover, $\sqrt{\Delta_1(\theta)}$ exists in $\mathbb{Q}_p$ if and only if the congruence $x^2 + 3 \equiv 0 (\operatorname{mod} p)$ has a solution (see [46]). The congruence $x^2 + 3 \equiv 0 (\operatorname{mod} p)$ is solvable if and only if $p \equiv 1 (\operatorname{mod} 6)$(see [46]). If equation (4.8) has four solutions, then equation (4.7) will also have four solutions. It follows that the system of equations (4.5) also has four solutions. □

Theorem 4.4. *Let N be the number of p-adic TIQGMs for the SOS model on the Cayley tree of order two. Then the following assertions hold:*

$$N = \begin{cases} 4, \ if \ \left(\frac{\theta_s}{p}\right) = 1, \ s \ is \ even \ and \ p \equiv 1 (\operatorname{mod} 6), \\ 0, \ otherwise. \end{cases} \tag{4.15}$$

Proof. Due to Lemma 4.3, the system (4.5) has four solutions, which implies the existence of four TIQGMs. □

Remark 4.5. The measures found in Theorem 4.4 are different from the measures found in Proposition 4.1.

5. Boundedness of the p-adic TIQGMs.

To prove the boundedness of the p-adic TIQGM, we state the following lemma:

Lemma 5.1. *Let* $\left(\frac{\theta_s}{p}\right)=1$*, s is even,* $p\equiv 1(\mathrm{mod}\,6), k=2$ *and* $m=2$*.* $z_i=(z_0^{(i)},z_1^{(i)})$*,* $i=\overline{1,4}$ *be the TI solutions of system* (3.5)*. Then the following relations hold:*

1. $|z_0^{(i)}|_p=|z_1^{(i)}|_p=1,\quad i=\overline{1,4};$
2. $|\theta^2z_0^{(i)}+\theta z_1^{(i)}+1|_p\le\frac{1}{p},\quad i=\overline{1,4}.$

Proof. 1. Let $z_i=(z_0^{(i)},z_1^{(i)}), i=\overline{1,4}$ be translation-invariant solutions of the system of equations (3.5) for $k=2$. Then from the solution of equation (4.5), we find the p-adic norm $|z_0^i|_p$ and $|z_1^i|_p$, where $z_0^i=x_i^2$ and $z_1^i=y_i^2$, $i=1,2,3,4$. At first, we find the p-adic norm of $|z_0^i|_p=|x_i^2|_p$. Using (4.14) we get

$$|x_{1,2}|_p=\left|\frac{\xi_1\pm\sqrt{\xi_1^2-4}}{2}\right|_p=\left|\frac{\frac{3\theta^2-1+\sqrt{\theta^4+2\theta^2-4\theta+1}}{2\theta^2}\pm\sqrt{\left(\frac{3\theta^2-1+\sqrt{\theta^4+2\theta^2-4\theta+1}}{2\theta^2}\right)^2-4}}{2}\right|_p, \tag{5.1}$$

$$|x_{3,4}|_p=\left|\frac{\xi_2\pm\sqrt{\xi_2^2-4}}{2}\right|_p=\left|\frac{\frac{3\theta^2-1-\sqrt{\theta^4+2\theta^2-4\theta+1}}{2\theta^2}\pm\sqrt{\left(\frac{3\theta^2-1-\sqrt{\theta^4+2\theta^2-4\theta+1}}{2\theta^2}\right)^2-4}}{2}\right|_p. \tag{5.2}$$

From (5.1) and the strong triangle equality we have

$$|x|_p=\left|\frac{\frac{3\theta^2-1+\sqrt{\theta^4+2\theta^2-4\theta+1}}{2\theta^2}+\frac{\sqrt{-3+o[1]}}{\theta^2}}{2}\right|_p=\left|\frac{1-3\theta^2+2\sqrt{-3}+o[1]}{4\theta^2}\right|_p$$

$$=\left|\frac{-1+\sqrt{-3}+o[1]}{2}\right|_p=1.$$

It implies that $|x_{1,2}|_p=1$. Observe that the p-adic norm $|x_{3,4}|_p=1$ and $|z_0^{(i)}|_p=|x_i^2|_p=1$, where $i=\overline{1,4}$.

Now, we find the p-adic norm of $|z_1^{(i)}|_p=|y_i^2|_p$. In (4.7) we find

$$|y_1|_p=\left|\frac{x_1}{\theta(x_1+1)}\right|_p=\left|\frac{\frac{-1+\sqrt{-3}}{2}+o[1]}{\frac{-1+\sqrt{-3}}{2}+1}\right|_p=\left|\frac{1+\sqrt{-3}+o[1]}{2}\right|_p=1.$$

It implies that $|y_{1,2}|_p=1$. Observe that the p-adic norm $|y_{3,4}|_p=1$ and $|z_1^{(i)}|_p=|y_i^2|_p=1$, where $i=\overline{1,4}$.

2. Using the solutions of equation (4.5), we determine $|\theta^2z_0^i+\theta z_1^i+1|_p$, where $i=\overline{1,4}$. According to Case 1 we know that $z_0^{(i)}=x_i^2$, and $z_1^{(i)}=y_i^2$, where $i=\overline{1,4}$ and from the strong triangle equality we have

$$|\theta^2z_0^{(i)}+\theta z_1^{(i)}+1|_p=\left|\theta^2\left(\frac{1+\sqrt{-3}+o[1]}{2}\right)^2+\theta\left(\frac{1+\sqrt{-3}+o[1]}{2}\right)^2+1\right|_p$$

$$=\left|\frac{-1-\sqrt{-3}+o[1]}{2}+\frac{-1+\sqrt{-3}+o[1]}{2}+1\right|_p=|o[1]|_p\le\frac{1}{p}.$$

□

Theorem 5.2. *Let* $\left(\frac{\theta_s}{p}\right) = 1$, *$s$ be even,* $p \equiv 1(\mathrm{mod}\, 6)$. *For the p-adic SOS model with three-state on a Cayley tree of order two p-adic TIQGM μ_{z_i} is unbounded iff $z_i \notin \mathcal{E}_p$, $i = \overline{1,4}$.*

Proof. Let $z_i, i = \overline{1,4}$ be translation-invariant solutions of (3.5) for $k = 2$. Then (see [36]) due to

$$\prod_{y \in S(x)} \sum_{j=0}^{m} \theta^{|i-j|_\infty} z_{j,y} + 1 = a(x) z_{i,x}, \qquad i = \overline{0, m-1},$$

for all $x \in V \setminus \{x_0\}$ we find

$$a(x) = \prod_{y \in S(x)} \left(\sum_{j=0}^{1} \theta^{|2-j|_\infty} z_{j,y} + 1 \right). \tag{5.3}$$

Since $\theta \in \mathcal{E}_p$ and $z_i \in Z_p^*$, due to the strong triangle inequality and Lemma 5.1 we obtain

$$|a(x)|_p = \begin{cases} 1, & \text{if } \ z_i \in \mathcal{E}_p; \\ \leq \frac{1}{p^2}, & \text{if } \ z_i \notin \mathcal{E}_p. \end{cases} \tag{5.4}$$

Hence, we get

$$|A_n(x)|_p = \prod_{y \in W_{n-1}} |a(y)|_p = \begin{cases} 1, & \text{if } \ z_i \in \mathcal{E}_p; \\ \leq p^{-2|W_{n-1}|}, & \text{if } \ z_i \notin \mathcal{E}_p. \end{cases} \tag{5.5}$$

Here we used the recursive formula $Z_{n,z} = A_{n-1} Z_{n-1,z}$. From (5.5) we get

$$|Z_n(x)|_p = \prod_{x \in V_{n-1}} |A_{n-1}(x)|_p = \begin{cases} 1, & \text{if } \ z_i \in \mathcal{E}_p; \\ \leq p^{-2|V_{n-1}|}, & \text{if } \ z_i \notin \mathcal{E}_p. \end{cases} \tag{5.6}$$

For any configuration $\sigma_n \in \Omega_{V_n}$ considering (5.6) we have

$$|\mu_{z_i}^{(n)}(\sigma)|_p = \frac{\left| \exp_p\{H_n(\sigma)\} \prod_{x \in W_n} z_{\sigma(x),x} \right|_p}{|Z_{n,z_i}|_p} =$$

$$= \frac{1}{|Z_{n,z_i^{(t)}}|_p} = \begin{cases} 1, & \text{if } \ z_i \in \mathcal{E}_p; \\ \geq p^{2|V_{n-1}|}, & \text{if } \ z_i \notin \mathcal{E}_p. \end{cases} \tag{5.7}$$

This implies that the measure $\mu_{z_i}^{(n)}(\sigma_n)$ is unbounded if and only if $z_i \notin \mathcal{E}_p$, $i = \overline{1,4}$. □

Summarising, we get the following result:

Theorem 5.3. *Let* $\left(\frac{\theta_s}{p}\right) = 1$, *$s$ be even and* $p \equiv 1(\mathrm{mod}\, 6)$. *Then for the p-adic SOS model with three-state on the Cayley tree of order two, a phase transition occurs.*

Acknowledgements. The author gratefully acknowledge the many helpful suggestions of supervisor Prof. Muzaffar M. Rahmatullaev and PhD. Akbarkhuja M. Tukhtabaev during the preparation of the paper. We also thank the referees for the careful reading of the manuscript and especially for a number of suggestions that have improved the paper.

References

[1] Marinari E., Parisi G., On the p-adic five-point function, Physics Letters B. 203(1-2), (1988), pp. 52–54.

[2] Areféva I.Ya., Dragovic B., Frampton P.H. and VolovichI.V., The wave function of the Universe and p-adic gravity, Int. J. Modern Phys. A 6 (1991) pp.4341–4358.

[3] Khrennikov A.Yu., p-Adic quantum mechanics with p-adic valued functions, J. Math. Phys., 32:4, (1991) pp. 932–936.

[4] Parisi G., Ricci-Tersenghi F., On the origin of ultrametricity, Journal of Physics A: Mathematical and General (2000), 33(1).

[5] Khrennikov A.Yu., p-adic valued distributions in mathematical physics, Kluwer, Dordrecht (1994).

[6] Khrennikov A.Yu., Kozyrev S.V., Replica symmetry breaking related to a general ultrametric space I: replica matrices and functionals, Physica A: Statistical Mechanics and its Applications. 359, (2006), pp.222–240.

[7] Khrennikov A.Yu., Kozyrev S.V., Replica symmetry breaking related to a general ultrametric space II: RSB solutions and the $n \to 0$ limit, Physica A: Statistical Mechanics and its Applications. 359, (2006) pp.241–266.

[8] Khrennikov A.Yu., Kozyrev S.V., Replica symmetry breaking related to a general ultrametric space III: The case of general measure, Physica A: Statistical Mechanics and its Applications. 378(2), (2007), pp.283–298.

[9] Rozikov U.A., What are the p-adic numbers? What are they used for?, Asia Pacific Mathematics Newsletter. 3(4), (2013), pp. 1–5.

[10] Avetisov V.A., Bikulov A.H. and Kozyrev S.V. Application of p-adic analysis to models of spontaneous breaking of the replica symmetry, J. Phys. A: Math. Gen. 32 (1999) pp.8785–8791.

[11] Khrennikov A.Yu., Kozyrev S.V. and Zuniga-Galindo W.A., Ultrametric Pseudodifferential Equations and Applications, Cambridge Univ. Press, (2018).

[12] Dragovich B., Khrennikov A., Kozyrev S.V. and Volovich I.V., On p-adic mathematical physics, p-Adic Numbers, Ultrametric Analysis, and Application, 1, (2009), pp.1–17.

[13] Rozikov U.A. and Khakimov O.N., p-Adic Gibbs measures and Markov random fields on countable graphs, Theoretical and Mathematical Physics, 175:1, (2013) pp.84–92.

[14] Rozikov U.A. and Khakimov O.N., Description of all translation-invariant p-adic Gibbs measures for the Potts model on a Cayley tree, Markov Process. Related Fields, 21:1, (2015) pp.177–204.

[15] Hans-Otto Georgii, Gibbs measures and phase transitions, Walter de Gruyter (2011) Vol. 9.

[16] Ising E., Beitrag zur Theorie des Ferromagnetismus, Z. Physik (1925).

[17] Rahmatullaev M.M. and Tukhtabaev A.M., Non periodic p-adic generalized Gibbs measure for the Ising model, p-Adic Numbers Ultrametric Anal.Appl. 11, (2019), pp.319–327.

[18] Rahmatullaev M.M. and Tukhtabaev A.M., On periodic p-adic generalized Gibbs measures for Ising model on a Cayley tree, Letters in Mathematical Physics (2022) 112:112.

[19] Rahmatullaev M.M., Rasulova M.A. and Asqarov J.N., Ground States and Gibbs Measures of Ising Model with Competing Interactions and an External Field on a Cayley Tree, Journal of Statistical Physics (2023) 190:116.

[20] Rahmatullaev M.M., Khakimov O.N. and Tukhtaboev A.M., A p-Adic generalized Gibbs measure for the Ising model on a Cayley tree, Theoretical and Mathematical Physics. 201(1), (2019), pp.1521–1530.

[21] Khamraev M., Mukhamedov F. and Rozikov U., On the Uniqueness of Gibbs Measures for p-Adic Nonhomogeneous λ-Model on the Cayley Tree. Letters in Mathematical Physics 70: (2004) pp. 17–28.

[22] Khakimov O.N., On a generalized p-adic Gibbs measure for Ising model on trees, p-Adic Numbers, Ultrametric Analysis, and Applications, 6, (2014), pp.207–217.

[23] Khakimov O. and Mukhammedov F., Translation-invariant generalized p-adic Gibbs measures for the Ising model on Cayley trees, Mathematical Methods in the Applied Sciences, (2021) 44.

[24] Potts R.B., Some generalized order-disorder transformations, Mathematical Proceedings of the Cambridge Philosophical Society. 48 (1): (1952) pp. 106–109.

[25] Mukhamedov F.M. and Rozikov U.A., On Gibbs measures of p-adic Potts model on the Cayley tree, Indag. Math., 15:1, (2004) pp.85–100.

[26] Rahmatullaev M.M. and Dekhkonov Zh.D., Potts model on a Cayley tree: a new class of Gibbs measures, Theoretical and Mathematical Physics, 215:1, (2023) pp. 150–162.

[27] Mukhamedov F.M., On dynamical systems and phase transitions for $q+1$-state p-adic Potts model on the Cayley tree, Math. Phys. Anal. Geom. 16 (2013) pp.49–87.

[28] Tukhtabaev A.M., On G_2-periodic quasi Gibbs measures of p-adic Potts model on a Cayley tree, p-Adic Numbers, Ultrametric Analysis and Applications, 13, (2021), pp.291–307.

[29] Mazel A.E. and Suhov Yu.M., Random surfaces with two-sided constraints: An application of the theory of dominant ground states, J. Stat. Phys. 64 (1991) pp. 111–134.

[30] Rozikov U.A. and Suhov Yu.M., Gibbs measures for SOS models on a Cayley tree, Infin. Dimens. Anal. Quan. Probab. Relat. Top., 9:3, (2006) pp.471–488.

[31] U.A.Rozikov, Gibbs Measures on Cayley Trees, World Sci., Singapore, (2013).

[32] Karshiboev O.Sh., Periodic Gibbs measures for the three-state SOS model on a Cayley tree with a translation-invariant external field, Theoretical and Mathematical Physics, 212(3): (2022) pp.1276–1283.

[33] Rahmatullaev M.M. and Karshiboev O.Sh., Gibbs measures for the three-state SOS model with external field on a Cayley tree. Positivity (2022) 26, 74.

[34] Rahmatullaev M.M. and Abraev B.U., Non-translation-invariant Gibbs measures of an SOS model on a Cayley tree, Reports on Mathematical Physics Vol. 86 Number 3, (2020) pp. 315–324.

[35] Rahmatullaev M.M. and Karshiboev O.Sh., Phase transition for the SOS model under inhomogeneous external field on a Cayley tree, Phase Transitions, 95:12, (2022) pp. 901–907.

[36] Khakimov O.N., On p-adic Solid-On-Solid model on a Cayley tree, Theor. Math.Phys. 193(3), (2017) pp.547–562.

[37] Ganikhodzhaev N.N., Mukhamedov F.M. and Rozikov U.A., Phase transitions of the Ising model on Z in the p-adic number field., Uzbek. Mat. Jour. No. 4, (1998) 23–29.

[38] Khakimov O., Mukhammedov F., On set of p-adic Gibbs measures for the countable state 1D SOS model, Journal of Physics A: Mathematical and Theoretical, 57, (2024) (15pp)

[39] Koblitz N., p-adic Numbers, p-adic Analysis, and Zeta-Functions, Springer, Berlin (1977).

[40] Vladimirov V.S., Volovich I.V. and Zelenov E.I., p-Adic analysis and Mathematical physics., World Sci., Singapoure, (1994).

[41] Mukhamedov F. and Khakimov O., On equation $x^k = a$ over Q_p and its applications, Izvestiya Math. 84, (2020) pp.348–360.

[42] Mukhamedov F., Omirov B. and Saburov M., On cubic equations over p-adic fields, Inter. J. Number Theory 10, (2014) pp. 1171–1190.

[43] Schikhof W.H., Ultrametric Calculus., Cambridge Univ. Press, Cambridge (1984).

[44] Rahmatullayev M.M., Akhmedov O.U., Tukhtabaev A.M., p-adic quasi Gibbs measures for the SOS model on the Cayley tree of order two., submitted to p-Adic Numbers, Ultrametric Analysis, and Application (2024).

[45] Mukhamedov F.M., On p-adic quasi Gibbs measures for $q+1$-state Potts model on the Cayley tree., p-Adic Numbers Ultrametric Anal. Appl., 2:3, (2010), pp.241–251.

[46] Rosen H.K., Elementary Number Theory and Its Applications, Addison-Westley, Canada, (1986).

[47] N.N. Ganikhodzhaev. The group representation and automorphisms on a Cayley tree. DAN RUz. Vol. 4. (1994) pp. 3–5.

[48] Ganikhodjayev N.N., Mukhamedov F.M., Rozikov U.A., Existence of phase transition for the Potts p-adic model on the set Z. Theoretical and Mathematical Physics. Vol.130, (2002), pp.425–431, .

O.U.Akhmedov,
Ferghana State University, 19, Murabbiylar str. Ferghana 150100, Uzbekistan.
e-mail: axmedovolimxon5@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.15-24
DOI: 10.29229/uzmj.2025-3-2

Construction of optimal quadrature formulas for Cauchy type singular integrals in the $L_2^{(m)}(-1,1)$ space

Akhmedov D.M.

Abstract. In the present paper in $L_2^{(m)}(-1,1)$ space an optimal quadrature formula is constructed for approximate calculation of the Cauchy type singular integral. Explicit formulas for the optimal coefficients are obtained.

Keywords: Cauchy type singular integral, quadrature formula, error functional, extremal function, optimal coefficients.

MSC (2020): 65D32

1. Introduction. Statement of the Problem

Many problems of science and engineering are naturally reduced to singular integral equations. Moreover plane problems are reduced to one dimensional singular integral equations (see [12]). The theory of one dimensional singular integral equations is given, for example, in [9, 13]. It is known that the solutions of such integral equations are expressed by singular integrals. Therefore approximate calculation of singular integrals with high exactness is actual problem of numerical analysis. For the singular integral of the Cauchy type we consider the following quadrature formula

$$\int_{-1}^{1} \frac{\varphi(x)}{x-t}dx \cong \sum_{\beta=0}^{N} C[\beta]\varphi(x_\beta) \tag{1.1}$$

with the error functional

$$\ell(x) = \frac{\varepsilon_{[-1,1]}(x)}{x-t} - \sum_{\beta=0}^{N} C[\beta]\delta(x-x_\beta), \tag{1.2}$$

where $-1 < t < 1$, $C[\beta]$ are the coefficients, x_β ($\in [-1,1]$) are the nodes, $N = 0,1,2,...$, $\varepsilon_{[-1,1]}(x)$ is the characteristic function of the interval $[-1,1]$, δ is the Dirac delta function, φ is a function of the space $L_2^{(m)}(-1,1)$. Here $L_2^{(m)}(-1,1)$ is the Sobolev space of functions with a square integrable m - th generalized derivative and equipped with the norm

$$\|\varphi|L_2^{(m)}(-1,1)\| = \left\{\int_{-1}^{1}(\varphi^{(m)}(x))^2dx\right\}^{1/2}$$

and $\left\{\int_{-1}^{1}(\varphi^{(m)}(x))^2dx\right\}^{1/2} < \infty$.

In order that the error functional (1.2) is defined on the space $L_2^{(m)}(-1,1)$ it is necessary to impose the following conditions (see [1])

$$(\ell(x), x^\alpha) = 0, \quad \alpha = 0,1,2,...,m-1. \tag{1.3}$$

Hence it is clear that for existence of the quadrature formulas of the form (1.1) the condition $N \geq m-1$ has to be met.

The difference

$$(\ell,\varphi) = \int_{-\infty}^{\infty} \ell(x)\varphi(x)dx = \int_{-1}^{1}\frac{\varphi(x)}{x-t}dx - \sum_{\beta=0}^{N} C[\beta]\varphi(x_\beta) \tag{1.4}$$

is called *the error* of the formula (1.1)

By the Cauchy-Schwarz inequality

$$|(\ell,\varphi)| \leq \left\|\varphi|L_2^{(m)}\right\| \cdot \left\|\ell|L_2^{(m)*}\right\|$$

the error (1.4) of the formula (1.1) on functions of the space $L_2^{(m)}(-1,1)$ is reduced to finding the norm of the error functional ℓ in the conjugate space $L_2^{(m)*}(-1,1)$.

Obviously the norm of the error functional ℓ depends on the coefficients and the nodes of the quadrature formula (1.1). The problem of finding the minimum of the norm of the error functional ℓ by coefficients and by nodes is called *the S.M. Nikol'skii problem*, and the obtained formula is called *the optimal quadrature formula in the sense of Nikol'skii.* This problem was first considered by S.M. Nikol'skii [14], and continued by many authors, see e.g. [15] and references therein. Minimization of the norm of the error functional ℓ by coefficients when the nodes are fixed is called *Sard's problem* and the obtained formula is called *the optimal quadrature formula in the sense of Sard.* First this problem was investigated by A. Sard [16].

There are several methods of construction of optimal quadrature formulas in the sense of Sard such as the spline method, ϕ- function method (see e.g. [5, 11]) and Sobolev's method which is based on construction of discrete analogs of a linear differential operator (see e.g. [21, 1]).

The main aim of the present paper is to construct optimal quadrature formulas in the sense of Sard of the form (1.1) in the space $L_2^{(m)}(-1,1)$ by the Sobolev method for approximate integration of the Cauchy type singular integral. This means to find the coefficients $C[\beta]$ which satisfy the following equality

$$\|\mathring{\ell}|L_2^{(m)*}\| = \inf_{C[\beta]} \|\ell|L_2^{(m)*}\|. \tag{1.5}$$

Thus, in order to construct optimal quadrature formulas in the form (1.1) in the sense of Sard we have to consequently solve the following problems.

Problem 1. Find the norm of the error functional (1.2) of the quadrature formula (1.1) in the space $L_2^{(m)*}(-1,1)$.

Problem 2. Find the coefficients $C[\beta]$ which satisfy equality (1.4).

Many works are devoted to the problem of approximate integration of Cauchy type singular integrals (see, for instance, [8, 9, 10, 4, 7, 10, 12, 18, 19, 20] and references therein).

The rest of the paper is organized as follows. In Section 2 using a concept of extremal function we find the norm of the error functional (1.2). Section 3 is devoted to a minimization of $\|\ell\|^2$ with respect to the coefficients $C[\beta]$. We obtain a system of linear equations for the coefficients of the optimal quadrature formula in Sard's sense of the form (1.1) in the space $L_2^{(m)}(-1,1)$. Moreover, the existence and uniqueness of the corresponding solution is proved. In Section 4 we give some definitions and known results which we use in the proof of the main results. In Section 5 we give the algorithm for construction of optimal quadrature formulas of the form (1.1). Finally, Explicit formulas for coefficients of the optimal quadrature formulas of the form (1.1) are found in Section 6.

2. The extremal function and the expression for the error functional norm

To solve Problem 1, i.e., for finding the norm of the error functional (1.2) in the space $L_2^{(m)}(-1,1)$ a concept of the extremal function is used [1]. The function $\psi_\ell(x)$ is said to be *the extremal function* of the error functional (1.2) if the following equality holds

$$(\ell,\psi_\ell) = \|\ell|L_2^{(m)*}\| \ \|\psi_\ell|L_2^{(m)*}\| \tag{2.1}$$

In the space $L_2^{(m)}$ the extremal function $\psi_\ell(x)$ of a functional $\ell(x)$ is found by S.L. Sobolev [21, 1]. This extremal function has the form

$$\psi_\ell = (-1)^m \ell(x) * G_m(x) + P_{m-1}(x), \tag{2.2}$$

where

$$G_m(x) = \frac{|x|^{2m-1}}{2\cdot(2m-1)!} \tag{2.3}$$

is a solution of the equation

$$\frac{d^{2m}}{dx^{2m}} G_m(x) = \delta(x), \tag{2.4}$$

$P_{m-1}(x)$ is a polynomial of degree $m-1$, and $*$ is the operation of convolution, and is defined as follows

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\infty} f(y)g(x-y)dy.$$

It is well known that for any functional $\ell(x)$ in $L_2^{(m)*}$ the equality

$$\|\ell|L_2^{(m)*}\|^2 = (\ell, \psi_\ell) = (\ell(x), (-1)^m \ell(x) * G_m(x)) = \int_{-\infty}^{\infty} \ell(x) \left((-1)^m \int_{-\infty}^{\infty} \ell(y) G_m(x-y)dy \right) dx$$

holds [1].

Applying this equality to the error functional (1.2) we obtain the following

$$\begin{aligned} \|\ell\|^2 &= (\ell, \psi_\ell) = (-1)^m \Bigg[\sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} C_\beta C_\gamma \frac{|x_\beta - x_\gamma|^{2m-1}}{2 \cdot (2m-1)!} \\ &\quad -2 \sum_{\beta=0}^{N} C[\beta] \int_{-1}^{1} \frac{|x - x_\beta|^{2m-1}}{2 \cdot (2m-1)!\ (x-t)} dx + \int_{-1}^{1} \int_{-1}^{1} \frac{|x-y|^{2m-1}}{2(2m-1)!(x-t)(y-t)} dx dy \Bigg]. \end{aligned} \tag{2.5}$$

Thus, Problem 1 is solved for quadrature formulas of the form (1.1) in the space $L_2^{(m)}(-1,1)$.

3. The system for optimal coefficients of the quadrature formula (1.1)

Assume that the nodes x_β of the quadrature formula (1.1) are fixed. The error functional (1.2) satisfies conditions (1.3). The norm of the error functional $\ell(x)$ is a multivariable function with respect to the coefficients $C[\beta]$ $(\beta = \overline{0, N})$. For finding the point of the conditional minimum of the expression (2.5), under the conditions (1.3), we apply the Lagrange method.

We denote $\mathbf{C} = (C[0], C[1], ..., C[N])$ and $\lambda = (\lambda_0, \lambda_1, ..., \lambda_{m-1})$. Consider the function

$$\Psi(\mathbf{C}, \lambda) = \|\ell\|^2 - 2(-1)^m \sum_{\alpha=0}^{m-1} \lambda_\alpha (\ell(x), x^\alpha).$$

Equating to zero the partial derivatives of $\Psi(\mathbf{C}, \lambda)$ by $C[\beta]$ $(\beta = \overline{0, N})$ and $\lambda_0, \lambda_1, ..., \lambda_{m-1}$, we get the following system of linear equations

$$\sum_{\gamma=0}^{N} C[\gamma] \frac{|x_\beta - x_\gamma|^{2m-1}}{2(2m-1)!} + \sum_{\alpha=0}^{m-1} \lambda_\alpha x_\beta^\alpha = f_m(x_\beta), \quad \beta = 0, 1, 2, ..., N, \tag{3.1}$$

$$\sum_{\gamma=0}^{N} C[\gamma] x_\gamma^\alpha = g_\alpha, \ \ \alpha = 0, 1, 2, ..., m-1, \tag{3.2}$$

where

$$\begin{aligned} f_m(x_\beta) &= \int_{-1}^{1} \frac{|x - x_\beta|^{2m-1}}{2(2m-1)!(x-t)} dx = \frac{(t - x_\beta)^{2m-1}}{2 \cdot (2m-1)!} \ln \left| \frac{1-t^2}{(x_\beta - t)^2} \right| \\ &\quad + \sum_{i=1}^{2m-1} \binom{2m-1}{i} \frac{(t - x_\beta)^{2m-1-i}}{2 \cdot (2m-1)! \cdot i!} \left((1-t)^i + (-1-t)^i - 2(x_\beta - t)^i \right) \end{aligned} \tag{3.3}$$

$$g_\alpha = \int_{-1}^{1} \frac{x^\alpha}{x-t} dx = \sum_{j=1}^{\alpha} \binom{\alpha}{i} \frac{t^{\alpha-j}}{j!} \left((1-t)^j - (-1-t)^j \right) + t^\alpha \ln \left| \frac{1-t}{-1-t} \right|, \tag{3.4}$$

and $C[\gamma]$, $\gamma = 0, 1, ..., N$ and λ_α, $\alpha = 0, 1, ..., m-1$ are unknowns.

4. Preliminaries

In this section we give some definitions and formulas that we need to prove the main results. Here we use the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given in [1, 21]. For completeness we give some definitions.

Assume that the nodes x_β are equally spaced, i.e., $x_\beta = h\beta$, $h = \frac{1}{N}$, $N = 1, 2, ...$, functions $\varphi(x)$ and $\psi(x)$ are real-valued and defined on the real line $\mathbb{R}$.

Definition 4.1. The function $\varphi(h\beta)$ is a *function of discrete argument* if it is given on some set of integer values of β.

Definition 4.2. *The inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

Definition 4.3. *The convolution of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product*

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

The Euler-Frobenius polynomials $E_k(x)$, $k = 1, 2, ...$ are defined by the following formula [6]

$$E_k(x) = \frac{(1-x)^{k+2}}{x} \left(x\frac{d}{dx} \right)^k \frac{x}{(1-x)^2}, \tag{4.1}$$

$E_0(x) = 1$.

For the Euler-Frobenius polynomials $E_k(x)$ the following identity holds

$$E_k(x) = x^k E_k\left(\frac{1}{x}\right), \tag{4.2}$$

and also the following is true

Lemma 4.4. (Lemma 3 of [6]). *Polynomial $Q_k(x)$ which is defined by the formula*

$$Q_k(x) = (x-1)^{k+1} \sum_{i=0}^{k+1} \frac{\Delta^i 0^{k+1}}{(x-1)^i} \tag{4.3}$$

is the Euler-Frobenius polynomial (4.1) of degree k, i.e. $Q_k(x) = E_k(x)$, where $\Delta^i 0^k = \sum_{l=1}^{i} (-1)^{i-l} C_i^l l^k$.

The following formula is valid [8]:

$$\sum_{\gamma=0}^{n-1} q^\gamma \gamma^k = \frac{1}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q}\right)^i \Delta^i 0^k - \frac{q^n}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q}\right)^i \Delta^i \gamma^k|_{\gamma=n}, \tag{4.4}$$

where $\Delta^i \gamma^k$ is the finite difference of order i of γ^k, q is the ratio of a geometric progression. When $|q| < 1$ from (4.4) we have

$$\sum_{\gamma=0}^{\infty} q^\gamma \gamma^k = \frac{1}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q}\right)^i \Delta^i 0^k. \tag{4.5}$$

In our computations we need the discrete analogue $D_m(h\beta)$ of the differential operator d^{2m}/dx^{2m} which satisfies the following equality

$$hD_m(h\beta) * G_m(h\beta) = \delta(h\beta), \tag{4.6}$$

where $G_m(h\beta) = \frac{|h\beta|^{2m-1}}{2(2m-1)!}$, $\delta(h\beta)$ is equal to 0 when $\beta \neq 0$ and is equal to 1. It should be noted that the operator $D_m(h\beta)$ was firstly introduced and investigated by S.L. Sobolev [1]. In [17] the discrete analogue $D_m(h\beta)$ of the differential operator d^{2m}/dx^{2m}, which satisfies equation (4.6), is constructed and the following result was proved.

Lemma 4.5. *The discrete analogue of the differential operator* d^{2m}/dx^{2m} *has the form*

$$D_m(h\beta) = p \begin{cases} \sum\limits_{k=1}^{m-1} A_k q_k^{|\beta|-1} & \text{for} \quad |\beta| \geq 2, \\ 1 + \sum\limits_{k=1}^{m-1} A_k & \text{for} \quad |\beta| = 1, \\ C + \sum\limits_{k=1}^{m-1} \frac{A_k}{q_k} & \text{for} \quad \beta = 0, \end{cases} \tag{4.7}$$

where

$$p = \frac{(2m-1)!}{h^{2m}}, \quad A_k = \frac{(1-q_k)^{2m+1}}{E_{2m-1}(q_k)}, \quad C = -2^{2m-1}, \tag{4.8}$$

$E_{2m-1}(q)$ *is the Euler-Frobenius polynomial of degree* $2m-1$, q_k *are the roots of the Euler-Frobenius polynomial* $E_{2m-2}(q)$, $|q_k| < 1$, h *is a small positive parameter.*

Furthermore several properties of the discrete argument function $D_m(h\beta)$ were proved in [17]. Here we give the following property of $D_m(h\beta)$ which we need in our computations.

Lemma 4.6. *The discrete argument function* $D_m(h\beta)$ *and the monomials* $(h\beta)^k$ *are related to each other as follows*

$$\sum_{\beta=-\infty}^{\infty} D_m(h\beta)(h\beta)^k = \begin{cases} 0 & \text{when} \quad 0 \leq k \leq 2m-1, \\ (2m)! & \text{when} \quad k = 2m, \end{cases}$$

where B_{2m} *is the Bernoulli number.*

5. The algorithm for computation of coefficients of optimal quadrature formula (1.1)

In the present section we give an algorithm for solution of system (3.1)-(3.2) when the nodes x_β are equally spaced, i.e., $x_\beta = h\beta - 1$, $h = \frac{2}{N}$, $N \geq m-1$. Here we use similar method suggested by S.L. Sobolev [21] for finding the coefficients of optimal quadrature formulas in the Sobolev space $L_2^{(m)}(0,1)$.

Suppose that $C[\beta] = 0$ when $\beta < 0$ and $\beta > N$. Using Definition 4.3, we rewrite system (3.1)-(3.2) in the convolution form:

$$G_m(h\beta) * C[\beta] + P_{m-1}(h\beta - 1) = f_m(h\beta), \quad \beta = 0, 1, ..., N, \tag{5.1}$$

$$\sum_{\beta=0}^{N} C[\beta] \cdot (h\beta - 1)^\alpha = g_\alpha, \quad \alpha = 0, 1, ..., m-1, \tag{5.2}$$

where $P_{m-1}(h\beta - 1) = \sum\limits_{\alpha=0}^{m-1} p_\alpha (h\beta - 1)^\alpha$.

Thus we have the following problem.

Problem 3. Find the discrete function $C[\beta]$ and polynomial $P_{m-1}(h\beta - 1)$ of degree $m-1$ which satisfy the system (5.1)-(5.2).

Further, we investigate Problem 3. Instead of $C[\beta]$ we introduce the following functions

$$v(h\beta) = G_m(h\beta) * C[\beta], \tag{5.3}$$

$$u(h\beta) = v(h\beta) + P_{m-1}(h\beta - 1). \tag{5.4}$$

In such statement the coefficients $C[\beta]$ are expressed by the function $u(h\beta)$, i.e. taking into account (4.6), (5.4) and Lemmas 4.5 and 4.6, for the coefficients we have

$$C[\beta] = hD_m(h\beta) * u(h\beta). \tag{5.5}$$

Thus, if we find the function $u(h\beta)$, then the coefficients $C[\beta]$ will be found from equality (5.5). To calculate the convolution (5.5) it is required to find the representation of the function $u(h\beta)$ for all integer values of β. From equality (5.1) we get that $u(h\beta) = f_m(h\beta)$ when $h\beta - 1 \in [-1, 1]$, i.e. $\beta = 0, 1, ..., N$. Now we need to find the representation of the function $u(h\beta)$ when $\beta < 0$ and $\beta > N$.

Since $C[\beta] = 0$ when $h\beta - 1 \notin [-1, 1]$ then $C[\beta] = hD_m(h\beta) * u(h\beta) = 0,\ \ h\beta - 1 \notin [-1, 1]$.

Now we calculate the convolution $v(h\beta) = G_m(h\beta) * C[\beta]$ when $\beta \leq 0$ and $\beta \geq N$. Suppose $\beta \leq 0$, then taking into account that $G_m(h\beta) = \frac{|h\beta|^{2m-1}}{2(2m-1)!}$ and equality (5.2), we have

$$\begin{aligned} v(h\beta) &= \sum_{\gamma=-\infty}^{\infty} C[\gamma] G_m(h\beta - h\gamma) = -\sum_{\gamma=0}^{N} C[\gamma] \sum_{k=0}^{2m-1} \frac{(h\beta-1)^{2m-1-k}(-1)^k (h\gamma-1)^k}{2 \cdot k! \cdot (2m-1-k)!} \\ &= -\sum_{k=0}^{m-1} \frac{(h\beta-1)^{2m-1-k}(-1)^k}{2 \cdot k! \cdot (2m-1-k)!} \sum_{\gamma=0}^{N} C[\gamma](h\gamma-1)^k - \sum_{k=m}^{2m-1} \frac{(h\beta-1)^{2m-1-k}(-1)^k}{2 \cdot k! \cdot (2m-1-k)!} \sum_{\gamma=0}^{N} C[\gamma](h\gamma-1)^k \\ &= -R_{2m-1}(h\beta-1) - Q_{m-1}(h\beta-1), \end{aligned}$$

where $R_{2m-1}(h\beta-1) = \sum_{k=0}^{m-1} \frac{(h\beta-1)^{2m-1-k}(-1)^k}{2\cdot k!\cdot(2m-1-k)!} g_k$ is the polynomial of degree $2m-1$ and $Q_{m-1}(h\beta-1) = \sum_{k=m}^{2m-1} \frac{(h\beta-1)^{2m-1-k}(-1)^k}{2\cdot k!\cdot(2m-1-k)!} \sum_{\gamma=0}^{N} C[\gamma](h\gamma-1)^k$ is an unknown polynomial of degree $m-1$ of $(h\beta-1)$.

Thus when $\beta \leq 0$ we get

$$v(h\beta) = -R_{2m-1}(h\beta-1) - Q_{m-1}(h\beta-1), \tag{5.6}$$

Similarly, in the case $\beta \geq N$ for the convolution $v(h\beta) = G_m(h\beta) * C[\beta]$ we obtain

$$v(h\beta) = R_{2m-1}(h\beta-1) + Q_{m-1}(h\beta-1). \tag{5.7}$$

We denote

$$P^-_{m-1}(h\beta-1) = P_{m-1}(h\beta-1) - Q_{m-1}(h\beta-1), \tag{5.8}$$

$$P^+_{m-1}(h\beta-1) = P_{m-1}(h\beta-1) + Q_{m-1}(h\beta-1), \tag{5.9}$$

where $P^-_{m-1}(h\beta-1) = \sum_{\alpha=0}^{m-1} p^-_\alpha \cdot (h\beta-1)^\alpha$, $P^+_{m-1}(h\beta-1) = \sum_{\alpha=0}^{m-1} p^+_\alpha \cdot (h\beta-1)^\alpha$.

Taking into account (5.4), (5.6) and (5.7) we get the following problem

Problem 4. Find the solution of the equation

$$hD_m(h\beta) * u(h\beta) = 0, \qquad h\beta \notin [0, 1] \tag{5.10}$$

having the form:

$$u(h\beta) = \begin{cases} -R_{2m-1}(h\beta-1) + P^-_{m-1}(h\beta-1), & \beta \leq 0, \\ f_m(h\beta), & 0 \leq \beta \leq N, \\ R_{2m-1}(h\beta-1) + P^+_{m-1}(h\beta-1), & \beta \geq N. \end{cases} \tag{5.11}$$

Here $P^-_{m-1}(h\beta-1)$ and $P^+_{m-1}(h\beta-1)$ are unknown polynomials of degree $m-1$ with respect to $(h\beta-1)$.

If we find $P^-_{m-1}(h\beta-1)$ and $P^+_{m-1}(h\beta-1)$ then from (5.8), (5.9) we have

$$P_{m-1}(h\beta-1) = \frac{1}{2}\left(P^+_{m-1}(h\beta-1) + P^-_{m-1}(h\beta-1)\right), \tag{5.12}$$

$$Q_{m-1}(h\beta-1) = \frac{1}{2}\left(P^+_{m-1}(h\beta-1) - P^-_{m-1}(h\beta-1)\right). \tag{5.13}$$

Unknowns $P^-_{m-1}(h\beta-1)$, $P^+_{m-1}(h\beta-1)$ can be found from equation (5.10), using the function $D_m(h\beta)$ defined by (4.7). Then we obtain explicit form of the function $u(h\beta)$ and from (5.5) we find the coefficients $C[\beta]$. Furthermore from (5.12) we get $P_{m-1}(h\beta-1)$.

Thus Problem 4 and respectively Problems 3 will be solved.

6. Computation of coefficients of the optimal quadrature formula (1.1)

In this section, using the above algorithm, we obtain explicit formulas for coefficients of the optimal quadrature formula (1.1). It should be noted that the quadrature formula (1.1) is exact for polynomials of degree $\leq m-1$.

The following holds

Theorem 6.1. *Coefficients of the optimal quadrature formulas (1.1), with equally spaced nodes in the space $L_2^{(m)}(-1,1)$, have the following form*

$$\begin{aligned}
C[0] &= hp\left[Cf_m(0)+f_m(h)+\sum_{\alpha=0}^{m-1}p^-_\alpha\cdot(-1-h)^\alpha+\sum_{\alpha=0}^{m-1}\frac{(1+h)^{2m-1-\alpha}g_\alpha}{2\alpha!(2m-1-\alpha)!}\right]+\\
&\quad+\sum_{k=1}^{m-1}\frac{A_k hp}{q_k}\left[\sum_{\gamma=0}^{N}q_k^\gamma f_m(h\gamma)+M_k+q_k^N N_k\right],\\
C[\beta] &= hp\Big[f_m(h\beta-h)+Cf_m(h\beta)+f_m(h\beta+h)\Big]+\\
&\quad+\sum_{k=1}^{m-1}\frac{A_k hp}{q_k}\left[\sum_{\gamma=0}^{N}q_k^{|\beta-\gamma|}f_m(h\gamma)+q_k^\beta M_k+q_k^{N-\beta}N_k\right],\quad \beta=1,2,...,N-1,\\
C[N] &= hp\left[Cf_m(2)+f_m(2-h)+\sum_{\alpha=0}^{m-1}p^+_\alpha\cdot(1+h)^\alpha+\sum_{\alpha=0}^{m-1}\frac{(1+h)^{2m-1-\alpha}(-1)^\alpha g_\alpha}{2\alpha!(2m-1-\alpha)!}\right]+\\
&\quad+\sum_{k=1}^{m-1}\frac{A_k hp}{q_k}\left[\sum_{\gamma=0}^{N}q_k^{N-\gamma}f_m(h\gamma)+q_k^N M_k+N_k\right],\\
p_\alpha &= \frac{1}{2}\left(p^+_\alpha+p^-_\alpha\right),\quad \alpha=0,1,...,m-1,
\end{aligned}$$

where

$$\begin{aligned}
M_k &= \sum_{\alpha=0}^{m-1}\frac{g_\alpha}{2\alpha!}\sum_{i=0}^{2m-1-\alpha}\frac{h^i}{i!(2m-1-\alpha-i)!}\sum_{j=0}^{i}\frac{q_k^j\Delta^j0^i}{(1-q_k)^{j+1}}+\\
&\quad+\sum_{\alpha=1}^{m-1}p^-_\alpha(-1)^\alpha\sum_{i=0}^{\alpha}\frac{\alpha!h^i}{i!(\alpha-i)!}\sum_{j=0}^{i}\frac{q_k^j\Delta^j0^i}{(1-q_k)^{j+1}}+\frac{p^-_0q_k}{1-q_k},
\end{aligned}\tag{6.1}$$

$$\begin{aligned}
N_k &= \sum_{\alpha=0}^{m-1}\frac{g_\alpha(-1)^\alpha}{2\alpha!}\sum_{i=0}^{2m-1-\alpha}\frac{h^i}{i!(2m-1-\alpha-i)!}\sum_{j=0}^{i}\frac{q_k^j\Delta^j0^i}{(1-q_k)^{j+1}}+\\
&\quad+\sum_{\alpha=1}^{m-1}p^+_\alpha\sum_{i=0}^{\alpha}\frac{\alpha!h^i}{i!(\alpha-i)!}\sum_{j=0}^{i}\frac{q_k^j\Delta^j0^i}{(1-q_k)^{j+1}}+\frac{p^+_0q_k}{1-q_k},
\end{aligned}\tag{6.2}$$

and p, C, A_k are defined by (4.8), q_k are roots of the Euler-Frobenius polynomial $E_{2m-2}(q)$, $|q_k|<1$, $\Delta^i0^\alpha=\sum_{l=1}^{i}(-1)^{i-l}\binom{i}{l}l^\alpha$, p^-_α, p^+_α, $\alpha=0,1,...,m-1$ are defined from system (6.6)-(6.7).

Proof. First we find the expressions for p_0^- and p_0^+. When $\beta=0$ and $\beta=N$ from (5.11) for p_0^- and p_0^+ we get

$$p_0^- = f_m(0)-\sum_{\alpha=0}^{m-1}\frac{g_\alpha}{2\alpha!(2m-1-\alpha)!}-\sum_{\alpha=1}^{m-1}p_\alpha^-(-1)^\alpha, \tag{6.3}$$

$$p_0^+ = f_m(2)-\sum_{\alpha=0}^{m-1}\frac{(-1)^\alpha g_\alpha}{2\alpha!(2m-1-\alpha)!}-\sum_{\alpha=1}^{m-1}p_\alpha^+. \tag{6.4}$$

Now we have $2m-2$ unknowns p_α^-, p_α^+, $\alpha=1,2,...,m-1$.

Taking into account (4.7), (5.11), (6.3) and (6.4), from (5.10) we get the following system

$$\begin{aligned}&\sum_{\alpha=1}^{m-1}p_\alpha^-(-1)^\alpha\sum_{j=1}^{\alpha}C_\alpha^j\sum_{\gamma=1}^{\infty}D_m(h\beta+h\gamma)(h\gamma)^j+\sum_{\alpha=1}^{m-1}p_\alpha^+\sum_{\gamma=1}^{\infty}D_m(2+h\gamma-h\beta)\sum_{j=1}^{\alpha}C_\alpha^j(h\gamma)^j\\&=-\sum_{\gamma=0}^{N}D_m(h\beta-h\gamma)f_m(h\gamma)+\sum_{\gamma=1}^{\infty}D_m(h\beta+h\gamma)\Big[-\sum_{\alpha=0}^{m-1}\frac{((h\gamma+1)^{2m-1-\alpha}-1)\,g_\alpha}{2\alpha!(2m-1-\alpha)!}-f_m(0)\Big]\\&-\sum_{\gamma=1}^{\infty}D_m(2+h\gamma-h\beta)\Big[\sum_{\alpha=0}^{m-1}\frac{((h\gamma+1)^{2m-1-\alpha}-1)\,(-1)^\alpha g_\alpha}{2\alpha!(2m-1-\alpha)!}+f_m(2)\Big],\end{aligned} \tag{6.5}$$

where $\beta=-1,-2,...,-(m-1)$ and $\beta=N+1,N+2,...,N+m-1$.

First we consider the cases $\beta=-1,-2,...,-(m-1)$. From (6.5) replacing β by $-\beta$ and using (4.7) and (4.5), after some calculations for $\beta=1,2,...,m-1$, we get the following system of $m-1$ linear equations

$$\sum_{\alpha=1}^{m-1}p_\alpha^-B_{\beta\alpha}^-+\sum_{\alpha=1}^{m-1}p_\alpha^+B_{\beta\alpha}^+=T_\beta,\quad \beta=1,2,...,m-1, \tag{6.6}$$

where

$$B_{\beta\alpha}^- = (-1)^\alpha\sum_{j=1}^{\alpha}C_\alpha^jh^j\left[\sum_{k=1}^{m-1}\frac{A_k}{q_k}\sum_{\gamma=1}^{\infty}q_k^{|\beta-\gamma|}\gamma^j+(\beta-1)^j+C\beta^j+(\beta+1)^j\right],$$

$$B_{\beta\alpha}^+ = \sum_{k=1}^{m-1}A_kq_k^{N+\beta-1}\sum_{j=1}^{\alpha}C_\alpha^jh^j\sum_{i=1}^{j}\frac{q_k^i\Delta^i0^j}{(1-q_k)^{i+1}},$$

$$\begin{aligned}T_\beta \;=\; &-\sum_{k=1}^{m-1}A_kq_k^{\beta-1}\sum_{\gamma=0}^{N}q_k^\gamma f_m(h\gamma)-\left(f_m(2)-\sum_{\alpha=0}^{m-1}\tfrac{(-1)^\alpha g_\alpha}{2\alpha!(2m-1-\alpha)!}\right)\sum_{k=1}^{m-1}\tfrac{A_kq_k^{N+\beta}}{1-q_k}+\\&+\left(\sum_{\alpha=0}^{m-1}\tfrac{g_\alpha}{2\alpha!(2m-1-\alpha)!}-f_m(0)\right)\left[\sum_{k=1}^{m-1}\tfrac{A_k}{q_k}\sum_{\gamma=1}^{\infty}q_k^{|\beta-\gamma|}+2+C\right]\\&-\sum_{k=1}^{m-1}\tfrac{A_k}{q_k}\sum_{\gamma=1}^{\infty}q_k^{|\gamma-\beta|}\sum_{\alpha=0}^{m-1}\tfrac{(1+h\gamma)^{2m-1-\alpha}g_\alpha}{2\alpha!(2m-1-\alpha)!}-\sum_{k=1}^{m-1}A_k\sum_{\gamma=1}^{\infty}q_k^{N+\gamma+\beta-1}\sum_{\alpha=0}^{m-1}\tfrac{(1+h\gamma)^{2m-1-\alpha}(-1)^\alpha g_\alpha}{2\alpha!(2m-1-\alpha)!}-\\&-\sum_{\alpha=0}^{m-1}\tfrac{g_\alpha}{2\alpha!(2m-1-\alpha)!}\Big((1+h(\beta-1))^{2m-1-\alpha}+C(1+h\beta)^{2m-1-\alpha}+(1+h(\beta+1))^{2m-1-\alpha}\Big).\end{aligned}$$

Here $\beta=1,2,...,m-1$ and $\alpha=1,2,...,m-1$.

Now we consider the cases $\beta=N+1,N+2,...,N+m-1$. From (6.5) replacing β by $N+\beta$ and using (4.7) and (4.5), after some calculations for $\beta=1,2,...,m-1$ we get the following system of $m-1$ linear equations

$$\sum_{\alpha=1}^{m-1}p_\alpha^-A_{\beta\alpha}^-+\sum_{\alpha=1}^{m-1}p_\alpha^+A_{\beta\alpha}^+=S_\beta,\quad \beta=1,2,...,m-1, \tag{6.7}$$

where

$$A^{-}_{\beta\alpha} = (-1)^\alpha \sum_{j=1}^{\alpha} C^j_\alpha h^j \sum_{k=1}^{m-1} A_k q_k^{N+\beta-1} \sum_{i=1}^{\alpha} \frac{q_k^i \Delta^i 0^\alpha}{(1-q_k)^{i+1}},$$

$$A^{+}_{\beta\alpha} = \sum_{j=1}^{\alpha} C^j_\alpha h^j \left[\sum_{k=1}^{m-1} \frac{A_k}{q_k} \sum_{\gamma=1}^{\infty} q_k^{|\beta-\gamma|}\gamma^j + (\beta-1)^j + C\beta^j + (\beta+1)^j\right],$$

$$S_\beta = \sum_{k=1}^{m-1} A_k q_k^{N+\beta-1}\left[-\sum_{\gamma=0}^{N} q_k^{-\gamma} f_m(h\gamma) - f_m(0)\frac{q_k}{1-q_k} + \sum_{\gamma=1}^{\infty} q_k^{\gamma} \sum_{\alpha=0}^{m-1} \frac{(1-(h\gamma+1)^{2m-1-\alpha})g_\alpha}{2\alpha!(2m-1-\alpha)!}\right]$$
$$- \sum_{k=1}^{m-1} \frac{A_k}{q_k} \sum_{\gamma=1}^{\infty} q_k^{|\beta-\gamma|}\left[\sum_{\alpha=0}^{m-1} \frac{((1+h\gamma)^{2m-1-\alpha}-1)(-1)^\alpha g_\alpha}{2\alpha!(2m-1-\alpha)!} + f_m(2)\right] - f_m(2)(2+C) -$$
$$- \sum_{\alpha=0}^{m-1} \frac{(-1)^\alpha g_\alpha}{2\alpha!(2m-1-\alpha)!}\Big((1+h(\beta-1))^{2m-1-\alpha} + C(1+h\beta)^{2m-1-\alpha} + (1+h(\beta+1))^{2m-1-\alpha} - 3\Big).$$

Here $\beta = 1, 2, ..., m-1$ and $\alpha = 1, 2, ..., m-1$.

Thus for the unknowns p^-_α, p^+_α, $\alpha = 1, 2, ..., m-1$ we obtained system (6.6), (6.7) of $2m-2$ linear equations. Since our optimal quadrature problem has a unique solution, the main matrix of this system is non singular. Unknowns p^-_α, p^+_α, $\alpha = 1, 2, ..., m-1$ can be found from system (6.6), (6.7). Then taking into account (5.12), using (6.3) and (6.4) we have

$$p_\alpha = \frac{1}{2}\left(p^+_\alpha + p^-_\alpha\right), \quad \alpha = 0, 1, ..., m-1.$$

Now we find the coefficients $C[\beta]$, $\beta = 0, 1, ..., N$. From (5.5), taking into account (4.7), we deduce

$$C[\beta] = h\Bigg[\sum_{\gamma=0}^{N} D_m(h\beta - h\gamma) f_m(h\gamma) + \sum_{\gamma=1}^{\infty} D_m(h\beta + h\gamma)\left(\sum_{\alpha=0}^{m-1} \frac{(h\gamma+1)^{2m-1-\alpha} g_\alpha}{2\alpha!(2m-1-\alpha)!} + \sum_{\alpha=0}^{m-1} p^-_\alpha(-1-h\gamma)^\alpha\right)$$
$$+ \sum_{\gamma=1}^{\infty} D_m(h(N+\gamma) - h\beta)\left(\sum_{\alpha=0}^{m-1} \frac{(h\gamma+1)^{2m-1-\alpha}(-1)^\alpha g_\alpha}{2\alpha!(2m-1-\alpha)!} + \sum_{\alpha=0}^{m-1} p^+_\alpha(1+h\gamma)^\alpha\right)\Bigg], \ \beta = 0, 1, ..., N.$$

From here, using (4.7) and formula (4.5), taking into account (6.1) and (6.2), after some calculations we arrive at the expressions of the coefficients $C[\beta]$, $\beta = 0, 1, ..., N$ which are given in the assertion of the theorem. Theorem 6.1 is proved. □

References

[1] Akhmedov D.M., Abdikayumova G.A.; Construction of optimal quadrature formulas with derivatives for Cauchy type singular integrals in the Sobolev space. Uzbek Mathematical Journal, 2020, Volume 64, Issue 1, pp.4-9, DOI: 10.29229/uzmj.2020-1-1

[2] Akhmedov D.M., Atamuradova B.M.; Construction of optimal quadrature formulas for Cauchy type singular integrals in the $W_2^{(1,0)}(01)$ space. Uzbek Mathematical Journal, 2022, Volume 66, Issue 2, pp. 5-9, DOI:10.29229/uzmj.2022-2-1

[3] Akhmedov D.M.; Optimal approximation of the Hadamard hypersingular integrals. Uzbek Mathematical Journal, 2023, Volume 67, Issue 3, pp. 5-12, DOI: 10.29229/uzmj.2023-3-1

[4] Belotserkovskii S.M., Lifanov I.K. Numerical Methods in Singular Integral Equations, Nauka, Moscow, 1985 (in Russian).

[5] Blaga P., Coman Gh. Some problems on optimal quadrature, Stud. Univ.Babe-Bolyai Math. 52 (4) (2007) 21-44.

[6] Frobenius F.G. On Bernoulli numbers and Euler polynomials, Berl,Ber. 1910 (July-December): 809-847, 1910 (in German).

[7] Gabdulkhaev B.G. Cubature formulas for many dimensional singular integrals I // Izvestiya Vuzov. Mathematics. 1975. 4. 3-13. (in Russian)

[8] Hamming R.W. Numerical Methods for Scientists and Engineers. NY, McGraw Bill Book Company, Inc, USA. 1962. 411p.

[9] Gakhov F.D. Boundary problems. Nauka, Moscow, 1977. -640 p. (in Russian)

[10] Israilov M.I.,Shadimetov Kh.M. Weight optimal quadrature formulas for singular integrals of the Cauchy type. Doklady AN RUz -1991, 8, 10-11.

[11] Lanzara F. On optimal quadrature formulae. J. Inequal. Appl. 5 (2000) 201-225.

[12] Lifanov I.K. The method of singular equations and numerical experiments. TOO "Yanus", Moscow, 1995. -520p. (in Russian)

[13] Muskhelishvili N.I. Singular Integral Equations. Nauka, Moscow, 1968. -512p. (in Russian)

[14] Nikol'skii S.M. Concerning estimation for approximate quadrature formulas, Uspekhi Mat.Nauk 5 (2(36)) (1950) 165-177 (in Russian).

[15] Nikol'skii S.M. Quadrature Formulas, Nauka, Moscow, 1988 (in Russian).

[16] Sard A. Best approximate integration formulas; best approximation formulas, Amer. J. Math. 71 (1949) 80-91.

[17] Shadimetov Kh.M. Optimal formulas of approximate integration for differentiable functions, Candidate dissertation, Novosibirsk, 1983, p. 140. arXiv:1005.0163v1 [NA.math].

[18] Shadimetov Kh.M. Optimal quadrature formulas for singular integrals of the Cauchy type. Doklady AN RUz -1987, 6, -Pp.9-11. (in Russian)

[19] Shadimetov Kh.M., Akhmedov D.M. Numerical Integration Formulas for Hypersingular Integrals. Numer. Math. Theor. Meth. Appl. Vol. 17, No. 3, pp. 805-826, doi: 10.4208/nmtma.OA-2024-0028

[20] Shadimetov Kh.M., Akhmedov D.M. Anoptimal approximate solution of the I kind Fredholm singular integral equations. Filomat 38:30 (2024), 1076510796 https://doi.org/10.2298/FIL2430765S

[21] Sobolev S.L. V.L.Vaskevich. The Theory of Cubature Formulas. Kluwer Academic Publishers Group, Dordrecht (1997)

[22] Sobolev S.L. Introduction to the Theory of Cubature Formulas, Nauka, Moscow, (in Russian) (1974).

Dilshod M. Akhmedov
V.I.Romanovskiy Institute of mathematics,
Uzbekistan Academy of Sciences,
Tashkent International University
e-mail: axmedovdilshod@mail.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.25-38
DOI: 10.29229/uzmj.2025-3-3

A Four-Parameter non-local Problem for a Fractional Wave Equation

Ashurov R., Nuraliyeva N.

Abstract. In this paper, we investigate a time-fractional diffusion-wave equation, where the classical second-order time derivative is replaced by a fractional derivative of order $\rho \in (1,2)$. We consider a class of non-local boundary value problems involving four real parameters α_1, α_2, β_1, and β_2 under general conditions.

Using the Fourier method in an abstract Hilbert space setting, we derive necessary and sufficient conditions for the well-posedness of the problem. We prove that the solution exists and is unique if certain algebraic conditions on the parameters are satisfied. In cases where these conditions fail, we describe the structure of the solution and show that uniqueness may be lost. For such cases, we also formulate orthogonality conditions that guarantee existence.

Keywords: Fourier method, abstract operator, complete system, non-local problems, Caputo derivatives.

MSC (2020): 35R11, 35L05, 26A33

1. Introduction

In this paper, we study a time-fractional diffusion-wave equation obtained by replacing the second-order time derivative in a classical wave equation with a fractional derivative of order $\rho \in (1,2)$. It is worth noting that this equation is rather specific in the following sense: for the subdiffusion equation, most properties of the solution resemble those of the classical diffusion equation. However, the solution of the diffusion-wave equation with a fractional time derivative exhibits characteristics of both the diffusion and wave equations. In a certain sense, it can be said that the time-fractional diffusion-wave equation interpolates between these two classical models the diffusion equation and the wave equation (see [1], [23]).

The objective of this paper is to investigate a class of non-local problems involving four parameters, $\alpha_1, \alpha_2, \beta_1, \beta_2$ $(\alpha_1^2+\alpha_2^2 \neq 0, \beta_1^2+\beta_2^2 \neq 0)$, for the time-fractional diffusion-wave equation, and to identify conditions under which these problems are well-posed. Additionally, in cases where the problem is ill-posed, we aim to find supplementary conditions that guarantee the existence of a solution (note that such a solution may not be unique), and to determine the explicit form of all possible solutions.

To achieve this goal, we employ the Fourier method, which requires that the elliptic part of the equation admits a complete orthonormal system of eigenfunctions. To address equations with varying elliptic parts, we formulate the problem in an abstract form within a separable Hilbert space H equipped with an inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $A: H \to H$ be an unbounded, positive, self-adjoint operator with domain $D(A)$. Suppose A has a complete orthonormal system of eigenfunctions $\{V_k\}$ in H and a countable set of positive eigenvalues $\{\lambda_k\}$, arranged in non-decreasing order:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty.$$

Let $C((a,b);H)$ denote the set of continuous functions H with value $h(t)$ defined in (a,b). For functions $h: \mathbb{R}_+ \to H$, the fractional analogues of integrals and derivatives are defined in the same manner as for scalar functions (see, for example, [20]). Recall that the fractional Caputo derivative of order $\rho > 0$ for a function $h(t)$ is defined as (see, e.g., [21]):

$$D_t^\rho h(t) = \frac{1}{\Gamma(n-\rho)} \int_0^t \frac{h^{(n)}(\xi)}{(t-\xi)^{\rho+1-n}}\, d\xi, \quad t>0, \quad n=[\rho]+1,$$

provided that the right-hand side exists. As usual, $[\rho]$ denotes the integer part of ρ, and $\Gamma(\sigma)$ is the Euler gamma function. Note that when ρ is an integer, the fractional derivative coincides with the classical derivative: $D_t^\rho h(t) = \frac{d^\rho}{dt^\rho} h(t)$.

Let $\rho \in (1,2)$ be a fixed number. Consider the following class of non-local boundary value problems involving real parameters: $\alpha_1, \alpha_2, \beta_1, \beta_2$ $(\alpha_1^2+\alpha_2^2 \neq 0, \beta_1^2+\beta_2^2 \neq 0)$:

$$\begin{cases} D_t^\rho u(t) + Au(t) = f, \quad 0 < t \leq T; \\ \alpha_1 u(0) + \alpha_2 u(\xi) = \varphi, \quad 0 < \xi \leq T; \\ \beta_1 u'(0) + \beta_2 u'(\xi) = \psi, \end{cases} \tag{1.1}$$

where $f, \varphi, \psi \in H$ are prescribed elements of H, and $\xi \in (0,T)$ is a fixed number. These problems are referred to as *forward problems.*

Definition 1.1. A function $u(t),\ u'(t) \in C([0,T];H)$ such that $D_t^\rho u(t),\ Au(t) \in C((0,T];H)$ and satisfying all conditions of problem (1.1) is called a solution to the non-local problem (1.1).

Non-local problems for various differential equations, especially the periodic case $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$ and $\varphi = \psi = 0$, due to their importance for applications, have been considered by many specialists (see, for example, fundamental monograph [19], Chapter 8, and papers [1],[3],[5],[7],[11],[12],[13],[17],[22]).

In the paper [5] a similar to (1.1) class of non-local problem

$$\begin{cases} D_t^\rho u(t) + Au(t) = f(t), \quad 0 < \rho < 1, \quad 0 < t \leq T; \\ u(\xi) = \alpha u(0) + \varphi, \quad 0 < \xi \leq T, \end{cases} \tag{1.2}$$

was considered for subdiffusion equations. The authors determined the values of the parameter α that ensure the well-posedness of the problem. For the remaining parameters α, it were found conditions for the orthogonality of f and φ to some eigenfunctions of the operator A, under which the solution of the problem exists (but there is no uniqueness). It was also considered inverse problems to determine the right-hand side of the equation and function φ in the non-local condition. Also, in the work [9], inverse problems were studied to determine the right-hand side of a fractional Schrödinger-type equation. Moreover, in [10], the authors have studied an inverse problem for systems of fractional pseudo-differential equations. We also note works [1] and [7], where similar forward and inverse problems are studied for the Rayleigh-Stokes equation $u_t(t) + (1+\gamma\partial_t^\rho)Au(t) = f(t)$, where $\gamma > 0$ and ∂_t^ρ is the Riemann-Liouville fractional derivative. Note that if $\alpha = 0$ in problem (1.2), then we get a well-known ill-posed backward problem, which was studied in detail in works [2],[15], [24] and [25].

2. Preliminaries

In this section, we introduce the Hilbert spaces of "smooth" functions associated with fractional powers of the operator A, and recall several important properties of the Mittag–Leffler functions, which will be used in the subsequent analysis.

For any real number τ, the fractional power of the operator A is defined by the formula

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k V_k,$$

where $h_k = (h, V_k)$ are the Fourier coefficients of a function $h \in H$. The domain of A^τ is given by

$$D(A^\tau) = \left\{ h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 < \infty \right\}.$$

If we introduce the inner product for elements $h, g \in D(A^\tau)$ as

$$(h,g)_\tau = \sum_{k=1}^{\infty} \lambda_k^{2\tau} h_k \overline{g_k} = (A^\tau h, A^\tau g),$$

then $D(A^\tau)$ becomes a Hilbert space with respect to this inner product. The corresponding norm is denoted by $\|h\|_\tau = \sqrt{(h,h)_\tau}$.

Let $\rho \in (1,2)$ be a fixed number. The two-parameter Mittag–Leffler function is defined as (see, e.g., [21], p. 12):

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \mu)},$$

where μ is an arbitrary complex number. In the special case $\mu = 1$, this reduces to the classical Mittag–Leffler function: $E_\rho(z) = E_{\rho,1}(z)$.

We will need the following properties of the Mittag–Leffler functions (see [16], p. 57):

$$E_{\rho,\mu}(z) = \frac{1}{\Gamma(\mu)} + zE_{\rho,\mu+\rho}(z),$$

and the estimate (see, e.g., [14], p. 136)

$$|E_{\rho,\mu}(-t)| \le \frac{c_1}{1+t}, \tag{2.1}$$

which essentially follows from the following asymptotic expansion (see, e.g., [14], p. 134):

$$E_{\rho,\mu}(-t) = -\sum_{k=1}^{n} \frac{(-t)^{-k}}{\Gamma(\mu - k\rho)} + O\big(t^{-n-1}\big). \tag{2.2}$$

Lemma 2.1 (see [16]). *Let $\lambda > 0$. Then, for all $t > 0$, the following identities hold:*

$$\frac{d}{dt}\left(E_{\rho,1}(-\lambda t^\rho)\right) = -\lambda t^{\rho-1} E_{\rho,\rho}(-\lambda t^\rho),$$

$$\frac{d}{dt}\left(E_{\rho,2}(-\lambda t^\rho)\right) = -\lambda t^{\rho-1}\left[E_{\rho,\rho+1}(-\lambda t^\rho) - E_{\rho,\rho+2}(-\lambda t^\rho)\right].$$

Lemma 2.2 (see [6]). *Let $1 < \rho < 2$ and $t \ge 0$. Then the following estimates hold:*

$$|E_{\rho,1}(-t^\rho)| \le 1,$$

$$|E_{\rho,\rho}(-t^\rho)| \le \frac{1}{\Gamma(\rho)},$$

$$|E_{\rho,2}(-t^\rho)| \le 1.$$

Throughout the following, we denote by C a constant that may vary from line to line.

3. Uniqueness Criterion

Let λ_k, $k \ge 1$, be the eigenvalues of the operator A, and let $\xi, \alpha_1, \alpha_2, \beta_1, \beta_2$ ($\alpha_1^2+\alpha_2^2 \ne 0, \beta_1^2+\beta_2^2 \ne 0$) be the constants from the non-local conditions in (1.1). Define, for $k \ge 1$,

$$\Delta_k = \alpha_1\beta_1 + (\alpha_1\beta_2 + \alpha_2\beta_1)E_{\rho,1}(-\lambda_k\xi^\rho) + \alpha_2\beta_2\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2$$

$$+\alpha_2\beta_2\lambda_k\xi^\rho E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,\rho}(-\lambda_k\xi^\rho).$$

Theorem 3.1. *If there exists a solution to problem (1.1), then it is unique if and only if the condition $\Delta_k \ne 0$ holds for all $k \in \mathbb{N}$.*

Proof. Assume $\Delta_k \ne 0$ for all $k \in \mathbb{N}$ and suppose there are two solutions $u_1(t)$ and $u_2(t)$. Since the problem is linear, the function $u(t) = u_1(t) - u_2(t)$ satisfies the homogeneous problem:

$$\begin{cases} D_t^\rho u(t) + Au(t) = 0, \quad 0 < t \le T; \\ \alpha_1 u(0) + \alpha_2 u(\xi) = 0, \quad 0 < \xi \le T; \\ \beta_1 u'(0) + \beta_2 u'(\xi) = 0, \end{cases} \tag{3.1}$$

Let $T_k(t)=(u,V_k)$. Since the operator A is self-adjoint, equation (3.1) yields:

$$D_t^\rho T_k(t)=(D_t^\rho u,V_k)=-(Au,V_k)=-\lambda_k(u,V_k)=-\lambda_k T_k(t).$$

Thus, for $T_k(t)$, $k\geq 1$, we obtain the following non-local problem:

$$\begin{cases} D_t^\rho T_k(t)+\lambda_k T_k(t)=0, & 0<t\leq T,\\ \alpha_1 T_k(0)+\alpha_2 T_k(\xi)=0, & 0<\xi\leq T,\\ \beta_1 T_k'(0)+\beta_2 T_k'(\xi)=0 \end{cases} \tag{3.2}$$

Let $a_k=T_k(0)$ and $b_k=T_k'(0)$. The solution of the corresponding Cauchy problem for (3.2) is given by (see [18], p.230):

$$T_k(t)=a_k E_{\rho,1}(-\lambda_k t^\rho)+b_k t E_{\rho,2}(-\lambda_k t^\rho). \tag{3.3}$$

To find the unknowns a_k and b_k, we substitute into the non-local conditions from (3.2), obtaining the system (see Lemma 2.1):

$$\begin{cases} \left(\alpha_1+\alpha_2 E_{\rho,1}(-\lambda_k\xi^\rho)\right)a_k+\alpha_2\xi E_{\rho,2}(-\lambda_k\xi^\rho)b_k=0,\\ \beta_2(-\lambda_k)\xi^{\rho-1}E_{\rho,\rho}(-\lambda_k\xi^\rho)a_k+\left(\beta_1+\beta_2 E_{\rho,1}(-\lambda_k\xi^\rho)\right)b_k=0. \end{cases} \tag{3.4}$$

This is a homogeneous linear system in a_k and b_k. Its determinant is:

$$\Delta_k=\alpha_1\beta_1+(\alpha_1\beta_2+\alpha_2\beta_1)E_{\rho,1}(-\lambda_k\xi^\rho)+\alpha_2\beta_2\left(E_{\rho,1}(-\lambda_k\xi^\rho)\right)^2$$
$$+\alpha_2\beta_2\lambda_k\xi^\rho E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,\rho}(-\lambda_k\xi^\rho).$$

Since $\Delta_k\neq 0$ by assumption, the system (3.4) has only the trivial solution $a_k=b_k=0$. Therefore, $T_k(t)\equiv 0$ for all k, and by completeness of $\{V_k\}$, we conclude $u(t)\equiv 0$.

Now suppose problem (3.1) has a unique solution and assume the contrary, i.e., $\Delta_k=0$ for some k_0 and some values of ξ, $\alpha_1,\alpha_2,\beta_1,\beta_2$. Then the system (3.4) becomes linearly dependent. Solving the first equation for a_{k_0} gives:

$$a_{k_0}=-b_{k_0}\frac{\alpha_2\xi E_{\rho,2}(-\lambda_{k_0}\xi^\rho)}{\alpha_1+\alpha_2 E_{\rho,1}(-\lambda_{k_0}\xi^\rho)},$$

where b_{k_0} is an arbitrary real number. Then, by (3.3), the function

$$u(t)=b_{k_0}\left(-\frac{\alpha_2\xi E_{\rho,2}(-\lambda_{k_0}\xi^\rho)}{\alpha_1+\alpha_2 E_{\rho,1}(-\lambda_{k_0}\xi^\rho)}E_{\rho,1}(-\lambda_k t^\rho)+tE_{\rho,2}(-\lambda_{k_0}t^\rho)\right)V_{k_0} \tag{3.5}$$

is a solution of $D_t^\rho u(t)+Au(t)=0$.

We verify that it satisfies the non-local conditions. Direct computation shows:

$$\alpha_1 u(0)+\alpha_2 u(\xi)=0,$$

$$\beta_1 u'(0)+\beta_2 u'(\xi)=0,$$

due to the vanishing of Δ_{k_0} and the structure of Mittag-Leffler terms. Thus, (3.5) is a nontrivial solution to problem (3.1), contradicting uniqueness. This proves the theorem. □

4. On the Auxiliary Problems

To solve problem (1.1), we decompose it into two auxiliary problems:

a Cauchy problem for an inhomogeneous equation:

$$\begin{cases} D_t^\rho v(t)+Av(t)=f(t), & 0<t\leq T;\\ v(0)=0;\\ v'(0)=0, \end{cases} \tag{4.1}$$

and a non-local problem for a homogeneous equation:

$$\begin{cases} D_t^\rho w(t) + Aw(t) = 0, \quad 0 < t \le T; \\ \alpha_1 w(0) + \alpha_2 w(\xi) = \varphi^*, \quad 0 < \xi \le T; \\ \beta_1 w'(0) + \beta_2 w'(\xi) = \psi^*, \end{cases} \tag{4.2}$$

where $\varphi^*, \psi^* \in H$ are given elementsand ξ is a fixed point in the interval $(0, T]$.

The problems (4.1) and (4.2) are special cases of the problem (1.1), and the solution to the problem (4.1) is defined analogously to Definition 1.1.

Lemma 4.1. *Let*

$$\varphi^* = \varphi - (\alpha_1 v(0) + \alpha_2 v(\xi)), \quad \psi^* = \psi - (\beta_1 v'(0) + \beta_2 v'(\xi)),$$

and let $v(t)$ and $w(t)$ be the solutions to problems (4.1) and (4.2), respectively. Then the function

$$u(t) = v(t) + w(t)$$

is a solution to problem (1.1).

Proof. The proof of this lemma has a very simple form and is carried out as a result of straightforward calculations (see, for example, [5, 8]). □

Hence, it is sufficient to solve the auxiliary problems to construct the solution to the original problem.

We begin by considering the Cauchy problem (4.1), for which the following result holds:

Theorem 4.2. *(see, [4]) Let $f \in H$. Then the solution of the Cauchy problem (4.1) is unique and has the form:*

$$v(t) = \sum_{k=1}^{\infty} f_k t^\rho E_{\rho,\rho+1}\big(-\lambda_k t^\rho\big) v_k.$$

where the series converges in H for $t \ge 0$.

Moreover, there exists a constant $C > 0$ such that the following coercive-type inequality holds:

$$\|D_t^\rho v(t)\|^2 + \|v(t)\|_1^2 \le C\|f\|^2, \quad 0 \le t \le T.$$

We now turn to the analysis of problem (4.2). Using the Fourier method, we seek the solution in the form of a formal series:

$$w(t) = \sum_{k=1}^{\infty} T_k(t) V_k. \tag{4.3}$$

It is straightforward to verify that each $T_k(t)$, for $k \ge 1$, satisfies the following non-local problem:

$$\begin{cases} D_t^\rho T_k(t) + \lambda_k T_k(t) = 0, \quad 0 < t \le T, \\ \alpha_1 T_k(0) + \alpha_2 T_k(\xi) = \varphi_k^*, \quad 0 < \xi \le T, \\ \beta_1 T_k'(0) + \beta_2 T_k'(\xi) = \psi_k^*, \end{cases}$$

where φ_k^* and ψ_k^* are the Fourier coefficients of φ^* and ψ^*, respectively.

The solution and its derivative to this problem can be expressed as (see (3.3)):

$$T_k(t) = a_k E_{\rho,1}(-\lambda_k t^\rho) + b_k t E_{\rho,2}(-\lambda_k t^\rho), \tag{4.4}$$

$$T_k'(t) = a_k(-\lambda_k) t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) + b_k E_{\rho,1}(-\lambda_k t^\rho),$$

where the unknown coefficients a_k and b_k satisfy the following system of equations (see (3.4)):

$$\begin{cases} (\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k \xi^\rho))\, a_k + \alpha_2 \xi E_{\rho,2}(-\lambda_k \xi^\rho) b_k = \varphi_k^*, \\ \beta_2(-\lambda_k)\xi^{\rho-1} E_{\rho,\rho}(-\lambda_k \xi^\rho) a_k + (\beta_1 + \beta_2 E_{\rho,1}(-\lambda_k \xi^\rho))\, b_k = \psi_k^*. \end{cases} \tag{4.5}$$

The determinant of this system is given by

$$\Delta_k = \alpha_1\beta_1 + (\alpha_1\beta_2 + \alpha_2\beta_1)E_{\rho,1}(-\lambda_k\xi^\rho) + \alpha_2\beta_2\left(E_{\rho,1}(-\lambda_k\xi^\rho)\right)^2$$
$$+\alpha_2\beta_2\lambda_k\xi^\rho E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,\rho}(-\lambda_k\xi^\rho).$$

and if $\Delta_k \neq 0$ for all $k \geq 1$, then the system admits a unique solution given by

$$\begin{cases} a_k = \dfrac{1}{\Delta_k}\left[\varphi_k^*\left(\beta_1 + \beta_2 E_{\rho,1}(-\lambda_k\xi^\rho)\right) - \psi_k^*\alpha_2\xi E_{\rho,2}(-\lambda_k\xi^\rho)\right], \\ b_k = \dfrac{1}{\Delta_k}\left[\psi_k^*\left(\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k\xi^\rho)\right) - \varphi_k^*\beta_2(-\lambda_k)\xi^{\rho-1}E_{\rho,\rho}(-\lambda_k\xi^\rho)\right]. \end{cases} \tag{4.6}$$

Remark 4.3. If $\Delta_k = 0$ for some k, then the necessary and sufficient condition for the existence of a solution is that the right-hand sides of equations (4.5) vanish, i.e., $\varphi_k^* = 0$ and $\psi_k^* = 0$. In this case, the coefficients a_k and b_k can be chosen arbitrarily.

5. Lower bounds for the denominator of the solution

In this section, we find out in which cases Δ_k can vanish, and in those cases when $\Delta_k \neq 0$, we establish lower bounds for

$$\Delta_k = \alpha_1\beta_1 + (\alpha_1\beta_2 + \alpha_2\beta_1)E_{\rho,1}(-\lambda_k\xi^\rho) + \alpha_2\beta_2\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2$$
$$+\alpha_2\beta_2\lambda_k\xi^\rho E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,\rho}(-\lambda_k\xi^\rho).$$

where $\xi \in (0, T]$, $k \in \mathbb{N}$ and $\rho \in (1, 2)$,

Lemma 5.1. *Let $\alpha_1\beta_1 \neq 0$.*

If numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy one of the following conditions for $c_1 > 0$, which is in (2.1),

$$\frac{\alpha_2\beta_2}{\alpha_1\beta_1} \geq 0, \quad 1 > c_1^2\frac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1\left|\frac{\alpha_2}{\alpha_1} + \frac{\beta_2}{\beta_1}\right| \tag{5.1}$$

or

$$\frac{\alpha_2\beta_2}{\alpha_1\beta_1} < 0, \quad 1 < -c_1^2\frac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1\left|\frac{\alpha_2}{\alpha_1} - \frac{\beta_2}{\beta_1}\right| \tag{5.2}$$

then for any $k \in \mathbb{N}$ we have the estimate

$$|\Delta_k| \geq \Delta_{01}, \quad \Delta_{01} = |\alpha_1\beta_1 - (c_1^2|\alpha_2\beta_2| + c_1(|\alpha_1\beta_2| + |\alpha_2\beta_1|)|.$$

If numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ don't satisfy any of the conditions (5.1) and (5.2) for $c_1 > 0$, then there exists a constant $0 < \sigma < 1$ and a number $k_0 = k_0(\sigma)$ such that for all $k > k_0$, the following estimate holds:

$$|\Delta_k| \geq |\alpha_1\beta_1|(1 - \sigma). \tag{5.3}$$

Proof. The idea of proving the lemma is based on the proof of Lemma 3.1 in the paper [13]. Let conditions (5.1) are satisfied. By virtue of properties of the Mittag-Leffler functions one has

$$|\Delta_k| = |\alpha_1\beta_1 + (\alpha_1\beta_2 + \alpha_2\beta_1)E_{\rho,1}(-\lambda_k\xi^\rho) + \alpha_2\beta_2\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2$$
$$+\alpha_2\beta_2\lambda_k\xi^\rho E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,\rho}(-\lambda_k\xi^\rho)|$$
$$\geq |\alpha_1\beta_1|\left|1 - \left|\frac{\alpha_2}{\alpha_1} + \frac{\beta_2}{\beta_1}\right|\frac{c_1}{(1+\lambda_k\xi^\rho)} + \frac{\alpha_2\beta_2}{\alpha_1\beta_1}\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2 - \frac{\alpha_2\beta_2}{\alpha_1\beta_1}\frac{c_1^2}{(1+\lambda_k\xi^\rho)^2}\lambda_k\xi^\rho\right|$$
$$\geq |\alpha_1\beta_1 - c_1|\alpha_1\beta_2 + \alpha_2\beta_1| - c_1^2|\alpha_2\beta_2| + \alpha_2\beta_2\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2| \geq \Delta_{01}, \quad k \geq 1.$$

Next step, let conditions (5.2) are satisfied.

$$|\Delta_k| = |\alpha_1\beta_1 + (\alpha_1\beta_2 + \alpha_2\beta_1)E_{\rho,1}(-\lambda_k\xi^\rho) + \alpha_2\beta_2\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2$$

$$+\alpha_2\beta_2\lambda_k\xi^\rho E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,\rho}(-\lambda_k\xi^\rho)|$$

$$\geq |\alpha_1\beta_1|\left|1-\left|\frac{\alpha_2}{\alpha_1}-\frac{\beta_2}{\beta_1}\right|\frac{c_1}{(1+\lambda_k\xi^\rho)}+\frac{\alpha_2\beta_2}{\alpha_1\beta_1}\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2+\frac{\alpha_2\beta_2}{\alpha_1\beta_1}\frac{c_1^2}{(1+\lambda_k\xi^\rho)^2}\lambda_k\xi^\rho\right|$$

$$\geq |\alpha_1\beta_1-c_1|\alpha_1\beta_2-\alpha_2\beta_1|-c_1^2|\alpha_2\beta_2|+\alpha_2\beta_2\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2|\geq \Delta_{01}, \quad k\geq 1.$$

Easy to see

(1) $-\dfrac{c_1}{1+\lambda_k\xi^\rho}\geq -c_1,$

(2) $\big(E_\alpha(-\lambda_k^2\xi^\rho)\big)^2\geq 0,$

(3) $-\dfrac{c_1^2}{(1+\lambda_k\xi^\rho)^2}\lambda_k\xi^\rho\geq -\dfrac{c_1^2\lambda_k\xi^\rho}{1+\lambda_k\xi^\rho}\geq -c_1^2.$

and from this, we obtain

$$|\Delta_k|\geq |\alpha_1\beta_1-(c_1^2|\alpha_2\beta_2|+c_1(|\alpha_1\beta_2|+|\alpha_2\beta_1|)|.$$

Now, suppose that $\alpha_1\beta_1\neq 0$. However, let neither of the conditions (5.1) nor (5.2) be satisfied for the given values of the parameters. Then, by using the asymptotic estimate (2.2), we have

$$|\Delta_k|=|\alpha_1\beta_1+(\alpha_1\beta_2+\alpha_2\beta_1)E_{\rho,1}(-\lambda_k\xi^\rho)+\alpha_2\beta_2\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2$$
$$+\alpha_2\beta_2\lambda_k\xi^\rho E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,\rho}(-\lambda_k\xi^\rho)|$$
$$=\left|\alpha_1\beta_1+(\alpha_1\beta_2+\alpha_2\beta_1)\left(-\frac{1}{\Gamma(1-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)+\alpha_2\beta_2\left(-\frac{1}{\Gamma(1-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)^2\right.$$
$$\left.+\alpha_2\beta_2\lambda_k\xi^\rho\left(-\frac{1}{\Gamma(2-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)\left(-\frac{1}{\Gamma(-\rho)\xi^{2\rho}}\frac{1}{\lambda_k^2}+O\left(\frac{1}{\lambda_k^3}\right)\right)\right|$$
$$\geq |\alpha_1\beta_1|\left|1+C(\alpha_1,\beta_1,\alpha_2,\beta_2,\rho,\xi)\left(\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)\right|.$$

Now take an arbitrary number $0<\sigma<1$. Since $\lambda_k\to\infty$ as $k\to\infty$, there exists a number k_0 such that for all $k>k_0$,

$$\left|C(\alpha_1,\beta_1,\alpha_2,\beta_2,\rho,\xi)\left(\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)\right|<\sigma,$$

which implies the desired inequality (5.3). □

Lemma 5.2. *Let numbers $\alpha_1,\alpha_2,\beta_1,\beta_2$ satisfy conditions:*

$$|\alpha_1|+|\beta_1|=0, \quad \alpha_2\beta_2\neq 0, \tag{5.4}$$

then there exists a number k_0 such that for all $k>k_0$, the following estimate holds:

$$|\Delta_k|\geq \frac{C}{\lambda_k^2} \tag{5.5}$$

Proof. Let condition (5.4) be satisfied. Then we have $\alpha_2\beta_2\neq 0$, and the parameters α_1,β_1 will be in the following case:

$$\alpha_1=0, \quad \beta_1=0.$$

In this case, according to the asymptotic estimate (2.2),

$$|\Delta_k|=|\alpha_2\beta_2\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2+\alpha_2\beta_2\lambda_k\xi^\rho E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,\rho}(-\lambda_k\xi^\rho)|$$

$$=\left|\alpha_2\beta_2\left(-\frac{1}{\Gamma(1-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)^2\right.$$

$$+\alpha_2\beta_2\lambda_k\xi^\rho\left(-\frac{1}{\Gamma(2-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)\left(-\frac{1}{\Gamma(-\rho)\xi^{2\rho}}\frac{1}{\lambda_k^2}+O\left(\frac{1}{\lambda_k^3}\right)\right)\Bigg|$$

$$\geq|\alpha_2\beta_2|\left|C(\rho,\xi)\left(\frac{1}{\lambda_k^2}+O\left(\frac{1}{\lambda_k^3}\right)\right)\right|\geq\frac{C}{\lambda_k^2}.$$

□

Lemma 5.3. *Let numbers $\alpha_1,\alpha_2,\beta_1,\beta_2$ satisfy conditions:*

$$\alpha_1\beta_1=0,\quad |\alpha_1|+|\beta_1|\neq 0,\quad \alpha_2\beta_2\neq 0, \tag{5.6}$$

or

$$\alpha_1\beta_1=0,\quad \alpha_2\beta_2=0,\quad |\alpha_1\beta_2|+|\alpha_2\beta_1|\neq 0 \tag{5.7}$$

then there exists a number k_0 such that for all $k>k_0$, the following estimate holds:

$$|\Delta_k|\geq\frac{C}{\lambda_k} \tag{5.8}$$

Proof. Let condition (5.6) be satisfied. Then we have $\alpha_2\beta_2\neq 0$, and the parameters α_1,β_1 fall into one of the following cases:

a) Let $\alpha_1=0$, $\beta_1\neq 0$. In this case, according to the asymptotic estimate (2.2),

$$|\Delta_k|=|\alpha_2\beta_1E_{\rho,1}(-\lambda_k\xi^\rho)+\alpha_2\beta_2\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2+\alpha_2\beta_2\lambda_k\xi^\rho E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,\rho}(-\lambda_k\xi^\rho)|$$

$$=\Bigg|\alpha_2\beta_1\left(-\frac{1}{\Gamma(1-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)+\alpha_2\beta_2\left(-\frac{1}{\Gamma(1-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)^2$$

$$+\alpha_2\beta_2\lambda_k\xi^\rho\left(-\frac{1}{\Gamma(2-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)\left(-\frac{1}{\Gamma(-\rho)\xi^{2\rho}}\frac{1}{\lambda_k^2}+O\left(\frac{1}{\lambda_k^3}\right)\right)\Bigg|$$

$$\geq\left|C(\beta_1,\alpha_2,\beta_2,\rho,\xi)\left(\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)\right|\geq\frac{C}{\lambda_k}$$

b) Let now, $\alpha_1\neq 0,\beta_1=0$. In this case, again according to the asymptotic estimate (2.2),

$$|\Delta_k|=|\alpha_1\beta_2E_{\rho,1}(-\lambda_k\xi^\rho)+\alpha_2\beta_2\big(E_{\rho,1}(-\lambda_k\xi^\rho)\big)^2+\alpha_2\beta_2\lambda_k\xi^\rho E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,\rho}(-\lambda_k\xi^\rho)|$$

$$=\Bigg|\alpha_1\beta_2\left(-\frac{1}{\Gamma(1-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)+\alpha_2\beta_2\left(-\frac{1}{\Gamma(1-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)^2$$

$$+\alpha_2\beta_2\lambda_k\xi^\rho\left(-\frac{1}{\Gamma(2-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)\left(-\frac{1}{\Gamma(-\rho)\xi^{2\rho}}\frac{1}{\lambda_k^2}+O\left(\frac{1}{\lambda_k^3}\right)\right)\Bigg|$$

$$\geq\left|C(\alpha_1,\alpha_2,\beta_2,\rho,\xi)\left(\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right)\right|\Delta_k\geq\frac{C}{\lambda_k}$$

Let condition (5.7) be satisfied. Then

$$|\Delta_k|=|cE_{\rho,1}(-\lambda_k\xi^\rho)|=\left|-\frac{1}{\Gamma(1-\rho)\xi^\rho}\frac{1}{\lambda_k}+O\left(\frac{1}{\lambda_k^2}\right)\right|\geq\frac{C}{\lambda_k}$$

holds for the large k. □

The above estimates (5.3),(5.5),(5.8) show that Δ_k is bounded from below for sufficiently large values of k. However, in some cases, it may happen that $\Delta_k=0$ for finitely many values of k. For example, suppose that $\alpha_1\beta_1\neq 0$. However, let neither of the conditions (5.1) nor (5.2) be satisfied for the given values of the parameters. In this case, Δ_k may vanish for certain indices k. Hence, we introduce the set

$$K_0=\{k\in\mathbb{N}:\Delta_k=0\}.$$

Lemma 5.4. *The set K_0 is either empty or contains only finitely many elements.*

Proof. From the proof of Lemma 5.1, it follows that if there exists an index $k \in K_0$, then necessarily $k \le k_0$. Therefore, K_0 is a finite set. Moreover, as mentioned in Section 1, the sequence $\{\lambda_k\}$ consists of discrete values. Hence, Δ_k can vanish only at isolated indices, and it is possible that no such index exists. In this case, the set K_0 is empty. This completes the proof of Lemma 5.4. □

6. The results for the problem (4.2)

We are now ready to study the problem (4.2). Assume that $\Delta_k \neq 0$ for all $k \ge 1$, substituting the coefficients from (4.6) into (4.4), we obtain

$$T_k(t) = a_k E_{\rho,1}(-\lambda_k t^\rho) + b_k t E_{\rho,2}(-\lambda_k t^\rho)$$

$$= \frac{\varphi_k^*}{\Delta_k}\left(\left(\beta_1 + \beta_2 E_{\rho,1}(-\lambda_k \xi^\rho)\right) E_{\rho,1}(-\lambda_k t^\rho) + \beta_2 \lambda_k \xi^{\rho-1} E_{\rho,\rho}(-\lambda_k \xi^\rho) t E_{\rho,2}(-\lambda_k t^\rho)\right)$$

$$+\frac{\psi_k^*}{\Delta_k}\left(-\alpha_2 \xi E_{\rho,2}(-\lambda_k \xi^\rho) E_{\rho,1}(-\lambda_k t^\rho) + \left(\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k \xi^\rho)\right) t E_{\rho,2}(-\lambda_k t^\rho)\right).$$

Then the formal solution of the problem (4.2) takes the form (see (4.3))

$$w(t) = \tag{6.1}$$

$$= \sum_{k=1}^{\infty}\left[\frac{\varphi_k^*}{\Delta_k}\left(\left(\beta_1 + \beta_2 E_{\rho,1}(-\lambda_k \xi^\rho)\right) E_{\rho,1}(-\lambda_k t^\rho) + \beta_2 \lambda_k \xi^{\rho-1} E_{\rho,\rho}(-\lambda_k \xi^\rho) t E_{\rho,2}(-\lambda_k t^\rho)\right)\right.$$

$$\left. + \frac{\psi_k^*}{\Delta_k}\left(-\alpha_2 \xi E_{\rho,2}(-\lambda_k \xi^\rho) E_{\rho,1}(-\lambda_k t^\rho) + \left(\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k \xi^\rho)\right) t E_{\rho,2}(-\lambda_k t^\rho)\right)\right] V_k.$$

Next, using the estimates of the denominator Δ_k (see Section 6), we investigate the well-posedness of problem (4.2) for various values of the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$.

We now relax the requirement of uniqueness of the solution and are only interested in its existence. If numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ do not satisfy any of the conditions (5.1) and (5.2), then the set

$$K_0 = \{k \in \mathbb{N} : \Delta_k = 0\}$$

is non-empty. As a result, for $k \in K_0$, in order for the function defined in (6.1) to be meaningful, it is necessary to impose the following orthogonality conditions:

$$\varphi_k^* = (\varphi^*, V_k) = 0, \quad \psi_k^* = (\psi^*, V_k) = 0, \quad k \in K_0. \tag{6.2}$$

If conditions (6.2) are satisfied, then for all $k \in K_0$, there exist solutions to equation (4.2) that are similar to the solution (3.5) of the homogeneous problem (3.1). Therefore, if K_0 is non-empty, the formal solution of problem (4.2) can be written as

$$w(t) = \sum_{k \in K_0} b_k \left[-\frac{\alpha_2 \xi E_{\rho,2}(-\lambda_k \xi^\rho)}{\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k \xi^\rho)} E_{\rho,1}(-\lambda_k t^\rho) + t E_{\rho,2}(-\lambda_k t^\rho)\right] V_k \tag{6.3}$$

$$+ \sum_{k \notin K_0}\left[\frac{\varphi_k^*}{\Delta_k}\left(\left(\beta_1 + \beta_2 E_{\rho,1}(-\lambda_k \xi^\rho)\right) E_{\rho,1}(-\lambda_k t^\rho) + \beta_2 \lambda_k \xi^{\rho-1} E_{\rho,\rho}(-\lambda_k \xi^\rho) t E_{\rho,2}(-\lambda_k t^\rho)\right)\right.$$

$$\left. + \frac{\psi_k^*}{\Delta_k}\left(-\alpha_2 \xi E_{\rho,2}(-\lambda_k \xi^\rho) E_{\rho,1}(-\lambda_k t^\rho) + \left(\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k \xi^\rho)\right) t E_{\rho,2}(-\lambda_k t^\rho)\right)\right] V_k.$$

where b_k are arbitrary constants. Hence, in this case, the solution is not unique. However, if K_0 is empty, then the solution is unique.

Theorem 6.1. *Let $\varphi^*, \psi^* \in H$, $\alpha_1\beta_1 \neq 0$ and one of the following conditions hold for $c_1 > 0$, which in (2.1)*

(1a) $\frac{\alpha_2\beta_2}{\alpha_1\beta_1} \geq 0, \quad 1 > c_1^2\frac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1\left|\frac{\alpha_2}{\alpha_1} + \frac{\beta_2}{\beta_1}\right|$;

(1b) $\frac{\alpha_2\beta_2}{\alpha_1\beta_1} < 0, \quad 1 < -c_1^2\frac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1\left|\frac{\alpha_2}{\alpha_1} - \frac{\beta_2}{\beta_1}\right|$;

Then problem (4.2) has a unique solution, and this solution has the form (6.1).

Moreover, there is a constant C, we also obtain a coercive inequality:

$$||D_t^\rho w(t)||^2 + ||w(t)||_1^2 \leq C(||\varphi^*||^2 + ||\psi^*||^2), \quad t > 0. \tag{6.4}$$

Proof. We will show that the function (6.1) satisfies all the conditions of Definition 1.1.

Let $S_j(t)$ denote the j-th partial sum of the series (6.1). Then

$$AS_j(t) =$$

$$= \sum_{k=1}^{j}\left[\frac{\varphi_k^*}{\Delta_k}\left((\beta_1 + \beta_2 E_{\rho,1}(-\lambda_k\xi^\rho))E_{\rho,1}(-\lambda_k t^\rho) + \beta_2\lambda_k\xi^{\rho-1}E_{\rho,\rho}(-\lambda_k\xi^\rho)tE_{\rho,2}(-\lambda_k t^\rho)\right)\right.$$
$$\left. + \frac{\psi_k^*}{\Delta_k}\left(-\alpha_2\xi E_{\rho,2}(-\lambda_k\xi^\rho)E_{\rho,1}(-\lambda_k t^\rho) + (\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k\xi^\rho))tE_{\rho,2}(-\lambda_k t^\rho)\right)\right]V_k.$$

By Parsevals identity and applying (2.1), Lemmas 2.2, we get:

$$\|AS_j(t)\|^2 \leq C\sum_{k=k_0}^{j}\frac{1}{|\Delta_k|}\left(|\varphi_k^*|^2 + |\psi_k^*|^2\right).$$

According to Lemma 5.1 we have $|\Delta_k| \geq \Delta_{01}$ for all k. Hence, the following inequalities hold:

$$\|Aw(t)\|^2 \leq C(\|\varphi^*\|^2 + \|\psi^*\|^2), \quad t \geq 0. \tag{6.5}$$

Therefore, if $\varphi^*, \psi^* \in H$, then $Aw(t) \in C((0,T], H)$. Since $\partial_t^\rho w(t) = -Aw(t)$, we conclude that $\partial_t^\rho w(t) \in C((0,T], H)$ and

$$\|\partial_t^\rho w(t)\|^2 \leq C(\|\varphi^*\|^2 + \|\psi^*\|^2), \quad t \geq 0. \tag{6.6}$$

Combining (6.5) and (6.6) give the desired inequality (6.4)

The uniqueness in this case follows from the condition $\Delta_k \neq 0$ for all $k \geq 1$, and is proved similarly to Theorem 3.1. □

Theorem 6.2. *Let one of the following conditions hold for parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ and elements φ^*, ψ^*:*

(1) *$\varphi^*, \psi^* \in H$ and the parameters do not satisfy any of the following conditions for any $c_1 > 0$, which in (2.1):*

(1a) $\frac{\alpha_2\beta_2}{\alpha_1\beta_1} \geq 0, \quad 1 > c_1^2\frac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1\left|\frac{\alpha_2}{\alpha_1} + \frac{\beta_2}{\beta_1}\right|$;

(1b) $\frac{\alpha_2\beta_2}{\alpha_1\beta_1} < 0, \quad 1 < -c_1^2\frac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1\left|\frac{\alpha_2}{\alpha_1} - \frac{\beta_2}{\beta_1}\right|$;

(2) *$\varphi^*, \psi^* \in D(A)$ and one of the following conditions hold:*

(2a) $\alpha_1\beta_1 = 0, \quad |\alpha_1| + |\beta_1| \neq 0, \quad \alpha_2\beta_2 \neq 0,$

(2b) $\alpha_1\beta_1 = 0, \quad \alpha_2\beta_2 = 0, \quad |\alpha_1\beta_2| + |\alpha_2\beta_1| \neq 0;$

(3) *$\varphi^*, \psi^* \in D(A^2)$ and the following condition hold:*

(3a) $|\alpha_1| + |\beta_1| = 0, \quad \alpha_2\beta_2 \neq 0.$

If the set K_0 is empty, then the problem (4.2) has a unique solution, given by the formula (6.1).

Moreover, there is a constant C, we also obtain the corresponding coercive inequalities for each case:

in case ***(1)*** *the inequality (6.4)*

in case ***(2)***

$$||D_t^\rho w(t)||^2 + ||w(t)||_1^2 \leq C(||\varphi^*||_1^2 + ||\psi^*||_1^2), \quad t > 0. \tag{6.7}$$

in case ***(3)***

$$||D_t^\rho w(t)||^2 + ||w(t)||_1^2 \le C(||\varphi^*||_2^2 + ||\psi^*||_2^2), \quad t > 0. \tag{6.8}$$

If the set K_0 is non-empty and the orthogonality condition (6.2) *is satisfied for indices $k \in K_0$, then the problem* (4.2) *has a solution, given in the form (6.3) with arbitrary coefficients b_k.*

Proof. Now, suppose $K_0 = \emptyset$. To prove the theorem, we need to show that the series (6.3) satisfies all the conditions of Definition 1.1. This follows directly from the above analysis for the Theorem 6.1. The series (6.3) consists of two parts: the first is a finite sum of smooth functions (as shown in Lemma 5.4), and the second part is handled similarly to the proof for the series (6.1).

In case **(1)**, according to Lemma 5.1 we have $|\Delta_k| \ge 1 - \sigma$ for all $k > k_0$. Like this, using Lemmas 5.2,5.3, in cases **(2),(3)** the estimates $|\Delta_k| \ge \frac{C}{\lambda_k}$ and $|\Delta_k| \ge \frac{C}{\lambda_k^2}$ hold for all $k > k_0$ respectively.

Hence, the following inequalities hold:

In case **(1)** the inequality (6.5)

In case **(2)**

$$\|Aw(t)\|^2 \le C(\|\varphi^*\|_1^2 + \|\psi^*\|_1^2), \quad t \ge 0. \tag{6.9}$$

In case **(3)**

$$\|Aw(t)\|^2 \le C(\|\varphi^*\|_2^2 + \|\psi^*\|_2^2), \quad t \ge 0. \tag{6.10}$$

Therefore, if in case **(1)** $\varphi^*, \psi^* \in H$, in case **(1)** $\varphi^*, \psi^* \in D(A)$, in case **(1)** $\varphi^*, \psi^* \in D(A^2)$, then $Aw(t) \in C((0,T], H)$. Since $\partial_t^\rho w(t) = -Aw(t)$, we conclude that $\partial_t^\rho w(t) \in C((0,T], H)$ and

In case **(1)** inequality (6.6)

In case **(2)**

$$\|\partial_t^\rho w(t)\|^2 \le C(\|\varphi^*\|_1^2 + \|\psi^*\|_1^2), \quad t \ge 0. \tag{6.11}$$

In case **(3)**

$$\|\partial_t^\rho w(t)\|^2 \le C(\|\varphi^*\|_2^2 + \|\psi^*\|_2^2), \quad t \ge 0. \tag{6.12}$$

Combining (6.5),(6.9),(6.10) and (6.6),(6.11),(6.12) give the desired inequalities (6.4),(6.7),(6.8).

The uniqueness in this case follows from the condition $\Delta_k \ne 0$ for all $k \ge 1$.

Let $K_0 \ne \emptyset$. In this case, we remove the special behavior in a finite number of terms using the orthogonality conditions (6.2), and apply the previous arguments to the rest of the functional series (6.3). As a result, the solution to the problem (4.2) exists, but its uniqueness is not guaranteed. □

7. The Results for the Main Problem

We now turn to the study of the main problem (1.1). Let $v(t)$ and $w(t)$ be the solutions to problems (4.1) and (4.2), respectively. Then, according to Lemma 4.1, the solution of problem (1.1) can be represented as $u(t) = v(t) + w(t)$. Therefore, if $\Delta_k \ne 0$ for all $k \ge 1$, the solution to problem (1.1) has the form

$$\begin{aligned} u(t) = \sum_{k=1}^{\infty} \Big[& \frac{\varphi_k - (\alpha_1 v_k(0) + \alpha_2 v_k(\xi))}{\Delta_k} \big((\beta_1 + \beta_2 E_{\rho,1}(-\lambda_k \xi^\rho)) E_{\rho,1}(-\lambda_k t^\rho) \\ & + \beta_2 \lambda_k \xi^{\rho-1} E_{\rho,\rho}(-\lambda_k \xi^\rho) t E_{\rho,2}(-\lambda_k t^\rho) \big) \\ & + \frac{\psi_k - (\beta_1 v_k'(0) + \beta_2 v_k'(\xi))}{\Delta_k} \big(-\alpha_2 \xi E_{\rho,2}(-\lambda_k \xi^\rho) E_{\rho,1}(-\lambda_k t^\rho) + \\ & (\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k \xi^\rho)) t E_{\rho,2}(-\lambda_k t^\rho) \big) + v_k(t) \Big] V_k. \end{aligned} \tag{7.1}$$

where

$$v_k(t) = \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta)\, d\eta.$$

The uniqueness of the solution $u(t)$ follows from the uniqueness of the solutions $v(t)$ and $w(t)$.

Furthermore, we note that if $\Delta_k \neq 0$ for some k then similar reasoning as in problem (4.2) can be applied to problem (1.1). In this case, the orthogonality conditions (6.2) for the functions $\varphi, \psi \in H$ take the form

$$\varphi_k^* = (\varphi^*, V_k) = (\varphi - (\alpha_1 v(0) + \alpha_2 v(\xi)), V_k) = 0,$$

$$\psi_k^* = (\psi^*, V_k) = (\psi - (\beta_1 v'(0) + \beta_2 v'(\xi)), V_k) = 0, \quad k \in K_0,$$

or equivalently,

$$(\varphi, V_k) = (\alpha_1 v(0) + \alpha_2 v(\xi), V_k), \quad (\psi, V_k) = (\beta_1 v'(0) + \beta_2 v'(\xi), V_k), \quad k \in K_0. \tag{7.2}$$

Remark 7.1. We emphasize that conditions (7.2) are both necessary and sufficient. However, since the function v and its derivative are involved, these conditions may be somewhat difficult to verify in practice. Therefore, we can formulate the following sufficient conditions:

$$\begin{cases} (\varphi, V_k) = 0, \quad (\psi, V_k) = 0, \quad k \in K_0, \\ (f, V_k) = 0, \quad k \in K_0, \end{cases} \tag{7.3}$$

These are easier to check and, when satisfied, imply the necessary and sufficient conditions (7.2).

If the orthogonality conditions (7.3) are satisfied, then by Lemma 4.1, the solution to problem (1.1) takes the form

$$u(t) = \sum_{k \in K_0} b_k \left[-\frac{\alpha_2 \xi E_{\rho,2}(-\lambda_k \xi^\rho)}{\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k \xi^\rho)} E_{\rho,1}(-\lambda_k t^\rho) + t E_{\rho,2}(-\lambda_k t^\rho) \right] V_k$$

$$+ \sum_{k \notin K_0} \Bigg[\frac{\varphi_k - (\alpha_1 v_k(0) + \alpha_2 v_k(\xi))}{\Delta_k} \big((\beta_1 + \beta_2 E_{\rho,1}(-\lambda_k \xi^\rho)) E_{\rho,1}(-\lambda_k t^\rho)$$

$$+ \beta_2 \lambda_k \xi^{\rho-1} E_{\rho,\rho}(-\lambda_k \xi^\rho) t E_{\rho,2}(-\lambda_k t^\rho) \big)$$

$$+ \frac{\psi_k - (\beta_1 v_k'(0) + \beta_2 v_k'(\xi))}{\Delta_k} (-\alpha_2 \xi E_{\rho,2}(-\lambda_k \xi^\rho) E_{\rho,1}(-\lambda_k t^\rho) +$$

$$\big(\alpha_1 + \alpha_2 E_{\rho,1}(-\lambda_k \xi^\rho)\big) t E_{\rho,2}(-\lambda_k t^\rho)) \Bigg] V_k + \sum_{k=1}^{\infty} v_k(t) V_k. \tag{7.4}$$

Thus, we arrive at the following results for the main problem (1.1):

Theorem 7.2. *Let $f, \varphi, \psi \in H$, $\alpha_1 \beta_1 \neq 0$ and one of the following conditions hold for $c_1 > 0$, which in (2.1)*

(a) $\frac{\alpha_2\beta_2}{\alpha_1\beta_1} \geq 0, \quad 1 > c_1^2 \frac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1 \left| \frac{\alpha_2}{\alpha_1} + \frac{\beta_2}{\beta_1} \right|$;

(b) $\frac{\alpha_2\beta_2}{\alpha_1\beta_1} < 0, \quad 1 < -c_1^2 \frac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1 \left| \frac{\alpha_2}{\alpha_1} - \frac{\beta_2}{\beta_1} \right|$;

Then problem (1.1) has a unique solution, and this solution has the form (7.1).

Moreover, there is a constant C, we also obtain a coercive inequality:

$$||D_t^\rho u(t)||^2 + ||u(t)||_1^2 \leq C(||f||^2 + ||\varphi||^2 + ||\psi||^2), \quad t > 0; \tag{7.5}$$

Theorem 7.3. *Let one of the following conditions hold for parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ and elements f, φ, ψ:*

(1) *$f, \varphi, \psi \in H$ and the parameters do not satisfy any of the following conditions for any $c_1 > 0$, which in (2.1):*

(1a) $\frac{\alpha_2\beta_2}{\alpha_1\beta_1} \geq 0, \quad 1 > c_1^2 \frac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1 \left| \frac{\alpha_2}{\alpha_1} + \frac{\beta_2}{\beta_1} \right|$;

(1b) $\frac{\alpha_2\beta_2}{\alpha_1\beta_1} < 0, \quad 1 < -c_1^2 \frac{\alpha_2\beta_2}{\alpha_1\beta_1} + c_1 \left| \frac{\alpha_2}{\alpha_1} - \frac{\beta_2}{\beta_1} \right|$;

(2) *$f, \varphi, \psi \in D(A)$ and one of the following conditions hold:*

(2a) $\alpha_1 \beta_1 = 0, \quad |\alpha_1| + |\beta_1| \neq 0, \quad \alpha_2 \beta_2 \neq 0,$

(2b) $\alpha_1\beta_1 = 0, \quad \alpha_2\beta_2 = 0, \quad |\alpha_1\beta_2| + |\alpha_2\beta_1| \neq 0$;
(3) $f, \varphi, \psi \in D(A^2)$ *and the following condition hold:*
(3a) $\quad |\alpha_1| + |\beta_1| = 0, \quad \alpha_2\beta_2 \neq 0.$

If the set K_0 *is empty, then the problem (1.1) has a unique solution, given by the formula (7.1).*
Moreover, there is a constant C*, we also obtain the corresponding coercive inequalities for each case:*
in case ***(1)*** *the inequality (7.5)*
in case ***(2)***

$$||D_t^\rho u(t)||^2 + ||u(t)||_1^2 \leq C(||f||_1^2 + ||\varphi||_1^2 + ||\psi||_1^2), \quad t > 0;$$

in case ***(3)***

$$||D_t^\rho u(t)||^2 + ||u(t)||_1^2 \leq C(||f||_2^2 + ||\varphi||_2^2 + ||\psi||_2^2), \quad t > 0;$$

If the set K_0 *is non-empty and the orthogonality condition* (7.3) *is satisfied for indices* $k \in K_0$*, then the problem (1.1) has a solution, given in the form (7.4) with arbitrary coefficients* b_k.

8. Acknowledgement.

The authors acknowledge financial support from the Ministry of Innovative Development of the Republic of Uzbekistan, Grant No F-FA-2021-424.

References

[1] Ashurov R., Mukhiddinova O., Umarov S.; A Non-Local Problem for the Fractional-Order RayleighStokes Equation. Fractal Fract.,–2023.–7.–490.

[2] Alimov S.A., Ashurov R.R.; On the Backward Problems in Time for Time-Fractional Subdiffusion Equations. Available online: `https://www.researchgate.net/publication/351575279` (accessed on [insert access date]).

[3] Alimov Sh.A., Khalmukhamedov A.R.; On a Non-Local Problem for a Boussinesq-Type Differential Equation. Lobachevskii J. Math.,–2022.–43.–4.–P.916–923.

[4] Ashurov R.R., Fayziyev Yu.E.; Inverse Problem for Finding the Order of the Fractional Derivative in the Wave Equation. Math. Notes,–2021.–110.–6.–P.842–852.

[5] Ashurov R.R., Fayziyev Yu.E.; On the Non-Local Problems in Time for Time-Fractional Subdiffusion Equations. Fractal Fract.,–2022.–6.–41.

[6] Ashurov R.R., Fayziyev Yu.E., Khudoykulova M.U.; On a Non-Local Problem for a Fractional Differential Equation of the Boussinesq Type. Bull. Karaganda Univ. Math. Ser.,–2024.–3(115).–P.34–45.

[7] Ashurov R., Vaisova N.; Backward and Non-Local Problems for the Rayleigh-Stokes Equation. Fractal Fract.,–2022.–6.–587.

[8] Ashurov R., Nuraliyeva N.; A Three-Parameter Problem for Fractional Differential Equation with an Abstract Operator. Lobachevskii J. Math.,–2024.–45(11).–P.5788–5801.

[9] Ashurov R., Shakarova M.; Time-dependent source identification problem for fractional Schrodinger type equations. Lobachevskii Journal of Mathematics,–2022.–43(2).–P.303–315.

[10] Ashurov R., Umarov S.; An inverse problem of determining orders of systems of fractional pseudo-differential equations. Fract. Calc. Appl. Anal.,–2022.–25.–P.109–127.

[11] Ashyralyev A.O., Hanalyev A., Sobolevskii P.E.; Coercive Solvability of Non-Local Boundary Value Problem for Parabolic Equations. Abstr. Appl. Anal.,–2001.–6.–P.53–61.

[12] Durdiev D.K., Rahmonov A.A., Bozorov Z.R.; A Two-Dimensional Diffusion Coefficient Determination Problem for the Time-Fractional Equation. Math. Methods Appl. Sci.,–2021.–44.–P.10753–10761.

[13] Durdiev D.K., Rahmonov A.A.; Inverse Coefficient Problem for a Fractional Wave Equation with Time-Non-Local and Integral Overdetermination Conditions. Bol. Soc. Mat. Mex.,–2023.–29.–50.

[14] Dzherbashian M.M.; Integral Transforms and Representation of Functions in the Complex Domain. Nauka, Moscow,–1966. (In Russian)

[15] Floridia G., Li Z., Yamamoto M.; Well-Posedness for the Backward Problems in Time for General Time-Fractional Diffusion Equation. Rend. Lincei Mat. Appl.,–2020.–31.–P.593–610.

[16] Gorenflo R., Kilbas A.A., Mainardi F., Rogozin S.V.; Mittag-Leffler Functions, Related Topics and Applications. Springer, Berlin/Heidelberg.

[17] Karimov E., Mamchuev M., Ruzhansky M.; Non-Local Initial Problem for Second-Order Time-Fractional and Space-Singular Equation. Hokkaido Math. J.,–2020.–49.–2.–P.349–361.

[18] Kilbas A.A., Srivastava H.M., Trujillo J.J.; Theory and Applications of Fractional Differential Equations. Elsevier, North-Holland Mathematics Studies,–2006.

[19] Krein S.G.; Linear Differential Equations in a Banach Space. Nauka, Moscow,–1966. (In Russian)

[20] Lizama C.; Abstract Linear Fractional Evolution Equations. In: Handbook of Fractional Calculus with Applications, Vol. 2, pp.465–497, De Gruyter, Berlin,–2019.

[21] Pskhu A.V.; Fractional Differential Equations. Nauka, Moscow,–2005. (In Russian)

[22] Ruzhansky M., Tokmagambetov N., Torebek B.; On a Non-Local Problem for a Multi-Term Fractional Diffusion-Wave Equation. Fract. Calc. Appl. Anal.,–2020.–23.–2.–P.324–355.

[23] Sabitov K.B.; Equations of Mathematical Physics. FIZMATLIT, Moscow,–2013.

[24] Liu J., Yamamoto M.; A Backward Problem for the Time-Fractional Diffusion Equation. Appl. Anal.,–2010.–89.–P.1769–1788.

[25] Sakamoto K., Yamamoto M.; Initial Value/Boundary Value Problems for Fractional Diffusion-Wave Equations and Applications to Some Inverse Problems. J. Math. Anal. Appl.,–2011.–382.–P.426–447.

Ashurov R.R. ,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Science,
Tashkent, Uzbekistan.
e-mail:ashurovr@gmail.com

Nuraliyeva N.Sh.,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Science,
Tashkent, Uzbekistan.
e-mail:n.navbahor2197@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.39-49
DOI: 10.29229/uzmj.2025-3-4

Lie affgebra structures on three-dimensional solvable Lie algebras

Berdalova Kh.R., Khudoyberdiyev A.Kh.

Abstract. In this work, the complex Lie affgebra structures on three-dimensional solvable Lie algebras are completely determined.

Keywords: Lie algebras, Lie affgebras, solvable Lie algebras, generalized derivations.
MSC (2020): 17B05; 17B30; 20N10.

1. Introduction

The study of vector space valued Lie brackets on affine spaces or *Lie affgebras* was initiated in [5] and developed and applied to the investigation of differential geometry or AV-geometry [6, 7]. In their definition of a Lie affgebra the authors of [5] rely on the existence of a vector space: both the antisymmetry and the Jacobi identity of a Lie bracket are formulated in this vector space. On the other hand, a vector space-independent definition of an affine space is available.

A new point of view and extension of Lie affgebras was proposed Tomasz Brzeziński and his collaborators in [1, 3, 4]. They gave an intrinsic definition of a Lie affgebra, removes the need for a specified element entirely and formulates the antisymmetry and Jacobi identity of the Lie bracket without invoking a neutral element or a vector space. In this approach, a vector space is an artefact rather than a fundamental ingredient of an affine space in the sense that any point of an affine space determines a vector space, the tangent space or the vector space fibre at this point.

In [3], various examples and properties of Lie affgebras are given, and it is shown how the affine Lie bracket reduces to the linear case. Lie affgebra structures on several classes of affine spaces of matrices are studied in [4]. It is shown that, when retracted to the underlying vector spaces, they correspond to classical matrix Lie algebras: general and special linear, anti-symmetric, anti-hermitian and special antihermitian Lie algebras, respectively. In [1], it is shown that any Lie affgebra, that is an algebraic system consisting of an affine space together with a bi-affine multiplication satisfying affine versions of the antisymmetry and Jacobi identity, is isomorphic to a Lie algebra together with an element and a specific generalized derivation. These Lie algebraic data can be taken for the construction of a Lie affgebra or, conversely, they can be uniquely derived for any Lie algebra fibre of the Lie affgebra. It is asserted that a homomorphism between Lie affgebras is given by a homomorphism between Lie algebra fibres and a constant. This allows for the formulation of clear criteria for isomorphisms between Lie affgebras. Using these assertions, the classification of all Lie affgebras with one-dimensional vector space fibres, non-abelian two-dimensional Lie algebra fibres and three-dimensional simple Lie algebra sl_2 is given.

In this work, we provide a complete classification of Lie affgebras with three-dimensional non-nilpotent solvable Lie algebras.

2. Preliminaries

Let X be a set, $\mathbb{F}$ be a field and $\langle -,-,-\rangle : X^3 \to X$ ternary operation.

Definition 2.1. [2] The set X with the ternary operation $\langle -,-,-\rangle : X^3 \to X$ is said to be abelian heap, if for all $x_i \in X$, $i = 1, \dots, 5$,

$$\langle x_1, x_2, x_3\rangle = \langle x_3, x_2, x_1\rangle,\ \langle x_1, x_1, x_2\rangle = x_2,\ \langle\langle x_1, x_2, x_3\rangle, x_4, x_5\rangle = \langle x_1, x_2, \langle x_3, x_4, x_5\rangle\rangle.$$

A homomorphism of heaps is a function $f : X \to Y$ preserving the operations in the sense that, $f(\langle x_1, x_2, x_3\rangle) = \langle f(x_1), f(x_2), f(x_3)\rangle$ for all $x_i \in X$.

Definition 2.2. By an $\mathbb{F}$-affine space, we mean algebraic system $\Big(X, \langle -,-,-\rangle, (-,-,-)\Big)$, where $\langle -,-,-\rangle : X^3 \to X$ and $(-,-,-) : \mathbb{F} \times X \times X \to X$, such that

a) $(X, \langle -,-,- \rangle)$ is an abelian heap;

b) For any $\alpha \in \mathbb{F}$ and $a \in X$, the map $(\alpha, a, -) : X \to X$ is a homomorphism of heaps;

c) For a fixed elements $a, b \in X$, the map $(-, a, b) : \mathbb{F} \to X$ is a homomorphism of heaps, where $\mathbb{F}$ is the heap with the operation $\alpha - \beta + \gamma$;

d) For all $\alpha, \beta \in \mathbb{F}$ and $a, b \in X$, $(\alpha\beta, a, b) = (\alpha, a, (\beta, a, b))$, $(1, a, b) = b$, $(0, a, b) = a$.

e) For all $\alpha \in \mathbb{F}$ and $a, b, c \in X$, $(\alpha, a, b) = \langle (\alpha, c, b), (\alpha, c, a), a \rangle$.

An affine map $f : X \to Y$ is a heap homomorphism preserving the actions in the sense that, for all $a, b, c \in X$ and $\alpha \in \mathbb{F}$,

$$f(\alpha, a, b) = (\alpha, f(a), f(b)).$$

The set of affine maps from X to Y is denoted by $\mathrm{Aff}(X, Y)$.

Let X be an affine space over $\mathbb{F}$. For a fixed element $e \in X$, we define a binary operation $+ : X \times X \to X$ and the map $\mathbb{F} \times X \to X$ as follows:

$$x + y := \langle x, e, y \rangle, \quad \alpha a := (\alpha, e, a).$$

Then the triple $(X, +, \alpha)$ forms a vector space, which is called the *tangent space* to X or the *vector space fibre* of X at the point e. This tangent space is usually denoted by $T_e(X)$.

Definition 2.3. [1] An affine space X with a binary operation $\{-,-\} : X \times X \to X$ is called a *Lie affgebra*, if the binary operation $\{-,-\}$ satisfies the following conditions:

a) for all $a \in X$, both $\{a, -\}$ and $\{-, a\}$ are affine map;

b) affine antisymmetry, that is, $\langle \{a, b\}, \{a, a\}, \{b, a\} \rangle = \{b, b\}$ for all $a, b \in X$;

c) the affine Jacobi identity, that is, for all $a, b, c \in X$,

$$\langle \{a, \{b, c\}\}, \{a, \{a, a\}\}, \{b, \{c, a\}\}, \{b, \{b, b\}\}, \{c, \{a, b\}\} \rangle = \{c, \{c, c\}\}.$$

The multiplication in a Lie affgebra is often referred to as an affine *Lie bracket*.

It is proven that any tangent space of a Lie affgebra inherits a natural Lie algebra structure.

Theorem 2.4. [1] *Let X be a Lie affgebra with a bracket $\{-,-\}$. Then, for all $e \in X$, the tangent space $T_e(X)$ is a Lie algebra with the multiplication*

$$[a, b] = \{a, b\} - \{a, e\} + \{e, e\} - \{e, b\}.$$

Let G be a Lie algebra. A linear map $f : G \to G$ is called a generalized derivation in the sense of Leger and Luks [9], if there exist linear maps f', f'' such that

$$[f(a), b] + [a, f'(b)] = f''([a, b]).$$

In the following theorem, the connection between the Lie affgebras and Lie algebras with the generalized derivation is established.

Theorem 2.5. [1] *Let G be a Lie algebra and $f, g \in End(G)$ be such that, for all $a, b \in G$,*

$$f([a, b]) = [f(a), b] + [a, f(b)] - [a, g(b)]. \tag{2.1}$$

Then, G is a Lie affgebra with the affine space structure

$$\langle a, b, c \rangle = a - b + c, \quad (\alpha, a, b) = (1 - \alpha)a + \alpha b \tag{2.2}$$

and the affine Lie bracket (for any fixed $s \in G$)

$$\{a, b\} = [a, b] + g(a) + f(b - a) + s. \tag{2.3}$$

We denote this Lie affgebra by $X(G; g, f, s)$.

Furthermore, for all $e \in G$, we have $T_e X(G; g, f, s) \cong G$.

Conversely, for any Lie affgebra X and any $e \in X$, there exist g, f necessarily satisfying (2.1) *and $s \in T_e X$, such that $X = X(T_e X; g, f, s)$.*

The criterion for the isomorphism of affgebras $X(G;g,f,s)$ and $X(G;g',f',s')$ is provided in the following theorem.

Theorem 2.6. [1] *A Lie affgebras $X(G;g,f,s)$ and $X(G;g',f',s')$ are isomorphic if and only if there exists a Lie algebra automorphism $\Psi : G \to G$ and an element $a \in G$ such that*

$$g' = \Psi g \Psi^{-1}, \quad f' = \Psi(f - \mathrm{ad}_a)\Psi^{-1}, \quad s' = \Psi(s + a - g(a)), \tag{2.4}$$

where ad_a *is an inner derivation, such that* $\mathrm{ad}_a(x) = [a,x]$.

3. Main result

In this work, we systematically construct all Lie affgebra structures on the three-dimensional complex non-nilpotent solvable Lie algebras. Here, we give the list of three-dimensional complex non-nilpotent solvable Lie algebras [8]:

$$\begin{array}{lll}
\mathbf{r}_3 & : & [e_1,e_2] = e_2, \quad [e_1,e_3] = e_2 + e_3, \\
\mathbf{r}_3(\lambda) & : & [e_1,e_2] = e_2, \quad [e_1,e_3] = \lambda e_3, \qquad \lambda \in \mathbb{C}^*, |\lambda| \le 1, \\
\mathbf{r}_2 \oplus \mathbb{C} & : & [e_1,e_2] = e_2.
\end{array}$$

3.1. **Lie affgebra structures on the algebra $\mathbf{r}_3$.** First, we present the description of the pair of linear transformations (f,g), that satisfy condition (2.1).

Proposition 3.1. *Any linear transformations f and g of the algebra $\mathbf{r}_3$ that satisfy condition* (2.1) *have the following form:*

$$\begin{array}{lll}
f(e_1) = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3, & f(e_2) = \beta_4 e_2, & f(e_3) = \beta_5 e_2 + \beta_4 e_3, \\
g(e_1) = \beta_1 e_1, & g(e_2) = \beta_1 e_2, & g(e_3) = \beta_1 e_3.
\end{array}$$

Proof. The proof follows directly from the definition through a routine verification. □

Applying Theorem 2.6, for any element $s \in \mathbf{r}_3$, we obtain Lie affgebra structure by the binary operation

$$\{x,y\} = [x,y] + g(x) + f(y-x) + s.$$

Considering $f - ad_a$ for $a = \beta_5 e_1 + (\beta_3 - \beta_2)e_2 - \beta_3 e_3$, instead of f, we can easily conclude that Lie affgebra over $\mathbf{r}_3$ is isomorphic to one with

$$\begin{array}{lll}
f(e_1) = \beta_1 e_1, & f(e_2) = \beta_4 e_2, & f(e_3) = \beta_4 e_3, \\
g(e_1) = \beta_1 e_1, & g(e_2) = \beta_1 e_2, & g(e_3) = \beta_1 e_3.
\end{array} \tag{3.1}$$

Thus, for any elements $x = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$ and $y = \eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3$, we obtain an affine Lie bracket

$$\{x,y\} = [x,y] + \beta_1 \eta_1 e_1 + \big(\beta_1 \xi_2 + \beta_4(\eta_2 - \xi_2)\big)e_2 + \big(\beta_1 \xi_3 + \beta_4(\eta_3 - \xi_3)\big)e_3 + s,$$

where $s = N_1 e_1 + N_2 e_2 + N_3 e_3$. Denote the Lie affgebra with this affine Lie bracket by $F(\beta_1, \beta_4, N_1, N_2, N_3)$.

Proposition 3.2. *Two Lie affgebras $F(\beta_1, \beta_4, N_1, N_2, N_3)$ and $F(\beta_1', \beta_4', N_1', N_2', N_3')$ are isomorphic if and only if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$, $\alpha_4 \in \mathbb{C}^*$, such that*

$$\begin{array}{lll}
\beta_1' = \beta_1, & \beta_4' = \beta_4, & N_2' = \alpha_1(N_1 - (\beta_1 - \beta_4)(1-\beta_1)) - \alpha_2(\beta_4 - \beta_1)(1-\beta_1) + \alpha_4 N_2 + \alpha_3 N_3, \\
N_1' = N_1, & & N_3' = \alpha_2(N_1 - (\beta_1 - \beta_4)(1-\beta_1)) + \alpha_4 N_3.
\end{array}$$

Proof. Let f, g and f', g' be linear operators of the form (3.1). Since $g = \beta_1\,\mathrm{id}$, it follows from (2.4) that $\beta_1' = \beta_1$. Since any automorphism of the algebra $\mathbf{r}_3$ has the form

$$\Psi(e_1) = e_1 + \alpha_1 e_2 + \alpha_2 e_3, \quad \Psi(e_2) = \alpha_4 e_2, \quad \Psi(e_3) = \alpha_3 e_2 + \alpha_4 e_3,$$

then, using the formulas

$$f' = \Psi(f - \mathrm{ad}_a)\Psi^{-1}, \quad s' = \Psi(s + a - g(a)),$$

for any element $a = C_1e_1 + C_2e_2 + C_3e_3$, we obtain

$$\begin{pmatrix} \beta_1' & 0 & 0 \\ 0 & \beta_4' & 0 \\ 0 & 0 & \beta_4' \end{pmatrix} = \begin{pmatrix} \beta_1 & 0 & 0 \\ \alpha_1(\beta_1 - \beta_4 + C_1) + \alpha_2 C_1 + \alpha_3 C_3 + \alpha_4(C_2 + C_3) & \beta_4 - C_1 & -C_1 \\ \alpha_2(\beta_1 - \beta_4 + C_1) + \alpha_4 C_3 & 0 & \beta_4 - C_1 \end{pmatrix}$$

and

$$\begin{aligned} N_1' &= C_1(1 - \beta_1) + N_1, \\ N_2' &= \alpha_1(C_1(1 - \beta_1) + N_1) + \alpha_3(C_3(1 - \beta_1) + N_3) + \alpha_4(C_2(1 - \beta_1) + N_2), \\ N_3' &= \alpha_2(C_1(1 - \beta_1) + N_1) + \alpha_4(C_3(1 - \beta_1) + N_3). \end{aligned}$$

Choosing

$$C_1 = 0, \quad C_3 = \frac{\alpha_2(\beta_4 - \beta_1)}{\alpha_4}, \quad C_2 = \frac{\alpha_1(\beta_4 - \beta_1) - (\alpha_3 + \alpha_4)C_3}{\alpha_4},$$

yields the required result, thereby completing the proof of the proposition. □

Theorem 3.3. *Any Lie affgebra structure on the algebra* $\mathbf{r}_3$ *is isomorphic to one of the following pairwise non-isomorphic Lie affgebras:*

$$\begin{aligned} &F_1(\beta_1, \beta_4, N_1, 0, 0), \quad F_2(\beta_1, \beta_4, (\beta_1 - \beta_4)(1 - \beta_1), 0, 1), \\ &F_3(\beta_1, \beta_1, 0, 1, 0), \qquad F_4(1, \beta_4, 0, 1, 0), \ \beta_4 \neq 1. \end{aligned}$$

Proof. Using Proposition 3.2, we consider the following cases.

- Let $N_1 \neq (\beta_1 - \beta_4)(1 - \beta_1)$, then taking

  $$\alpha_1 = \frac{\alpha_3 N_3 + \alpha_4 N_2 - \alpha_2(\beta_4 - \beta_1)(1 - \beta_1)}{(\beta_1 - \beta_4)(1 - \beta_1) - N_1}, \quad \alpha_2 = \frac{\alpha_4 N_3}{(\beta_1 - \beta_4)(1 - \beta_1) - N_1},$$

  we get $N_2' = 0$, $N_3' = 0$. Hence, the corresponding Lie affgebra is $F_1(\beta_1, \beta_4, N_1, 0, 0)$ with $N_1 \neq (\beta_1 - \beta_4)(1 - \beta_1)$.

- Let $N_1 = (\beta_1 - \beta_4)(1 - \beta_1)$, then $N_2' = \alpha_2 N_1 + \alpha_4 N_2 + \alpha_3 N_3$ and $N_3' = \alpha_4 N_3$.

  - Let $N_3 \neq 0$, then taking $\alpha_4 = \frac{1}{N_3}$, $\alpha_3 = -\frac{\alpha_2 N_1 + \alpha_4 N_2}{N_3}$, we obtain $N_3' = 1$, $N_2' = 0$. Thus, we get the Lie affgebra $F_2(\beta_1, \beta_4, (\beta_1 - \beta_4)(1 - \beta_1), 0, 1)$.
  - Let $N_3 = 0$, then $N_2' = \alpha_2 N_1 + \alpha_4 N_2$.
    * If $N_1 \neq 0$, then $\beta_1 \neq \beta_4$ and $\beta_1 \neq 1$. Taking $\alpha_4 = -\frac{\alpha_2 N_1}{N_2}$, we get $N_2' = 0$ and obtain the Lie affgebra $F_1(\beta_1, \beta_4, N_1, 0, 0)$ with $N_1 = (\beta_1 - \beta_4)(1 - \beta_1) \neq 0$.
    * If $N_1 = 0$, then $N_2' = \alpha_4 N_2$, and $\beta_1 = \beta_4$ or $\beta_1 = 1$.
      · If $N_2 = 0$, then we get the Lie affgebras $F_1(\beta_1, \beta_1, 0, 0, 0)$ and $F_1(1, \beta_4, 0, 0, 0)$.
      · If $N_2 \neq 0$, then taking $\alpha_4 = \frac{1}{N_2}$, we have $N_2' = 1$ and obtain the Lie affgebras $F_3(\beta_1, \beta_1, 0, 1, 0)$ and $F_4(1, \beta_4, 0, 1, 0)$.

□

3.2. **Lie affgebra structures on the algebra $\mathbf{r}_3(\lambda)$.** In the following proposition, we present the description of the pair of linear transformations (f, g), for the algebra $\mathbf{r}_3(\lambda)$, that satisfy condition (2.1).

Proposition 3.4. *Any linear transformations f and g of the algebra $\mathbf{r}_3(\lambda)$ that satisfy condition* (2.1) *have the following form:*

$$\begin{array}{llll} \lambda \neq 1: & f(e_1) = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3, & f(e_2) = \beta_4 e_2, & f(e_3) = \beta_5 e_3, \\ & g(e_1) = \beta_1 e_1, & g(e_2) = \beta_1 e_2, & g(e_3) = \beta_1 e_3. \\ \lambda = 1: & f(e_1) = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3, & f(e_2) = \beta_4 e_2 + \beta_6 e_3, & f(e_3) = \beta_7 e_2 + \beta_5 e_3, \\ & g(e_1) = \beta_1 e_1, & g(e_2) = \beta_1 e_2, & g(e_3) = \beta_1 e_3. \end{array}$$

Proof. The proof of the proposition follows directly from the straightforward verification. □

It is not difficult to get that, any automorphism of the algebra $\mathbf{r}_3(\lambda)$ has the form

$$\begin{array}{llll} \lambda \neq 1: & \Psi(e_1) = e_1 + \alpha_1 e_2 + \alpha_2 e_3, & \Psi(e_2) = \alpha_3 e_2, & \Psi(e_3) = \alpha_4 e_3. \\ \lambda = 1: & \Psi(e_1) = e_1 + \alpha_1 e_2 + \alpha_2 e_3, & \Psi(e_2) = \alpha_3 e_2 + \alpha_5 e_3, & \Psi(e_3) = \alpha_4 e_3 + \alpha_6 e_2. \end{array}$$

Case $\lambda \neq 1$. Considering $f - ad_a$ for $a = \beta_4 e_1 - \beta_2 e_2 - \frac{\beta_3}{\lambda} e_3$, instead of f, we can easily conclude that Lie affgebra over $\mathbf{r}_3(1)$ is isomorphic to one with

$$\begin{array}{lll} f(e_1) = \beta_1 e_1, & f(e_2) = 0, & f(e_3) = \beta_5 e_3, \\ g(e_1) = \beta_1 e_1, & g(e_2) = \beta_1 e_2, & g(e_3) = \beta_1 e_3. \end{array} \tag{3.2}$$

Thus, for any elements $x = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$ and $y = \eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3$, we obtain an affine Lie bracket

$$\{x, y\} = [x, y] + \beta_1 \eta_1 e_1 + \beta_1 \xi_2 e_2 + (\beta_1 \xi_3 + \beta_5 \eta_3 - \beta_5 \xi_3) e_3 + s,$$

where $s = N_1 e_1 + N_2 e_2 + N_3 e_3$. Denote the Lie affgebra on the algebra $\mathbf{r}_3(\lambda)$ with this affine Lie bracket by $H(\beta_1, \beta_5, N_1, N_2, N_3)$.

Proposition 3.5. *Two Lie affgebras $H(\beta_1, \beta_5, N_1, N_2, N_3)$ and $H(\beta_1', \beta_5', N_1', N_2', N_3')$ are isomorphic if and only if there exist $\alpha_1, \alpha_2 \in \mathbb{C}$, $\alpha_3, \alpha_4 \in \mathbb{C}^*$, such that*

$$\begin{array}{lll} \beta_1' = \beta_1, & \beta_5' = \beta_5, & N_2' = \alpha_3 N_2 + \alpha_1 \big(N_1 - \beta_1(1 - \beta_1)\big), \\ N_1' = N_1, & & N_3' = \alpha_4 N_3 + \alpha_2 \big(N_1 - \frac{(\beta_1 - \beta_5)(1 - \beta_1)}{\lambda}\big). \end{array}$$

Proof. The proof is straightforward and follows similarly to the proof of Proposition 3.2. □

Proposition 3.6. *Any Lie affgebra structure on the algebra $\mathbf{r}_3(\lambda)$ with $\lambda \neq 1$, is isomorphic to one of the following pairwise non-isomorphic Lie affgebras:*

$$\begin{array}{ll} H_1(\beta_1, \beta_5, N_1, 0, 0), & H_2(\beta_1, \beta_5, \beta_1(1 - \beta_1), 1, 0), \\ H_3(\beta_1, \beta_5, \frac{(\beta_1 - \beta_5)(1 - \beta_1)}{\lambda}, 0, 1), & H_4(1, \beta_5, 0, 1, 1), \ \beta_5 \neq 1 - \lambda, \\ H_5(\beta_1, \beta_1(1 - \lambda), \beta_1(1 - \beta_1), 1, 1). & \end{array}$$

Proof. Using Proposition 3.5, we consider the following cases.

- $N_1 \neq \beta_1(1 - \beta_1)$ and $N_1 \neq \frac{(\beta_1 - \beta_5)(1 - \beta_1)}{\lambda}$, then taking $\alpha_1 = \frac{\alpha_3 N_2}{\beta_1(1 - \beta_1) - N_1}$, $\alpha_2 = \frac{\alpha_4 \lambda N_3}{(\beta_1 - \beta_5)(1 - \beta_1) - \lambda N_1}$, we get $N_2' = N_3' = 0$ and obtain the Lie affgebra $H_1(\beta_1, \beta_5, N_1, 0, 0)$.
- $N_1 = \beta_1(1 - \beta_1)$ and $N_1 \neq \frac{(\beta_1 - \beta_5)(1 - \beta_1)}{\lambda}$, then we have $\beta_1 \neq 1$, $\beta_5 \neq \beta_1(1 - \lambda)$ and taking $\alpha_2 = \frac{\alpha_4 \lambda N_3}{(\beta_1 - \beta_5)(1 - \beta_1) - \lambda N_1}$, we obtain $N_3' = 0$ and $N_2' = \alpha_3 N_2$. Thus, in the case of $N_2 = 0$, we obtain the Lie affgebra H_1 with $N_1 = \beta_1(1 - \beta_1)$. In the case of $N_2 \neq 0$, we can get $N_2' = 1$ and obtain the Lie affgebra $H_2(\beta_1, \beta_5, \beta_1(1 - \beta_1), 1, 0)$.

- $N_1 \neq \beta_1(1-\beta_1)$ and $N_1 = \frac{(\beta_1-\beta_5)(1-\beta_1)}{\lambda}$, then taking $\alpha_1 = \frac{\alpha_3 N_2}{\beta_1(1-\beta_1)-N_1}$, we obtain $N_2' = 0$ and $N_3' = \alpha_4 N_3$. Thus, in the case of $N_3 = 0$, we obtain the affgebra H_2 with $N_1 = \frac{(\beta_1-\beta_5)(1-\beta_1)}{\lambda}$. In the case of $N_3 \neq 0$, we can suppose $N_3' = 1$ and obtain the affgebra $H_3(\beta_1, \beta_5, \frac{(\beta_1-\beta_5)(1-\beta_1)}{\lambda}, 0, 1)$.

- $N_1 = \beta_1(1-\beta_1)$ and $N_1 = \frac{(\beta_1-\beta_5)(1-\beta_1)}{\lambda}$. Then we have $\beta_1 = 1$ or $\beta_5 = \beta_1(1-\lambda)$ and $N_2' = \alpha_3 N_2$, $N_3' = \alpha_4 N_3$. If $N_2 N_3 = 0$, then we obtain one of the affgebras from H_1, H_2, H_3. In the case of $N_2 N_3 \neq 0$, we can suppose $N_2' = N_3' = 1$ and obtain the affgebras $H_4(1, \beta_5, 0, 1, 1)$ and $H_5(\beta_1, \beta_1(1-\lambda), \beta_1(1-\beta_1), 1, 1)$.

□

Case $\lambda = 1$. Considering $f - ad_a$ for $a = \mu e_1 - \beta_2 e_2 - \beta_3 e_3$, instead of f, we can easily conclude that Lie affgebra over $\mathbf{r}_3(\lambda)$ is isomorphic to one with

$$\begin{array}{lll} f(e_1) = \beta_1 e_1, & f(e_2) = (\beta_4 - \mu)e_2 + \beta_6 e_3, & f(e_3) = \beta_7 e_2 + (\beta_5 - \mu)e_3, \\ g(e_1) = \beta_1 e_1, & g(e_2) = \beta_1 e_2, & g(e_3) = \beta_1 e_3. \end{array}$$

By selecting μ as one of the eigenvalues of the matrix $\begin{pmatrix} \beta_4 & \beta_6 \\ \beta_7 & \beta_5 \end{pmatrix}$, without loss of generality we can assume that the matrix has a zero eigenvalue.

From the formulas $f' = \Psi(f - \mathrm{ad}_a)\Psi^{-1}$, and the general form of the automorphism Ψ, it follows that the matrix of the operator f' can be reduced to one of the following Jordan forms:

$$\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta_5 \end{pmatrix}, \quad \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & 0 & \beta_6 \\ 0 & 0 & 0 \end{pmatrix}, \ \beta_6 \neq 0.$$

If the Jordan form of the matrix corresponds to the first case, then the situation is analogous to that of $\lambda \neq 1$ and we obtain the affgebras H_1, H_2, H_3, H_4 and H_5 with $\lambda = 1$.

If the Jordan form of the matrix corresponds to the second case, then we get

$$\begin{array}{lll} f(e_1) = \beta_1 e_1, & f(e_2) = \beta_6 e_3, & f(e_3) = 0, \\ g(e_1) = \beta_1 e_1, & g(e_2) = \beta_1 e_2, & g(e_3) = \beta_1 e_3, \end{array} \tag{3.3}$$

and for any elements $x = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$, $y = \eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3$, we obtain an affine Lie bracket

$$\{x, y\} = [x, y] + \beta_1 \eta_1 e_1 + \beta_1 \xi_2 e_2 + (\beta_1 \xi_3 + \beta_6 \eta_2 - \beta_6 \xi_2)e_3 + s,$$

where $s = N_1 e_1 + N_2 e_2 + N_3 e_3$. Denote the Lie affgebra on the algebra $\mathbf{r}_3(\lambda)$ with this affine Lie bracket by $K(\beta_1, \beta_6, N_1, N_2, N_3)$.

Proposition 3.7. *Two Lie affgebras $K(\beta_1, \beta_6, N_1, N_2, N_3)$ and $K(\beta_1', \beta_6', N_1', N_2', N_3')$ are isomorphic if and only if there exist $\alpha_1, \alpha_2 \in \mathbb{C}$, $\alpha_3, \alpha_4 \in \mathbb{C}^*$, such that*

$$\begin{array}{lll} \beta_1' = \beta_1, & \beta_6' = \frac{\alpha_4 \beta_6}{\alpha_3}, & N_2' = \alpha_3 N_2 + \alpha_1\big(N_1 - \beta_1(1-\beta_1)\big), \\ N_1' = N_1, & & N_3' = \alpha_4 N_3 + \alpha_2\big(N_1 - \beta_1(1-\beta_1)\big) + \alpha_5 N_2 + \frac{\alpha_1 \alpha_4 \beta_6 (1-\beta_1)}{\alpha_3}. \end{array}$$

Proof. The proof is straightforward and follows similarly to the proof of Proposition 3.2. □

Proposition 3.8. *Any Lie affgebra from the class $K(\beta_1, \beta_6, N_1, N_2, N_3)$ is isomorphic to one of the following pairwise non-isomorphic Lie affgebras:*

$$K_1(\beta_1, 1, N_1, 0, 0), \quad K_2(\beta_1, 1, \beta_1(1-\beta_1), 1, 0), \quad K_3(1, 1, 0, 0, 1).$$

Proof. By taking $\alpha_3 = \alpha_4 \beta_6$, in Proposition 3.7, without loss of generality we may assume $\beta_6 = 1$ and $\alpha_3 = \alpha_4$. We now consider the following cases:

- $N_1 \neq \beta_1(1-\beta_1)$, then taking $\alpha_1 = \frac{\alpha_3 N_2}{\beta_1(1-\beta_1)-N_1}$, $\alpha_2 = \frac{\alpha_4 N_3+\alpha_5 N_2+\alpha_1(1-\beta_1)}{\beta_1(1-\beta_1)-N_1}$, we get $N_2' = N_3' = 0$ and obtain the affgebra $K_1(\beta_1, 1, N_1, 0, 0)$.
- $N_1 = \beta_1(1-\beta_1)$, then $N_2' = \alpha_4 N_2$, $N_3' = \alpha_4 N_3 + \alpha_5 N_2 + \alpha_1(1-\beta_1)$.
  - If $N_2 \neq 0$, then choosing $\alpha_4 = \frac{1}{N_2}$, $\alpha_5 = -\frac{\alpha_4 N_3+\alpha_1(1-\beta_1)}{N_2}$, we have $N_2' = 1$, $N_3' = 0$ and obtain the affgebra $K_2(\beta_1, 1, \beta_1(1-\beta_1), 1, 0)$.
  - If $N_2 = 0$, then $N_3' = \alpha_4 N_3 + \alpha_1(1-\beta_1)$. If $\beta_1 \neq 1$ or $N_3 = 0$, then by choosing appropriate values for α_1 and α_4, we have $N_3' = 0$ and obtain the affgebra K_1 with $N_1 = \beta_1(1-\beta_1)$. If $\beta_1 = 1$ and $N_3 \neq 0$, then taking $\alpha_4 = \frac{1}{N_3}$, we have $N_3' = 1$ and obtain the affgebra $K_3(1, 1, 0, 0, 1)$.

□

Summarizing the results for the cases $\lambda \neq 1$ and $\lambda = 1$, we obtain the following theorem.

Theorem 3.9. *Any Lie affgebra structure on the algebra* $\mathbf{r}_3(\lambda)$ *is isomorphic to one of the following pairwise non-isomorphic Lie affgebras:*

$$H_1(\beta_1, \beta_5, N_1, 0, 0), \qquad H_2(\beta_1, \beta_5, \beta_1(1-\beta_1), 1, 0),$$
$$H_3(\beta_1, \beta_5, \tfrac{(\beta_1-\beta_5)(1-\beta_1)}{\lambda}, 0, 1), \qquad H_4(1, \beta_5, 0, 1, 1),\ \beta_5 \neq 1-\lambda,$$
$$H_5(\beta_1, \beta_1(1-\lambda), \beta_1(1-\beta_1), 1, 1),$$

and

$$K_1(\beta_1, 1, N_1, 0, 0), \quad K_2(\beta_1, 1, \beta_1(1-\beta_1), 1, 0), \quad K_3(1, 1, 0, 0, 1).$$

Note that in the case of $\lambda = 1$, we have $H_2(\beta_1, 0, \beta_1(1-\beta_1), 1, 0) \simeq H_3(\beta_1, 0, \beta_1(1-\beta_1), 0, 1)$.

3.3. **Lie affgebra structures on the algebra $\mathbf{r}_2 \oplus \mathbb{C}$.** First, we present the description of the pair of linear transformations (f, g), that satisfy condition (2.1).

Proposition 3.10. *Any linear transformations f and g of the algebra $\mathbf{r}_2 \oplus \mathbb{C}$, that satisfy condition* (2.1) *have the following form:*

$$f(e_1) = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3, \quad f(e_2) = \beta_4 e_2, \quad f(e_3) = \beta_5 e_3,$$
$$g(e_1) = \beta_1 e_1 + \gamma_1 e_3, \quad g(e_2) = \beta_1 e_2 + \gamma_2 e_3, \quad g(e_3) = \gamma_3 e_3.$$

Proof. The proof of the proposition follows directly from the straightforward verification. □

For any element $s = N_1 e_1 + N_2 e_2 + N_3 e_3 \in \mathbf{r}_2 \oplus \mathbb{C}$, a Lie affgebra structure is defined by the binary operation

$$\{x, y\} = [x, y] + g(x) + f(y - x) + s,$$

which is denoted by $X(\mathbf{r}_2 \oplus \mathbb{C}; g, f, s)$.

Since any automorphism of the algebra $\mathbf{r}_2 \oplus \mathbb{C}$ has the form

$$\Psi(e_1) = e_1 + \alpha_1 e_2 + \alpha_2 e_3, \quad \Psi(e_2) = \alpha_3 e_2, \quad \Psi(e_1) = \alpha_4 e_3,$$

we obtain the following proposition.

Proposition 3.11. *Two Lie affgebras $X(\mathbf{r}_2 \oplus \mathbb{C}; g, f, s)$ and $X\mathbf{r}_2 \oplus \mathbb{C}; g', f', s')$ are isomorphic if and only if there exist $\alpha_1, \alpha_2, C_1, C_2, C_3 \in \mathbb{C}$ and $\alpha_3, \alpha_4 \in \mathbb{C}^*$, such that*

$$\beta_1' = \beta_1, \quad \beta_2' = \alpha_1(\beta_1 + \beta_4 - C_1) + \alpha_3(\beta_2 + C_2), \quad \beta_3' = \alpha_2(\beta_1 - \beta_5) + \alpha_4 \beta_3,$$
$$\beta_5' = \beta_5, \quad \beta_4' = \beta_4 - C_1,$$
$$\gamma_3' = \gamma_3, \quad \gamma_1' = \alpha_2(\beta_1 - \gamma_3) + \alpha_4(\gamma_1 - \tfrac{\alpha_1 \gamma_2}{\alpha_3}), \qquad \gamma_2' = \tfrac{\alpha_4 \gamma_2}{\alpha_3},$$

and

$$N_1' = C_1(1-\beta_1) + N_1,$$
$$N_2' = \alpha_1(C_1(1-\beta_1) + N_1) + \alpha_3(C_2(1-\beta_1) + N_2),$$
$$N_3' = \alpha_2(C_1(1-\beta_1) + N_1) + \alpha_4(C_3(1-\gamma_3) + C_1\gamma_1 + C_2\gamma_2 + N_3).$$

Proof. The proof is obtained by straightforward computation using Theorem2.6. □

Choosing $C_1 = \beta_4$ and $C_2 = -\frac{\alpha_3\beta_2+\alpha_1\beta_1}{\alpha_3}$ in Proposition 3.11, we obtain $\beta_2' = \beta_4' = 0$. Therefore, without loss of generality, we may assume $\beta_2 = \beta_4 = 0$, $C_1 = 0$, $C_2 = -\frac{\alpha_1\beta_1}{\alpha_3}$. Hence, we derive the following restrictions:

$$\beta_3' = \alpha_2(\beta_1 - \beta_5) + \alpha_4\beta_3, \quad \gamma_1' = \alpha_2(\beta_1 - \gamma_3) + \alpha_4(\gamma_1 - \tfrac{\alpha_1\gamma_2}{\alpha_3}), \quad \gamma_2' = \tfrac{\alpha_4\gamma_2}{\alpha_3}, \tag{3.4}$$

and

$$\begin{aligned} N_1' &= N_1, \\ N_2' &= \alpha_3N_2 + \alpha_1\big(N_1 - \beta_1(1-\beta_1)\big), \\ N_3' &= \alpha_4N_3 + \alpha_2N_1 + \alpha_4\big(C_3(1-\gamma_3) - \tfrac{\alpha_1\beta_1}{\alpha_3}\gamma_2\big). \end{aligned} \tag{3.5}$$

Hence, Lie affgebra structures on the algebra $\mathbf{r}_2 \oplus \mathbb{C}$ depend on the parameters $\beta_1, \beta_3, \beta_5, \gamma_1, \gamma_2, \gamma_3, N_1, N_2, N_3$. We denote this class of Lie affgebras by

$$L(\beta_1, \beta_3, \beta_5, \gamma_1, \gamma_2, \gamma_3, N_1, N_2, N_3).$$

Theorem 3.12. *Any Lie affgebra structure on the algebra $\mathbf{r}_2 \oplus \mathbb{C}$ is isomorphic to one of the following pairwise non-isomorphic Lie affgebras:*

$L_1(\beta_1, 0, \beta_5, 0, 1, \gamma_3, N_1, 0, 0)$, $L_2(\beta_1, 0, \beta_5, 0, 1, \gamma_3, N_1, 1, 0)$,
$L_3(\beta_1, 0, \beta_5, 0, 1, 1, N_1, N_2, 1)$, $L_4(\beta_1, 0, \beta_5, 0, 0, \gamma_3, N_1, 0, 0)$,
$L_5(\beta_1, 0, \beta_5, 1, 0, \gamma_3, N_1, 0, 0)$, $L_6(\beta_1, 0, \beta_5, \gamma_1, 0, 1, N_1, 0, 1)$,
$L_7(\beta_1, 0, \beta_5, 0, 0, \gamma_3, \beta_1(1-\beta_1), 1, 0)$, $L_8(\beta_1, 0, \beta_5, 1, 0, \gamma_3, \beta_1(1-\beta_1), 1, 0)$,
$L_9(\beta_1, 0, \beta_5, \gamma_1, 0, 1, \beta_1(1-\beta_1), 1, 1)$, $L_{10}(\beta_1, 1, \beta_1, 0, 1, \gamma_3, N_1, 0, 0)$,
$L_{11}(\beta_1, 1, \beta_1, 0, 1, \beta_1, N_1, N_2, 0), \beta_1 \neq 1$, $L_{12}(\beta_1, 1, \beta_1, 0, 1, \gamma_3, \beta_1(1-\beta_1), N_2, 0),\ \gamma_3 \neq 1$,
$L_{13}(\beta_1, 1, \beta_1, 0, 1, 1, N_1, N_2, 0)$, $L_{14}(\beta_1, \beta_3, \beta_1, 0, 1, 1, \beta_1(1-\beta_1), N_2, 1)$,
$L_{15}(\beta_1, 1, \beta_1, 0, 0, \gamma_3, N_1, 0, 0)$, $L_{16}(\beta_1, \beta_3, \beta_1, 1, 0, \beta_1, N_1, 0, 0),\ \beta_1 \neq 1$,
$L_{17}(\beta_1, \beta_3, \beta_1, 1, 0, 1, N_1, 0, 0)$, $L_{18}(\beta_1, \beta_3, \beta_1, 0, 0, 1, 0, 0, 1)$,
$L_{19}(\beta_1, 1, \beta_1, 0, 0, \gamma_3, \beta_1(1-\beta_1), 1, 0)$, $L_{20}(\beta_1, \beta_3, \beta_1, 1, 0, \beta_1, \beta_1(1-\beta_1), 1, 0),\ \beta_1 \neq 1$,
$L_{21}(\beta_1, \beta_3, \beta_1, 0, 0, 1, \beta_1(1-\beta_1), 0, 1)$, $L_{22}(\beta_1, \beta_3, \beta_1, 0, 0, 1, \beta_1(1-\beta_1), 1, 1)$,
$L_{23}(1, \beta_3, 1, 1, 0, 1, 0, 0, N_3)$, $L_{24}(1, \beta_3, 1, 1, 0, 1, 0, 1, N_3)$

Proof. Consider the following cases.

- Let $\beta_1 \neq \beta_5$ and $\gamma_2 \neq 0$, then taking $\alpha_2 = \frac{\alpha_4\beta_3}{\beta_5-\beta_1}$, $\alpha_4 = \frac{\alpha_3}{\gamma_2}$ and $\alpha_1 = \alpha_2(\beta_1 - \gamma_3) + \alpha_4\gamma_1$, we obtain $\beta_3' = 0$, $\gamma_1' = 0$, $\gamma_2' = 1$. Thus, we get that $N_2' = \alpha_3N_2$, $N_3' = \alpha_3(C_3(1-\gamma_3) + N_3)$.
  - Let $\gamma_3 \neq 1$, then taking $C_3 = \frac{N_3}{\gamma_3-1}$, we obtain $N_3' = 0$. Hence, we get the Lie affgebras $L_1(\beta_1, 0, \beta_5, 0, 1, \gamma_3, N_1, 0, 0)$ and $L_2(\beta_1, 0, \beta_5, 0, 1, \gamma_3, N_1, 1, 0)$ depending on whether $N_2 = 0$ or not.
  - Let $\gamma_3 = 1$, then we get $N_2' = \alpha_3N_2$, $N_3' = \alpha_3N_3$.
    - ∗ If $N_3 \neq 0$, then taking $\alpha_3 = \frac{1}{N_3}$, we get $N_3' = 1$, and obtain the affgebra $L_3(\beta_1, 0, \beta_5, 0, 1, 1, N_1, N_2, 1)$.
    - ∗ If $N_3 = 0$, then we obtain the affgebras $L_1(\beta_1, 0, \beta_5, 0, 1, 1, N_1, 0, 0)$ and $L_2(\beta_1, 0, \beta_5, 0, 1, 1, N_1, 1, 0)$ depending on whether $N_2 = 0$ or not.

- Let $\beta_1 \neq \beta_5$ and $\gamma_2 = 0$, then $\gamma_2' = 0$ and taking $\alpha_2 = \frac{\alpha_4\beta_3}{\beta_5-\beta_1}$, we obtain $\beta_3' = 0$. Thus, we get that

$$\gamma_1' = \alpha_4\gamma_1, \quad N_2' = \alpha_1(N_1 - \beta_1(1-\beta_1)) + \alpha_3N_2, \quad N_3' = \alpha_4(C_3(1-\gamma_3) + N_3).$$

- Let $N_1 \neq \beta_1(1-\beta_1)$ and $\gamma_3 \neq 1$, then taking $\alpha_1 = \frac{\alpha_3 N_2}{\beta_1(1-\beta_1)-N_1}$, $C_3 = \frac{N_3}{\gamma_3 - 1}$, we obtain $N_2' = 0$, $N_3' = 0$ and $\gamma_1' = \alpha_4\gamma_1$. Thus, in this case we get the affgebras $L_4(\beta_1, 0, \beta_5, 0, 0, \gamma_3, N_1, 0, 0)$ and $L_5(\beta_1, 0, \beta_5, 1, 0, \gamma_3, N_1, 0, 0)$ depending on whether $\gamma_1 = 0$ or not.
- Let $N_1 \neq \beta_1(1-\beta_1)$ and $\gamma_3 = 1$, then taking $\alpha_1 = \frac{\alpha_3 N_2}{\beta_1(1-\beta_1)-N_1}$, we obtain $N_2' = 0$, and $N_3' = \alpha_4 N_3$, $\gamma_1' = \alpha_4\gamma_1$.
  - If $N_3 = 0$, then $N_3' = 0$, and we get the affgebras $L_4(\beta_1, 0, \beta_5, 0, 0, 1, N_1, 0, 0)$ and $L_5(\beta_1, 0, \beta_5, 1, 0, 1, N_1, 0, 0)$ depending on whether $\gamma_1 = 0$ or not.
  - If $N_3 \neq 0$, then taking $\alpha_4 = \frac{1}{N_3}$, we get $N_3' = 1$ and obtain the affgebra $L_6(\beta_1, 0, \beta_5, \gamma_1, 0, 1, N_1, 0, 1)$.
- Let $N_1 = \beta_1(1-\beta_1)$ and $\gamma_3 \neq 1$, then taking $C_3 = \frac{N_3}{\gamma_3-1}$, we obtain $N_3' = 0$ and $N_2' = \alpha_3 N_2$, $\gamma_1' = \alpha_4\gamma_1$.
  - If $N_2 = 0$, then $N_2' = 0$, and we get the affgebras $L_4(\beta_1, 0, \beta_5, 0, 0, 1, \beta_1(1-\beta_1), 0, 0)$ and $L_5(\beta_1, 0, \beta_5, 1, 0, 1, \beta_1(1-\beta_1), 0, 0)$ depending on whether $\gamma_1 = 0$ or not.
  - If $N_2 \neq 0$, then taking $\alpha_3 = \frac{1}{N_2}$, we get $N_2' = 1$ and obtain the affgebras $L_7(\beta_1, 0, \beta_5, 0, 0, \gamma_3, \beta_1(1-\beta_1), 1, 0)$ and $L_8(\beta_1, 0, \beta_5, 1, 0, \gamma_3, \beta_1(1-\beta_1), 1, 0)$ depending on whether $\gamma_1 = 0$ or not.
- Let $N_1 = \beta_1(1-\beta_1)$ and $\gamma_3 = 1$, then we get $N_2' = \alpha_3 N_2$, $N_3' = \alpha_4 N_3$, $\gamma_1' = \alpha_4\gamma_1$.
  - If $N_2 = 0$, $N_3 = 0$, then $N_2' = 0$, $N_3' = 0$, and we obtain the affgebras $L_4(\beta_1, 0, \beta_5, 0, 0, 1, \beta_1(1-\beta_1), 0, 0)$ and $L_5(\beta_1, 0, \beta_5, 1, 0, 1, \beta_1(1-\beta_1), 0, 0)$ depending on whether $\gamma_1 = 0$ or not.
  - If $N_2 = 0$, $N_3 \neq 0$, then $N_2' = 0$, and taking $\alpha_4 = \frac{1}{N_3}$, we get $N_3' = 1$. Thus, in this case we obtain the affgebra $L_6(\beta_1, 0, \beta_5, \gamma_1, 0, 1, \beta_1(1-\beta_1), 0, 1)$.
  - If $N_2 \neq 0$, $N_3 = 0$, then $N_3' = 0$, and taking $\alpha_3 = \frac{1}{N_2}$, we get $N_2' = 1$. Thus, in this case, we obtain the affgebras $L_7(\beta_1, 0, \beta_5, 0, 0, 1, \beta_1(1-\beta_1), 1, 0)$, and $L_8(\beta_1, 0, \beta_5, 1, 0, 1, \beta_1(1-\beta_1), 1, 0)$ depending on whether $\gamma_1 = 0$ or not.
  - If $N_2 \neq 0$, $N_3 \neq 0$, taking $\alpha_3 = \frac{1}{N_2}$, $\alpha_4 = \frac{1}{N_3}$, we get $N_2' = N_3' = 1$. Hence, we get the affgebra $L_9(\beta_1, 0, \beta_5, \gamma_1, 0, 1, \beta_1(1-\beta_1), 1, 1)$.

- Let $\beta_1 = \beta_5$ and $\gamma_2 \neq 0$, then taking $\alpha_1 = \alpha_4\gamma_1 + \alpha_2(\beta_1 - \gamma_3)$ and $\alpha_4 = \frac{\alpha_3}{\gamma_2}$, we obtain $\gamma_1' = 0$, $\gamma_2' = 1$. Thus, without loss of generality, we may assume $\gamma_1 = 0$, $\gamma_2 = 1$, then $\alpha_1 = \alpha_2(\beta_1 - \gamma_3)$, $\alpha_4 = \alpha_3$, and we have

$$\beta_3' = \alpha_3\beta_3, \quad N_2' = \alpha_3 N_2 + \alpha_2(\beta_1 - \gamma_3)(N_1 - \beta_1(1-\beta_1)),$$

$$N_3' = \alpha_3(C_3(1-\gamma_3) + N_3) + \alpha_2(N_1 - \beta_1(\gamma_3 - \beta_1)).$$

  - Let $\gamma_3 \neq 1$ and $(\beta_1 - \gamma_3)(N_1 - \beta_1(1-\beta_1)) \neq 0$, then taking

$$C_3 = \frac{\alpha_3 N_3 + \alpha_2(N_1 - \beta_1(\gamma_3 - \beta_1))}{\alpha_3(\gamma_3 - 1)}, \quad \alpha_2 = \frac{\alpha_3 N_2}{(\gamma_3 - \beta_1)(N_1 - \beta_1(1-\beta_1))},$$

  we obtain $N_2' = 0$, $N_3' = 0$.
    - If $\beta_3 = 0$, then we get the affgebra $L_1(\beta_1, 0, \beta_1, 0, 1, \gamma_3, N_1, 0, 0)$.
    - If $\beta_3 \neq 0$, we get the affgebra $L_{10}(\beta_1, 1, \beta_1, 0, 1, \gamma_3, N_1, 0, 0)$.
  - Let $\gamma_3 \neq 1$ and $(\beta_1 - \gamma_3)(N_1 - \beta_1(1-\beta_1)) = 0$, then taking $C_3 = \frac{\alpha_3 N_3 + \alpha_2(N_1 - \beta_1(\gamma_3 - \beta_1)}{\alpha_3(\gamma_3 - 1)}$, we obtain $N_3' = 0$. Thus, we have $\beta_3' = \alpha_3\beta_3$, and $N_2' = \alpha_3 N_2$.
    - If $\beta_3 = 0$, then we get the affgebras $L_1(\beta_1, 0, \beta_1, 0, 1, \gamma_3, N_1, 0, 0)$ and $L_2(\beta_1, 0, \beta_1, 0, 1, \gamma_3, N_1, 1, 0)$ depending on whether $N_2 = 0$ or not.

  * If $\beta_3 \neq 0$, then we can suppose $\beta_3' = 1$, and obtain the affgebras $L_{11}(\beta_1, 1, \beta_1, 0, 1, \beta_1, N_1, N_2, 0)$ and $L_{12}(\beta_1, 1, \beta_1, 0, 1, \gamma_3, \beta_1(1-\beta_1), N_2, 0)$.

- Let $\gamma_3 = 1$ and $N_1 \neq \beta_1(1-\beta_1)$, then taking $\alpha_2 = \frac{\alpha_3 N_3}{\beta_1(1-\beta_1)-N_1}$, we have $N_3' = 0$ and obtain $\beta_3' = \alpha_3\beta_3$, $N_2' = \alpha_3 N_2$.
  * If $\beta_3 = 0$, then we get the affgebras $L_1(\beta_1, 0, \beta_1, 0, 1, 1, N_1, 0, 0)$ and $L_2(\beta_1, 0, \beta_1, 0, 1, 1, N_1, 1, 0)$ depending on whether $N_2 = 0$ or not.
  * If $\beta_3 \neq 0$, then we can suppose $\beta_3' = 1$, and obtain the affgebra $L_{13}(\beta_1, 1, \beta_1, 0, 1, 1, N_1, N_2, 0)$.
- Let $\gamma_3 = 1$ and $N_1 = \beta_1(1-\beta_1)$, then we get $\beta_3' = \alpha_3\beta_3$, $N_2' = \alpha_3 N_2$ and $N_3' = \alpha_3 N_3$.
  * If $N_3 = 0$, then we get the affgebras L_1, L_2 and L_{13}, with $\beta_3 = \beta_1$, $\gamma_3 = 1$, $N_1 = \beta_1(1-\beta_1)$.
  * If $N_3 \neq 0$, then we may assume $N_3' = 1$ and obtain the affgebra $L_{14}(\beta_1, \beta_3, \beta_1, 0, 1, 1, \beta_1(1-\beta_1), N_2, 1)$.

- Let $\beta_1 = \beta_5$ and $\gamma_2 = 0$, then $\gamma_2' = 0$. Thus, we get that

$$\beta_3' = \alpha_4\beta_3, \quad \gamma_1' = \alpha_2(\beta_1 - \gamma_3) + \alpha_4\gamma_1,$$

$$N_2' = \alpha_3 N_2 + \alpha_1(N_1 - \beta_1(1-\beta_1)), \quad N_3' = \alpha_2 N_1 + \alpha_4 N_3 + \alpha_4 C_3(1-\gamma_3)$$

  - Let $N_1 \neq \beta_1(1-\beta_1)$ and $\gamma_3 \neq 1$, then taking $\alpha_1 = \frac{\alpha_3 N_2}{\beta_1(1-\beta_1)-N_1}$, $C_3 = \frac{\alpha_2 N_1 + \alpha_4 N_3}{\alpha_4(\gamma_3 - 1)}$, we obtain $N_2' = 0$, $N_3' = 0$.
    * If $\gamma_3 \neq \beta_1$, then taking $\alpha_2 = \frac{\alpha_4\gamma_1}{\gamma_3 - \beta_1}$, we get $\gamma_1' = 0$ and obtain the affgebras $L_4(\beta_1, 0, \beta_1, 0, 0, \gamma_3, N_1, 0, 0)$ and $L_{15}(\beta_1, 1, \beta_1, 0, 0, \gamma_3, N_1, 0, 0)$, depending on whether $\beta_3 = 0$ or not.
    * If $\gamma_3 = \beta_1$, then in case of $\gamma_1 = 0$, we obtain the affgebras L_4 and L_{15} with $\gamma_3 = \beta_3 = \beta_1$. In the case of $\gamma_1 \neq 0$, we obtain the affgebra $L_{16}(\beta_1, \beta_3, \beta_1, 1, 0, \beta_1, N_1, 0, 0)$.
  - Let $N_1 \neq \beta_1(1-\beta_1)$ and $\gamma_3 = 1$, then taking $\alpha_1 = \frac{\alpha_3 N_2}{\beta_1(1-\beta_1)-N_1}$, we obtain $N_2' = 0$.
    * If $N_1 \neq 0$, then taking $\alpha_2 = -\frac{\alpha_4 N_3}{N_1}$, we have $N_3' = 0$ and $\beta_3' = \alpha_4\beta_3$, $\gamma_1' = \alpha_4\gamma_1$.
      · If $\gamma_1 = 0$, then we get the affgebras L_4 and L_{15} with $\gamma_3 = 1$, $\beta_3 = \beta_1$.
      · If $\gamma_1 \neq 0$, then we may assume $\gamma_1' = 1$, and obtain the affgebra $L_{17}(\beta_1, \beta_3, \beta_1, 1, 0, 1, N_1, 0, 0)$.
    * If $N_1 = 0$, then $\beta_1(1-\beta_1) \neq 0$. Taking $\alpha_2 = \frac{\alpha_4\gamma_1}{1-\beta_1}$, we have $\gamma_1' = 0$ and $\beta_3' = \alpha_4\beta_3$, $N_3' = \alpha_4 N_3$.
      · If $N_3 = 0$, then we get the affgebras L_4 and L_{15} with $\beta_3 = \beta_1$, $N_1 = 0$.
      · $N_3 \neq 0$, then we may assume $N_3' = 1$, and obtain the affgebra $L_{18}(\beta_1, \beta_3, \beta_1, 0, 0, 1, 0, 0, 1)$.
  - Let $N_1 = \beta_1(1-\beta_1)$ and $\gamma_3 \neq 1$, then taking $C_3 = \frac{\alpha_2 N_1 + \alpha_4 N_3}{\alpha_4(\gamma_3 - 1)}$, we obtain $N_3' = 0$.
    * If $\gamma_3 \neq \beta_1$, then taking $\alpha_2 = \frac{\alpha_4\gamma_1}{\gamma_3 - \beta_1}$, we may suppose $\gamma_1' = 0$.
      · If $\beta_3 = 0$, $N_2 = 0$, then we obtain the affgebra L_4 with $N_1 = \beta_1(1-\beta_1)$.
      · If $\beta_3 = 0$, $N_2 \neq 0$, then we obtain the affgebra L_7 with $N_1 = \beta_1(1-\beta_1)$.
      · If $\beta_3 \neq 0$, $N_2 = 0$, then we obtain the affgebra L_{15} with $N_1 = \beta_1(1-\beta_1)$.
      · If $\beta_3 \neq 0$, $N_2 \neq 0$, then we obtain the affgebra $L_{19}(\beta_1, 1, \beta_1, 0, 0, \gamma_3, \beta_1(1-\beta_1), 1, 0)$.
    * If $\gamma_3 = \beta_1$, then we have $\beta_3' = \alpha_4\beta_3$, $\gamma_1' = \alpha_4\gamma_1$, $N_2' = \alpha_3 N_2$.
      · If $\gamma_1 = 0$, then similarly to the previous case we obtain the affgebras L_4, L_7, L_{15} and L_{19} with $\gamma_3 = \beta_3 = \beta_1$ and $N_1 = \beta_1(1-\beta_1)$.

· If $\gamma_1 \neq 0$, then we can suppose $\gamma_1' = 1$, and obtain the affgebras $L_{16}(\beta_1, \beta_3, \beta_1, 1, 0, \beta_1, \beta_1(1-\beta_1), 0, 0)$ and $L_{20}(\beta_1, \beta_3, \beta_1, 1, 0, \beta_1, \beta_1(1-\beta_1), 1, 0)$ depending on whether $N_2 = 0$ or not.

– Let $N_1 = \beta_1(1-\beta_1)$ and $\gamma_3 = 1$.

∗ Let $\beta_1 \neq 1$, then taking $\alpha_2 = \frac{\alpha_4\gamma_1}{1-\beta_1}$, we get that $\gamma_1' = 0$.

· If $N_3 = 0$, then similarly to the previous case, we get the affgebras L_4, L_7, L_{15} and L_{19} with $\gamma_3 = 1$, $\beta_3 = \beta_1$ and $N_1 = \beta_1(1-\beta_1)$.

· If $N_3 \neq 0$, then we may suppose $N_3' = 1$, and obtain the algebras $L_{21}(\beta_1, \beta_3, \beta_1, 0, 0, 1, \beta_1(1-\beta_1), 0, 1)$ and $L_{22}(\beta_1, \beta_3, \beta_1, 0, 0, 1, \beta_1(1-\beta_1), 1, 1)$ depending on whether $N_2 = 0$ or not.

∗ Let $\beta_1 = 1$.

· If $\gamma_1 = 0$, then we get the affgebras L_4, L_7, L_{15}, L_{19}, L_{21} and L_{22}, with $\gamma_3 = \beta_3 = \beta_1 = 1$ and $N_1 = 0$.

· If $\gamma_1 \neq 0$, then we may suppose $\gamma_1' = 1$, and obtain the affgebras $L_{23}(1, \beta_3, 1, 1, 0, 1, 0, 0, N_3)$ and $L_{24}(1, \beta_3, 1, 1, 0, 1, 0, 1, N_3)$ depending on whether $N_2 = 0$ or not.

□

References

[1] Andruszkiewicz R.R., Brzeziński T., Radziszewski K., Lie Affgebras Vis-á-Vis Lie Algebras. Results in Mathematics. 2025, 80 (2), 1–27.

[2] Baer R., Zur Einführung des Scharbegriffs. J. Reine Angew. Math. 1929, 160, 199–207.

[3] Brzeziński T., Papworth J., Lie and Nijenhuis brackets on affine spaces. Bulletin of the Belgian Mathematical Society, Simon Stevin. 2023, 30 (5), 683–704.

[4] Brzeziński T., Radziszewski K. On matrix Lie affgebras. arXiv:2403.05142.

[5] Grabowska K., Grabowski J., Urbański P., Lie brackets on affine bundles. Annals of Global Analysis and Geometry. 2003, 24 (2), 101–130.

[6] Grabowska K., Grabowski J., Urbański P., AV-differential geometry: Poisson and Jacobi structures, Journal of Geometry and Physics. 2004, 52 (4), 398–444.

[7] Grabowska K., Grabowski J., Urbański P., AV-differential geometry: Euler-Lagrangian equations. Journal of Geometry and Physics. 2007, 57 (10), 1984–1998.

[8] Jacobson N., Lie algebras, Interscience, New York, 1962.

[9] Leger G.F., Luks E.M. Generalized derivations of Lie algebras. Journal of Algebra. 2000, 228 (1), 165–203.

Berdalova Kh.R.,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences
email: xursanoyberdalova@gmail.com

Khudoyberdiyev A.Kh.,
V.I. Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
National University of Uzbekistan,
Tashkent, Uzbekistan
email: khabror@mail.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.50-63
DOI: 10.29229/uzmj.2025-3-5

Kernel identification problem in a time-fractional wave equation

Durdiev D. K., Subhonova Z. A.

Abstract. In this paper, the inverse problem of determining convolution kernel in the time-fractional wave equation with the Caputo derivative is studied. To express the solution of the Cauchy problem, the fundamental solution of the corresponding equation is systematically formulated, with a detailed investigation into the properties of this solution. The fundamental solution contains a Fox's function, which is widely used in the theory of diffusion-wave equation. Using the formulas of asymptotic expansions for the fundamental solution and its derivatives, an estimate for the solution of the direct problem is obtained. A priori estimate contains the norm of the unknown kernel function and it was used for studying the inverse problem. The inverse problem is reduced to the equivalent integral equation, By the fixed point argument in suitable functional classes the local solvability is proven. The global uniqueness results and also the stability estimate for solution to the inverse problem are established.

Keywords: Gerasimov-Caputo fractional derivative, Fox's H-function, Mittag-Leffler function, integral equation.

MSC (2020): 35A01, 65L10, 65L12, 65L20, 65L70

1. Introduction

The determination of the kernel in the fractional derivatives-wave equation is an important step in understanding the behavior of waves in non-local and memory-dependent media, and it has applications in various fields such as signal processing, viscoelastic materials, and electromagnetic wave propagation.

Over the last years Fractional Calculus has provided new and better methods to describe the behavior of several systems. Novel applications of fractional partial differential equations in physics, engineering, signal processing, chaos, viscoelastic materials, electrical circuits, and so forth, were developed [1, 2, 3, 4, 5]. Tomovski in [6] solved the fractional wave equation with frictional memory kernel of Mittag-Leffler type via the Liouville-Caputo fractional derivative. The method of separation of variables and Laplace transform were used to solve the equations. Delic [7] studied the time-fractional wave equation with Dirac delta distribution and with homogeneous initial-boundary conditions. The rate of convergence in special discrete energetic Sobolev norms is obtained. In [8], Liu et al.considered a fractional diffusion-wave equation with damping using the Liouville-Caputo derivative. They derived the analytical solution for the equation using the method of separation of variables and constructed an implicit differences method of approximation. Ferreira and Vieira, in [10], studied the multidimensional time-fractional diffusion-wave equation via the Liouville-Caputo derivative. The authors obtained an integral representation of the fundamental solution of the time-fractional diffusion-wave operator. In [11], the authors studied the telegraph equation considering the topological generalization of the elementary circuit used in transmission line modeling in order to include the effects of charge accumulation along the line. The Laplace transform technique is used in obtaining the analytical solution of signal propagation along the line. Tomovski and Sandev, in [12], considered the wave equation for a vibrating string in the presence of a fractional friction with power-law memory kernel. Exact solutions were obtained in terms of the Mittag-Leffler type functions and a generalized integral operator containing a four-parameter Mittag-Leffler function in the kernel. Mainardi [13] pointed out that the fractional wave equation governs the propagation of mechanical diffusive waves in viscoelastic media. Kochubei [14, 15] applied the semigroup theory in Banach spaces, and Eidelman and Kochubei [16] constructed the fundamental solution in R^d and proved the maximum principle for the Cauchy problem. Mainardi [13], [17] solved a fractional diffusion-wave equation using the Laplace transform in a one-dimensional bounded domain. (see, also [18]). Gejji and Jafari [19] solved a nonhomogeneous fractional diffusion-wave equation in a one-dimensional bounded domain.

Inverse problems for classical differential equations of heat and wave conduction have been studied quite widely. Inverse problems for identification source functions and coefficients of equations by different overdetermination conditions are most often encountered (see, for example, [20, 21]). In these works authors discussed the unique solvability and stability of solution as well as numerical approach for solving some problems. The works [22, 23] are devoted to study the memory recovery problems for hyperbolic integro-differential equations of the second order with convolution type integral term. The article [24] is concerned with the study of unique solvability of an inverse coefficient problem of determining the coefficient at the lower term of a fractional diffusion equation. The existence and uniqueness theorems of solving the inverse problem are obtained. In addition, a numerical algorithm based on a finite difference scheme is proposed for the exact calculation of the inverse problem of simultaneous determination of the time-dependent coefficient in the fractional diffusion equation together with its solution. In [25], the authors studied an inverse problem of reconstructing the time-dependent source function for the population model with population density nonlocal boundary conditions and an integral over-determination measurement. Huntul [26] identified the unknown time-dependent coefficient in the third-order equation from nonlocal integral observation. Various statements of inverse problems on determination of thermal coefficient in one-dimensional heat equation were studied in [27, 28, 29]. In papers [27, 28], the time-dependent thermal coefficient is determined from the heat moment. In [30, 31, 32, 33, 34], the unique solvability of the nonlocal direct problems and inverse source problems for the various fractional diffusion wave equations with Caputo and Riemann-Liouville integral-differential operators were investigated.

The remainder of this paper is organized as follows. In the next section, Section 2, we present the mathematical formulations of the direct and inverse problems and an equivalent transformed auxiliary problem. In Section 3, we give well known definations, assertions and formulas that will be used for proof of results. Section 4 is devoted to the investigation of the direct problem. In section 5, the inverse problem is studied. Finally, conclusions are presented in Section 6.

2. Formulation of problem and auxiliary constructions

We consider the time-fractional wave equation with convolution integral

$$(\mathbb{D}_t^{(\alpha)}u)(x,t) - u_{xx}(x,t) = \int_0^t k(\tau)u(x,t-\tau)d\tau + f(x,t), \quad (x,t) \in Q_0^T, \tag{2.1}$$

the solution of which satisfies the initial conditions

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \quad x \in \mathbb{R}, \tag{2.2}$$

where $1 < \alpha < 2$, $\mathbb{D}_t^{(\alpha)}$ is Caputo-Dzhrbashyan fractional derivative, that is

$$(\mathbb{D}_t^{(\alpha)}u)(x,t) = \frac{1}{\Gamma(2-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\alpha+1}u'_\tau(\tau,x)d\tau - t^{-\alpha+1}\frac{u'_t(0,x)}{\Gamma(2-\alpha)},$$

and $\varphi(x)$, $\psi(x)$ are given smooth functions, $Q_0^T := \{(x,t) : x \in \mathbb{R},\ 0 < t \leq T\}$.

For the given function $k(t)$, $t \in [0,T]$, we will call the problem of finding the function $u(x,t)$, $(x,t) \in \mathbb{R} \times [0,T]$ from the equations (2.1) and (2.2) as **Cauchy problem.**

We pose the *inverse problem* as follows: find the function $k(t)$, $t \geq 0$ in (2.1), if the solution of the Cauchy problem (2.1),(2.2) satisfies

$$u|_{x=0} = g(t), \quad t \in [0,T], \tag{2.3}$$

where $g(t)$ is a given function.

Definition 2.1. A function $u(x,t)$ is called a classical solution to the Cauchy problem (2.1) and (2.2) if:

(i) twice continuously differentiable in x for each $t > 0$;

(ii) for each $x \in \mathbb{R}$ is continuously differentiable in (x,t) on $\mathbb{R} \times [0,T]$, and the fractional integral

$$\left(I_{0+}^{2-\alpha}u\right)(x,t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{-\alpha+1}u_\tau(x,\tau)d\tau$$

is continuously differentiable in t for $t > 0$;

(iii) satisfies the equation (2.1) and initial conditions (2.2).

We proceed to define the functional spaces that will be employed throughout the analysis. Let $C^{m,2-\alpha}(Q_0^T)$ be the class of the m times continuously differentiable with respect to $x \in \mathbb{R}$ variable, continuous in t and its fractional integral of the order $2-\alpha$ is continuously differentiable in t on $[0,T]$ functions. Everywhere in this paper, we will denote by $H^l(\mathbb{R})$ the locally Holder continuous and bounded functions with exponent $l \in (0,1]$. The space $H^{m+l}(\mathbb{R})$ (m is nonnegative integer) and norms $|\cdot|^l$, $|\cdot|^{m+l}$ are defined from ([36], p. 16-27). By $C(H^l(\mathbb{R}),[0,T])$ we denote a class of continuous functions with respect to t on the segment $[0,T]$ with values in $H^l(\mathbb{R})$. The norm of a function $f(x,t)$ in $C(H^l(\mathbb{R}),[0,T])$ is defined by the equality

$$\|f\|^l := \max_{t\in[0,\,T]} \left[|f|^l\,(t) \right].$$

Let us denote by $C_b^m(\mathbb{R})$ the space of functions $f : \mathbb{R} \to \mathbb{R}$ that are m-times continuously differentiable and such that all derivatives up to order m are bounded. This space is equipped with the norm

$$\|f\|_{C_b^m(\mathbb{R})} = \sum_{k=0}^{m} \sup_{x\in\mathbb{R}} |f^{(k)}(x)|.$$

Assuming $u(x,t) \in C^{3,2-\alpha}(Q_0^T) \cap C_t^1(\bar{Q}_0^T)$ is a classical solution of the problem, and that the functions f, φ, ψ, and g are sufficiently smooth, we proceed to transform the inverse problem given by equations (2.1)-(2.3).

Lemma 2.2. *Let $u_x(x,t)$ be a classical solution to the problem (2.1)-(2.2) with $\varphi(0) = g(0)$, $\psi(0) = g'(0)$ and $\varphi'(x), \psi'(x), f_x(x,t)$. Moreover, let, $v(x,t) = u_x(x,t)$. Then, the problem (2.1)-(2.3) is equivalent to the problem of determining the functions $v \in C^{2,2-\alpha}(Q_0^T) \cap C_t^1(\bar{Q}_0^T)$ and $k(t) \in C[0,T]$ from the system of equations:*

$$(\mathbb{D}_t^{(\alpha)} v)(x,t) - v_{xx}(x,t) = \int_0^t k(\tau) v(x,t-\tau)d\tau + f_x(x,t), \quad x \in \mathbb{R}, \quad t \in (0,T], \tag{2.4}$$

$$v|_{t=0} = \varphi'(x), \quad v_t|_{t=0} = \psi'(x), \quad x \in \mathbb{R}, \tag{2.5}$$

$$v|_{x=0} = (\mathbb{D}_t^{(\alpha)} g)(t) - \int_0^t k(\tau) g(t-\tau)d\tau - f(0,t). \tag{2.6}$$

Proof. Denote for this purpose the first derivative of $u(x,t)$ with respect to x by $v(x,t)$, i.e. $v(x,t) := u_x(x,t)$. Differentiating (2.1) and (2.2) once in x, we get

$$(\mathbb{D}_t^{(\alpha)} v)(x,t) - v_{xx}(x,t) = \int_0^t k(\tau) v(x,t-\tau)d\tau + f_x(x,t), \quad x \in \mathbb{R}, \quad t \in (0,T], \tag{2.7}$$

$$v|_{t=0} = \varphi'(x), \quad v_t|_{t=0} = \psi'(x), \quad x \in \mathbb{R}. \tag{2.8}$$

To obtain an additional condition for the function $v(x,t)$, we note that the second term in (2.1) is $v_x(x,t)$. Setting $x = 0$ in (2.1) and using equality (2.3), we obtain

$$v_x|_{x=0} = (\mathbb{D}_t^{(\alpha)} g)(t) - \int_0^t k(\tau) g(t-\tau)d\tau - f(0,t). \tag{2.9}$$

Thus, if the problem (2.1)-(2.3) has a solution $(u,\, k)$, then the problem (2.7)-(2.9) also has a solution (v,k) with the same k, and $v(x,t) = u_x(x,t)$.

Conversely, let (v,k) satisfy (2.7)-(2.9). Let us show that there exists, a unique solution (u,k) of system (2.1)-(2.3) with given k. The uniqueness follows from the uniqueness of the solution of the Cauchy problem (2.1)-(2.2). To prove existence, we note that if (u,k) is a solution of (2.1)-(2.3), then $v(x,t) = u_x(x,t)$; hence

$$u(x,t) = \int_0^x v(\xi,t)d\xi + \Phi(t). \tag{2.10}$$

Let us find $\Phi(t)$ such that (2.1)-(2.3) are satisfied. From (2.3), we obtain that $\Phi(t) = g(t)$. Hence $u(x,0) = \int_0^x v(\xi,0)d\xi + g(0) = \varphi(x) - \varphi(0) - g(0) = \varphi(x)$, i.e., (2.2) is valid. In this manner, it is not difficult to show that $u_t(x,0) = \psi(x)$, i.e. (2.2) is valid. We need to verify that (2.1) holds. From (2.7) and (2.9) it follows that

$$\mathbb{D}_t^\alpha u(x,t) - u_{xx}(x,t) - \int_0^t k(\tau)u(x,t-\tau)d\tau$$
$$= \int_0^x \underbrace{\left(\mathbb{D}_t^\alpha v(\xi,t) - v_{\xi\xi}(\xi,t) - \int_0^t k(\tau)v(\xi,t-\tau)\right)}_{=f_\xi(\xi,t)} d\xi$$
$$- v_x(0,t) + \mathbb{D}_t^\alpha g(t) - \int_0^t k(\tau)g(t-\tau)d\tau = f(x,t).$$

So, the equivalence of (2.1)-(2.3) and (2.7)-(2.9) is proved. □

Given functions $k(t)$, $f(x,t)$, $\varphi(x)$, $\psi(x)$ and a number $\alpha \in (1,2)$, the problem of finding the solution to the Cauchy type problem (2.7)-(2.8) we call as **auxiliary Cauchy problem**.

We now present well-known definitions, assertions, and formulas that will be used in the proofs of the main results.

3. Preliminaries

In this section, we present well known definitions, lemmas and theorems that will be used for the proofs of main results and they will mainly deal with fractional calculus.

Fox's $H-$function. The Fox's H-function is one of the so-called special functions of the fractional calculus and contains, as particular case the Mittag-Leffer function. The $H-$function was introduced by Fox [37] as generalization of the Meijer function. Here we adopt the definition and properties mentioned in [38] with minimal modifications regarding notation. Moreover, the $H-$function is defined by means of the Mellin-Barnes type integral in the following form

$$H_{p,q}^{m,n}(z) \equiv H_{p,q}^{m,n}\left[z\Big|_{(b_1,\ B_1),\ldots,(b_q,\ B_q)}^{(a_1,\ A_1),\ldots,(a_p,\ A_p)}\right] = \frac{1}{2\pi i}\int_\Omega \mathcal{H}_{p,q}^{m,n}(s)z^{-s}ds,$$

where

$$\mathcal{H}_{p,q}^{m,n}(z) = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j + B_j s)\prod\limits_{l=1}^{n}\Gamma(1 - a_l - A_l s)}{\prod\limits_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)\prod\limits_{l=n+1}^{p}\Gamma(a_l + A_l s)} \tag{3.1}$$

$i = (-1)^{1/2}, z \neq 0$ and $z^{-s} = \exp[-s\{ln|z| + iargz\}]$. We note that $ln|z|$ represents the natural logarithm of $|z|$ and $argz$ is not necessarily the principal argument. In (3.5), an empty product is always interpreted as unity, $m,n,p,q \in \mathbb{N}$, $0 \le n \le p, 1 \le m \le q, A_k, B_k \in \mathbb{R}^+ := (0,+\infty)$; $a_k, b_j \in \mathbb{C}(\mathbb{R})$, $k = 1,...,p, j = 1,...,q$. The contour Ω starting at the point $p - i\infty$ and going to $p + i\infty$ such that all the poles of $\Gamma(b_j + B_j s), j = 1,...,m$ are separated from those of $\Gamma(1 - a_k - A_k s), k = 1,...,n, p \in \mathbb{R}$. The integral is convergent in the following cases:

(1) $\alpha > 0$, $|argz| < \frac{1}{2}\pi\alpha$ va $z \neq 0$;

(2) $\alpha = 0$, $\sigma\mu + \Re(\delta) < -1$, $argz = 0$ va $z \neq 0$,

where

$$\alpha := \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j,$$

$$\mu := \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j, \quad \delta := \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}.$$

A more detailed study about the H-function can be found in [38]. We mentioned some of the properties and its Hankel transform that will be used in this paper.

Properties of $H-$function ([38], pp. 11-13). We have the following reduction formulas:

$$H^{m,n}_{p,q}\left[z\Big|^{(a_1,A_1),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(a_1,A_1)}\right]=H^{m,n-1}_{p-1,q-1}\left[z\Big|^{(a_2,A_2),\ldots,(a_p,A_p)}_{(b_1,B_1),\ldots,(b_{q-1},B_{q-1})}\right], \tag{3.2}$$

$$H^{m,n}_{p,q}\left[z\Big|^{[a_p,A_p]}_{[b_q,B_q]}\right]=H^{n,m}_{q,p}\left[\frac{1}{z}\Big|^{[1-b_q,B_q]}_{[1-a_p,A_p]}\right],\quad n\geq 1,\ q>m. \tag{3.3}$$

Proposition 3.1. *Let $\mu\geq 0$ and $A_l(b_j+i)\neq B_j(a_l-i'-1)$ be satisfied. Then the H-function has the asymptotic expansion at zero given by* [38]

$$H^{m,n}_{p,q}(z)=O(z^c),\ |z|\to 0,$$

where

$$c:=\min_{1\leq j\leq m}\left[\frac{\Re(b_j)}{B_j}\right],$$

$\Re(b_j)$ *denotes the real part of the complex number* b_j.

In this article, we also need the asymptotic behaviour of $H-$ functions for $z\to\infty$. It will need only one result of this kind, for a specific class of $H-$ functions, for a real argument, with only the leading term of the asymptotic expansion:

Proposition 3.2. *The $H-$function has the asymptotic expansion at given by*

$$H^{m,0}_{p,m}\left[z\Big|^{(a_j,A_j)_1^p}_{(b_j,B_j)_1^m}\right]\sim Cz^{\frac{1-\varepsilon}{\rho}}exp\left(-\chi^{\frac{1}{\rho}}\rho z^{\frac{1}{\rho}}\right),\ z\to\infty,$$

where $C=const$, $\varepsilon:=\sum\limits_{j=1}^{p}a_j-\sum\limits_{j=1}^{m}b_j+\frac{1}{2}(m-p+1)$, $\chi:=\prod\limits_{j=1}^{p}A_j^{A_j}\prod\limits_{j=1}^{m}B_j^{-B_j}$, $\rho:=\sum\limits_{j=1}^{m}B_j-\sum\limits_{j=1}^{p}A_j$.

Now we give formulas for differentiating the H-functions of a special form. For the proofs and further details, see [38, 40]. They have the following forms:

$$\frac{d}{dz}H^{m,n}_{p,q}\left[z\Big|^{(a_1,A_1)}_{(b_1,B_1),}\right]=-\frac{1}{z}H^{m+1,n}_{p+1,q+1}\left[z\Big|^{(a_1,A_1),(0,1)}_{(1,1),(b_1,B_1)}\right], \tag{3.4}$$

$$\left(\frac{d}{dz}\right)^k\left\{z^{\omega}H^{m,n}_{p,q}\left[Cz^{\sigma}\Big|^{(a_1,A_1)}_{(b_1,B_1),}\right]\right\}=z^{\omega-k}H^{m+1,n}_{p+1,q+1}\left[Cz^{\sigma}\Big|^{(-\omega,\sigma),(a_1,A_1)}_{(b_1,B_1),(k-\omega,\sigma)}\right]. \tag{3.5}$$

Mittag-Leffler function. A two-parameter Mittag-Lefller (M-L) function is defined as ([41], pp 40-45)

$$E_{\alpha,\beta}(z):=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)},$$

where $\alpha,\beta,z\in\mathbb{C},\Re(\alpha)>0$.

The relationship between the two-parameter Mittag-Leffler function and the Fox H-function is noted below

$$E_{\alpha,\beta}(z)=H^{1,1}_{1,2}\left[z\Big|^{(0,1)}_{(0,1),(1-\beta,\alpha)}\right]. \tag{3.6}$$

4. Investigation of auxiliary Cauchy problem

First we consider the Cauchy problem

$$\mathbb{D}_t^{(\alpha)}v(x,t)-v_{xx}(x,t)=F(x,t),\quad t>0,\quad x\in\mathbb{R}^d \tag{4.1}$$

with the initial condition

$$v|_{t=0}=\eta_1(x),\quad v_t|_{t=0}=\eta_2(x),\quad x\in\mathbb{R}^d. \tag{4.2}$$

In the work [9] the solution to the problem (3.9)-(4.1)

$$v(x,t)=\int_{\mathbb{R}^d} Z_1(x-\xi,t)\eta_1(\xi)d\xi+\int_{\mathbb{R}^d} Z_2(x-\xi,t)\eta_2(\xi)d\xi$$
$$+\int_0^t\int_{\mathbb{R}^d} Y(x-\xi,t-\tau)F(\xi,\tau)d\xi d\tau, \quad t>0, \quad x\in\mathbb{R}^d, \quad (4.3)$$

where

$$Z_j(x,t)=\frac{t^{j-1}}{\pi^{\frac{d}{2}}|x|^d}H^{2,0}_{1,2}\left[\frac{|x|^2}{4t^\alpha}\left|^{(j,\alpha)}_{(\frac{d}{2},1),(1,1)}\right.\right], \ j=1,2,$$
$$Y(x,t)=\frac{t^{\alpha-1}}{\pi^{\frac{d}{2}}|x|^d}H^{2,0}_{1,2}\left[\frac{|x|^2}{4t^\alpha}\left|^{(\alpha,\alpha)}_{(\frac{d}{2},1),(1,1)}\right.\right] \quad (4.4)$$

are the fundamental solutions of the one dimensional fractional diffusion equation.

For the Fox's H-function, based on Proposition 3.2, we get the estimate:

$$\left|H^{2,0}_{1,2}\left[z\left|^{(j,\alpha)}_{(\frac{1}{2},1),(1,1)}\right.\right]\right|\le C|z|^{\frac{3}{2}-j}exp(-\alpha^{\frac{\alpha}{2-\alpha}}(2-\alpha)|z|^{\frac{1}{2-\alpha}}), \quad z\to\infty. \quad (4.5)$$

In (4.3), substituting the expression $\varphi'(x)$, $\psi'(x)$ and $\int_0^t k(\tau)v(x,t-\tau)d\tau+f_x(x,t)$ instead of $\eta_1(x)$, $\eta_2(x)$ and $F(x,t)$ respectively, then we obtain the integral equation for solution to the direct problem (2.7)-(2.8)

$$v(x,t)=v_0(x,t)-\int_0^t\int_{\mathbb{R}} Y(x-\xi,t-\tau)\int_0^\tau k(\zeta)v(\xi,\tau-\zeta)d\zeta d\xi d\tau, \quad (4.6)$$

where

$$v_0(x,t):=\int_{\mathbb{R}} Z_1(x-\xi,t)\varphi'(\xi)d\xi+\int_{\mathbb{R}} Z_2(x-\xi,t)\psi'(\xi)d\xi$$
$$+\int_0^t\int_{\mathbb{R}} Y(x-\xi,t-\tau)f_\xi(\xi,\tau)d\xi d\tau. \quad (4.7)$$

The following assertion holds:

Lemma 4.1. *If* $k(t)\in C[0,T], f(x,t)\in C(C_b^2(\mathbb{R}),[0,T])$, $\varphi(x)\in H^{\gamma+2}(\mathbb{R})$, $\psi(x)\in H^{l+1}(\mathbb{R})$, $l\in(0,1]$, $\gamma\in(\frac{2}{\alpha}-1,1]$, *then there exists a unique solution of the integral equation (4.6) such that* $v(x,t)\in C^{2,2-\alpha}(Q_0^T)\cap C^1_t(\overline{Q}_0^T)$.

Proof. For proof of Lemma 4.1 we use the method of successive approximations and consider the sequence of functions:

$$v_n(x,t)=-\int_0^t\int_R Y(x-\xi,t-\tau)\int_0^\tau k(\zeta)v_{n-1}(\xi,\tau-\zeta)d\zeta d\xi d\tau, \quad n=1,2,..., \quad (4.8)$$

where $v_0(x,t)$ is determined by the equality (4.7).

Further, we need estimations for functions $Z_1(x,t)$, $Z_2(x,t)$, $Y(x,t)$ and their some derivatives (see [35]):

$$\left|\frac{\partial^m}{\partial x^m}Z_1(x,t)\right|\le Ct^{-\frac{\alpha}{2}(m+1)}\exp\left\{-\sigma(t^{-\alpha/2}|x|)^{\frac{2}{2-\alpha}}\right\}, \quad (4.9)$$

$$\left|\frac{\partial^m}{\partial x^m}Z_2(x,t)\right|\le Ct^{-\frac{\alpha}{2}(m+1)+1}\exp\left\{-\sigma(t^{-\alpha/2}|x|)^{\frac{2}{2-\alpha}}\right\}, \quad (4.10)$$

$$\left|\frac{\partial^m}{\partial x^m}Y(x,t)\right|\le Ct^{-\frac{\alpha}{2}(m-1)-1}\exp\left\{-\sigma(t^{-\alpha/2}|x|)^{\frac{2}{2-\alpha}}\right\}, \quad |m|\le 3; \quad (4.11)$$

$$\left|\frac{\partial}{\partial t}Z_1(x,t)\right| \le Ct^{-1-\frac{\alpha}{2}}\exp\left\{-\sigma(t^{-\alpha/2}|x|)^{\frac{2}{2-\alpha}}\right\}, \tag{4.12}$$

$$\left|\frac{\partial}{\partial t}Z_2(x,t)\right| \le Ct^{-\frac{\alpha}{2}}\exp\left\{-\sigma(t^{-\alpha/2}|x|)^{\frac{2}{2-\alpha}}\right\}, \tag{4.13}$$

$$\left|\frac{\partial}{\partial t}Y(x,t)\right| \le Ct^{\frac{\alpha}{2}-2}\exp\left\{-\sigma(t^{-\alpha/2})|x|^{\frac{2}{2-\alpha}}\right\}, \tag{4.14}$$

here and below the letters C, σ will denote various positive constants. We also note that in accordance with the construction of the functions $Z_1(x,t)$, $Z_2(x,t)$, $Y(x,t)$, the following equalities are valid:

$$\int_{\mathbb{R}} Z_1(x,t)dx = 1, \quad t\in[0,T], \tag{4.15}$$

$$\int_{\mathbb{R}} Z_2(x,t)dx = t, \quad t\in[0,T], \tag{4.16}$$

$$\int_{\mathbb{R}} Y(x,t)dx = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t\in[0,T]. \tag{4.17}$$

Introduce the notations

$$\varphi_0 := |\varphi|^{2+\gamma}, \quad \psi_0 := |\psi|^{1+l}, f_0 := \|f\|_{C(C_b^2(\mathbb{R});[0,T])}.$$

We estimate the modulus of $v_0(x,t)$ in the domain $\overline{Q}_0^T$ as follows

$$|v_0(x,t)| \le \varphi_0 + t\psi_0 + \frac{f_0}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-1}d\tau \le \varphi_0 + T\psi_0 + \frac{f_0T^\alpha}{\Gamma(\alpha+1)} =: C_0.$$

Similary way from (4.8) for $n=1,2$, we obtain

$$|v_1(x,t)| \le C_0\|k\|\int_0^t (t-\tau)^{\alpha-1}d\tau \le \frac{C_0\|k\|\Gamma(\alpha)}{\Gamma(1+\alpha)}T^\alpha, \tag{4.18}$$

$$|v_2(x,t)| \le \frac{C_0\|k\|}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^t (t-\tau)^{\alpha-1}\tau^\alpha d\tau \le \frac{C_0(\|k\|\Gamma(\alpha))^2}{\Gamma(1+2\alpha)}T^{2\alpha}.$$

For arbitrary $n=1,2,...$, we have

$$v_n(x,t) \le \frac{C_0(\|k\|\Gamma(\alpha))^n}{\Gamma(n\alpha+1)}T^{n\alpha}.$$

It follows from the above estimates that the series

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t).$$

converges uniformly in $\bar{Q}_0^T$, since it can be majorized in $\bar{Q}_0^T$ by the convergent numerical series

$$|v(x,t)| \le C_0\sum_{n=0}^{\infty}\frac{(\|k\|\Gamma(\alpha))^n}{\Gamma(n\alpha+1)}T^{n\alpha} = C_0E_{\alpha,1}(\|k\|\Gamma(\alpha)T^\alpha). \tag{4.19}$$

where $E_{\alpha,1}(\cdot)$ is the one-parameter Mittag-Leffler function of a nonnegative real argument (see, for example, [41], pp.40-45).

Under the assumptions of Lemma 4.1 and on the bases of estimates (4.9)- (4.14), one has the inclusion $v_n(x,t) \in C^{2,2-\alpha}(Q_0^T) \cap C_t^1(\overline{Q}_0^T)$, $n=1,2,\dots$. According to the general theory of integral equations, this implies that the same property will be possessed by the function $v(x,t)$ in $\overline{Q}_0^T$. As usual, this function is a solution of the integral equation (4.6). Thus, Lemma 4.1 is proven. □

Now we will obtain an estimate for the norm of the difference between the solution of the original integral equation (4.6) and the solution of this equation with perturbed functions $\tilde{k}$, $\tilde{\varphi}'$, $\tilde{\psi}'$ and $\tilde{f}_x$.

Let $\tilde{v}(x,t)$ be a solution of the integral equation (4.6) corresponding to the functions $\tilde{k}$, $\tilde{\varphi}'$, $\tilde{\psi}'$ and $\tilde{f}_x$ i.e.,

$$\tilde{v}(x,t) = \tilde{v}_0(x,t) - \int_0^t \int_{\mathbb{R}} Y(x-\xi, t-\tau) \int_0^\tau \tilde{k}(\zeta)\tilde{v}(\xi, \tau-\zeta) d\zeta d\xi d\tau, \tag{4.20}$$

where

$$\tilde{v}_0(x,t) := \int_{\mathbb{R}} Z_1(x-\xi,t)\tilde{\varphi}'(\xi)d\xi + \int_{\mathbb{R}} Z_2(x-\xi,t)\tilde{\psi}'(\xi)d\xi + \int_0^t \int_{\mathbb{R}} Y(x-\xi,t-\tau)\tilde{f}_\xi(\xi,\tau)d\xi d\tau. \tag{4.21}$$

Composing the difference $v - \tilde{v}$ with the help of the equations (4.6)-(4.20)and introducing the notations $v - \tilde{v} = \bar{v},\ v_0 - \tilde{v}_0 = \bar{v}_0,\ k - \tilde{k} = \bar{k}$, we get the integral equation for $\bar{v}$

$$\bar{v}(x,t) = \bar{v}_0(x,t) - \int_0^t \int_{\mathbb{R}} Y(x-\xi,t-\tau) \int_0^\tau \bar{k}(\zeta)\tilde{v}(\xi,\tau-\zeta)d\zeta d\xi d\tau - \int_0^t \int_{\mathbb{R}} Y(x-\xi,t-\tau) \int_0^\tau \tilde{k}(\zeta)\bar{v}(\xi,\tau-\zeta)d\zeta d\xi d\tau, \tag{4.22}$$

from which is derived the following linear integral inequality for $|\bar{v}(x,t)|$:

$$\begin{aligned} |\bar{v}(x,t)| &\le |\bar{v}_0(x,t)| + C_1 C_0 E_{\alpha,1}(\|\tilde{k}\|\Gamma(\alpha)T^\alpha)\|\bar{k}\| \int_0^t (t-\tau)^{\alpha-1}\tau d\tau \\ &\quad + ||\tilde{k}|| \int_0^t d\tau \int_R Y(x-\xi,t-\tau) \int_0^\tau |\bar{v}(\xi,\tau-\zeta)| d\zeta d\xi \\ &= |\bar{v}_0(x,t)| + C_2 C_0 E_{\alpha,1}(\|\tilde{k}\|\Gamma(\alpha)T^\alpha)T^{\alpha+1}\|\bar{k}\| \\ &\quad + \|\tilde{k}\| \int_0^t d\tau \int_{\mathbb{R}} Y(x-\xi,t-\tau) \int_0^\tau |\bar{v}(\xi,\tau-\zeta)| d\zeta d\xi, \end{aligned} \tag{4.23}$$

where $C_i > 0$ various constants depending on α, σ. It follows from the equalities (4.6) and (4.21) the estimate

$$|\bar{v}_0(x,t)| \le |\bar{\varphi}|^{2+\gamma} + T|\bar{\psi}|^{1+l} + \frac{T^\alpha}{\Gamma(1+\alpha)}\|\bar{f}\|_{C(C_b^2(\mathbb{R});[0,T])},$$

where $\bar{\varphi} = \varphi - \tilde{\varphi}$, $\bar{\psi} = \psi - \tilde{\psi}$, $\bar{f} = f - \tilde{f}$.

Introduce notation

$$\theta := max\left\{1, T, \frac{T^\alpha}{\Gamma(1+\alpha)}, C_2 C_0 E_{\alpha,1}(\|\tilde{k}\|\Gamma(\alpha)T^\alpha)T^{\alpha+1}\right\}.$$

Applying the method of successive approximations to inequality (4.23), by the aid of the scheme

$$|\bar{v}(x,t)|_0 \le \theta\left[|\bar{\varphi}|^{2+\gamma} + |\bar{\psi}|^{1+l} + ||\bar{f}|| + \|\bar{k}\|\right],$$

$$|\bar{v}(x,t)|_n \le \|\tilde{k}\| \int_0^t d\tau \int_{\mathbb{R}^n} Y(x-\xi,t-\tau) \int_0^\tau |\bar{v}(\xi,\tau-\zeta)|_{n-1} d\zeta d\xi, \quad n = 1, 2, ...$$

We arrive at the estimate

$$|\bar{v}(x,t)| \le \theta\left[|\bar{\varphi}|^{2+\gamma} + |\bar{\psi}|^{1+l} + \|\bar{f}\|_{C(C_b^2(\mathbb{R});[0,T])} + \|\bar{k}\|\right], \tag{4.24}$$

which will be used in the next section. Indeed, the expression (4.24) is the stability estimate for the solution to auxiliary Cauchy problem.. In particular, the uniqueness for this solution follows from (4.24).

5. Investigation of Inverse problem

In this section, we study the inverse problem (2.7)-(2.9) using the contraction mapping principle. First, we differentiate equation (4.6) with respect to x, set $x = 0$, and equate the result to equation (2.9).

$$\int_{\mathbb{R}} Z_1(y,1)\varphi''(t^{\alpha/2}y)d\xi + \int_{\mathbb{R}} Z_2(\xi,t)\psi''(\xi)d\xi + \int_0^t \int_{\mathbb{R}} Y(\xi,t-\tau)f_{\xi\xi}(\xi,\tau)d\xi d\tau$$
$$- \int_0^t \int_{\mathbb{R}} Y(\xi,t-\tau) \int_0^\tau k(\zeta)v_\xi(\xi,\tau-\zeta)d\zeta d\xi d\tau$$
$$= \partial_t^\alpha g(t) - \int_0^t k(\tau)g(t-\tau)d\tau - f(0,t), \quad (5.1)$$

where

$$\partial_t^\alpha g(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{g''(s)}{(t-s)^{\alpha-1}} ds.$$

Next, we differentiate both sides of the resulting equation (5.1) with respect to the variable t. This yields the following integral equation for determining $k(t)$:

$$k(t) = k_0(t) + \frac{1}{g(0)} \left[\int_0^t \int_{\mathbb{R}} Y_t(\xi,t-\tau) \int_0^\tau k(\zeta)v_\xi(\xi,\tau-\zeta)d\zeta d\xi d\tau - \int_0^t k(\tau)g'(t-\tau)d\tau \right], \quad (5.2)$$

where

$$k_0(t) = \frac{1}{g(0)} \Bigg[\frac{d}{dt}\Big(\partial_t^\alpha g(t)\Big) - f_t(0,t) - \frac{\alpha}{2}t^{\alpha/2-1} \int_{\mathbb{R}} Z_1(y,1)y\varphi'''(t^{\alpha/2}y)dy - \int_{\mathbb{R}} Z_{2t}(\xi,t)\psi''(\xi)d\xi$$
$$- \int_0^t \int_{\mathbb{R}} Y_t(\xi,t-\tau)f_{\xi\xi}(\xi,\tau)d\xi d\tau \Bigg].$$

Theorem 5.1. *If $f \in C^1\big(C_b^2(\mathbb{R}),[0,T]\big)$, $\varphi(x) \in H^{\gamma+3}(\mathbb{R})$, $\psi(x) \in H^{l+2}(\mathbb{R})$, $l \in (0,1]$, $\gamma \in (\frac{2}{\alpha}-1,1]$, $g(t) \in C^3\big([0,T]\big)$, with $g''(0)=0$, and the conditions of agreement $\varphi(0) = g(0) \neq 0$, $\psi(0) = g'(0)$. Then there exists a number $T^* \in (0,T)$, such that there exists a unique solution $k(t) \in C[0,T^*]$ of the inverse problem (2.7)-(2.9).*

Remark 5.2. The $g(t) \in C^3\big([0,T]\big)$ and $g''(0) = 0$ conditions imposed on g in Theorem 5.1 ensure that $\frac{d}{dt}\Big(\partial_t^\alpha g(t)\Big) \in C[0,T]$.

Proof. Let us write equations (4.6) and (5.2) in the form of a closed system of integral equations of the Volterra type of the second kind. To do this, we introduce into consideration the vector function $\nu = \big(\nu_1(x,t), \nu_2(x,t), \nu_3(x,t)\big)$ by specifying their components by the equalities

$$\nu_1(x,t) = v(x,t), \quad \nu_2(x,t) := v_x(x,t), \quad \nu_3(x,t) := \nu_3(t) = k(t).$$

Then the system of equations (4.6) and(5.2) takes the operator form

$$\nu = \mathcal{A}\nu, \quad (5.3)$$

where the operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, in accordance with the right-hand sides of equations (4.6) and(5.2) is defined by the equalities

$$\mathcal{A}_1\nu = \nu_{01} - \int_0^t \int_{\mathbb{R}} Y(x-\xi,t-\tau) \int_0^\tau \nu_3(\lambda)\nu_1(\xi,\tau-\lambda)d\lambda d\xi d\tau, \quad (5.4)$$

$$\mathcal{A}_2\nu = \nu_{02} - \int_0^t \int_{\mathbb{R}} Y(x-\xi, t-\tau) \int_0^\tau \nu_3(\lambda)\nu_2(\xi, \tau-\lambda) d\lambda d\xi d\tau, \tag{5.5}$$

$$\mathcal{A}_3\nu = \nu_{03} + \frac{1}{g(0)} \int_0^t \int_{\mathbb{R}} Y_t(x-\xi, t-\tau) \int_0^\tau \nu_3(\lambda)\nu_1(\xi, \tau-\lambda) d\lambda d\xi d\tau - \frac{1}{g(0)} \int_0^t \nu_3(\tau) g(t-\tau) d\tau. \tag{5.6}$$

These formulas use the notation vector function $\nu_0 = (\nu_{01}, \nu_{02}, \nu_{03})$:

$$\nu_{01}(x,t) = \int_{\mathbb{R}} Z_1(x-\xi,t)\varphi'(\xi)d\xi + \int_{\mathbb{R}} Z_2(x-\xi,t)\psi'(\xi)d\xi + \int_0^t \int_{\mathbb{R}} Y(x-\xi,t-\tau) f_\xi(\xi,\tau) d\xi d\tau,$$

$$\nu_{02}(x,t) = \int_{\mathbb{R}} Z_1(\xi,t)\varphi''(x-\xi)d\xi + \int_{\mathbb{R}} Z_2(\xi,t)\psi''(x-\xi)d\xi + \int_0^t \int_{\mathbb{R}} Y(\xi,t-\tau) f_{\xi\xi}(x-\xi,\tau) d\xi d\tau,$$

$$\nu_{03}(t) = \frac{1}{g(0)} \left[\frac{d}{dt}\left(\partial_t^\alpha g(t)\right) - f_t(0,t) - \frac{\alpha}{2} t^{\alpha/2-1} \int_{\mathbb{R}} Z_1(y,1) y \varphi'''(t^{\alpha/2}y) dy - \int_{\mathbb{R}} Z_{2t}(\xi,t)\psi''(\xi)d\xi \right.$$
$$\left. - \int_0^t \int_{\mathbb{R}} Y_t(\xi,t-\tau) f_{\xi\xi}(\xi,\tau) d\xi d\tau \right].$$

Now, we will show that the functions $\nu_{01}, \nu_{02}, \nu_{03}$, defined above, have finite absolute values. To do this, let $\varphi \in H^{l+3}(\mathbb{R})$, $\psi \in C_b^2(\mathbb{R})$ and $f \in C(C_b^2(\mathbb{R}); [0,T])$ be given, then we have

$$|\nu_{01}| \le \frac{2C}{\sigma^{\frac{2-\alpha}{\alpha}}} \Gamma\left(\frac{4-\alpha}{2}\right) \left[\|\varphi'\|_{C_b(\mathbb{R})} + T\|\psi'\|_{C_b(\mathbb{R})} + T^{\alpha/2}\|f'\|_{C(C_b(\mathbb{R}),[0,T])} \right] =: \Phi_{01},$$

$$|\nu_{02}| \le \frac{2C}{\sigma^{\frac{2-\alpha}{\alpha}}} \Gamma\left(\frac{4-\alpha}{2}\right) \left[\|\varphi''\|_{C_b(\mathbb{R})} + T\|\psi''\|_{C_b(\mathbb{R})} + T^{\alpha/2}\|f''\|_{C(C_b(\mathbb{R}),[0,T])} \right] =: \Phi_{02}$$

$$|\nu_{03}| = \left| \frac{1}{g(0)} \left[\frac{d}{dt}\left(\partial_t^\alpha g(t)\right) - f_t(0,t) - \frac{\alpha}{2} t^{\alpha/2-1} \int_{\mathbb{R}} Z_1(y,1) y \varphi'''(t^{\alpha/2}y) dy - \int_{\mathbb{R}} Z_{2t}(\xi,t)\psi''(\xi)d\xi \right.\right.$$
$$\left.\left. - \int_0^t \int_{\mathbb{R}} Y_t(\xi,t-\tau) f_{\xi\xi}(\xi,\tau) d\xi d\tau \right] \right| \le g_0 + f_0 + \tilde{C}_1 T^{\alpha-1} \left[\varphi'''\right]^l + \tilde{C}_2 \|\psi''\|_{C_b(\mathbb{R})} + \tilde{C}_3 T^{\alpha-1} = \Phi_{03},$$

where

$$g_0 := \max\left[\max_{t\in[0,T]} \left| \frac{\frac{d}{dt}\left(\partial_t^\alpha g(t)\right)}{g(0)} \right|, \max_{t\in[0,T]} \left| \frac{g'(t)}{g(0)} \right| \right], \quad f_0 = \max_{t\in[0,T]} \left| \frac{f_t(0,t)}{g(0)} \right|.$$

We obtain

$$\nu_0 = (\nu_{01}, \nu_{02}, \nu_{03}).$$

Fix a number $r > 0$ and consider the ball

$$B[\nu_0, r] := \{\nu \in Y : \|\nu - \nu_0\| \le r\},$$

where $Y := C(\bar{Q}_0^T) \times C(\bar{Q}_0^T) \times C[0,T]$.

First we prove that for an enough small $T > 0$ the operator $\mathcal{A}$ maps the ball $B[\nu_0, r]$ into itself, i.e. the condition $\nu \in B[\nu_0, r]$ implies that $\mathcal{A}\nu \in B[\nu_0, r]$. It is easy to see that for $\nu \in B[\nu_0, r]$ the estimate

$$\|\nu\| \le \|\nu_0\| + r := r_0 \tag{5.7}$$

holds. Thus, r_0 is a known number.

Having estimated the norm of the differences, we observe that

$$\|\mathcal{A}_1\nu - \nu_{01}\| = \max_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}} Y(x-\xi,t-\tau)\int_0^\tau \nu_3(\lambda)\nu_1(\xi,\tau-\lambda)d\lambda d\xi d\tau\right|$$
$$\leq CT^{\frac{\alpha^2}{4}+\alpha}\Gamma(1-\frac{\alpha}{2})r_0^2 =: \beta_1(T)$$

$$\|\mathcal{A}_2\nu - \nu_{02}\| = \max_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}} Y(x-\xi,t-\tau)\int_0^\tau \nu_3(\lambda)\nu_2(\xi,\tau-\lambda)d\lambda d\xi d\tau\right|$$
$$\leq CT^{\frac{\alpha^2}{4}+\alpha}\Gamma(1-\frac{\alpha}{2})r_0^2 =: \beta_2(T)$$

$$\|\mathcal{A}_3\nu - \nu_{03}\| = \frac{1}{|g(0)|}\max_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}} Y_t(x-\xi,t-\tau)\int_0^\tau \nu_3(\lambda)\nu_1(\xi,\tau-\lambda)d\lambda d\xi d\tau - \int_0^t \nu_3(\tau)g(t-\tau)d\tau\right|$$
$$\leq \frac{1}{|g(0)|}\left[\frac{\Gamma(\alpha)}{\Gamma(\alpha+2)}r_0^2T^{\alpha+1} + g_0r_0T\right] =: \beta_3(T).$$

It is straightforward to observe that β_i, $(i=1,2,3)$ are increasing functions of T and pass through the origin. Additionally, the system of equations $\beta_i(T) = r$ has solutions, and we denote them by $T = T_i, i = 1,2,3$ respectively. Let T_0 be the smallest of these solutions. Then, for every T in the interval $[0,T_0]$, we have $\mathcal{A}B[\nu_0 r] \subset B[\nu_0 r]$.

Now let $\nu(t)$, $\tilde{\nu}(t)$ be two arbitrary elements in $B[\nu_0, r]$. In this case, using the obvious inequalities

$$|\nu_k\nu_s - \tilde{\nu}_k\tilde{\nu}_s| \leq |\nu_k(\nu_s-\tilde{\nu}_s) + \tilde{\nu}_s(\nu_k-\tilde{\nu}_k)| \leq 2r_0||\nu-\tilde{\nu}||$$

and estimates for the integrals similar to those given above, we obtain

$$||(\mathcal{A}_1\nu - \mathcal{A}_1\tilde{\nu})|| = \max_{(x,t)\in\bar{D}_T}\left|\int_0^t\int_{\mathbb{R}} Y(x-\xi,t-\tau)\int_0^\tau(\nu_3(\lambda)\nu_1(\lambda) - \tilde{\nu}_3(\lambda)\tilde{\nu}_1(\lambda))d\lambda d\xi d\tau\right|$$
$$\leq \frac{4r_0C}{\sigma^{\frac{2-\alpha}{2}}}\frac{\Gamma(4-\frac{\alpha}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2}+2)}T^{\alpha/2+1}\|\nu-\tilde{\nu}\| =: \tilde{\beta}_1(T)\|\nu-\tilde{\nu}\|$$
$$||(\mathcal{A}_2\nu - \mathcal{A}_2\tilde{\nu})|| = \max_{(x,t)\in\bar{D}_T}\left|\int_0^t\int_{\mathbb{R}} Y(x-\xi,t-\tau)\int_0^\tau(\nu_3(\lambda)\nu_2(\lambda) - \tilde{\nu}_3(\lambda)\tilde{\nu}_2(\lambda))d\lambda d\xi d\tau\right|$$
$$\leq \frac{4r_0C}{\sigma^{\frac{2-\alpha}{2}}}\frac{\Gamma(4-\frac{\alpha}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2}+2)}T^{\alpha/2+1}\|\nu-\tilde{\nu}\| =: \tilde{\beta}_2(T)\|\nu-\tilde{\nu}\|$$

$$||(\mathcal{A}_3\nu - \mathcal{A}_3\tilde{\nu})|| = \frac{1}{|g(0)|}\max_{(x,t)\in\bar{D}_T}\left|\int_0^t\int_{\mathbb{R}} Y_t(x-\xi,t-\tau)\int_0^\tau(\nu_3(\lambda)\nu_1(\lambda) - \tilde{\nu}_3(\lambda)\tilde{\nu}_1(\lambda))d\lambda d\xi d\tau\right.$$
$$\left. + \frac{1}{g(0)}\int_0^t g(t-\tau)(\nu_3-\tilde{\nu}_3)(\tau)d\tau\right|$$
$$\leq \frac{C\Gamma(\alpha-1)}{\Gamma(\alpha+1)g(0)}T^\alpha\|\nu-\tilde{\nu}\| + \frac{|g_0|T}{|g(0)|}\|\nu-\tilde{\nu}\| =: \tilde{\beta}_3(T)\|\nu-\tilde{\nu}\|$$

Suppose that T_{00} is the smallest number among T_i, $(i=1,2,3)$ satisfying the inequalities $\beta_i(T) = \rho < 1$. Then, for all $T \in [0,T_{00}]$ and $x \in \mathbb{R}$, the operator $\mathcal{A}$ is contractive on the set $B[\nu_0, r]$. Let us denote by T^* the minimum of T_0 and T_{00}. As a result, equation (5.3) has a unique solution for all $(x,t) \in \bar{Q}_0^{T^*}$. Theorem 5.1 is proven. □

Remark 5.3. According to Theorem 5.1, the solution to the inverse problem can be uniquely found on the segment $[0,T^*]$. Then the solution to **Cauchy problem** is found in the same way as it was found in the case of **auxiliary Cauchy problem** .

Conclusion

The inverse problem of determining convolution kernel in the time-fractional wave equation with the Caputo derivative was considered. Direct problem is Cauchy problem for this equation and first it was studied this problem. The fundamental solution of the time-fractional wave equation is constructed and properties of this solution are investigated. The fundamental solution contains a Fox's function, which is widely used in the theory of diffusion-wave equation. Using the formulas of asymptotic expansions for the fundamental solution and its derivatives, an estimate for the solution of the direct problem is obtained. A priori estimate contains the norm of the unknown kernel function and it was used for studying the inverse problem. The inverse problem is reduced to the equivalent integral equation. By the fixed point argument in suitable functional classes the local solvability is proven. The global uniqueness results and also the stability estimate for solution to the inverse problem are established.

In applications, more important is an equation of the form (2.1), when under the integral sign the kernel $k(t)$ is multiplied by u_{xx}. The study of direct and inverse problems for this equation similar to those in this work is an open problem.

References

[1] Agila A., Baleanu D., Eid R., Irfanoglu B.; A freely damped oscillating fractional dynamic system modeled by fractional Euler-Lagrange equations. J. Vib. Control., **24**(7) (2018), 1228–1238.

[2] Atangana A., Ga Omez-Aguilar J.F.; A new derivative with normal distribution kernel: Theory, methods and applications. Stat. Mech. Appl. **476** (2017), 1–14.

[3] Choo K.Y., Muniandy S.V., Woon K.L., Gan M.T., Ong D.S., Modeling anoma- lous charge carrier transport in disordered organic semiconductors using the fractional drift-diffusion equation. Org. Electron. **41** (2017), 157–165.

[4] Povstenko Y., Kyrylych T., Rygal G.; Fractional diffusion in a solid with mass absorption Entropy. 19, 203.

[5] Singh J., Kumar D., Nieto J.J.; A reliable algorithm for a local fractional tricomi equation arising in fractal transonic flow. Entropy **2016**, 18(6), 206.

[6] Tomovski Z., Sandev T.; Generalized space time fractional diffusion equation with composite fractional time derivative. Appl. Math. Comput. **391**(8), (2012), 2527–2542.

[7] Delic A., Jovanovic B.S.; Finite difference approximation of fractional wave equation with concentrated capacity. Comput. Methods Appl. Math. 17, 33 (2017).

[8] Ghen J., Liu F., Anh V., Shen S., Liu Q., Turner I.; Numerical techniques for the variable order time fractional diffusion equation. Appl. Math. Comput. **218**(22), (2012), 10861–10870.

[9] A.N. Kochubei. Cauchy problem for fractional diffusion-wave equations with variable coefficients, Appl. Anal. Int. J. 93 (2014).

[10] Ferreira M., Rodrigues M.M., Vieira N.; Fundamental solution of the multi-dimensional time fractional telegraph equation. J. Theo. Appl. **20**(4),(2017), 868–894.

[11] Cveticanin S.M., Zorica D., Rapaic M.R.; Generalized time-fractional telegrapheras equation in transmission line modeling. Nonlinear Dyn. **88**(2),(2017), 1453–1472.

[12] Tomovski Z., Sandev T.; The general time fractional Fokker-Planck equation with a constant external force. Comput. Math. Appl. (2011), 27–39.

[13] Mainardi F.; Fractional diffusive waves in viscoelastic solids. Comput. Acous. **9**(4), (2011), 1417–1436.

[14] Kochubei A.N.; A Cauchy problem for evolution equations of fractional order. J. Differential Equations 25, 967–974, (1989).

[15] Kochubei A.N.; Fractional order diffusion, J. Differ. Uravn., **26**(4), (1990), 660–670.

[16] Eidelman S.D., Kochubei A.N.; Cauchy problem for fractional diffusion equations. J. Differential Equations **199**(2), (2004), 211–255.

[17] Mainardi F.; The time fractional diffusion-wave equation. Radiophys. and Quantum Electronics **38**, (1995), 13–24.

[18] Mainardi F.; On the initial value problem for the fractional diffusion-wave equation, in: Rionero S., Ruggeri T; Waves and Stability in Continuous Media, World Scientific, Singapore, **23**, (1994), 246–251.

[19] Gejji V.D., Jafari H.; Boundary value problems for fractional diffusion-wave equation. Aust. J. Math. Anal. Appl. **3**, (2006), 1–8.

[20] Ashurov R. R, and Fayziev Yu. E.; Uniqueness and existence for inverse problem of determining an order of time-fractional derivative of subdiffusion equation. Lobachevskii J. Math. **42**, (2021), 508–516.

[21] Elesin A. V, and Kadyrova A. Sh.; The inverse coefficient problem for equations of three-phase flow in porous medium, Lobachevskii J. Math. **40**, (2019), 724–729.

[22] Durdiev D. K. and Rahmonov A. A.; Inverse problem for a system of integro-differential equations for SH waves in a visco-elastic porous medium: Global solvability. Theor. Math. Phys. **195**(3), (2018), 491–506.

[23] Durdiev D. K. and Rahmonov A. A.; The problem of determining the 2D-kernel in a system of integrodifferential equations of a viscoelastic porous medium. J. Appl. Ind. Math. **14**, (2020), 281–295.

[24] Durdiev D. K., Durdiev D.D.; An inverse problem of finding a time-dependent coefficient in a fractional diffusion equation. Turkish Journal of Mathematics, **47**(5), (2023), 1437–1452.

[25] Hazanee A., Lesnic D., Ismailov M. I., Kerimov N. B.; Inverse time-dependent source problems for the heat equation with nonlocal boundary conditions. Appl Math Comput, **346**, (2019), 800–815.

[26] Huntul M. J.; Identifying an unknown heat source term in the third-order pseudo-parabolic equation from nonlocal integral observation. Int Commun Heat Mass Transfer, **128**, (2021).

[27] Ivanchov M. I., and Pabyrivska N. V.; Simultaneous determination of two coefficients in a parabolic equation in the case of nonlocal and integral conditions. Ukrainian Mathematical Journal, **53**(5), (2001), 674–684.

[28] Kanca F. and ismailov M. I.; The inverse problem of finding the time-dependent diffusion coefficient of the heat equation from integral overdetermination data, Inverse Problems in Science and Engineering, **20**(4), (2012), 463–476.

[29] Liao W., Denghan M., and Mohebbi A.; Direct numerical method for an inverse problem of a parabolic partial differential equation. Journal of Computational and Applied Mathematics, **232**(2), (2009), 351–360.

[30] Durdiev D.K.; Inverse coefficient problem for the time-fractional diffusion equation. Eurasian Journal of Mathematical and Computer Applications, **9**(1), (2021), 44–54.

[31] Turdiev H.H; Inverse coefficient problems for a time-fractional wave equation with the generalized Riemann–Liouville time derivative. Russian Mathematics (Izvestiya VUZ. Matematika), **10**, (2023), 46–59.

[32] Durdiev D.K., Turdiev H.H.; Inverse coefficient problem for fractional wave equation with the generalized Riemann–Liouville time derivative. Indian J. Pure Appl. Math. (2023), https://doi.org/10.1007/s13226-023-00517-9.

[33] Durdiev U.D.; Problem of determining the reaction coefficient in a fractional diffusion equation. Differential Equations, **57**(9), (2021), 1195–1204.

[34] Durdiev D.K., Turdiev H.H.; Inverse coefficient problem for a time-fractional wave equation with initial-boundary and integral type overdetermination conditions. Math. Meth. Appl. Sci. (2023), https://doi.org/10.1002/mma.9867.

[35] Kochubei A.N., A Cauchy problem for evolution equations of fractional order, Differential Equations. **25**(8), (1989), 967–974.

[36] Ladyzhenskaja O.A., Solonnikov V.A.; Linear and Quasi-linear Equations of Parabolic Type. Trans. Math. Monog.**23** (1968), 648.

[37] Fox C.; The G and H-functions as symmetrical Fourier kernels. Trans. Am. Math. Soc. **98**, (1961), 395–398.

[38] Mathai A. M., Saxena R. K., and Haubold H. J.; The H-function: Theory and Application. Springer, Berlin, (2010).

[39] Li C., Deng W.H., Shen X.Q. Exact solutions and their asymptotic behaviors for the averaged generalized fractional elastic models. Commun. Theor. Phys. 2014, 62, 443-450.

[40] Srivastava H.M., Gupta K. C., Goyal S.P. The H-functions of one and two variables with applications. South Asian publishers, New York, (1982).

[41] Kilbas A. A., Srivastava H. M., Trujillo J. J.; Theory and application of fractional differential equations. North Holland Mathematical Studies, Amsterdam: Elsevier, (2006).

[42] Tomovski Z., Hilfer R., Srivastava N.M.; Fractional and operational calculus with generalized fractional derivatives operators and Mittag-Leffler type functions. Integr. Transform. Spec. Funct., **21**(11), (2010), 797–814.

[43] Henry D.; Geometric Theory of Semi linear Parabolic Equations. Berlin. Germany. 1981.

Durdiev Durdimurod Kalandarovich
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan.
Bukhara State University,
Bukhara, Uzbekistan
email: durdievdd@gmal.com

Subhonova Ziyoda Anvar qizi
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan.
Bukhara State University,
Bukhara, Uzbekistan
email: zsubhonova24@mail.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.64-72
DOI: 10.29229/uzmj.2025-3-6

The inverse problem of determining the right-hand side of a fourth-order differential equation

Dudiev U.D., Odinayev R.R.

Abstract. This article studies the inverse problem of finding a multiplier on the right-hand side, depending on the spatial variable x. In the direct problem, an initial-boundary value problem for a fourth-order differential equation is considered. Using the Fourier method, the solution to the initial-boundary value problem is constructed, and its properties are investigated. Sufficient conditions for the existence of a solution to the direct problem are obtained, which will be used in the study of the inverse problem. Theorems on local existence and global uniqueness are proven, and an estimate of the conditional stability of the solution to both the direct and inverse problems is provided.

Keywords: Inverse problem, existence, uniqueness, Cauchy problem, spectral problem, initial-boundary value problems, Fourier method.

MSC (2020): 34K29

1. Introduction

Differential equations are commonly used to model real-world problems in science and engineering, which often involve multiple parameters and variable dependencies. Solving these problems typically requires addressing initial and boundary conditions, meaning the solutions must satisfy specific constraints and data. However, modeling real-world scenarios is a complex task that can take various forms, and finding an exact solution is often challenging.

The study of vibrations in rods, beams, and plates holds significant importance in structural design, stability analysis of rotating shafts, as well as in understanding the vibration behavior of ships and pipelines. These problems often involve differential equations of orders higher than the second, reflecting the complexity inherent in analyzing such dynamic systems [1]–[4]. In recent years, there has been a growing interest in the study of both linear and nonlinear initial boundary value problems, and inverse problems involving the equation governing the vibration of a beam [5]–[14]. Additionally, an initial-boundary problem for the inhomogeneous heat equation, which includes a higher-order derivative alongside an initial condition, was examined in [13].

Inverse problems in mathematical physics have been extensively investigated across various classes of differential equations. Inverse problems associated with the simplest hyperbolic-type equation were discussed in detail in the monograph [15]. The papers [16]–[20] and other sources have delved into methods for establishing local existence and uniqueness theorems, as well as uniqueness and conditional stability theorems, for solutions of inverse dynamic problems. Additionally, numerical methods for discovering solutions have been explored.

Boundary value problems for the Laplace, Poisson, and Helmholtz equations with boundary conditions involving higher-order derivatives have been explored in the works of Bavrin [21], Karachik [22]–[25], and Sokolovskii [26].

Now we reconsider the following equation

$$u_{tt} + u_{xxxx} = p(x)q(t), \quad (x,t) \in D, \tag{1.1}$$

in the domain $D := \{(x,t);\ 0 < x < 1,\ \ 0 < t \le T\}$ with initial conditions

$$\frac{\partial^k u}{\partial t^k}(x,0) = \varphi(x), \quad \frac{\partial^{k+1} u}{\partial t^{k+1}}(x,0) = \psi(x), \quad 0 \le x \le 1, \tag{1.2}$$

and boundary conditions

$$u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad 0 \le t \le T, \tag{1.3}$$

where $k \ge 2$ is a fixed natural number.

In the direct problem, it is required to find a function $u(x,t) \in C^{4,2}_{x,t}(D) \cap C^{2,k+1}_{x,t}(\overline{D})$ satisfying equalities (1.1)–(1.3) for given number T and sufficiently smooth functions $p(x), q(t), \varphi(x)$ and $\psi(x)$.

The inverse problem consists in finding the function $p(x)$, from the available additional information about the solution to the direct problem (1.1)–(1.3) :

$$u(x,T) = g(x),\ 0 \le x \le 1, \tag{1.4}$$

where $g(x)$ is a given sufficiently smooth function.

2. Study of the direct problem

The solution of the direct problem (1.1)–(1.3) will be found in the following form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) X_n(x), \tag{2.1}$$

where $X_n(x)$ is the solution to the following problem:

$$X^{(4)}(x) + \lambda X(x) = 0,$$
$$X(0) = X(1) = X''(0) = X''(1) = 0. \tag{2.2}$$

The solution to equation (2.2) is obtained by:

$$X_n(x) = \sqrt{2} \sin \mu_n x, \quad \lambda_n = -\mu_n^4 = -(\pi n)^4 . \tag{2.3}$$

It is known [5], the function system (2.3) is orthonormal and complete in $L_2[0,1]$. Let us introduce the function:

$$u_n(t) = \sqrt{2} \int_0^1 u(x,t) \sin \mu_n x \, dx.$$

By applying the formal scheme of the Fourier method and using equations (1.1) and (1.3), we obtain the folowing result:

$$u_n''(t) + \mu_n^4\, u_n(t) = p_n q(t),\ 0 < t \le T, \ \ n = 1, 2, ..., \tag{2.4}$$

$$u_n^{(k)}(0) = \varphi_n, \qquad u_n^{(k+1)}(0) = \psi_n, \tag{2.5}$$

where

$$p_n = \int_0^1 p(x) X_n(x)\, dx,\ \varphi_n = \int_0^1 \varphi(x) X_n(x)\, dx,\ \psi_n = \int_0^1 \psi(x) X_n(x)\, dx. \tag{2.6}$$

The solution to problem (2.4) and (2.5) is represented as

$$\begin{aligned} u_n(t) = {} & \frac{\varphi_n}{\mu_n^{2k}} \cos\left(\mu_n^2 t - \frac{\pi k}{2}\right) + \frac{\psi_n}{\mu_n^{2k+2}} \sin\left(\mu_n^2 t - \frac{\pi k}{2}\right) \\ & - \sum_{i=0}^{[\frac{k+1}{2}]-1} (-1)^i \mu_n^{4i-2k-2}\ p_n q^{(k-1-2i)}(0) \sin\left(\mu_n^2 t - \frac{\pi k}{2}\right) \\ & - \sum_{i=0}^{[\frac{k}{2}]-1} (-1)^i \mu_n^{4i-2k}\ p_n q^{(k-2-2i)}(0) \cos\left(\mu_n^2 t - \frac{\pi k}{2}\right) + \frac{p_n}{\mu_n^2} \int_0^t q(t-\tau) \sin \mu_n^2 \tau d\tau. \end{aligned} \tag{2.7}$$

Using (2.7) we find the following derivatives of $u_n(t)$:

$$u_n^{(k)}(t) = \varphi_n \cos \mu_n^2 t + \frac{\psi_n}{\mu_n^2} \sin \mu_n^2 t + \frac{p_n}{\mu_n^2} \int_0^t q^{(k)}(t-\tau) \sin \mu_n^2 \tau d\tau, \tag{2.8}$$

$$u_n^{(k+1)}(t) = -\mu_n^2 \varphi_n \sin \mu_n^2 t + \psi_n \cos \mu_n^2 t + \frac{p_n}{\mu_n^2} q^{(k)}(0) \sin \mu_n^2 t + \frac{p_n}{\mu_n^2} \int_0^t q^{(k+1)}(t-\tau) \sin \mu_n^2 \tau d\tau. \tag{2.9}$$

Now we show that the solution to problem (1.1)–(1.3) exists and is unique.

Lemma 2.1. *The estimates*

$$|u_n(t)| \leq \frac{1}{\mu_n^{2k}}|\varphi_n| + \frac{1}{\mu_n^{2k+2}}|\psi_n| + \frac{|p_n|||q||}{\mu_n^4-1}\left(\mu_n^{4[\frac{k+1}{2}]-2k-2} + \mu_n^{4[\frac{k}{2}]-2k}\right) + \frac{|p_n|\,||q||}{\mu_n^2}T, \tag{2.10}$$

$$\left|u_n^{(k)}(t)\right| \leq |\varphi_n| + \frac{1}{\mu_n^2}|\psi_n| + \frac{|p_n|\,||q||}{\mu_n^2}T, \tag{2.11}$$

$$\left|u_n^{(k+1)}(t)\right| \leq \mu_n^2|\varphi_n| + |\psi_n| + \frac{|p_n|||q||}{\mu_n^2}(1+T), \tag{2.12}$$

hold for any $t \in [0,T]$.

Proof. Estimating $u_n(t)$ for any $t \in [0,T]$, we obtain:

$$|u_n(t)| \leq \frac{1}{\mu_n^{2k}}|\varphi_n| + \frac{1}{\mu_n^{2k+2}}|\psi_n| + \frac{|p_n|||q||}{\mu_n^{2k+2}}\sum_{i=0}^{[\frac{k+1}{2}]-1}\mu_n^{4i} + \frac{|p_n|||q||}{\mu_n^{2k}}\sum_{i=0}^{[\frac{k}{2}]-1}\mu_n^{4i} + \frac{|p_n|||q||}{\mu_n^2}T$$

$$\leq \frac{1}{\mu_n^{2k}}|\varphi_n| + \frac{1}{\mu_n^{2k+2}}|\psi_n| + \frac{||q||\,|p_n|}{\mu_n^4-1}\left(\mu_n^{4[\frac{k+1}{2}]-2k-2} + \mu_n^{4[\frac{k}{2}]-2k}\right) + \frac{|p_n|\,||q||}{\mu_n^2}T,$$

where

$$\|q\| = \max_{0\leq i\leq k+1}\left\{\max_{t\in[0.T]}|q^{(i)}(t)|\right\}.$$

Estimating the functions (2.8) and (2.9) for $t \in [0,T]$, we have inequalities (2.11) and (2.12).
Lemma 2.1 is proven. □

Formally, differentiating (2.1), we obtain the series:

$$u_{xxxx}(x,t) = \sum_{n=1}^{\infty}\mu_n^4 u_n(t)X_n(x), \tag{2.13}$$

$$\frac{\partial^k u(x,t)}{\partial t^k} = \sum_{n=1}^{\infty}u_n^{(k)}(t)X_n(x), \tag{2.14}$$

$$\frac{\partial^{k+1} u(x,t)}{\partial t^{k+1}} = \sum_{n=1}^{\infty}u_n^{(k+1)}(t)X_n(x). \tag{2.15}$$

Next, we need to prove the absolute and uniform convergence of the series (2.13)–(2.15).

Lemma 2.2. *If the conditions:*

$$\varphi(x) \in C^2[0,1],\ \varphi^{(3)}(x) \in L_2(0,1),\quad \varphi^{(j)}(0) = \varphi^{(j)}(1) = 0,\ j = 0,2,$$

$$\psi(x) \in C[0,1],\ \psi'(x) \in L_2(0,1),\quad \psi(0) = \psi(1) = 0,$$

$$p(x) \in C^2[0,1],\quad p^{(3)}(x) \in L_2(0,1),\quad p^{(j)}(0) = p^{(j)}(1) = 0,\ j = 0,2,$$

are satisfied, then the representations:

$$\varphi_n = -\frac{1}{\mu_n^3}\varphi_n^{(3)},\quad \psi_n = \frac{1}{\mu_n}\psi_n',\quad p_n = -\frac{1}{\mu_n^3}p_n^{(3)}, \tag{2.16}$$

are valid, where

$$\varphi_n^{(3)} = \sqrt{2}\int_0^1 \varphi^{(3)}(x)\cos\mu_n x dx,$$

$$\psi_n' = \sqrt{2}\int_0^1 \psi'(x)\cos\mu_n x dx,$$

$$p_n^{(3)} = \sqrt{2}\int_0^1 p^{(3)}(x)\cos\mu_n x dx,$$

with the following estimates holding true :

$$\sum_{n=1}^{\infty}\left|\varphi_n^{(3)}\right|^2 \le \left\|\varphi^{(3)}\right\|^2_{L_2(0,1)}, \quad \sum_{n=1}^{\infty}|\psi_n'|^2 \le \|\psi'\|^2_{L_2(0,1)},$$

$$\sum_{n=1}^{\infty}\left|p_n{}^{(3)}\right|^2 \le \left\|p^{(3)}\right\|^2_{L_2(0,1)}. \tag{2.17}$$

Proof. By integrating by parts φ_n and p_n three times and ψ_n once, considering the conditions of Lemma 2.1, we obtain the representations in (2.16). The inequalities in (2.17) are Bessel inequalities for the coefficients of the Fourier series expansions of the functions $\varphi_n^{(3)}$, ψ_n' and $p_n^{(3)}$ in the cosine system on the interval $[0,1]$.

Lemma 3.1 is completely proven. □

Theorem 2.3. *Let $q(t) \in C^{k+1}[0,T]$ and the functions $\varphi(x)$, $\psi(x)$, $p(x)$ satisfy the assumptions of Lemma 3.1, then there exists a unique solution of problem* (1.1)-(1.3)*, that can be represented by the series* (2.1) *with the coefficients given by relation* (2.7)*.*

Proof. The series (2.1), (2.13), (2.14) and (2.15) for any $(x,t) \in \overline{D}$ are majorized by the following numerical series:

$$\sum_{n=1}^{\infty}\left(\frac{1}{\mu_n^{2k}}|\varphi_n| + \frac{1}{\mu_n^{2k+2}}|\psi_n| + \frac{||q||\,|p_n|}{\mu_n^4-1}\left(\mu_n^{4[\frac{k+1}{2}]-2k-2} + \mu_n^{4[\frac{k}{2}]-2k}\right) + \frac{|p_n|||q||}{\mu_n^2}T\right),$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{\mu_n^{2k-4}}|\varphi_n| + \frac{1}{\mu_n^{2k-2}}|\psi_n| + \frac{|p_n|\|q\|}{\mu_n^4-1}\left(\mu_n^{4[\frac{k+1}{2}]-2k+2} + \mu_n^{4[\frac{k}{2}]-2k+4}\right) + \mu_n^2|p_n|\|q\|T\right),$$

$$\sum_{n=1}^{\infty}\left(|\varphi_n| + \frac{1}{\mu_n^2}|\psi_n| + \frac{|p_n|\,||q||}{\mu_n^2}T\right),$$

$$\sum_{n=1}^{\infty}\left(\mu_n^2|\varphi_n| + |\psi_n| + \frac{p_n||q||}{\mu_n^2}(1+T)\right).$$

If the functions $\varphi(x)$, $\psi(x)$ and $p(x)$ satisfy the conditions of lemma 3.1, then by virtue of the representations (2.16) and (2.17) the series (2.1), (2.13), (2.14) and (2.15) converge uniformly in the rectangle $\overline{D}$, hence the function $u(x,t)$ satisfies relations (1.1)–(1.3). Here, the following estimates hold true :

$$|u(x,t)| \le C_1\left(\|\varphi\|_{L_2(0,1)} + \|\psi\|_{L_2(0,1)} + \|q\|\,\|p\|_{L_2(0,1)}\right),$$

$$|u_{xxxx}(x,t)| \le C_2\left(\|\varphi'\|_{L_{2(0,1)}} + \|\psi\|_{L_2(0,1)} + \|q\|\left\|p^{(3)}\right\|_{L_2(0,1)}\right),$$

$$\left|\frac{\partial^k u(x,t)}{\partial t^k}\right| \le C_3\left(\|\varphi'\|_{L_2(0,1)} + \|\psi\|_{L_2(0,1)} + \|q\|\,\|p\|_{L_2(0,1)}\right),$$

$$\left|\frac{\partial^{k+1} u(x,t)}{\partial t^{k+1}}\right| \le C_4\left(\left\|\varphi^{(3)}\right\|_{L_2(0,1)} + \|\psi'\|_{L_2(0,1)} + \|q\|\,\|p\|_{L_2(0,1)}\right).$$

$C_i, i = \overline{1,5}$ are positive constants here and throughout the following.

Uniqueness. Assume that the given problem has two solutions, $u_1(x,t)$ and $u_2(x,t)$. We will prove that $u(t) = u_1(t) - u_2(t) = 0$. By utilizing the linearity of the problem's conditions for determining $u(x,t)$, we have the equation:

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}=0, \tag{2.18}$$

$$\frac{\partial^k u}{\partial t^k}(x,0)=0, \quad \frac{\partial^{k+1} u}{\partial t^{k+1}}(x,0)=0, \quad 0\le x\le 1. \tag{2.19}$$

Let $u(x,t)$ be a solution to this problem. Then, from the problem (2.18)–(2.19), we have:

$$u_n''(t)+\mu_n^4\ u_n(t)=0,$$

$$u_n^{(k)}(t)\big|_{t=0}=0, \qquad u_n^{(k+1)}(t)\big|_{t=0}=0.$$

It follows that $u_n=0$ for all $n\in N$. From the completeness of the eigenfunction system. $X_n(x)$ we conclude that $u(x,t)=0$.

Theorem 2.3 is completely proven. □

Now, we establish the stability of the solution of the problem posed under perturbations of the initial data $\varphi(x)$ and $\psi(x)$ and the right-hand side $p(x)q(t)$. Let us derive an estimate for the norm of the difference between the solution of problem (1.1)- (1.3) and the solution $\widetilde{u}_n(t)$ of the problem with perturbed functions $\widetilde{\varphi}_n$, $\widetilde{\psi}_n$, $\widetilde{p}_n$ and $\widetilde{q}(t)$. Let $\widetilde{u}_n(t)$ be the solution of this problem corresponding to the functions $\widetilde{\varphi}_n$, $\widetilde{\psi}_n$, $\widetilde{p}_n$ and $\widetilde{q}(t)$, then $\widetilde{u}_n(t)$ has the form:

$$\widetilde{u}_n(t)=\frac{\widetilde{\varphi}_n}{\mu_n^{2k}}\cos\left(\mu_n^2 t-\frac{\pi k}{2}\right)+\frac{\widetilde{\psi}_n}{\mu_n^{2k+2}}\sin\left(\mu_n^2 t-\frac{\pi k}{2}\right)$$
$$-\sum_{i=0}^{[\frac{k+1}{2}]-1}(-1)^i\mu_n^{4i-2k-2}\ \widetilde{p}_n\widetilde{q}^{(k-1-2i)}(0)\sin\left(\mu_n^2 t-\frac{\pi k}{2}\right)$$
$$-\sum_{i=0}^{[\frac{k}{2}]-1}(-1)^i\mu_n^{4i-2k}\ \widetilde{p}_n\widetilde{q}^{(k-2-2i)}(0)\cos\left(\frac{\pi k}{2}-\mu_n^2 t\right)+\frac{\widetilde{p}_n}{\mu_n^2}\int_0^t\widetilde{q}(t-\tau)\sin\mu_n^2\tau d\tau. \tag{2.20}$$

Composing the difference $u_n(t)-\widetilde{u}_n(t)$ with the help of equations (2.7), (2.20) and introducing the notation $\overline{u}(t)=u_n(t)-\widetilde{u}_n(t)$, $\overline{\varphi}(t)=\varphi_n(t)-\widetilde{\varphi}_n(t)$, $\overline{\psi}(t)=\psi_n(t)-\widetilde{\psi}_n(t)$, $\overline{p}_n=p_n-\widetilde{p}_n$ and $\overline{q}(t)=q(t)-\widetilde{q}(t)$, we obtain the next formula:

$$\overline{u}_n(t)=\frac{\overline{\varphi}_n}{\mu_n^{2k}}\cos\left(\mu_n^2 t-\frac{\pi k}{2}\right)+\frac{\overline{\psi}_n}{\mu_n^{2k+2}}\sin\left(\mu_n^2 t-\frac{\pi k}{2}\right)$$
$$-\sum_{i=0}^{[\frac{k+1}{2}]-1}(-1)^i\mu_n^{4i-2k-2}\ \left(p_nq^{(k-1-2i)}-\widetilde{p}_n\widetilde{q}^{(k-1-2i)}(0)\right)\sin\left(\mu_n^2 t-\frac{\pi k}{2}\right)$$
$$-\sum_{i=0}^{[\frac{k}{2}]-1}(-1)^i\mu_n^{4i-2k}\ \left(p_nq^{(k-2-2i)}-\widetilde{p}_n\widetilde{q}^{(k-2-2i)}(0)\right)\cos\left(\mu_n^2 t-\frac{\pi k}{2}\right)$$
$$+\frac{1}{\mu_n^2}\int_0^t\left(p_nq(t-\tau)-\overline{p}_n\overline{q}(t-\tau)\right)\sin\mu_n^2\tau d\tau.$$

Using the next formula

$$p_nq(t)-\widetilde{p}_n\widetilde{q}(t)=p_nq(t)+\widetilde{p}_nq(t)-\widetilde{p}_nq(t)-\widetilde{p}_n\widetilde{q}(t)$$
$$=q(t)\left(p_n-\widetilde{p}_n\right)+\widetilde{p}_n\left(q(t)-\widetilde{q}(t)\right)=q(t)\overline{p}_n+\widetilde{p}_n\overline{q}(t),$$

we obtain

$$\overline{u}_n(t)=\frac{\overline{\varphi}_n}{\mu_n^{2k}}\cos\left(\mu_n^2 t-\frac{\pi k}{2}\right)+\frac{\overline{\psi}_n}{\mu_n^{2k+2}}\sin\left(\mu_n^2 t-\frac{\pi k}{2}\right)$$

$$-\sum_{i=0}^{[\frac{k+1}{2}]-1}(-1)^i\mu_n^{4i-2k-2}\ \overline{p}_n q^{(k-1-2i)}(0)\sin\left(\mu_n^2 t-\frac{\pi k}{2}\right)$$

$$-\sum_{i=0}^{[\frac{k+1}{2}]-1}(-1)^i\mu_n^{4i-2k-2}\ \widetilde{p}_n \overline{q}^{(k-1-2i)}(0)\sin\left(\mu_n^2 t-\frac{\pi k}{2}\right)$$

$$-\sum_{i=0}^{[\frac{k}{2}]-1}(-1)^i\mu_n^{4i-2k}\ \overline{p}_n q^{(k-2-2i)}(0)\cos\left(\mu_n^2 t-\frac{\pi k}{2}\right)$$

$$-\sum_{i=0}^{[\frac{k}{2}]-1}(-1)^i\mu_n^{4i-2k}\ \widetilde{p}_n \overline{q}^{(k-2-2i)}(0)\cos\left(\mu_n^2 t-\frac{\pi k}{2}\right)$$

$$+\frac{\overline{p}_n}{\mu_n^2}\int_0^T q(t-\tau)\sin\mu_n^2\tau d\tau+\frac{\widetilde{p}_n}{\mu_n^2}\int_0^T \overline{q}(t-\tau)\sin\mu_n^2\tau d\tau.$$

From the previous expression, applying Lemma 3.1, we obtain the following estimate:

$$|\overline{u}(t)|\le C_5\left(\|\overline{\varphi}\|_{L_2(0,1)}+\|\overline{\psi}\|_{L_2(0,1)}+\|\overline{p}\,\|_{L_2(0,1)}||q||+\|\widetilde{p}_{\,n}\|_{L_2(0,1)}\,\|\overline{q}\|\right).$$

3. Study of the inverse problem

The main result of this work is the following statement:
Substituting $t=T$, into equation (2.7), and using the additional condition (1.4), we obtain:

$$g_n=\frac{\varphi_n}{\mu_n^{2k}}\cos\left(\mu_n^2 T-\frac{\pi k}{2}\right)+\frac{\psi_n}{\mu_n^{2k+2}}\sin\left(\mu_n^2 T-\frac{\pi k}{2}\right)$$

$$-\sum_{i=0}^{[\frac{k+1}{2}]-1}(-1)^i\mu_n^{4i-2k-2}\ p_n q^{(k-1-2i)}(0)\sin\left(\mu_n^2 T-\frac{\pi k}{2}\right)$$

$$-\sum_{i=0}^{[\frac{k}{2}]-1}(-1)^i\mu_n^{4i-2k}\ p_n q^{(k-2-2i)}(0)\cos\left(\mu_n^2 T-\frac{\pi k}{2}\right)+\frac{p_n}{\mu_n^2}\int_0^T q(T-\tau)\sin\mu_n^2\tau d\tau. \tag{3.1}$$

where g_n is defined as:

$$g_n=\sqrt{2}\int_0^1 g(x)\sin\mu_n x dx.$$

From equation (3.1), we can express $p_n$as:

$$p_n=\frac{1}{W_n(T)}\left(g_n-\frac{\varphi_n}{\mu_n^{2k}}\cos\left(\mu_n^2 T-\frac{\pi k}{2}\right)-\frac{\psi_n}{\mu_n^{2k+2}}\sin\left(\mu_n^2 T-\frac{\pi k}{2}\right)\right), \tag{3.2}$$

where

$$W_n(T)=-\sum_{i=0}^{[\frac{k+1}{2}]-1}(-1)^i\mu_n^{4i-2k-2}\ q^{(k-1-2i)}(0)\sin\left(\mu_n^2 T-\frac{\pi k}{2}\right)$$

$$-\sum_{i=0}^{[\frac{k}{2}]-1}(-1)^i\mu_n^{4i-2k}\ q^{(k-2-2i)}(0)\cos\left(\mu_n^2 T-\frac{\pi k}{2}\right)+\frac{1}{\mu_n^2}\int_0^T q(T-\tau)\sin\mu_n^2\tau d\tau. \tag{3.3}$$

Thus, we can determine $p(x)$ in the form of a series:

$$p(x)=\sqrt{2}\sum_{n=1}^{\infty}\frac{1}{W_n(T)}\left(g_n-\frac{\varphi_n}{\mu_n^{2k}}\cos\left(\mu_n^2 T-\frac{\pi k}{2}\right)-\frac{\psi_n}{\mu_n^{2k+2}}\sin\left(\mu_n^2 T-\frac{\pi k}{2}\right)\right)\sin\mu_n x. \tag{3.4}$$

Theorem 3.1. *Let $q(t) = 1$, and the conditions of Theorem 2.3, are satisfied, additionally, the following conditions hold:*

$$W(T) = \frac{1}{\mu_n^4}, \quad g(x) \in C^4[0,1],$$

$$g^{(5)}(x) \in L_2(0,1),\; g^{(j)}(0) = g^{(j)}(1) = 0,\; j = 0,2,4.,$$

then the representation

$$g_n^{(5)} = \frac{1}{\mu_n^5} g_n^{(5)},$$

where

$$g_n^{(5)} = \sqrt{2} \int_0^1 g^{(5)}(x) \cos \mu_n x dx,$$

holds, and the following series converges:

$$\sum_{n=1}^{\infty} \left| g_n^{(5)} \right|^2 \le \left\| g^{(5)} \right\|^2_{L_2(0,1)}.$$

Thus, there exists a unique solution to the inverse problem defined by equations (1.1)-(1.4).

Proof. We analyze the zeros of the function $W_n(T)$ defined by equation (3.3).

$$W_n(T) = \left(-\frac{q^{(k-1)}(0)}{\mu_n^{2k+2}} + \frac{q^{(k-3)}(0)}{\mu_n^{2k-2}} - \frac{q^{(k-5)}(0)}{\mu_n^{2k-6}} \cdots + (-1)^{[\frac{k+1}{2}]} \frac{q(0)}{\mu_n^4} \right) \sin \left(\mu_n^2 T - \frac{\pi k}{2} \right)$$
$$- \left(\frac{q^{(k-2)}(0)}{\mu_n^{2k}} - \frac{q^{(k-4)}(0)}{\mu_n^{2k-4}} + \frac{q^{(k-6)}(0)}{\mu_n^{2k-8}} \cdots + (-1)^{[\frac{k}{2}]-1} \frac{q'(0)}{\mu_n^6} \right) \cos \left(\mu_n^2 T - \frac{\pi k}{2} \right)$$
$$+ \frac{1}{\mu_n^2} \int_0^T q(T-\tau) \sin \mu_n^2 \tau d\tau,$$

if k is odd number

$$W_n(T) = \left(-\frac{q^{(k-1)}(0)}{\mu_n^{2k+2}} + \frac{q^{(k-3)}(0)}{\mu_n^{2k-2}} - \frac{q^{(k-5)}(0)}{\mu_n^{2k-6}} \cdots + (-1)^{[\frac{k+1}{2}]} \frac{q'(0)}{\mu_n^6} \right) \sin \left(\mu_n^2 T - \frac{\pi k}{2} \right)$$
$$- \left(\frac{q^{(k-2)}(0)}{\mu_n^{2k}} - \frac{q^{(k-4)}(0)}{\mu_n^{2k-4}} + \frac{q^{(k-6)}(0)}{\mu_n^{2k-8}} \cdots + (-1)^{[\frac{k}{2}]-1} \frac{q(0)}{\mu_n^4} \right) \cos \left(\mu_n^2 T - \frac{\pi k}{2} \right)$$
$$+ \frac{1}{\mu_n^2} \int_0^T q(T-\tau) \sin \mu_n^2 \tau d\tau,$$

if k is even number. Let $q(t) = 1$, and considering that $\forall n \in N,\ \ \forall T > 0$, we have the following results:

$$W_n(T) = \frac{(-1)^{[\frac{k+1}{2}]}}{\mu_n^4} \sin \left(\mu_n^2 T - \frac{\pi k}{2} \right) + \frac{1}{\mu_n^2} \int_0^T \sin \mu_n^2 \tau d\tau,$$

if k is odd number,

$$W_n(T) = \frac{(-1)^{[\frac{k}{2}]}}{\mu_n^4} \cos \left(\mu_n^2 T - \frac{\pi k}{2} \right) + \frac{1}{\mu_n^2} \int_0^T \sin \mu_n^2 \tau d\tau,$$

if k is even number.

We get the same result for even and odd values of k,

$$W_n(T) = \frac{\cos \mu_n^2 T}{\mu_n^4} + \frac{1}{\mu_n^2} \int_0^T \sin \mu_n^2 \tau d\tau = \frac{1}{\mu_n^4}.$$

Next, we prove the existence of the solution to the inverse problem defined by equations (1.1)(1.4) expressed in (3.4). The series (3.4) for any $(x,t) \in \overline{D}$ are majorized by the following numerical series:

$$\sum_{n=1}^{\infty} \left|\mu_n^4\right| \left(|g_n| + \left|\frac{\varphi_n}{\mu_n^{2k}}\right| + \left|\frac{\psi_n}{\mu_n^{2k+2}}\right| \right).$$

Applying the conditions of Theorem 3.1, we obtain the following estimate:

$$|P(x)| \leq C_5 \Big(\left\|g^{(5)}\right\|_{L_2(0,1)} + \|\varphi'\|_{L_2(0,1)} + \|\psi\|_{L_2(0,1)} \Big). \tag{3.5}$$

□

Theorem 3.1 is completely proved.

3.1. **Conclusion.** This article studies the inverse problem related to determining the right-hand side of a fourth-order differential equation. To represent the solution of the direct problem, a fundamental solution of this equation was constructed, and its properties were analyzed. Sufficient conditions for the existence of a solution to the direct problem were derived, which were then used in the investigation of the inverse problem. Results on local existence and global uniqueness of the solution were proven, and conditional stability estimates for the obtained solutions were provided.

References

[1] Strutt, J., Baron Rayleigh, The Theory of Sound, London: Macmillan, 1877. Translated under the title Teoriya zvuka, Moscow: Gosudarstv. Izdat. Tekhn.-Teor. Lit., 1955, vol. 1.

[2] Tikhonov A. N. About the boundary conditions containing derivatives of an order, exceeding an equation order, Mat. Sb., 1950. vol. 26(1), pp. 3556 (In Russian).

[3] Tikhonov, A.N. and Samarskii, A.A., Uravneniya matematicheskoi fiziki (Equations of Mathematical Physics), Moscow: Nauka, 1966.

[4] Krylov, A.N., Vibratsiya sudov (Ship Oscillations), Moscow, 2012.

[5] Sabitov, K.B. A Remark on the Theory of Initial-Boundary Value Problems for the Equation of Rods and Beams, // Differential Equations, 2017, Vol. 53, No. 1, pp. 8698. https://doi.org/10.1134/S0012266117010086.

[6] Sabitov, K.B., Fadeeva, O.V. Initial-boundary value problem for the equation of forced vibrations of a cantilevered beam, // Vestn. Samarsk.Gos. Tekh. Univ. Ser. Fiz.-Mat.2021, vol. 25, pp. 51–66. https://doi.org/10.1134/S0012266117010086.

[7] Sabitov K.B. Inverse Problems of Determining the Right-Hand Side and the Initial Conditions for the Beam Vibration Equation, // Differential equations. 2020, vol. 6, issue 56, pp. 771–774. https://doi.org/10.1134/S0012266120060099.

[8] Kasimov S. G., Madrakhimov U. S. Initial-boundary value problem for the beam vibration equation in the multidimensional case // Differ. Equations, 552019, vol. 55, pp. 1336–1348.

[9] Durdiev U.D. Inverse problem of determining an unknown coefficient in the beam vibration equation, // Differential Equations, 2022, vol. 58, issue 1, pp. 37–44. https://doi.org/10.1134/S0012266122010050.

[10] Durdiev U.D. Inverse problem of determining the unknown coefficient in the beam vibration equation in an infinite domain, // Differ. Equations, 2023, vol. 59, issue 4, pp. 462–472. https://doi.org/10.1134/S0012266123040031

[11] Durdiev U.D. A time-nonlocal inverse problem for the beam vibration equation with an integral condition, // Differ. Equations, 2023, vol. 59 issue 3, pp. 358–367. https://doi.org/10.1134/S0012266123030060

[12] Durdiev U.D. Inverse problem of determining the time-dependent beam stiffness coefficient in the beam vibration equation using the finite difference method, // Eurasian Journal of Mathematical and Computer Applications, 2024, vol. 12, issue 1, pp. 41–56. https://doi.org/10.32523/2306-6172-2024-12-1-41-56

[13] Amanov D., Kilichov O.Sh. On a Generalization of the Initial-Boundary Problem for the Beam Equation, //Uzbek Mathematical Journal, 2022, vol. 66, issue 1, pp. 51–63 https://doi.org/10.29229/uzmj.2022-1-5

[14] Amanov, D. On a generalization of the first initial-boundary value problem for the heat conduction equation. Contemp. Anal. Appl. Math. 2014, 2, 88-97.

[15] Romanov, V.G., Obratnye zadachi matematicheskoi fiziki (Inverse Problems of Mathematical Physics), Moscow: Akad. Nauk SSSR, 1984.

[16] Durdiev D.K. and Totieva, Zh.D. The problem of determining the one-dimensional kernel of the electroviscoelasticity equation // Sib.mat., 2017, vol. 58, no. 3, pp. 427-444. https://doi.org/10.1134/S0037446617030077

[17] Durdiev D.K. and Rakhmonov A.A. The problem of determining the 2D kernel in a system of integro-differential equations of a viscoelastic porous medium, // J. Appl. Ind. Math., 2020, vol. 14, issue 2, pp. 281–295. https://doi.org/10.1134/S1990478920020076

[18] Durdiev D.K., Turdiev H.H. Inverse coefficient problem for a time-fractional wave equation with initial-boundary and integral type over-determination conditions, // Mathematical Methods in the Applied Sciences, 2024, vol. 47, issue 6, pp. 5329–5340. https://doi.org/10.1002/mma.9867

[19] Durdiev D.K., Turdiev H.H. Inverse Problem for a First-Order Hyperbolic System with Memory, Differential Equations, 2020, vol. 56, issue 12, pp. 1634–1643. https://doi.org/10.1134/S00122661200120125

[20] Durdiev D.K., Zhumaev Zh.Zh. Memory kernel reconstruction problems in the integro-differential equation of rigid heat conductor. *Math. Meth. Appl. Sci.,* 2022, vol. 45, no. 14, pp. 8374-8388. https://doi.org/10.1002/mma.7133

[21] Bavrin I. I.Operators for harmonic functions and applications, Differential equations, 1985. vol. 21(1), pp. 915 (In Russian).

[22] Karachik V. V. About solvability of a boundary value problem for Helmholtzs equation with high order normal derivative on a boundary, Differ. Uravneniya, 1992. vol. 28 (5), pp. 907909 (In Russian).

[23] Karachik V. V. About a problem for Poissons equation with high order normal derivative on boundary, Differ. Uravneniya, 1996. vol. 32, no. 3, pp. 15011503 (In Russian).

[24] Karachik V. V. Generalized Neumann problem for harmonic functions in space, Differ. Uravneniya, 1999. vol. 35, no. 7, pp. 16 (In Russian).

[25] Karachik V. V, Turmetov B. H. About a problem for harmonic equation, Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk, 1990. vol. 4, pp. 1721 (In Russian).

[26] Sokolovskiy V. B.On a generalization of Neumann problem, Differ. Uravneniya, 1998. vol. 24, no. 4, pp. 714716 (In Russian).

[27] Strutt J., Rayleigh B. The Theory of Sound. London: Macmillan, 1877

[28] Yi-Ren Wang, Zhi-Wei Fang. Vibrations in an elastic beam with nonlinear supports at both ends, 2015, vol. 56, issue 3, pp. 337–346. https://doi.org/10.1134/S0021894415020200.

Durdiev U.D.,
V.I. Romanovsky Institute of Mathematics,
Uzbekistan Academy of Sciences
Department of Differential Equations,
Bukhara State University, Uzbekistan,
email:umidjan93@list.ru

Odinayev R.R.,
Department of Differential Equations,
Bukhara State University, Uzbekistan,
email:odinayevrashid@mail.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.73-82
DOI: 10.29229/uzmj.2025-3-7

Lauricella hypergeometric function $F_A^{(n)}$ with applications to the solving Dirichlet problem for three-dimensional degenerate elliptic equation

Hasanov A., Ergashev T.G., Djuraev N.

Abstract. In this paper, hypergeometric function of Lauricella $F_A^{(n)}$ has been investigated. The new properties of which are established and applied to the solution of the Dirichlet problem for the three-dimensional degenerate elliptic equation. Fundamental solutions of the named equation are expressed through the Lauricella hypergeometric function in three variables and an explicit solution of the Dirichlet problem in the first octant is written out through the Appell hypergeometric function F_2. A limit theorem for calculating the value of a function of many variables is proved, and formulas for their transformation are established. These results are used to determine the order of singularity of fundamental solutions and to prove the truth of the solution to the Dirichlet problem. The uniqueness of the solution to the Dirichlet problem is proved by the extremum principle for elliptic equations.

Keywords: Appell and Lauricella hypergeometric functions, three-dimensional degenerate elliptic equation, PDE-systems of hypergeometric type, fundamental solution, Dirichlet problem

MSC (2020): 35A08, 35J25, 35J70, 35J75

1. Introduction

It is known, that a special functions are used for solving many problems of mathematical physics (see [4, 18]). These include the Gauss hypergeometric series, Bessel and Hermite functions, Lauricella hypergeometric functions, etc. The Hermite functions are actively applied in algorithms and information systems that are used in medical diagnostics [16]. The Bessel functions are used in solving a number of problems of hydrodynamics, radiophysics, and thermal conductivity [14, Part 2]. Some functions that are used in astronomy can be arranged in hypergeometric series [20, Chapter 3]. Multidimensional hypergeometric functions are used in the superstrings theory [5].

The study of boundary value problems for degenerate equations is one of the important directions of the modern theory of partial differential equations. It is known that in the formulation and construction of local and nonlocal boundary value problems solutions, the main role is played by fundamental solutions. Fundamental solutions of the two-dimensional degenerate elliptic equations are expressed by the Appell function F_2, and when the dimension of the equation exceeds two – by the Lauricella hypergeometric function $F_A^{(n)}$ with three and more variables.

In this work, the established properties of the Lauricella function are applied to solving the Dirichlet problem for the three-dimensional degenerate elliptic equation

$$y^m z^k u_{xx} + x^n z^k u_{yy} + x^n y^m u_{zz} = 0, m > 0, n > 0, k > 0 \tag{1.1}$$

in the domain $\Omega = \{(x, y, z) : x > 0,\ y > 0,\ z > 0\}$.

A degenerate elliptic equation (1.1) is related to an elliptic equation with the singular coefficients

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y + \frac{2\gamma}{z}u_z = 0,\ 0 < 2\alpha, 2\beta, 2\gamma < 1. \tag{1.2}$$

Namely, if in the region of ellipticity the equation (1.1) is reduced to a canonical form, then we obtain equation (1.2). Using the fundamental solutions constructed in [9], the main boundary value problems for the equation (1.2) in the finite (first octant of the ball) were solved in explicit forms [10, 11, 22], and local and nonlocal boundary value problems for the equation (1.2) by the Fourier method in special infinite domains were investigated [12, 13].

Few works are devoted to the study of boundary value problems for the two-dimensional analogue of the equation (1.1). In works [1, 19], for the two-dimensional degenerate elliptic equation

$$y^m u_{xx} + x^n u_{yy} = 0, m > 0, n > 0$$

solutions of the Dirichlet and Neumann problems in the bounded and unbounded domains were found in explicit forms.

2. Multiple hypergeometric functions and their some new properties

The Gauss hypergeometric function can be represented by the following series [6, p.56, Eq. 2.1(2)]

$$F(a,b;c;x)=\sum_{m=0}^{\infty}\frac{(a)_m\,(b)_m}{(c)_m}\frac{x^m}{m!},\quad |x|<1, \tag{2.1}$$

where $(z)_n$ is a Pochhammer symbol: $(z)_n=z(z+1)...(z+n-1),\ n=1,2,...;\ (z)_0=1$.

The great success of the theory of hypergeometric function in one variable has stimulated the development of corresponding theory in two or more variables. Appell [2] has defined four functions F_1 to F_4, which are all analogues to Gauss' $F(a,b;c;x)$. For instance, the Appell function F_2 has a form

$$F_2\left[\begin{matrix}a,b_1,b_2;\\ c_1,c_2;\end{matrix}\ x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\,(b_1)_m\,(b_2)_n}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|+|y|<1, \tag{2.2}$$

which satisfies the following system of partial differential equations [6, p. 234, Eq. 5.9(10)]:

$$\begin{cases} x(1-x)u_{xx}-xyu_{xy}+\left[c_1-(a+b_1+1)\,x\right]u_x-b_1yu_y-ab_1u=0,\\ y(1-y)u_{yy}-xyu_{xy}-b_2xu_x+\left[c_2-(a+b_2+1)\,y\right]u_y-ab_2u=0.\end{cases} \tag{2.3}$$

Lauricella hypergeometric function[15] (see also [21, p. 33])

$$F_A^{(n)}\left[\begin{matrix}a,\mathbf{b};\\ \mathbf{c};\end{matrix}\ \mathbf{x}\right]=\sum_{|\mathbf{k}|=0}^{\infty}(a)_{|\mathbf{k}|}\prod_{i=1}^{n}\frac{(b_i)_{k_i}}{(c_i)_{k_i}}\frac{x_i^{k_i}}{k_i!},\ |x_1|+...+|x_n|<1$$

is a natural generalization of the classical Gauss hypergeometric function (2.1) and the Appell function (2.2) to the case of many complex variables and their corresponding complex parameters. Hereinafter

$$\mathbf{b}:=(b_1,...,b_n)\,,\ \mathbf{c}:=(c_1,...,c_n)\,,\ \mathbf{x}:=(x_1,...,x_n)\,,$$

$$\mathbf{k}:=(k_1,...,k_n)\,,\ |\mathbf{k}|:=k_1+...+k_n,\ \ k_1\geq 0,...,k_n\geq 0.$$

Let us list some properties of the Lauricella hypergeometric function $F_A^{(n)}$:

1) tranformation formula [3, p. 116, Eq. (9)]:

$$F_A^{(n)}\left[\begin{matrix}a,\mathbf{b};\\ \mathbf{c};\end{matrix}\ \mathbf{x}\right]=(1-X)^{-a}\,F_A^{(n)}\left[\begin{matrix}a,\mathbf{c}-\mathbf{b};\\ \mathbf{c};\end{matrix}\ \frac{\mathbf{x}}{X-1}\right],\ X:=\sum_{j=1}^{n}x_j; \tag{2.4}$$

2) differentation formula:

$$\frac{\partial}{\partial x_k}F_A^{(n)}\left[\begin{matrix}a,\mathbf{b};\\ \mathbf{c};\end{matrix}\ \mathbf{x}\right]=\frac{ab_k}{c_k}F_A^{(n)}\left[\begin{matrix}a+1,\mathbf{b}_k+1;\\ \mathbf{c}_k+1;\end{matrix}\ \mathbf{x}\right], \tag{2.5}$$

where the vectors $\mathbf{b}_k+1$ and $\mathbf{c}_k+1$ appear, the k-th component of which is one greater than the corresponding components of the vectors $\mathbf{b}$ and $\mathbf{c}$, respectively:

$$\mathbf{b}_k+1:=(b_1,...,b_{k-1},b_k+1,b_{k+1},...,b_n)\,,\ \mathbf{c}_k+1:=(c_1,...,c_{k-1},c_k+1,c_{k+1},...,c_n)\,,\ k=\overline{1,n}.$$

The Lauricella hypergeometric function of n variables satisfies the system with n equations and this system has 2^n linearly independent solutions (for details, see [3, pp. 117, 118]). In our further studies, we use the following system corresponding to a function of three variables $u=F_A^{(3)}\left[\begin{matrix}a,b_1,b_2,b_3;\\ c_1,c_2,c_3;\end{matrix}\ x,y,z\right]$:

$$\begin{cases} x(1-x)u_{xx}-xyu_{xy}-xzu_{xz}+[c_1-(a+b_1+1)x]u_x-b_1yu_y-b_1zu_z-ab_1u=0,\\ y(1-y)u_{yy}-xyu_{xy}-yzu_{yz}-b_2xu_x+[c_2-(a+b_2+1)y]u_y-b_2zu_z-ab_2u=0,\\ z(1-z)u_{zz}-xzu_{xz}-yzu_{yz}-b_3xu_x-b_3yu_y+[c_3-(a+b_3+1)z]u_z-ab_3u=0.\end{cases} \tag{2.6}$$

The PDE-system (2.6) has 8 linearly independent solutions [3, pp. 117, 118] :

$$1\left\{F_A^{(3)}\left[\begin{matrix} a,b_1,b_2,b_3; \\ c_1,c_2,c_3;\end{matrix}\ x,y,z\right],\right. \tag{2.7}$$

$$3\begin{cases} x^{1-c_1}F_A^{(3)}\left[\begin{matrix} a+1-c_1,b_1+1-c_1,b_2,b_3; \\ 2-c_1,c_2,c_3;\end{matrix}\ x,y,z\right],\\ y^{1-c_2}F_A^{(3)}\left[\begin{matrix} a+1-c_2,b_1,b_2+1-c_2,b_3; \\ c_1,2-c_2,c_3;\end{matrix}\ x,y,z\right],\\ z^{1-c_3}F_A^{(3)}\left[\begin{matrix} a+1-c_3,b_1,b_2,b_3+1-c_3; \\ c_1,c_2,2-c_3;\end{matrix}\ x,y,z\right],\end{cases} \tag{2.8}$$

$$3\begin{cases} x^{1-c_1}y^{1-c_2}F_A^{(3)}\left[\begin{matrix} a+2-c_1-c_2,b_1+1-c_1,b_2+1-c_2,b_3; \\ 2-c_1,2-c_2,c_3;\end{matrix}\ x,y,z\right],\\ y^{1-c_2}z^{1-c_3}F_A^{(3)}\left[\begin{matrix} a+2-c_2-c_3,b_1,b_2+1-c_2,b_3+1-c_3; \\ c_1,2-c_2,2-c_3;\end{matrix}\ x,y,z\right],\\ x^{1-c_1}z^{1-c_3}F_A^{(3)}\left[\begin{matrix} a+2-c_1-c_3,b_1+1-c_1,b_2,b_3+1-c_3; \\ 2-c_1,c_2,2-c_3;\end{matrix}\ x,y,z\right],\end{cases} \tag{2.9}$$

$$1\left\{x^{1-c_1}y^{1-c_2}z^{1-c_3}F_A^{(3)}\left[\begin{matrix} a+3-c_1-c_2-c_3,b_1+1-c_1,b_2+1-c_2,b_3+1-c_3; \\ 2-c_1,2-c_2,2-c_3;\end{matrix}\ x,y,z\right].\right. \tag{2.10}$$

It can also be shown by direct calculations that the functions (2.7) – (2.10) satisfy the system (2.6).

3. Fundamental solutions of a degenerate three-dimensional elliptic equation

Let (x,y,z) and (ξ,η,ζ) be two points of the domain Ω. We are looking for a solution of the equation (1.1) in the form

$$u=r^{-2\alpha-2\beta-2\gamma-1}\omega\left(\rho,\sigma,\theta\right), \tag{3.1}$$

where ω is a new unknown function,

$$\alpha=\frac{n}{2\left(n+2\right)},\ \ \beta=\frac{m}{2\left(m+2\right)},\ \ \gamma=\frac{k}{2\left(k+2\right)};\ \ q=\frac{n+2}{2},\ \ p=\frac{m+2}{2},\ \ l=\frac{k+2}{2};$$

$$\rho=-\frac{4x^q\xi^q}{q^2r^2},\ \sigma=-\frac{4y^p\eta^p}{p^2r^2},\ \theta=-\frac{4z^l\zeta^l}{l^2r^2},\ \ r^2=\frac{1}{q^2}\left(x^q-\xi^q\right)^2+\frac{1}{p^2}\left(y^p-\eta^p\right)^2+\frac{1}{l^2}\left(z^l-\zeta^l\right)^2.$$

It is obvious that

$$0<2\alpha<1,\ \ 0<2\beta<1,\ \ 0<2\gamma<1;\quad q>1,\ \ p>1,\ \ l>1.$$

Substituting (3.1) into equation (1.1), we obtain a system of differential equations of hypergeometric type

$$\begin{cases} \rho\left(1-\rho\right)\omega_{\rho\rho}-\rho\sigma\omega_{\rho\sigma}-\rho\theta\omega_{\rho\theta}+\\ \qquad+\left[2\alpha-\left(2\alpha+\beta+\gamma+\frac{3}{2}\right)\rho\right]\omega_\rho-\alpha\sigma\omega_\sigma-\alpha\theta\omega_\theta-\alpha\left(\alpha+\beta+\gamma+\frac{1}{2}\right)\omega=0,\\ \sigma\left(1-\sigma\right)\omega_{\sigma\sigma}-\rho\sigma\omega_{\rho\sigma}-\sigma\theta\omega_{\sigma\theta}+\\ \qquad+\left[2\beta-\left(\alpha+2\beta+\gamma+\frac{3}{2}\right)\sigma\right]\omega_\sigma-\beta\rho\omega_\rho-\beta\theta\omega_\theta-\beta\left(\alpha+\beta+\gamma+\frac{1}{2}\right)\omega=0,\\ \theta\left(1-\theta\right)\omega_{\theta\theta}-\rho\theta\omega_{\rho\theta}-\sigma\theta\omega_{\sigma\theta}+\\ \qquad+\left[2\gamma-\left(\alpha+\beta+2\gamma+\frac{3}{2}\right)\theta\right]\omega_\theta-\gamma\rho\omega_\rho-\gamma\sigma\omega_\sigma-\gamma\left(\alpha+\beta+\gamma+\frac{1}{2}\right)\omega=0.\end{cases} \tag{3.2}$$

Comparing the system (3.2) with the system (2.6) which has 8 particular solutions, we obtain [9]

$$q_0\left(x,y,z;\xi,\eta,\zeta\right)=k_0r^{-2\alpha-2\beta-2\gamma-1}F_A^{(3)}\left[\begin{matrix} 1/2+\alpha+\beta+\gamma,\alpha,\beta,\gamma; \\ 2\alpha,2\beta,2\gamma;\end{matrix}\ \rho,\sigma,\theta\right], \tag{3.3}$$

$$q_1(x,y,z;\xi,\eta,\zeta)=k_1x\xi r^{2\alpha-2\beta-2\gamma-3}F_A^{(3)}\left[\begin{matrix}3/2-\alpha+\beta+\gamma,1-\alpha,\beta,\gamma;\\2-2\alpha,2\beta,2\gamma;\end{matrix}\ \rho,\sigma,\theta\right],\tag{3.4}$$

$$q_{11}(x,y,z;\xi,\eta,\zeta)=k_{11}y\eta r^{-2\alpha+2\beta-2\gamma-3}F_A^{(3)}\left[\begin{matrix}3/2+\alpha-\beta+\gamma,\alpha,1-\beta,\gamma;\\2\alpha,2-2\beta,2\gamma;\end{matrix}\ \rho,\sigma,\theta\right],\tag{3.5}$$

$$q_{12}(x,y,z;\xi,\eta,\zeta)=k_{12}z\zeta r^{-2\alpha-2\beta+2\gamma-3}F_A^{(3)}\left[\begin{matrix}3/2+\alpha+\beta-\gamma,\alpha,\beta,1-\gamma;\\2\alpha,2\beta,2-2\gamma;\end{matrix}\ \rho,\sigma,\theta\right],\tag{3.6}$$

$$q_2(x,y,z;\xi,\eta,\zeta)=k_2xy\xi\eta r^{2\alpha+2\beta-2\gamma-5}F_A^{(3)}\left[\begin{matrix}5/2-\alpha-\beta+\gamma,1-\alpha,1-\beta,\gamma;\\2-2\alpha,2-2\beta,2\gamma;\end{matrix}\ \rho,\sigma,\theta\right],\tag{3.7}$$

$$q_{21}(x,y,z;\xi,\eta,\zeta)=k_{21}xz\xi\zeta r^{2\alpha-2\beta+2\gamma-5}F_A^{(3)}\left[\begin{matrix}5/2-\alpha+\beta-\gamma,1-\alpha,\beta,1-\gamma;\\2-2\alpha,2\beta,2-2\gamma;\end{matrix}\ \rho,\sigma,\theta\right],\tag{3.8}$$

$$q_{22}(x,y,z;\xi,\eta,\zeta)=k_{22}yz\eta\zeta r^{-2\alpha+2\beta+2\gamma-5}F_A^{(3)}\left[\begin{matrix}5/2+\alpha-\beta-\gamma,\alpha,1-\beta,1-\gamma;\\2\alpha,2-2\beta,2-2\gamma;\end{matrix}\ \rho,\sigma,\theta\right],\tag{3.9}$$

$$q_3(x,y,z;\xi,\eta,\zeta)=k_3xyz\xi\eta\zeta r^{2\alpha+2\beta+2\gamma-7}F_A^{(3)}\left[\begin{matrix}7/2-\alpha-\beta-\gamma,1-\alpha,1-\beta,1-\gamma;\\2-2\alpha,2-2\beta,2-2\gamma;\end{matrix}\ \rho,\sigma,\theta\right],\tag{3.10}$$

where k_0,..., k_3 are constants, which are determined when solving boundary value problems for the equation (1.1).

It is easy to see that the each of three particular solutions q_1, q_{11} and q_{12} are symmetrical to each other with respect to the numerical parameters of the Lauricella function. Hence, in further studies there is no need to consider the functions q_{11} and q_{12}, i.e. we omit them and study only the function q_1. Similar propositions can be made about the second trio of particular solutions q_2, q_{21} and q_{22}: we study only q_2, and omit the functions q_{21} and q_{22}.

It is easy to see that the constructed functions q_0, q_1, q_2 and q_3 have the following properties:

$$\left.\frac{\partial}{\partial x}q_0\right|_{x=0}=0,\ \left.\frac{\partial}{\partial y}q_0\right|_{y=0}=0,\ \left.\frac{\partial}{\partial z}q_0\right|_{z=0}=0;\quad \left.q_1\right|_{x=0}=0,\ \left.\frac{\partial}{\partial y}q_1\right|_{y=0}=0,\ \left.\frac{\partial}{\partial z}q_1\right|_{z=0}=0,$$

$$\left.q_2\right|_{x=0}=0,\ \left.q_2\right|_{y=0}=0,\ \left.\frac{\partial}{\partial z}q_2\right|_{z=0}=0;\quad \left.q_3\right|_{x=0}=0,\ \left.q_3\right|_{y=0}=0,\ \left.q_3\right|_{z=0}=0.$$

Note, these properties will be used in solving four (Neumann, two Dirichlet-Neumann and Dirichlet) boundary value problems for the equation (1.1).

Lemma 3.1. *If* $0<2\alpha,2\beta,2\gamma<1$*, then every function* q_k *(*$k=\overline{0,3}$*) has a singularity of order* $\frac{1}{r}$ *as* $r\to0$.

Proof. To give an example, we consider function q_0. The order of singularity of the remaining functions is determined similarly.

In the case of three variables, the transformation formula (2.4) takes the form

$$F_A^{(3)}\left[\begin{matrix}a,b_1,b_2,b_3;\\c_1,c_2,c_3;\end{matrix}\ x,y,z\right]=(1-x-y-z)^{-a}\times$$
$$\times F_A^{(3)}\left[\begin{matrix}a,c_1-b_1,c_2-b_2,c_3-b_3;\\c_1,c_2,c_3;\end{matrix}\ \frac{x}{x+y+z-1},\frac{y}{x+y+z-1},\frac{z}{x+y+z-1}\right].\tag{3.11}$$

Using the transformation formula (3.11), the function q_0 defined in (3.3) can be reduced to the form

$$q_0(x,y,z;\xi,\eta,\zeta)=\frac{1}{r}\cdot q_0^*(x,y,z;\xi,\eta,\zeta),\tag{3.12}$$

where

$$q_0^*(x,y,z;\xi,\eta,\zeta)=k_0\varrho^{-2\alpha-2\beta-2\gamma}F_A^{(3)}\left[\begin{matrix}\alpha+\beta+\gamma+1/2,\alpha,\beta,\gamma;\\2\alpha,\ 2\beta,\ 2\gamma;\end{matrix}\ \frac{4x^q\xi^q}{q^2\varrho^2},\frac{4y^p\eta^p}{p^2\varrho^2},\frac{4z^l\zeta^l}{l^2\varrho^2}\right],\tag{3.13}$$

$$\varrho^2 = \frac{1}{q^2}\left(x^q+\xi^q\right)^2 + \frac{1}{p^2}\left(y^p+\eta^p\right)^2 + \frac{1}{l^2}\left(z^l+\zeta^l\right)^2.$$

We must show that the value of $q_0^*\left(x,y,z;\xi,\eta,\zeta\right)$ as $r\to 0$, i.e. $x\to\xi$, $y\to\eta$, $z\to\zeta$, is bounded.

According to the theory of Lauricella hypergeometric functions [3, Chap. VII], if the sum of the absolute values of the variables is less than one, then the function $F_A^{(n)}$ is bounded for any values of the numerical parameters. In the case of three variables, this statement looks like

$$\left|F_A^{(3)}\left[\begin{matrix} a, b_1, b_2, b_3; \\ c_1, c_2, c_3; \end{matrix}\; x, y, z\right]\right| < \infty, |x|+|y|+|z|<1. \tag{3.14}$$

By virtue of (3.14), it is obvious that in (3.13):

$$\frac{4x^q\xi^q}{q^2\varrho^2} + \frac{4y^p\eta^p}{p^2\varrho^2} + \frac{4z^l\zeta^l}{l^2\varrho^2} < 1,$$

therefore the following inequality is true

$$\left|q_0^*\left(x,y,z;\xi,\eta,\zeta\right)\right| \le \frac{C}{R^{2\alpha+2\beta+2\gamma}}, \quad r\to 0, \tag{3.15}$$

where $C = const > 0$ and

$$R^2 = \frac{1}{q^2}x^{2q} + \frac{1}{p^2}y^{2p} + \frac{1}{l^2}z^{2l}. \tag{3.16}$$

Now from (3.12) and (3.15) follows that the function q_0 has a singularity of order $\frac{1}{r}$ as $r\to 0$. The Lemma 3.1 is proved. □

Based on the Lemma 3.1, we conclude that the particular solutions defined in (3.3) – (3.10) are fundamental solutions of the equation (1.1).

4. Statement of the Dirichlet problem and the uniqueness theorem

Dirichlet problem. Find a solution $u\left(x,y,z\right)$ of the equation (1.1) with the regularity $C\left(\overline{\Omega}\right)\bigcap C^2\left(\Omega\right)$ that satisfies the conditions

$$u\left(x,y,z\right)\big|_{z=0} = \tau_1\left(x,y\right), \quad 0\le x,y<\infty, \tag{4.1}$$

$$u\left(x,y,z\right)\big|_{y=0} = \tau_2\left(x,z\right), \quad 0\le x,z<\infty, \tag{4.2}$$

$$u\left(x,y,z\right)\big|_{x=0} = \tau_3\left(y,z\right), \quad 0\le y,z<\infty, \tag{4.3}$$

$$\lim_{R\to\infty} u\left(x,y,z\right) = 0, \tag{4.4}$$

where $\overline{\Omega} = \left\{(x,y,z):\, x\ge 0, y\ge 0,\, z\ge 0\right\}$; R is defined in (3.16); $\tau_1\left(y,z\right), \tau_2\left(x,z\right), \tau_3\left(x,y\right)$ are given continuous functions in a closed domain and have representations

$$\tau_1\left(x,y\right) = \frac{\tilde{\tau}_1\left(x,y\right)}{\left(1+\frac{1}{q^2}x^{2q}+\frac{1}{p^2}y^{2p}\right)^{\varepsilon_1}}, \quad \tilde{\tau}_1(x,y)\in C(0\le x,y<\infty), \tag{4.5}$$

$$\tau_2\left(x,z\right) = \frac{\tilde{\tau}_2\left(x,z\right)}{\left(1+\frac{1}{q^2}x^{2q}+\frac{1}{l^2}z^{2l}\right)^{\varepsilon_2}}, \quad \tilde{\tau}_2(x,z)\in C(0\le x,z<\infty), \tag{4.6}$$

$$\tau_3\left(y,z\right) = \frac{\tilde{\tau}_3\left(y,z\right)}{\left(1+\frac{1}{p^2}y^{2p}+\frac{1}{l^2}z^{2l}\right)^{\varepsilon_3}}, \quad \tilde{\tau}_3(y,z)\in C(0\le y,z<\infty), \tag{4.7}$$

where $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$ are a real numbers with $\alpha+\beta+\gamma<\varepsilon_1,\,\varepsilon_2,\,\varepsilon_3<2$.

In addition, the functions $\tau_1(x,y)$, $\tau_2(x,z)$ and $\tau_3(y,z)$ satisfy the matching conditions at the origin: $\tau_1(0,0)=\tau_2(0,0)=\tau_3(0,0)$ and at the lateral edges of the domain Ω:

$$\tau_1(x,0)=\tau_2(x,0),\ \tau_1(0,y)=\tau_3(y,0),\ \ \tau_2(0,z)=\tau_3(0,z),\ x,y,z\in\overline{\Omega}.$$

Theorem 4.1. *The Dirichlet problem can have at most one solution.*

Proof. To prove Theorem 4.1, it suffices to show that the corresponding homogeneous Dirichlet problem has a trivial solution. For this purpose, the finite part of the domain Ω, bounded by the planes $x=0, y=0,\ z=0$ and the sphere σ_0 :

$$\frac{1}{q^2}x^{2q}+\frac{1}{p^2}y^{2p}+\frac{1}{l^2}z^{2l}=R^2,\ x>0,\ y>0,\ z>0,$$

we denote by Ω_R. Let

$$\tau_1\left(y,z\right)=\tau_2\left(x,z\right)=\tau_3\left(x,y\right)=0. \tag{4.8}$$

Then the validity of Theorem 4.1 follows from the extremum principle for elliptic equations [17, p. 12]. Indeed, the function $u\left(x,y,z\right)$ in the domain $\bar{\Omega}_R$, by virtue of (4.8), can reach its positive maximum and negative minimum only at σ_0.

Let (x,y,z) be an arbitrary point in D_R. We take an arbitrary small number $\varepsilon>0$ and, considering (4.8), we choose R large enough that $\left|u\left(x,y,z\right)\right|<\varepsilon$ on σ_0. For R large enough, this point falls in D_R and therefore $\left|u\left(x,y,z\right)\right|<\varepsilon$. Since ε is arbitrary, we have $u\left(x,y,z\right)=0$. Then $u\left(x,y,z\right)\equiv 0$ in D. The Theorem 4.1 is proved. □

5. Existence of a solution to the Dirichlet problem

Consider a function

$$u\left(x,y,z\right)=\int\limits_0^\infty\int\limits_0^\infty t^n s^m \tau_1\left(t,s\right)\left.\frac{\partial}{\partial\zeta}q_3\left(x,y,z;t,s,\zeta\right)\right|_{\zeta=0}dtds+$$

$$+\int\limits_0^\infty\int\limits_0^\infty t^n s^k \tau_2\left(t,s\right)\left.\frac{\partial q_3}{\partial\eta}\right|_{\eta=0}dtds+\int\limits_0^\infty\int\limits_0^\infty t^m s^k \tau_3\left(t,s\right)\left.\frac{\partial q_3}{\partial\xi}\right|_{\xi=0}dtds, \tag{5.1}$$

where $q_3\left(x,y,z;\xi,\eta,\zeta\right)$ is a fundamental solution defined in (3.10). Applying a differential formula (2.5), from (5.1) we get the following function:

$$u\left(x,y,z\right)=u_1\left(x,y,z\right)+u_2\left(x,y,z\right)+u_3\left(x,y,z\right), \tag{5.2}$$

where

$$u_1\left(x,y,z\right)=k_3xyz\int\limits_0^\infty\int\limits_0^\infty\frac{\tau_1\left(t,s\right)t^{n+1}s^{m+1}}{r_1^{2\delta}}F_2\left[\begin{matrix}\delta,\,1-\alpha,1-\beta; \\ 2-2\alpha,\,2-2\beta;\end{matrix}\ -\frac{4x^qt^q}{q^2r_1^2},-\frac{4y^ps^p}{p^2r_1^2}\right]dtds, \tag{5.3}$$

$$u_2\left(x,y,z\right)=k_3xyz\int\limits_0^\infty\int\limits_0^\infty\frac{\tau_2\left(t,s;\right)t^{n+1}s^{k+1}}{r_2^{2\delta}}F_2\left[\begin{matrix}\delta,\,1-\alpha,1-\gamma; \\ 2-2\alpha,\,2-2\gamma;\end{matrix}\ -\frac{4x^qt^q}{q^2r_2^2},-\frac{4z^ls^l}{l^2r_2^2}\right]dtds, \tag{5.4}$$

$$u_3\left(x,y,z\right)=k_3xyz\int\limits_0^\infty\int\limits_0^\infty\frac{\tau_3\left(t,s;\right)t^{m+1}s^{k+1}}{r_3^{2\delta}}F_2\left[\begin{matrix}\delta,\,1-\beta,1-\gamma; \\ 2-2\beta,\,2-2\gamma;\end{matrix}\ -\frac{4y^pt^p}{p^2r_3^2},-\frac{4z^ls^l}{l^2r_3^2}\right]dtds, \tag{5.5}$$

$$k_3=\frac{1}{2\pi}q^{-2+2\alpha}p^{-2+2\beta}l^{-2+2\gamma}\frac{\Gamma\left(1-\alpha\right)\Gamma\left(1-\beta\right)\Gamma\left(1-\gamma\right)\Gamma\left(6-2\alpha-2\beta-2\gamma\right)}{\Gamma\left(2-2\alpha\right)\Gamma\left(2-2\beta\right)\Gamma\left(2-2\gamma\right)\Gamma\left(3-\alpha-\beta-\gamma\right)}, \tag{5.6}$$

$$\delta=\frac{7}{2}-\alpha-\beta-\gamma;\qquad r_1^2=\frac{1}{q^2}\left(x^q-t^q\right)^2+\frac{1}{p^2}\left(y^p-s^p\right)^2+\frac{1}{l^2}z^{2l},$$

$$r_2^2=\frac{1}{q^2}\left(x^q-t^q\right)^2+\frac{1}{p^2}y^{2p}+\frac{1}{l^2}\left(z^l-s^l\right)^2,\ \ r_3^2=\frac{1}{q^2}x^{2q}+\frac{1}{p^2}\left(y^p-t^p\right)^2+\frac{1}{l^2}\left(z^l-s^l\right)^2.$$

Here F_2 is Appell hypergeometric function defined in (2.2).

Lemma 5.1. *If the function* $\tau_1(x,y)$ *can be represented as* (4.5), *then the function* $u_1(x,y,z)$ *defined by equality* (5.3) *is a regular solution of equation* (1.1) *in the domain* Ω *satisfying the conditions* (4.4) *and*

$$u_1(x,y,0)=\tau_1(x,y),\ \ u_1(x,0,z)=0,\ \ u_1(0,y,z)=0. \tag{5.7}$$

Proof. First let us prove that the function (5.3) satisfies the degenerate elliptic equation (1.1). For this purpose, we consider the auxiliary function

$$W(x,y,z;t,s)=xyzr_1^{-2\delta}\omega(\vartheta,\varsigma), \tag{5.8}$$

where

$$\omega(\vartheta,\varsigma):=F_2\left[\begin{matrix}\delta,1-\alpha,1-\beta;\\ 2-2\alpha,\ 2-2\beta;\end{matrix}\ \vartheta,\varsigma\right],\ \ \vartheta=-\frac{4xt}{r_1^2},\ \varsigma=-\frac{4ys}{r_1^2}.$$

We calculate the necessary derivatives of the auxiliary function W with respect to the variables x, y, z and substitute them into the degenerate elliptic equation (1.1). As a result, we obtain the relation

$$\begin{gathered}y^mz^kW_{xx}+x^nz^kW_{yy}+x^ny^mW_{zz}=\\ =\vartheta yzr_1^{-2\mu}\left\{\vartheta(1-\vartheta)\omega_{\vartheta\vartheta}-\vartheta\varsigma\omega_{\vartheta\varsigma}+\left[2(1-\alpha)-(2-\alpha+\delta)\,\vartheta\right]\omega_\vartheta-(1-\alpha)\delta\omega\right\}\\ +x\varsigma zr_1^{-2\mu}\left\{\varsigma(1-\varsigma)\omega_{\varsigma\varsigma}-\vartheta\varsigma\omega_{\vartheta\varsigma}+\left[2(1-\beta)-(2-\beta+\delta)\varsigma\right]\omega_\varsigma-(1-\beta)\delta\omega\right\}=0,\end{gathered}$$

which is equivalent to the following system of hypergeometric equations

$$\begin{cases}\vartheta(1-\vartheta)\omega_{\vartheta\vartheta}-\vartheta\varsigma\omega_{\vartheta\varsigma}+\left[2(1-\alpha)-(2-\alpha+\delta)\,\vartheta\right]\omega_\vartheta-(1-\alpha)\delta\omega=0,\\ \varsigma(1-\varsigma)\omega_{\varsigma\varsigma}-\vartheta\varsigma\omega_{\vartheta\varsigma}+\left[2(1-\beta)-(2-\beta+\delta)\varsigma\right]\omega_\varsigma-(1-\beta)\delta\omega=0.\end{cases}$$

Comparing the last system of equations with the system of equations (2.3) for the Appell function F_2, we can conclude that the function (5.8) is a solution of the corresponding degenerate elliptic equation. Consequently, the function $u_1(x,\ y,\ z)$ defined by (5.3) satisfies the degenerate elliptic equation (1.1).

Now we prove that the function $u_1(x,y,z)$ satisfies the boundary conditions (5.7). Indeed, introducing in the integrand in (5.3) instead of t and s new variables

$$\mu=\frac{l\left(t^q-x^q\right)}{qz^l},\nu=\frac{l\left(s^p-y^p\right)}{pz^l},$$

we obtain

$$\begin{gathered}u_1(x,y,z)=l^{2\delta-2}k_3xyz^{2l(\alpha+\beta-2)}\int\limits_{-\frac{lx^q}{qz^l}}^{\infty}\int\limits_{-\frac{ly^p}{pz^l}}^{\infty}\frac{\left(x^q+\mu qz^l/l\right)\left(y^p+\nu qz^l/l\right)}{\left(1+\mu^2+\nu^2\right)^\delta}\times\\ \times F_2\left[\begin{matrix}\delta,\ 1-\alpha,1-\beta;\\ 2-2\alpha,\ 2-2\beta;\end{matrix}\ -\frac{4l^2x^q\left(x^q+\mu qz^l/l\right)}{q^2z^{2l}\left(1+\mu^2+\nu^2\right)},-\frac{4l^2y^p\left(y^p+\nu pz^l/l\right)}{p^2z^{2l}\left(1+\mu^2+\nu^2\right)}\right]\times\\ \times\tau_1\left[\left(x^q+\mu qz^l/l\right)^{1/q},\left(y^p+\nu pz^l/l\right)^{1/p}\right]d\mu d\nu.\end{gathered}$$

Taking the expression (5.6) into account for the coefficient k_3, considering the well-known formula for calculating the double improper integral [8, p. 633, Eq. 4.623]

$$\int\limits_0^\infty\int\limits_0^\infty\varphi\left(a^2x^2+b^2y^2\right)dxdy=\frac{\pi}{4ab}\int\limits_0^\infty\varphi(x)dx$$

and Legendre's duplication formula [6, p. 5, Eq. 1.2(15)],

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

we obtain

$$\lim_{z\to 0} u_1(x,y,z) = \tau_1(x,y). \tag{5.9}$$

Using the similar transformations, we have

$$\lim_{x\to 0} u_1(x,y,z) = 0, \ \ \lim_{y\to 0} u_1(x,y,z) = 0. \tag{5.10}$$

Therefore, based on equalities (5.9) and (5.10) we conclude that the function $u_1(x,y,z)$, defined by (5.3), satisfies conditions (5.7).

Let us show that if given function τ_1 has representation (4.5), then the function $u_1(x,y,z)$ defined in (5.3) tends to zero at infinity.

Using the transformation formula for Appell function F_2 [6, p. 240, Eq. 5.11(8)]

$$F_2\left[\begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} x, y\right] = (1-x-y)^{-a} F_2\left[\begin{matrix} a, c_1-b_1, c_2-b_2; \\ c_1, c_2; \end{matrix} \frac{x}{x+y-1}, \frac{y}{x+y-1}\right],$$

we write the function (5.3) in the form

$$u_1(x,y,z) = k_3 xyz \int\limits_0^\infty \int\limits_0^\infty \frac{\tau_1(t,s)\, t^{n+1} s^{m+1}}{\rho^{2\delta}} F_2\left[\begin{matrix} \delta, 1-\alpha, 1-\beta; \\ 2-2\alpha,\, 2-2\beta; \end{matrix} \frac{4x^q t^q}{q^2\rho^2}, \frac{4y^p s^p}{p^2\rho^2}\right] dtds, \tag{5.11}$$

where

$$\rho^2 = \frac{1}{q^2}(x^q+t^q)^2 + \frac{1}{p^2}(y^p+s^p)^2 + \frac{1}{l^2} z^{2l}.$$

It is easy to see that in (5.11) the following inequality holds

$$\frac{4x^q t^q}{q^2\rho^2} + \frac{4y^p s^p}{p^2\rho^2} < 1, \ \ x>0, y>0, z>0, t>0, s>0.$$

Let us prove that when the point (x,y,z) tends to infinity, i.e. when $R\to\infty$, the function (5.11) tends to zero. It known from the theory of Appell functions [2], that, if $|x|+|y|<1$, then for any values of the numerical parameters the Appell hypergeometric function F_2 is bounded:

$$|\ F_2(a, b_1, b_2; c_1, c_2; x, y)\ | \le C_1, \ \ |x|+|y|<1.$$

Next, applying the representation (4.5) for given function $\tau_1(x,y)$, we obtain

$$|u_1| \le C_2 xyz \int\limits_0^\infty \int\limits_0^\infty \frac{t^{n+1} s^{m+1} dtds}{\left(1+\frac{1}{q^2}t^{2q}+\frac{1}{p^2}s^{2p}\right)^{\varepsilon_1} \left[\frac{1}{q^2}(x^q+t^q)^2+\frac{1}{p^2}(y^p+s^p)^2+\frac{1}{l^2}z^{2l}\right]^{7/2-\alpha-\beta-\gamma}}. \tag{5.12}$$

Substituting t and s for

$$\mu = \frac{1}{qR} t^q, \ \ \nu = \frac{1}{pR} s^p$$

in the last double improper integral (5.12), we get

$$|u_1| \le \frac{qpC_3}{R^{2\varepsilon_1-2\alpha-2\beta-2\gamma}} \cdot \frac{x}{R} \cdot \frac{y}{R} \cdot \frac{z}{R} \cdot K(x,y;R), \tag{5.13}$$

where $\varepsilon_1 > \alpha+\beta+\gamma$ (see condition in (4.5)) and

$$K(x,y;R) = \int\limits_0^\infty \int\limits_0^\infty \frac{\mu\nu d\mu d\nu}{\left(\mu^2+\nu^2+\frac{1}{R^2}\right)^{\varepsilon_1} \left(1+\mu^2+\nu^2+\frac{2x^q}{qR}+\frac{2y^p}{pR}\right)^{7/2-\alpha-\beta-\gamma}}. \tag{5.14}$$

It is easy to show that the double improper integral on the right-hand side (5.14) is bounded as $R\to\infty$. Indeed, using the formula [7]

$$\underbrace{\int\limits_0^{+\infty}...\int\limits_0^{+\infty}}_{n} \frac{x_1^{p_1-1}...x_n^{p_n-1}dx_1...dx_n}{\left[(r_1x_1)^{q_1}+...+(r_nx_n)^{q_n}\right]^t\left[1+(r_1x_1)^{q_1}+...+(r_nx_n)^{q_n}\right]^s} =$$

$$= \frac{\Gamma\left(p_1/q_1\right)...\Gamma\left(p_n/q_n\right)\Gamma\left(P-t\right)\Gamma\left(s+t-P\right)}{q_1q_2...q_nr_1^{p_1q_1}...r_n^{p_nq_n}\Gamma\left(P\right)\Gamma\left(s\right)}, \ P:=\frac{p_1}{q_1}+...+\frac{p_n}{q_n},$$

where p_k, q_k, r_k and s are positive numbers $(k=\overline{1,n})$, $0<P-t<s$, and passing in (5.14) to the limit as $R\to\infty$, we obtain

$$\lim_{R\to\infty} K\left(x,y;R\right)=\frac{\Gamma\left(2-\varepsilon_1\right)\Gamma\left(3/2-\alpha-\beta-\gamma+\varepsilon_1\right)}{4\Gamma\left(7/2-\alpha-\beta-\gamma\right)}, \ \alpha+\beta+\gamma<\varepsilon_1<2. \tag{5.15}$$

Thus, by virtue of (5.13) and (5.15) the following estimate is valid:

$$|u_1|\leq\frac{C_4}{R^{2(\varepsilon_1-\alpha-\beta-\gamma)}}, \alpha+\beta+\gamma<\varepsilon_1<2, \ R\to\infty. \tag{5.16}$$

Considering (5.16), we conclude that the function (5.3) vanishes at infinity. Lemma 5.1 is proved. □

Remark 5.2. Repeating the arguments given in Lemma 5.1, one can prove two lemmas concerning the functions $u_2(x,y,z)$ and $u_3(x,y,z)$ defined by equalities (5.4) and (5.5), respectively. Thus, if the representations (4.6) and (4.7) are valid for the given functions $\tau_2(x,z)$ and $\tau_3(y,z)$, then each of the functions $u_2(x,y,z)$ and $u_3(x,y,z)$ is a solution to the degenerate elliptic equation (1.1) that vanishes at infinity and satisfies the set of conditions

$$u_2(x,y,0)=0, \quad u_2(x,0,z)=\tau_2(x,z), \quad u_2(0,y,z)=0,$$

$$u_3(x,y,0)=0, \quad u_3(x,0,z)=0, \quad u_3(0,y,z)=\tau_3(y,z),$$

respectively.

Theorem 5.3. *If given functions $\tau_1(x,y)$, $\tau_2(x,z)$ and $\tau_3(y,z)$ have the representations* (4.5), (4.6) *and* (4.7), *respectively, then the function $u(x,y,z)$ defined in* (5.2) *is a regular solution of the equation* (1.1) *in the domain Ω satisfying the conditions* (4.1) – (4.4).

Proof of Theorem 5.3 follows from Lemma 5.1 and Remark 5.2.

References

[1] Amanov D. Some boundary value problems for a degenerate elliptic equation in an unbounded domain. Izv.AN UzSSR, Ser.Fiz.-Mat.Nauki, –1984. –1.– P. 8 – 13.

[2] Appell P. Sur les séries hypergéométriques de deux variables, et sur des équations différentielles linéaires aux dérivées partielles C.R. Acad. Sci., Paris, – 1880.–90. – P. 296 – 298.

[3] Appell P. and Kampe de Feriet J. Fonctions Hypergeometriques et Hyperspheriques; Polynomes d'Hermite, Gauthier - Villars. Paris. – 1926.

[4] Bers L. Mathematical Aspects of Subsonic and Transonic Gas Dynamics Wiley New York. – 1958.

[5] Candelas P., de la Ossa X., Greene P., Parkes L., A pair of Calabi-Yau manifolds as an exactly soluble super conformal theory. Nucl. Phys., – 1991. – B539. – 21 – 74.

[6] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G. Higher Transcendental Functions 1, McGraw-Hill, New York, Toronto, London. – 1953.

[7] Ergashev T. G., Tulakova Z. R. The Neumann problem for a multidimensional elliptic equation with several singular coefficients in an infinite domain. Lobachevskii Journal of Mathematics,– 2022. – 43(1). – P. 199 – 206.

[8] Gradshteyn I. S., Ryzhik I. M Table of integrals, series, and products Academic Press Amsterdam. – 2007.

[9] Hasanov A., Karimov E. T. Fundamental solutions for a class of three-dimensional elliptic equations with singular coefficients. Applied Mathematics Letters,– 2009. – 22. – P. 1828 – 1832.

[10] Karimov E.T. A boundary value problem for 3D elliptic equation with singular coefficients. Progress in analysis and its applications, – 2010. P. 619 – 625.

[11] Karimov E. T. On the Dirichlet problem for a three-dimensional elliptic equation with singular coefficients. Dokl.AN Uz, – 2010. – 2. – P. 9 – 11.

[12] Karimov K. T. Nonlocal problem for an elliptic equation with singular coefficients in a semi-infinite parallelepiped. Lobachevskii Journal of Mathematics, 2020. – 41(1). – P. 46 – 57.

[13] Karimov K. T. Boundary value problems in a semi-infinite parallelepiped for an elliptic equation with three singular coefficients. Lobachevskii Journal of Mathematics, – 2021.– 42(3).– P. 560 – 571.

[14] Korenev B.G. Introduction to the theory of Bessel functions. Nauka, Moscow, 1971 (in Russian).

[15] Lauricella G. Sulle funzioni ipergeometriche a piu variabili Rend. Circ. Mat. Palermo, – 1893. – 7. – P. 111 – 158.

[16] Mamayev N.V., Lukin A.S., Yurin D.V., Glazkova M.A., Sinitsin V.E. Algorithm of nonlocal mean based on decompositions via Hermite functions in problems of computer tomography. Proceedings of the 23rd Inter. Conf. on Comp. Graphics and Vision GraphiCon2013, Vladivostok, Russia. (2013) Sept 1620, P. 254–258 (in Russian).

[17] Miranda C. Partial Differential Equations of Elliptic Type Berlin Springer. – 1970.

[18] Niukkanen A. W. Generalised hypergeometric series arising in physical and quantum chemical applications. J. Phys. A: Math. Gen., – 1983. – 16.– P. 1813 – 1825.

[19] Salakhitdinov M. C., Hasanov A. Tricomi problem for a mixed type equation with a non-smooth degeneracy line. Diff. Uravn., – 1983. – 19(1).– P. 110–119.

[20] Smart U.M., Celestial mechanics. Longmans, Green and Co, London - New York - Toronto, 1953

[21] Srivastava H. M. Karlsson P. W. Multiple Gaussian hypergeometric series New York, Chichester, Brisbane and Toronto Halsted Press (Ellis Horwood Limited, Chichester), Wiley. – 1985.

[22] Tulakova Z.R. Spatial mixed problems and Neumann problem for the three-dimensional elliptic equation with the two singular coefficients. Uzbek Math. Journal, – 2024. – 68(3). P. 150–157.

Hasanov A. ,
Department of Differential Equations and Applications, V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan
Department of Mathematics, Analysis, Logic and Discrete Mathematics, Ghent University, Belgium
email: anvarhasanov@yahoo.com
Ergashev T.G. ,
Department of Differential Equations and Applications, V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan
Department of Higher Mathematics, National Research University "TIIAME" Tashkent, Uzbekistan
Department of Mathematics, Analysis, Logic and Discrete Mathematics, Ghent University, Belgium
email: ergashev.tukhtasin@gmail.com
Djuraev N.,
Department of Higher Mathematics, Karshi Engineering Economics Institute, Karshi, Uzbekistan
email: norqul.djurayev@mail.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.83-92
DOI: 10.29229/uzmj.2025-3-8

Optimal pursuit game of two pursuers and one evader with the Grönwall type constraints on controls

Ibragimov G., Yusupov I.

Abstract. We study a differential game of two pursuers and one evader, whose dynamics are described by linear differential equations, in $\mathbb{R}^n$. The control functions of pursuers and evader are subjected to the Grönwall type constraints. The game is said to be completed when the state of the evader coincides with the state of any pursuer. The pursuers aim to complete game as soon as possible, while the evader tries to either evade capture or prolong the time until capture. We construct optimal strategies of players and find an equation for the optimal pursuit time.

Keywords: Differential game, Grönwall constraints, pursuer, evader, optimal pursuit time, optimal strategy

MSC (2020): 91A23, 49N70, 49J15

1. Introduction

Isaacs used the concept of differential games for the first time [18]. Then many researchers such as Azamov [2], Blagodatskikh [6], Pashkov et al. [21], Petrosjan [22], Pshenichnii [23], Subbotin [25] and others developed differential games theory.

Differential games of many players are considered one of the most important current discussions (see for example, [3, 7, 8, 9, 10, 12, 13, 14, 19, 20, 23, 26, 27, 28, 29]). Most of the literature considers differential games when the pursuers move faster than the evader to complete the game. The paper [8] is devoted to simultaneous multiple capture of the rigidly coordinated evaders by several pursuers.

Optimal pursuit-evasion games are another difficult and significant branch of differential games (for instance, see [1], [5], [15, 16, 17, 20]). To construct optimal strategies for players and to find optimal pursuit-evasion times are the main problems for such games. In paper [15], the optimal pursuit game described by infinite system of differential equations is considered. The papers [16, 17] by Ibragimov, investigate the optimal pursuit differential games of many pursuers. In the games, optimal strategies of players were constructed.

The differential game problems are studied usually under geometric, integral and mixed constraints. However, in the recent works on differential games, control functions are subjected to Grönwall type constraints [1, 4, 24]. The papers by Samatov et al. [4, 24] were considered optimal pursuit-evasion and "Life line" differential games of a pursuer and an evader with Grönwall type constraints.

In the paper [1], a simple motion differential game of two pursuers and one evader is studied. The players' control functions are subject to the Gronwall-type constraints. While the present paper is devoted to a linear differential game of optimal pursuit of two pursuers and one evader when the control functions of players are subjected to Grönwall type constraints. We find optimal pursuit time in terms of reachability sets and construct optimal strategies of players. To prove the main theorem, we consider an auxiliary differential game in a half plane.

1.1. Statement of problem.

Let the dynamics of two pursuers x_1, x_2 and one evader y be described in $\mathbb{R}^n$ by the following differential equations:

$$\begin{array}{ll} \dot{x}_i = ax_i + u_i, & x_i(0) = x_{i0},\ i = 1, 2, \\ \dot{y} = ay + v, & y(0) = y_0, \end{array} \tag{1.1}$$

where $x_i, y, x_{i0}, y_0 \in \mathbb{R}^n$, u_i and v stand for the control parameters of the i-th pursuer x_i, $i = 1, 2$, and evader y, respectively, a is given positive number.

Definition 1.1. Measurable functions $u_i(t) = (u_{i1}(t), u_{i2}(t))$ and $v(t) = (v_1(t), v_2(t))$, $t \geq 0$, that satisfy the following constraints

$$|u_i(t)|^2 \leq \rho_i{}^2 + 2k \int_0^t |u_i(s)|^2 ds, \quad t \geq 0, \tag{1.2}$$

$$|v(t)|^2 \leq \sigma^2 + 2k \int_0^t |v(s)|^2 ds, \quad t \geq 0, \tag{1.3}$$

are called admissible controls of pursuers x_i, $i = 1, 2$, and evader y, respectively, where ρ_1, ρ_2, σ ($\rho_i > \sigma$ i=1,2) and k are given positive numbers.

We let $\mathbb{U}_1$, $\mathbb{U}_2$ and $\mathbb{V}$ denote the set of all admissible controls of pursuers x_1, x_2, and evader y, respectively.

The trajectories of pursuers and evader corresponding to admissible controls $u_i(\cdot)$ and $v(\cdot)$ are defined by the following equations

$$x_i(t) = x_{i0}e^{at} + \int_0^t e^{a(t-s)} u_i(s) ds, \ i = 1, 2, \quad y(t) = y_0 e^{at} + \int_0^t e^{a(t-s)} v(s) ds, \tag{1.4}$$

respectively. We need the following statement.

Lemma 1.2. [11] *If, for the positive numbers ρ and k,*

$$|\omega(t)|^2 \leq \rho^2 + 2k \int_0^t |\omega(s)|^2 ds,$$

then $|\omega(t)| \leq \rho e^{kt}$, where $\omega(t)$, $t \geq 0$, is a measurable function.

By Lemma 1.2, for the admissible controls $u_i(\cdot) \in \mathbb{U}$ and $v(\cdot) \in \mathbb{V}$, we have

$$|u_i(t)| \leq \rho_i e^{kt}, \quad |v(t)| \leq \sigma e^{kt}, \quad t \geq 0. \tag{1.5}$$

It should be noted that (2.2) doesn't imply (1.2) and (2.1). It is not difficult to verify that if

$$|u_i(t)| = \rho_i e^{kt}, \quad |v(t)| = \sigma e^{kt}, \quad t \geq 0, \tag{1.6}$$

then equations (1.2) and (2.1) are satisfied, respectively.

Next, we give definitions for the optimal strategies of players and optimal pursuit time.

1.2. **Guaranteed pursuit time.** Let $H(x, r)$ (respectively, $S(x, r)$) denote the ball (sphere) of radius r and centered at x, and let O be the origin.

Definition 1.3. We call the function

$$U_i(x_{i0}, y_0, t, v), \quad U_i : \mathbb{R}^2 \times \mathbb{R}^2 \times [0, \infty) \times H(O, \sigma e^{kt}) \to H(O, \rho_i e^{kt}), \quad i \in \{1, 2\},$$

strategy of the pursuer x_i, if for any $v(\cdot) \in \mathbb{V}$ and for $u_i = U_i(x_{i0}, y_0, t, v(t))$, the initial value problem (1.1) has a unique solution $(x_i(t), y(t))$, and

$$|U_i(x_{i0}, y_0, t, v(t))|^2 \leq \rho_i^2 + 2k \int_0^t |U_i(x_{i0}, y_0, s, v(s))|^2 ds, \ t \geq 0.$$

In other words, the pursuer x_i uses information about the initial states x_{i0}, y_0, and the value of the control parameter $v(t)$ at the current time t.

Definition 1.4. We say that the strategies $U_i = U_i(x_{i0}, y_0, t, v(t))$, $i = 1, 2$, ensures the completion of the game for the time $T(U_1, U_2)$ if, for any $v(\cdot) \in \mathbb{V}$, we have $x_i(\tau) = y(\tau)$ for some $i \in \{1, 2\}$ and $\tau \in [0, T(U_1, U_2)]$, where $(x_1(t), x_2(t), y(t))$ is the solution of initial value problem (1.1) with $u_i = U_i(x_{i0}, y_0, t, v(t))$, $i = 1, 2$.

We call the number $T(U_1, U_2)$ a guaranteed pursuit time. It should be noted that any time T', $T' \geq T(U_1, U_2)$, is also a guaranteed pursuit time corresponding to the same strategies U_1, U_2. Let $T^*(U_1, U_2)$ denote the infimum of the numbers $T(U_1, U_2)$ corresponding to the strategies U_1, U_2.

The pursuers try to minimize $T^*(U_1, U_2)$ by choosing their strategies U_1, U_2, and the evader tries to maximize $T^*(U_1, U_2)$ by choosing $v(\cdot) \in \mathbb{V}$. If, for some strategies U_{10}, U_{20} of pursuers, $\inf\limits_{U_1, U_2} T^*(U_1, U_2) = T^*(U_{10}, U_{20})$, then U_{10}, U_{20} are called optimal strategies of pursuers and the number $T^*(U_{10}, U_{20})$ is called a guaranteed pursuit time in the game.

1.3. **Guaranteed evasion time.**

Definition 1.5. A continuous function

$$V(x_{10}, x_{20}, y_0, t, x_1, x_2, y), \quad V : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times [0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \ H(O, \sigma e^{kt}),$$

is called a strategy of the evader if, for any $u_i(\cdot) \in \mathbb{U}_i$, $i = 1, 2$, and for $v = V(x_{10}, x_{20}, y_0, t, x_1, x_2, y)$, the initial value problem (1.1) has a unique solution $(x_1(t), x_2(t), y(t))$ and along this solution

$$|V(x_{10}, x_{20}, y_0, t, x_1(t), x_2(t), y(t))|^2 \leq \sigma^2 + 2k \int_0^t |V(x_{10}, x_{20}, y_0, s, x_1(s), x_2(s), y(s))|^2 ds, \ t \geq 0.$$

Definition 1.6. We say that the strategy V guarantees the evasion on the interval of time $[0, T(V))$ if, for any $u_i(\cdot) \in \mathbb{U}_i$, $i = 1, 2$, we have $x_i(t) \neq y(t)$, for all $i = 1, 2$ and $t \in [0, T(V))$. We let $T_*(V)$ denote the supremum of the numbers $T(V)$ corresponding to the strategy V. Also, we call the number $T_*(V)$ a guaranteed evasion time corresponding to the strategy V.

The evader tries to maximize the number $T_*(V)$ by choosing the strategy V, and the pursuers try to minimize the number $T_*(V)$ by choosing the controls $u_i(\cdot) \in \mathbb{U}_i$, $i = 1, 2$.

Definition 1.7. If for some strategy V_0 of the evader $\sup\limits_V T_*(V) = T_*(V_0)$, then V_0 is called optimal strategy of the evader, and the number $T_*(V_0)$ is called a guaranteed evasion time in the game. If $T^*(U_{10}, U_{20}) = T_*(V_0)$, then this number is called optimal pursuit time in the game (1.1).

Problem. Construct optimal strategies U_{10}, U_{20} of the pursuers and that V_0 of evader, and find the optimal pursuit time in game (1.1).

2. Main result

In this section, we demonstrate the main result of the paper.

Theorem 2.1. *The number*

$$\theta = \min \left\{ t \geq 0 \mid H(y_0 e^{at}, r(t)) \subset H(x_{10} e^{at}, R_1(t)) \cup H(x_{20} e^{at}, R_2(t)) \right\} \tag{2.1}$$

is the optimal pursuit time in game (1.1).

Let $\xi_i = x_{i0} + \int\limits_0^t e^{-as} u(s) ds$ and $\eta = y_i + \int\limits_0^t e^{-as} v(s) ds$. By (1.4) the equality $\xi_i(t) = \eta(t)$ is equivalent to $x_i(t) = y(t)$. Therefore, it is sufficient for us to show that $\xi_i(t) = \eta(t)$. Therefore, we consider the following game instead of (1.1):

$$\begin{aligned} &\dot{\xi}_i = e^{-at} u_i, \quad \xi_i(0) = x_{i0}, \ i = 1, 2, \\ &\dot{\eta} = e^{-at} v, \quad\ \ \eta(0) = y_0, \end{aligned} \tag{2.2}$$

where $\xi_i, \eta, x_{i0}, y_0 \in \mathbb{R}^2$, u_i and v stand for the control parameters of the i-th pursuer ξ_i, $i = 1, 2$, and evader η, respectively.

Let

$$R_i(t) = \rho_i \int_0^t e^{(k-a)s} ds, \; i = 1, 2, \; r(t) = \sigma \int_0^t e^{(k-a)s} ds.$$

It's easy to establish that the set of all points the pursuer ξ_i (and respectively the evader η) can reach from the starting point x_{i0} (respectively y_0) at time $t = 0$ up to time t forms the ball $H(x_{i0}, R_i(t))$ (and respectively, $H(y_0, r(t))$).

2.1. **A differential game in the half plane.** To prove the theorem, we examine an auxiliary differential game involving a single pursuer x and a single evader y, governed by the following equations:

$$\begin{aligned} &\dot{\xi} = e^{-at}u, \quad \xi(0) = x_0 = (x_{10}, x_{20}), \quad |u(t)|^2 \le \rho^2 + 2k\int_0^t |u(s)|^2 ds, \quad t \ge 0, \\ &\dot{\eta} = e^{-at}v, \quad \eta(0) = y_0 = (y_{10}, y_{20}), \quad |v(t)|^2 \le \sigma^2 + 2k\int_0^t |v(s)|^2 ds, \quad t \ge 0. \end{aligned} \tag{2.3}$$

Assume that $\rho > \sigma$. Define $R(t) = \rho \int_0^t e^{(k-a)s} ds$, and suppose the circumferences $S(x_0, R(\theta_0))$ and

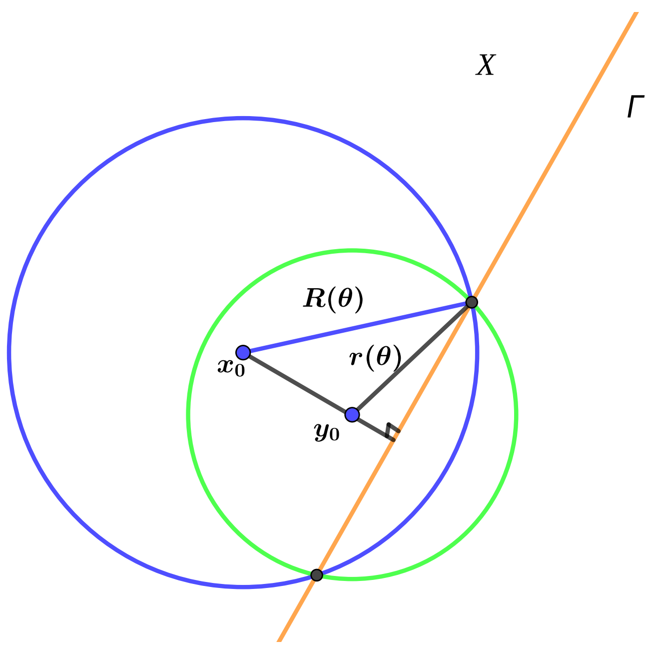


FIGURE 1. Game in the half plane X.

$S(y_0, r(\theta_0))$ intersect for some $\theta_0 > 0$. We pass a straight line Γ perpendicular to the vector $y_0 - x_0$ through the intersection points of these circumferences (see Figure 1). We label the half-plane bounded by Γ and containing the point x_0 by X. Note that the half-plane X may not include the point y_0. It is assumed that the evader must be within the half-plane X at the time θ_0, while the pursuer aims to achieve the condition $\xi(t) = \eta(t)$ as soon as possible.

Lemma 2.2. *If the position of the evader $y(\theta_0)$ belongs to X, then θ_0 is the guaranteed pursuit time in game* (2.3).

Proof. The pursuer uses following strategy

$$u = v - (v, e)e + e\sqrt{(\rho^2 - \sigma^2)e^{2kt} + (v, e)^2}, \quad e = \frac{y_0 - x_0}{|y_0 - x_0|}. \tag{2.4}$$

We assume, without loss of generality, that X is the upper half-plane bounded by the x-axis. Consequently, we have $x_{10} = y_{10}$. It can be easily shown that

$$\theta_0 = \frac{1}{k-a} \ln \left(1 + (k-a)\sqrt{\frac{x_{20}^2 - y_{20}^2}{\rho^2 - \sigma^2}} \right).$$

Admissibility of strategy (3.1) can be easily verified from the obvious equation: $|u(t)|^2 = (\rho^2 - \sigma^2)\, e^{2kt} + |v(t)|^2$ and (1.2).

Since $y_0 - x_0$ is perpendicular to the x-axis, $e = \frac{y_0 - x_0}{|y_0 - x_0|} = (0, -1)$, and consequently, strategy (3.1) is simplified to the form

$$u_1 = v_1, \quad u_2 = -\sqrt{(\rho^2 - \sigma^2)\, e^{2kt} + v_2^2}. \tag{2.5}$$

The condition $\eta(\theta_0) \in X$ can be written as follows

$$\int_0^{\theta_0} e^{-as} v_2(s)ds \geq -y_{20}. \tag{2.6}$$

By (3.2) $u_1(t) = v_1(t)$, $t \geq 0$, and hence $\xi_1(t) = \eta_1(t)$, for all $t \geq 0$. Therefore, it suffices to show that $\xi_2(\tau) = \eta_2(\tau)$ at some τ, $0 < \tau \leq \theta_0$. To this end, we consider the following vector function $f(t) = \left(\sqrt{\rho^2 - \sigma^2} e^{(k-a)t}, e^{-at} v_2(t)\right)$, $t \geq 0$. Then

$$\begin{aligned} \xi_2(\theta_0) - \eta_2(\theta_0) &= x_{20} - y_{20} - \int_0^{\theta_0} e^{-as}\sqrt{(\rho^2 - \sigma^2)e^{2ks} + v_2^2(s)}ds - \int_0^{\theta_0} e^{-as} v_2(s)ds \\ &= x_{20} - y_{20} - \int_0^{\theta_0} |f(s)|ds - \int_0^{\theta_0} e^{-as} v_2(s)ds \end{aligned}$$

Since $\int_0^{\theta_0} |f(s)|ds \geq \left|\int_0^{\theta_0} f(s)ds\right|$, then

$$\begin{aligned} \xi_2(\theta_0) - \eta_2(\theta_0) &\leq x_{20} - y_{20} - \left|\int_0^{\theta_0} f(s)ds\right| - \int_0^{\theta_0} e^{-as} v_2(s)ds \\ &= x_{20} - y_{20} - \left|\left(\frac{\sqrt{\rho^2 - \sigma^2}}{k-a}\left(e^{(k-a)\theta_0} - 1\right), \int_0^{\theta_0} e^{-as} v_2(s)ds\right)\right| - \int_0^{\theta_0} e^{-as} v_2(s)ds \\ &= x_{20} - y_{20} - \left(\frac{\rho^2 - \sigma^2}{(k-a)^2}\left(e^{(k-a)\theta_0} - 1\right)^2 + \left(\int_0^{\theta_0} e^{-as} v_2(s)ds\right)^2\right)^{1/2} \\ &- \int_0^{\theta_0} e^{-as} v_2(s)ds. \end{aligned} \tag{2.7}$$

By letting $\int_0^{\theta_0} e^{-as} v_2(s)ds = \beta$ on the right-hand side of (3.6), we examine the following function:

$$f(\beta) = x_{20} - y_{20} - \sqrt{\frac{\rho^2 - \sigma^2}{(k-a)^2}\left(e^{(k-a)\theta_0} - 1\right)^2 + \beta^2} - \beta,$$

where by (3.3) $\beta \geq -y_{20}$. For the derivative of $f(\beta)$, we have

$$f'(\beta) = -\frac{\beta}{\sqrt{\frac{\rho^2-\sigma^2}{(k-a)^2}\left(e^{(k-a)\theta_0}-1\right)^2+\beta^2}} - 1 < 0.$$

Thus, the function $f(\beta)$ is decreasing, hence, it reaches its maximum value at $\beta = -y_{20}$. Consequently, by using this in (3.6), we obtain:

$$\begin{aligned}\xi_2(\theta_0) - \eta_2(\theta_0) &\leq x_{20} - y_{20} - \left(\frac{\rho^2-\sigma^2}{(k-a)^2}\left(e^{(k-a)\theta_0}-1\right)^2 + (-y_{20})^2\right)^{1/2} + y_{20} \\ &= x_{20} - \left(\frac{\rho^2-\sigma^2}{(k-a)^2}\left(e^{(k-a)\theta_0}-1\right)^2 + y_{20}^2\right)^{1/2} = 0.\end{aligned}$$

By combining this inequality with the condition $\xi_2(0) - \eta_2(0) > 0$ and noting that $\xi_2(t) - \eta_2(t)$ is continuous, we conclude that there exists some $0 < \tau < \theta_0$ such that $\xi_2(\tau) - \eta_2(\tau) = 0$.

Recalling that $\xi_1(t) = \eta_1(t)$ for all $t \geq 0$, which specifically implies $\xi_1(\tau) = \eta_1(\tau)$, we have $\xi(\tau) = \eta(\tau)$. Consequently, θ_0 serves as a guaranteed pursuit time in game (2.3). This completes the proof of Lemma 2.2. □

Next, we prove Theorem 5.2.

Proof. We will begin by demonstrating that θ is a guaranteed pursuit time in game (1.1). To do this, we suggest that the pursuers use the following strategies:

$$u_i = v - (v, e_i)e_i + e_i\sqrt{(\rho_i^2 - \sigma^2)e^{2kt} + (v, e_i)^2}, \quad e_i = \frac{y_0 - x_{i0}}{|y_0 - x_{i0}|}, \quad i = 1, 2. \tag{2.8}$$

It is not difficult to see that strategies (3.2) are admissible.

The following two cases are considered for the time θ:

Case 1. $H(y_0, r(\theta)) \subset H(x_{i_0 0}, R_{i_0}(\theta))$ for some $i_0 \in \{1, 2\}$ (see Figure 2).

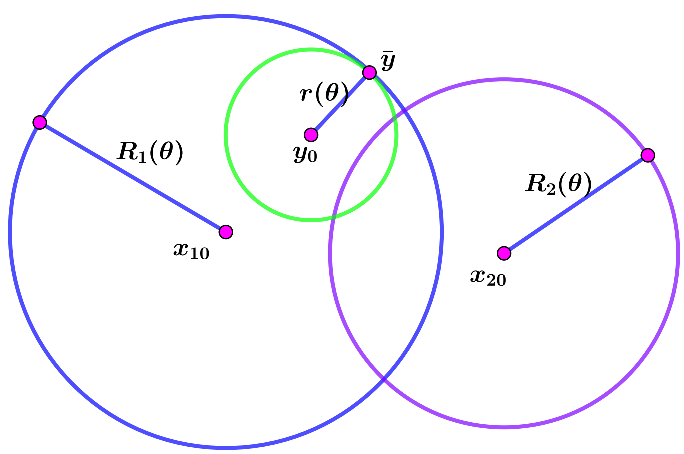


FIGURE 2. Case 1: $H(y_0, r(\theta)) \subset H(x_{10}, R_1(\theta))$

In this case, one can prove that strategies (3.7) guarantee the completion of pursuit for the time

$$\theta = \frac{1}{k-a}\ln\left(1 + (k-a)\frac{|x_{i_0 0} - y_0|}{\rho - \sigma}\right).$$

More specifically, only the pursuer x_{i_0} is capable of finishing the game within the time θ. Figure 2 depicts Case 1, where $i_0 = 1$, indicating that only the first pursuer x_1 can complete the game by the time θ [24].

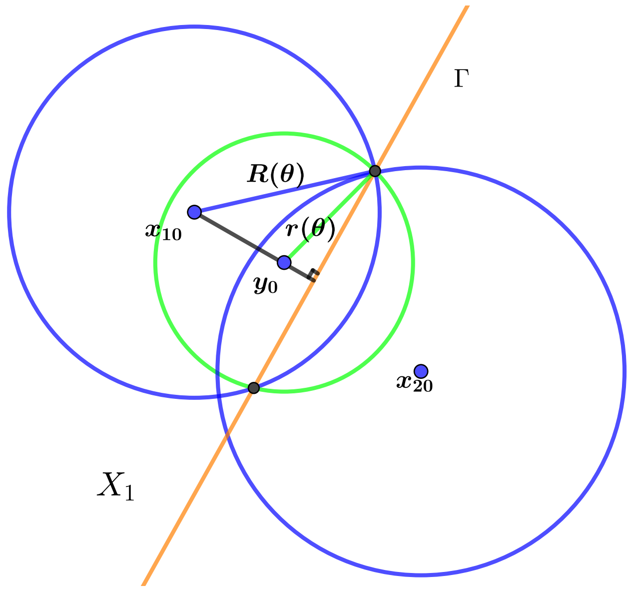


FIGURE 3. Case 2: $H(y_0, r(\theta)) \subset H(x_{10}, R_1(\theta)) \cup H(x_{20}, R_2(\theta))$

Case 2. (see Figure 3)

$$H(y_0, r(\theta)) \nsubseteq H(x_{i0}, R_i(\theta)), \quad i = 1, 2, \quad H(y_0, r(\theta)) \subset H(x_{10}, R_1(\theta)) \cup H(x_{20}, R_2(\theta)). \tag{2.9}$$

In Case 2, by the definition of θ we have the following relation

$$H(y_0, r(t)) \not\subset H(x_{10}, R_1(t)) \cup H(x_{20}, R_2(t)), \quad 0 \le t < \theta. \tag{2.10}$$

We show that θ is a guaranteed pursuit time in game (1.1). Indeed, for some $\bar{y} \in S(y_0, r(\theta))$, we have

$$\begin{aligned} \bar{y} &\in S(x_{10}, R_1(\theta)) \cap S(x_{20}, R_2(\theta)), \\ \bar{y} &\notin H(x_{10}, R_1(t)) \cup H(x_{20}, R_2(t)) \end{aligned} \tag{2.11}$$

for all $0 \le t < \theta$.

We draw straight lines Γ_i from the point $\bar{y}$, perpendicular to the vectors $y_0 - x_{i0}$, $i = 1, 2$. For example, Figure 3 illustrates the straight line Γ_1. The half-plane bounded by the straight line Γ_i that includes the point x_{i0} is denoted as X_i, where $i = 1, 2$.

It can be shown similar to Assertion 4 (Appendix, [17]) that $H(y_0, r(\theta)) \subset X_1 \cup X_2$. By combining this inclusion with the fact that $y(\theta) \in H(y_0, r(\theta))$ we conclude that $y(\theta)$ must be in either X_1 or X_2. If $y(\theta) \in X_1$, then by applying Lemma 2.2, we obtain $x_1(\tau_1) = y(\tau_1)$ at some $0 \le \tau_1 \le \theta$; similarly, if $y(\theta) \in X_2$, then Lemma 2.2 implies $x_2(\tau_2) = y(\tau_2)$ at some $0 \le \tau_2 \le \theta$. This concludes that θ is a guaranteed pursuit time in game (1.1).

Next, we demonstrate that θ is a guaranteed evasion time in game (1.1) for both Case 1 and Case 2. We let the evader employ the following strategy:

$$V(t) = \frac{\bar{y} - y_0}{|\, \bar{y} - y_0 \,|} \sigma e^{kt}, \quad t \ge 0, \tag{2.12}$$

where $\bar{y}$ is defined as above in Case 2, and $\bar{y} \in S(y_0, r(\theta)) \cap S(x_{10}, R_1(\theta))$ in Case 1. Strategy (4.1) is admissible. Indeed,

$$|V(t)| = \left| \frac{\bar{y} - y_0}{|\ \bar{y} - y_0\ |} \sigma e^{kt} \right| = \sigma e^{kt},$$

and so it satisfies the condition (2.1). Also, since $|\bar{y} - y_0| = \int\limits_0^\theta \sigma e^{(k-a)s} ds$, we have

$$\eta(\theta) = y_0 + \int\limits_0^\theta e^{(k-a)s} v(s) ds = y_0 + \int\limits_0^\theta \frac{\bar{y} - y_0}{|\bar{y} - y_0|} \sigma e^{(k-a)s} ds = y_0 + \bar{y} - y_0 = \bar{y},$$

that is the evader reaches the point $\bar{y}$ at the time θ.

What remains is to demonstrate that $\xi_i(t) \neq \eta(t)$ for all $0 \le t < \theta$ and $i = 1, 2$. The following reasoning applies to the definition of $\bar{y}$ in both Case 1 and Case 2. Suppose the contrary, let $\xi_{i_0}(\tau) = \eta(\tau)$ at some $\tau < \theta$ and $i_0 \in \{1, 2\}$ when the evader applies (4.1). For specificity, assume $i_0 = 1$, that is, $\xi_1(\tau) = \eta(\tau)$. This implies $\int\limits_0^\tau e^{-as}(u_1(s) - v(s)) ds = y_0 - x_{10}$. Then using $\eta(\theta) = \bar{y}$, we have

$$\begin{aligned}
|\bar{y} - x_{10}| &= \left| y_0 + \int\limits_0^\theta e^{-as} v(s) ds - x_{10} \right| \\
&= \left| (y_0 - x_{10}) + \int\limits_0^\theta e^{-as} v(s) ds \right| \\
&= \left| \int\limits_0^\tau e^{-as}(u(s) - v(s)) ds + \int\limits_0^\theta e^{-as} v(s) ds \right| \\
&= \left| \int\limits_\tau^\theta e^{-as} v(s) ds + \int\limits_0^\tau e^{-as} u_1(s) ds \right| \\
&\le \int\limits_\tau^\theta e^{-as} |v(s)| ds + \int\limits_0^\tau e^{-as} |u_1(s)| ds \\
&\le \sigma \int\limits_\tau^\theta e^{-as} e^{ks} ds + \rho \int\limits_0^\tau e^{-as} e^{ks} ds \\
&< \rho \int\limits_0^\theta e^{(k-a)s} ds = R_1(\theta).
\end{aligned}$$

This implies $\bar{y}$ lies within the interior of the ball $H(x_{10}, R_1(\theta))$, and therefore $\bar{y} \in H(x_{10}, R_1(t_1))$ for some $t_1 < \theta$. This contradicts condition (2.11). Consequently, $\xi_i(t) \neq \eta(t)$ for all $0 \le t < \theta$ and $i = 1, 2$, meaning that θ is a guaranteed evasion time. Therefore, θ is the optimal pursuit time. The proof of the theorem is now complete.

□

3. Conclusions

We have studied a differential game described by a linear differential equations involving two pursuers and one evader in $\mathbb{R}^2$. The control functions of players are subjected to the Grönwall type constraints. We have found an equation for the optimal pursuit time and constructed optimal strategies of players. The optimal strategies of pursuers are defined by the equation (3.7) and the optimal strategy of the evader is defined by (4.1). The optimal strategy of the evader (4.1) satisfies the equation $|v(t)| = \sigma e^{kt}$. Also, according to the Grönwall type constraint (2.1) we have $|v(t)| \le \sigma e^{kt}$. Therefore, we can say that the evader moves with its maximal speed. The equation $|v(t)| = \sigma e^{kt}$ and (3.7) imply that $|u_i(t)| = \rho_i e^{kt}$, $i = 1, 2$, meaning that the pursuers move with maximal speeds as well.

References

[1] Ibragimov G., Yusupov I., Ferrara M. (2023). Optimal pursuit game of two pursuers and one evader with the Grönwall-type constraints on controls. Mathematics. 11(2). https://doi.org/10.3390/math11020374.

[2] Azamov A. (1986). O zadache kachestva dlya igr prostogo presledovaniya s ogranicheniem [On the quality problem for simple pursuit games with constraint]. *Sofia, Serdica Bulgariacae Math.* Publ., 12(1): 38–43.

[3] Blagodatskikh A. I., Petrov N. N. (2009). *Konfliktnoe vzaimodejstvie grupp upravlyaemyh ob'ektov* [*Conflict interaction of groups of controlled objects*]. Izhevsk, Udmurt State University Press, 266 p.

[4] Samatov B., Akbarov A. (2024). Differential game with a "Life line" under the Grönwall constraint on controls. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes.* 265–280.

[5] Samatov B. T. (2013). On a pursuit-evasion problem under a linear change of the pursuer resource. *Siberian Advances in Mathematics*, vol. 23, no. 10, pp. 294–302.

[6] Blagodatskikh V. I. (2001). *Vvedenie v optimal'noe upravlenie* [*Introduction to optimal control*]. Moscow, Vysshaya Shkola Publ., 239 p.

[7] Bakolas E., Tsiotras P. (2011). On the relay pursuit of a maneuvering target by a group of pursuers. In 50th IEEE Conference on Decision and Control and European Control Conference, Orlando, FL, pp. 4270–4275.

[8] Blagodatskikh A.I., Petrov N.N. (2019). Simultaneous Multiple Capture of Rigidly Coordinated Evaders Dynamic Games and Applications 9: 594–613.

[9] Ganebny S.A., Kumkov S.S., Le Ménec S., Patsko V.S. (2012). Model problem in a line with two pursuers and one evader. Dyn. Games Appl. 2(2): 228–257.

[10] Garcia E., Casbeer D.W., Von Moll A., Pachter M. (2021). Multiple pursuer multiple evader differential games. IEEE Transactions on Automatic Control, 66(5): 2345–2350, May 2021, doi: 10.1109/TAC.2020.3003840.

[11] Grönwall T.H.(1919) Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. Ann. Math. 20(2): 293–296.

[12] Huang H., Zhang W., Ding J., Stipanovic D.M., Tomlin C.J. (2011). Guaranteed decentralized pursuit-evasion in the plane with multiple pursuers. Proc. 50th IEEE Conf. Decis. Control Eur. Control Conf., pp. 4835–4840.

[13] Ibragimov G.I., Ferrara M., Ruziboev M., Pansera B.A. Linear evasion differential game of one evader and several pursuers with integral constraints. International Journal of Game Theory. https://doi.org/10.1007/s00182-021-00760-6. 11 February 2021.

[14] Ibragimov G.I., Ferrara M., Kuchkarov A.Sh., Pansera B.A. (2018). Simple motion evasion differential game of many pursuers and evaders with integral constraints. Dynamic Games and Applications. 8: 352–378. https://doi.org/10.1007/s13235-017-0226-6.

[15] Ibragimov G.I. (2013). The optimal pursuit problem reduced to an infinite system of differential equation. Journal of Applied Mathematics and Mechanics. 77(5): 470–476. http://dx.doi.org/10.1016/j.jappmathmech.2013.12.002

[16] Ibragimov G.I. (1998). A game of optimal pursuit of one object by several. J. Appl. Maths Mechs, 62(2): 187-192. (Prikladnaya Matematika i Mekhanika, 62(2): 199–205, 1998.)

[17] Ibragimov G.I. (2005) Optimal pursuit with countably many pursuers and one evader, Differential Equations, 41(5), 627–635.

[18] Isaacs R. (1965) Differential games. John Wiley and Sons, New York.

[19] Jang J.S., Tomlin C.J. (2005). Control strategies in multi-player pursuit and evasion game. In: AIAA guidance, navigation, and control conference and exhibit, San Francisco, CA, AIAA Paper 2005–6239.

[20] Makkapati V.R., Tsiotras P. (2019). Optimal evading strategies and task allocation in multi-player pursuit-evasion problems. Dyn Games Appl 9, 1168–1187. https://doi.org/10.1007/s13235-019-00319-x

[21] Pashkov A.G., Teorekov S.D. (1983). A game of optimal pursuit of one evader by two pursuers, Prikl. Mat. Mekh., 47(6): 898–903.

[22] Petrosjan L.A. (1993). Differential games of pursuit. Series on optimization, Vol.2. World Scientific Poblishing, Singapore.

[23] Pshenichnii B.N. (1976). Simple pursuit by several objects. Cybernetics and System Analysis. 12(3): 484-485. DOI 10.1007/BF01070036

[24] Samatov B.T., Ibragimov G.I., Khodjibayeva I.V. (2020) Pursuit-evasion differential games with Grönwall type constraints on controls. Ural Mathematical Journal, 6(2): 95–107, DOI: 10.15826/umj.2020.2.010.

[25] Subbotin A.I. (1984) Generalization of the main equation of differential game theory. Journal of Optimization Theory and Applications, 43(1): 103–133.

[26] Sun W., Tsiotras P., Lolla T., Subramani D.N., and Lermusiaux P.F.J. (2017). Multiple-pursuer/one-evader pursuit-evasion game in dynamic flowfields. JGCD, 40(7), https://doi.org/10.2514/1.G002125

[27] Von Moll A., Casbeer D., Garcia E., Milutinovic D. (2018). Pursuit-evasion of an evader by multiple pursuers. Proc. Int. Conf. Unmanned Aircr. Syst., 133–142, 2018.

[28] Ibragimov G.I., Kuchkarov A.Sh. (2013). Fixed Duration Pursuit-Evasion Differential Game with Integral Constraints. Journal of Physics: Conference Series 435 (2013) 012017, doi:10.1088/1742-6596/435/1/012017, 2013.

[29] Kuchkarov A.Sh., Ibragimov G.I., Massimiliano Ferrara. (2016). Simple motion pursuit and evasion differential games with many pursuers on manifolds with Euclidean metric. Differential Games and Discrete Dynamics in Applied Sciences. Discrete Dynamics in Nature and Society. Vol. 2016 (2016), ID 1386242, 8 p. 10.1155/2016/1386242.

Ibragimov G.I. ,
V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 100174, Tashkent, Uzbekistan & Department of Econometrics, Tashkent State University of Economics, 100066, Tashkent, Uzbekistan.
email: gofurjon.ibragimov@tsue.uz

Yusupov I.M. ,
Department of Mathematics, Andijan State University, Andijan, Uzbekistan & Department of Mathematics and Information technology, Andijon Pedagogical institute, Andijan, Uzbekistan
email: yusupov@adu.uz

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.93-99
DOI: 10.29229/uzmj.2025-3-9

Construction of cubature formulas for regions symmetrical with respect to coordinate axes

Ismatullaev G.P.

Abstract. In approximate integration, an interesting and relevant problem is the construction of formulas with a minimum number of nodes for a given algebraic degree of accuracy.

Keywords: approximate integration formula, cubature formula, orthogonal polynomial, n-dimensional spherical shell.

MSC (2020): 65D30, 65D32

1. Introduction

The literature [1, 2, 3, 4, 5, 6] provides a theory for constructing approximate integration formulas, and a number of quadrature and cubature formulas are presented. Cubature formulas are constructed mainly for standard domains: C_n is an n-dimensional cube, C_n^{hull} is an n-dimensional cubic hull, S_n is an n-dimensional ball, S_n^{hull} is an n-dimensional spherical hull, U_n is a sphere (the surface of an n-dimensional ball), T_n is an n-dimensional simplex, $E_n^{r^2}$ is an n-dimensional space with the weight function $\exp\left(-x_1^2 - x_2^2 - \cdots - x_n^2\right)$, E_n^r is an n-dimensional space with the weight function $\exp\left(-\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\right)$, etc.

In the works [7, 8, 9, 10, 11] applications of orthogonal polynomials to the construction of cubature formulas are shown, and in [12, 13, 14] cubature formulas with a minimum number of nodes for a given degree of accuracy for some domains are constructed.

In this work cubature formulas of the fifth and seventh degrees of accuracy are constructed for domains symmetric with respect to the coordinate axes. The point (0,0) the origin of coordinates is the center of symmetry, which belongs to the domain of integration. Cubature formulas of the fifth degree of accuracy are constructed using theorem 11.6 [5] p. 234, and the formula of the seventh degree of accuracy is constructed using a modified method of the reproducing kernel, adapted for domains with central symmetry [5] p. 245. The number of nodes of the constructed cubature formulas is minimal.

In recent years, a number of results have been obtained on the construction of optimal formulas for the approximate calculation of definite integrals (see [15, 16, 17, 18, 19, 20]).

1.1. **Construction of a cubature formula of the fifth degree of accuracy.** Let us write out the main orthogonal polynomials of the third degree of the region under consideration:

$$P_{30} = x^3 - \frac{\mu_{40}}{\mu_{20}}x, P_{12} = xy^2 - \frac{\mu_{22}}{\mu_{20}}x, P_{21} = x^2y - \frac{\mu_{22}}{\mu_{02}}y, P_{03} = y^3 - \frac{\mu_{04}}{\mu_{02}}y.$$

Consider the following linear combination:

$$f_1 = \alpha P_{21} + \gamma P_{03} = y\left(\alpha x^2 + \gamma y^2 - \frac{\alpha\mu_{22} + \gamma\mu_{04}}{\mu_{02}}\right),$$

$$g_1 = P_{30} - \frac{\mu_{40}}{\mu_{22}}P_{12} = x(x^2 - \frac{\mu_{40}}{\mu_{22}}). \tag{1.1}$$

Let us list the intersection points of the curves $f_1 = 0$, $g_1 = 0$: $(0,0)$ is triple point,

$$\left(0, \pm\sqrt{\frac{\alpha\mu_{21} + \beta\mu_{04}}{\mu_{02}}}\right) = (0, \pm c), \quad \left(\pm\sqrt{\frac{\mu_{22}(\alpha\mu_{22} + \gamma\mu_{04})}{\mu_{02}(\alpha\mu_{40} + \gamma\mu_{22})}}, \quad \pm\sqrt{\frac{\mu_{22}(\alpha\mu_{22} + \gamma\mu_{04})}{\mu_{02}(\alpha\mu_{40} + \gamma\mu_{22})}}\right) = (\pm a, \pm b)$$

Curves $f_1 = 0$, $g_1 = 0$ intersect at nine points taking into account multiplicity, therefore, by theorem 11.6 [5] p. 234, the following cubature formula exists:

$$\iint\limits_{\Omega} F(x,y)\,dx\,dy \approx C_{00}F(0,0) + C_{20}\frac{\partial^2 F(0,0)}{\partial x^2} + C_{02}\frac{\partial^2 F(0,0)}{\partial y^2}$$
$$+A\sum_{1}^{4} F(\pm a, \pm b) + B\sum_{1}^{2} F(0, \pm c) \tag{1.2}$$

algebraic degree of accuracy $s \le 5$.

The terms containing the values of the derivatives F_x, F_y, F_{xy} at the point $(0,0)$ are not written out here, since it is known in advance that the coefficients of these derivatives are equal to zero. This follows from the symmetry of the region Ω and the arrangement of the nodes of the cubature formula (1.2).

The values of the parameters a, b, c are defined as follows: In the cubature formula (1.2), assuming $F = f_1 \cdot y$, we obtain that the integral is equal to zero due to the orthogonality of the polynomial f_1, and in the cubature sum only the term $-2C_{02}\frac{\alpha\mu_{22}+\gamma\mu_{04}}{\mu_{02}}$ remains, since all nodes lie on the curve $f_1 = 0$. From this we conclude that $C_{02} = 0$ for any $\alpha \neq 0$, $\beta \neq 0$.

Next, consider the following orthogonal polynomial:

$$f_2 = \alpha P_{30} + \gamma P_{12} = x(\alpha x^2 + \gamma y^2 - \frac{\alpha\mu_{40} + \gamma\mu_{22}}{\mu_{20}}). \tag{1.3}$$

We define the values of α and γ so that the second moments of the polynomials f_1 and f_2 are equal. This is equivalent to the equality:

$$\alpha\frac{\mu_{22}}{\mu_{02}} + \gamma\frac{\mu_{04}}{\mu_{20}} = \alpha\frac{\mu_{40}}{\mu_{20}} + \beta\frac{\mu_{22}}{\mu_{20}}.$$

From this we get:

$$\alpha = \mu_{20}\mu_{04} - \mu_{02}\mu_{22}, \quad \beta = \mu_{02}\mu_{40} - \mu_{20}\mu_{22}. \tag{1.4}$$

Thus we have:

$$f_1 = y\left(\alpha(x^2 + y^2) - \Delta\right), \quad f_2 = x\left(\alpha(x^2 + y^2) - \Delta\right),$$

where $\Delta = \mu_{04}\mu_{40} - \mu_{22}^2 > 0$, since Δ is the Gram determinant composed of monomials x^2, y^2. For $F = f_2 \cdot x$, from (1.2) we obtain that the integral is equal to zero due to the orthogonality of the polynomial f_2, and in the cubature sum only the term $-2C_{20}\Delta$ remains, since all nodes of the cubature formula (1.2) also lie on the curve $f_2 = 0$. From this we conclude that $C_{20} = 0$.

Now we present the values of the nodes, taking into account equalities (1.4)

$$(\pm a, \pm b) = \left(\pm\sqrt{\frac{\mu_{40}}{\mu_{20}}}, \pm\sqrt{\frac{\mu_{22}}{\mu_{20}}}\right), \quad (0, \pm c) = \left(0, \pm\sqrt{\frac{\Delta}{\gamma}}\right)$$

Let us write the final form of the cubature formula (1.2):

$$\iint\limits_{\Omega} F(x,y)\,dx\,dy \cong C_{00}F(0,0) + A_1\sum_{1}^{4} F(\pm a, \pm b) + B_1\sum_{1}^{2} F(0, \pm c) \tag{1.5}$$

where

$$A = \frac{\mu_{20}}{4\mu_{40}}, \quad B = \frac{\gamma^2}{2\mu_{40}\cdot\Delta}, \quad C_{00} = \mu_{00} - \frac{\alpha\mu_{20} + \gamma\mu_{02}}{\Delta}$$

and they are determined from (1.5) with $F = x^2, y^2, 1$.

If we take the following polynomials as orthogonal polynomials:

$$f_2 = \alpha P_{30} + \gamma P_{12} = x(\alpha x^2 + \gamma y^2 - \Delta),$$
$$g_2 = P_{21} - \frac{\mu_{04}}{\mu_{22}} = y(x^2 - \frac{\mu_{04}}{\mu_{22}}y^2) \tag{1.6}$$

and having performed all the previous calculations, we arrive at the following cubature formula with seven nodes

$$\iint_{\Omega} F(x,y)\,dx\,dy \cong C_{00}F(0,0) + A_1\sum_{1}^{4} F(\pm a_1, \pm b_1) + B_1\sum_{1}^{2} F(\pm c_1, 0) \tag{1.7}$$

where:

$$a_1 = \sqrt{\frac{\mu_{04}}{\mu_{02}}}, \quad b_1 = \frac{\mu_{22}}{\mu_{02}}, \quad c_1 = \sqrt{\frac{\Delta}{\alpha}},$$

$$A_1 = \frac{\mu_{02}^2}{4\mu_{04}}, \quad B_1 = \frac{\alpha^2}{2\mu_{04}\cdot\Delta}, \quad C_{00} = \mu_{00} - \frac{\alpha\mu_{20} + \gamma\mu_{02}}{\Delta}.$$

Note that for other values of the parameters α and γ, different from (1.4), the values of the partial derivatives $F_{xx}(0,0)$ and $F_{yy}(0,0)$, respectively, will participate in the cubature formulas (1.5) and (1.7).

The number of nodes of the cubature formulas (1.5) and (1.7) $N = 7$ is minimal in the class of cubature formulas that have among their nodes the origin of coordinates - the center of symmetry of the domain Ω [5], Theorem 9.1, p. 196.

Now we will conduct a comparative analysis of the obtained results, without performing computational work on constructing a cubature formula of the fifth degree of accuracy with the Radon method of the reproducing kernel for $k = 2$. This is explained by the fact that we have already obtained all the necessary information.

(1) The values of the parameters α and γ (1.4) are the same as in the Radon method for the considered domain Ω.

(2) The nodes of the cubature formula (1.5) are the common zeros of two orthogonal polynomials.

$$f_1 = y(\alpha x^2 + \gamma y^2 - \Delta), \quad g_1 = x\left(x^2 - y^2\frac{\mu_{40}}{\mu_{22}}\right),$$

and in the Radon method, the nodes are the common zeros of three orthogonal polynomials:

$$f_1 = y(\alpha x^2 + \gamma y^2 - \Delta), \quad f_2 = -x(\alpha x^2 + \gamma y^2 - \Delta), \quad g_1 = x\left(x^2 - y^2\frac{\mu_{40}}{\mu_{22}}\right).$$

For clarity of further presentation, we present Lemma 8.2 from [5], p. 178.

Lemma 8.2. If $L(x)$ is a polynomial of degree one such that $L(a) = 0$, then the polynomial

$$L(x)\cdot K_k(x), \quad k \in \mathbb{N}$$

is orthogonal to all polynomials of degree at most $k-1$. In particular, if a is the common root of all orthogonal polynomials of degree at most $k+1$, then the polynomial $L(x)\cdot K_k(x)$ is an orthogonal polynomial of degree $k+1$.

3. According to this lemma, the point $a^{(1)} = (0,0)$ is the common root of all orthogonal polynomials of degree three, which means that the polynomials

$$f_1 = y(\alpha x^2 + \gamma y^2 - \Delta), \quad f_2 = x(\alpha x^2 + \gamma y^2 - \Delta)$$

differ with an accuracy of a non-zero constant factor from the polynomials

$$yK_2(0,0;xy) \quad \text{and} \quad xK_2(0,0;xy)$$

So we get

$$K_2(a^{(1)};xy) = c_1(\alpha x^2 + \gamma y^2 - \Delta), \quad c_1 \neq 0$$

Point $a^{(2)}$ is determined from the condition $K_2(a^{(1)}, a^{(2)}) = 0$. It follows that $a^{(2)} = \left(0, \sqrt{\frac{\Delta}{\gamma}}\right)$, and the corresponding kernel have the form

$$K_2(a^{(2)}, x, y) = c_2 \left(x^2 - \frac{\mu_{40}}{\mu_{22}} y^2\right), \quad c_2 \neq 0.$$

The curves determined by the reproducing kernels $K_2(a^{(i)}, xy)$ and $i = 1, 2$ intersect at four different real points:

$$\left(\pm\sqrt{\frac{\mu_{40}}{\mu_{20}}}, \pm\sqrt{\frac{\mu_{22}}{\mu_{20}}}\right) = (\pm a, \pm b),$$

then by Theorem 12.2 [5], p. 245 there exists a cubature formula

$$\iint\limits_{\Omega} F(x,y)\,dx\,dy \cong C_{00}F(0,0) + B\sum_{1}^{2}\left(F\left(0, \sqrt{\frac{\gamma}{\delta}}\right) + F\left(0, -\sqrt{\frac{\gamma}{\delta}}\right)\right) + A\sum_{1}^{4} F(\pm a, \pm b),$$

where

$$C_{00} = \frac{1}{b_1}, \quad B = \frac{1}{b_2}, \quad A = \frac{\mu_{20}^2}{4\mu_{40}},$$

$$b_1 = K_2^{(1)}(a^{(1)}; a^{(1)}) > 0, \quad b_2 = K_2^{(2)}(a^{(2)}; a^{(2)}) > 0,$$

so that $C_{00} > 0$, $B > 0$.

2. Cubature formula of the seventh degree of accuracy

We will construct a cubature formula of the seventh degree of accuracy using a modified method of the reproducing kernel. According to Theorem 12.2 [5], p. 245. In this section, we assume that the boundaries of one integral depend on the variable of the other integral, while maintaining the symmetry of the domain Ω.

Let us present orthonormal polynomials of odd degree $k \leq 3$ of the considered domain Ω:

$$f_1 = \sqrt{\frac{1}{\mu_{20}}} x, \quad f_2 = \sqrt{\frac{1}{\mu_{02}}} y, \quad f_3 = \sqrt{\frac{\mu_{20}}{\Delta_1}} \left(x^3 - \frac{\mu_{40}}{\mu_{20}} x\right),$$

$$f_4 = A_4 \left(xy^2 + \frac{\sigma_1}{\Delta_1} x^3 - \frac{\sigma_2}{\Delta_1} x\right), \quad f_5 = \sqrt{\frac{\mu_{02}}{\Delta_2}} \left(y^3 - \frac{\mu_{04}}{\mu_{02}} y\right),$$

$$f_6 = A_6 \left(x^2 y + \frac{\sigma_3}{\Delta_2} y^3 - \frac{\sigma_4}{\Delta_1} y\right),$$

where

$$\Delta_1 = \mu_{20}\mu_{60} - \mu_{40}^2 > 0, \quad \Delta_2 = \mu_{02}\mu_{60} - \mu_{04}^2 > 0, \quad \sigma_1 = \mu_{40}\mu_{22} - \mu_{20}\mu_{42},$$

$$\sigma_2 = \mu_{60}\mu_{22} - \mu_{40}\mu_{42}, \quad \sigma_3 = \mu_{04}\mu_{22} - \mu_{02}\mu_{24}, \quad \sigma_4 = \mu_{06}\mu_{22} - \mu_{04}\mu_{24}.$$

A_4, A_6 are normalizing factors that do not need to be calculated, this is found out quite simply.

As the point $a^{(1)}$ we take the point $a^{(1)} = (c, 0)$, so that the monomial xy^2 is absent in the reproduced kernel. For this, it is sufficient to satisfy the equality:

$$F_4(c, 0) = 0 \tag{2.1}$$

From (2.1) we obtain:

$$\frac{\sigma_1}{\Delta_1} c^3 - \frac{\sigma_2}{\Delta_1} c = 0$$

$$c_1 = 0, \quad c_2 = \sqrt{\frac{\sigma_2}{\sigma_1}}, \quad c_3 = -\sqrt{\frac{\sigma_2}{\sigma_1}}$$

Point $(0,0) \in V_3$, and points $\left(\pm\sqrt{\frac{\sigma_2}{\sigma_1}}, 0\right) \notin V_3$, where V_3 is the set of common zeros of the basic orthogonal polynomials of degree three.

Let $a^{(1)} = (c, 0), \quad c = \sqrt{\frac{\sigma_2}{\sigma_1}}$.

Let us write the reproducing kernel of the point $a^{(1)}$:

$$K_3^{(1)}(a^{(1)}, xy) = F_1(a^{(1)})F_1(x,y) + F_3(a^{(1)})F_3(x,y) = \frac{1}{\mu_{20}}cx + \frac{\mu_{20}}{\Delta_1}(c^3 - \frac{\mu_{40}}{\mu_{20}}c)(x^3 - \frac{\mu_{40}}{\mu_{20}}x)$$

After some simplifications we write:

$$K_3^{(1)}(a^{(1)}; x, y) = \frac{c\mu_{22}}{\sigma_1}\left(x^3 - \frac{\mu_{42}}{\mu_{22}}x\right). \tag{2.2}$$

We take same point on the ordinate axis $a^{(2)} = (0, d)$ so that the reproduced kernel does not contain the monomial xy^2. For this, it is sufficient to satisfy the equality:

$$F_6(0, d) = 0 \tag{2.3}$$

From (2.3) it follows:

$$\frac{\sigma_3}{\Delta_2}y^2 - \frac{\sigma_4}{\Delta_2}y = 0,$$

$$d_1 = 0, \quad d_2 = \sqrt{\frac{\sigma_4}{\sigma_3}}, \quad d_3 = -\sqrt{\frac{\sigma_4}{\sigma_3}}.$$

Point $(0,0) \in V_3$, and points $\left(\pm\sqrt{\frac{\sigma_4}{\sigma_3}}, 0\right) \notin V_3$,
where V_3 is the set of two zeros of the basic orthogonal polynomials of the third degree.

Let $a^{(2)} = (0, d), \quad d = \sqrt{\frac{\sigma_4}{\sigma_3}}$

The reproducing kernel of the point $a^{(2)}$ is of the form:

$$K_3^{(2)}(a^{(2)}, xy) = \frac{d\mu_{22}}{\sigma_3}\left(y^3 - \frac{\mu_{24}}{\mu_{22}}y\right) \tag{2.4}$$

Now we define the common zeros of the polynomials determined by the reproducing kernels:

$$(0,0), \left(\pm\sqrt{\frac{\mu_{42}}{\mu_{22}}}, 0\right), \left(0, \pm\sqrt{\frac{\mu_{24}}{\mu_{22}}}\right),$$

$$\left\{\begin{matrix} x^3 - \dfrac{\mu_{42}}{\mu_{22}}x = 0 \\ y^3 - \dfrac{\mu_{24}}{\mu_{22}}y = 0 \end{matrix}\right\} \Rightarrow \left(\pm\sqrt{\frac{\mu_{42}}{\mu_{22}}}, \pm\sqrt{\frac{\mu_{24}}{\mu_{22}}}\right).$$

According to Theorem 12.2.1 [5], p. 215, there exists the following cubature formula of the seventh degree of accuracy:

$$\iint\limits_{\Omega} f(x,y)\,dxdy \cong \sum_{i=1}^{2}\frac{1}{2b_i}\left[f(a^{(i)}) + f(-a^{(i)})\right] + A\sum_{1}^{4} f\left(\pm\sqrt{\frac{\mu_{42}}{\mu_{22}}}, \pm\sqrt{\frac{\mu_{24}}{\mu_{22}}}\right) +$$

$$+B\sum_{1}^{2} f\left(\pm\sqrt{\frac{\mu_{42}}{\mu_{22}}}, 0\right) + C\sum_{1}^{2} f\left(0, \pm\sqrt{\frac{\mu_{24}}{\mu_{22}}}\right) + 2Df(0,0), \tag{2.5}$$

where:

$$b_1 = K_3^{(1)}(a^{(1)}, a^{(1)}) = \frac{\sigma_2(\sigma_2\mu_{22} - \sigma_1\mu_{42})}{\sigma_1^3} > 0,$$

$$b_2 = K_3^{(2)}(a^{(2)}, a^{(2)}) = \frac{\sigma_4(\sigma_4\mu_{22} - \sigma_3\mu_2)}{\sigma_3^3} > 0$$

$$A = \frac{\mu_{22}^3}{4\mu_{42}\mu_{24}}, \quad B = \frac{\mu_{22}}{\mu_{42}}\left(\mu_{20} - \frac{1}{b_1}\frac{\sigma_2}{\sigma_1}\right) - 2A,$$

$$C = \frac{\mu_{22}}{\mu_{24}}\left(\mu_{02} - \frac{1}{b_2}\frac{\sigma_4}{\sigma_3}\right) - 2A, \quad D = \mu_{00} - 4A - 2B - 2C.$$

The cubature formula (2.5) is exact for algebraic polynomials of degree $s \leq 7$ and the number of nodes $N = 13$ is minimal according to Theorem 9.1 p. 196 [5] in the class of cubature formulas that have among their nodes the origin of coordinates the center of symmetry of the domain Ω.

3. Cubature formula of medium accuracy taking into account the values of second-order derivatives.

We will construct such a cubature formula using the method of indefinite parameters, arranging the nodes according to the symmetry of the region and comparing the same values of the coefficients to each group of symmetric nodes. Let us write the cubature formula in the following form:

$$\iint\limits_{\Omega} f(x,y)\,dxdy = A\sum_1^4 f(\pm a, \pm b) + B\sum_1^2 f(\pm c, 0) + D\sum_1^2 f(0, \pm d) +$$

$$+ C_0 f(0,0) + C_{20}\frac{\partial^2 f(0,0)}{\partial x^2} + C_{20}\frac{\partial^2 f(0,0)}{\partial y^2}, \tag{3.1}$$

where the undefined parameters are $A, B, C, C_{00}, C_{20}, C_{02}, a, b, c, d,$ the number of which is equal to nine. We will find them by requiring that the cubature formula (3.1) has an algebraic degree of accuracy $s \leq 7$. For this, the accuracy of equality (3.1) is sufficient for

$$f = 1, x^2, y^2, x^4, y^4, x^2y^2, x^6, y^6, x^2y^4, x^4y^2.$$

This follows from the symmetry of the domain and the arrangement of the nodes. The number of equations is equal to nine, i.e. the same as the number of undefined parameters.

For $f = x^2, y^2, x^4y^2, x^2y^4$ from (3.1) we obtain that

$$a = \sqrt{\frac{\mu_{42}}{\mu_{22}}}, \quad b = \sqrt{\frac{\mu_{24}}{\mu_{22}}}, \quad A = \frac{\mu_{22}^3}{4\mu_{42}\mu_{24}}.$$

Next, putting in (3.1) $f = x^4, x^6$ we obtain:

$$\begin{cases} 4Aa^4 + 2Bc^4 = \mu_{40}, \\ 4Aa^6 + 2Bc^6 = \mu_{60} \end{cases} \Rightarrow C = \sqrt{\frac{\mu_{60} - 4Aa^6}{\mu_{40} - 4Aa^4}}, \quad B = \frac{\left(\mu_{40} - 4Aa^4\right)^3}{2\left(\mu_{60} - 4Aa^6\right)^2}.$$

For $f = x^2$ we get:

$$4Aa^2 + 2Bc^2 + 2C_{20} = \mu_{20},$$

from which we have:

$$C_{20} = \frac{\mu_{20}}{2} - 2Aa^2 - Bc^2$$

Similarly, in (3.1), setting $f = y^4, y^6, y^2$, we get:

$$d = \sqrt{\frac{\mu_{06} - 4Ab^6}{\mu_4 - 4Ab^4}}, \quad D = \frac{(\mu_{04} - 4Ab^4)^3}{2(\mu_{06} - 4Ab^6)^2},$$

$$C_{02} = \frac{\mu_{02}}{2} - 2Ab^2 - Dd^2.$$

And finally, for $f = 1$ from (3.1) we get:

$$C_{00} = \mu_{00} - 4A - 2B - 2D.$$

References

[1] Krylov V.I. Approximate calculation of integrals. Nauka, — Moscow: 1967.

[2] Krylov V. I., Shulgina L. T. Reference book on numerical integration. Nauka, — Moscow: 1966.

[3] Stroud A. H. Approximate calculation of multiple integrals. — Englewood Cliffs, New Jersey: Prentice-Hall, 1977.

[4] Sobolev S. L. Introduction to the theory of cubature formulas. Nauka, — Moscow: 1974.

[5] Misovskikh I. P. Interpolation cubature formulas. Nauka, — Moscow: 1981.

[6] Nikol'skii S. M. Quadrature formulas. Nauka, — Moscow: 1988.

[7] Misovskikh I. P. Cubature formulas and orthogonal polynomials. — JVM i MF, 1969, Vol. 9, No. 2, pp. 419428.

[8] Misovskikh I.P. Application of Orthogonal Polynomials to the Construction of Cubature Formulas. — JVM i MF, 1972, Vol. 12, No. 2, pp. 467475.

[9] Misovskikh I.P. Interpolation Cubature Formulas and Orthogonal Polynomials. — In: Problems of Comput. and Applied Mathematics. Tashkent, 1972. Issue 14, pp. 99-102.

[10] Cools R. Constructing Cubature Formulas: The Science Behind the Art. *Acta Numerica*, Vol. 6, Cambridge University Press, Cambridge, 1997, pp. 1-54.

[11] Cools R., Mysovskikh I.P., Schmid H.J. Cubature formulas and orthogonal polynomials. *Journal of Computational and Applied Mathematics*, 127 (2001), 121-152.

[12] Ismatullaev G.P., Bakhromov S.A., Mirzakobilov R.N. Construction of cubature formulas with minimal number of nodes. *AIP Conference Proceedings*, Vol. 2365, 020019 (2021).

[13] Ismatullaev G.P., Bakhromov S.A., Mirzakobilov R.N. Construction of cubature formulas with fourth and fifth degrees by the reproducing kernel and Radon methods. *Uzbek Mathematical Journal*, 64(2). (2020), pp. 76-84.

[14] Ismatullaev G.P., Mirzakobilov R.N., Karimov R.S. Cubature formulas in the Parabolic Domain. *AIP Conference Proceedings*, Vol. 3004, 060040 (2024).

[15] Akhmedov D.M., Abdikayumova G.A. Construction of optimal quadrature formulas with derivatives for Cauchy type singular integrals in the Sobolev space. *Uzbek Mathematical Journal* 64(1). (2020), pp. 4-9.

[16] Akhmedov D.M.,Atamuradova B.M. Construction of optimal quadrature formulas for Cauchy type singular integrals in the $W_2^{(10)}(0,1)$ space. *Uzbek Mathematical Journal* 66(2). (2022), pp. 59.

[17] Akhmedov D. Optimal approximation of the Hadamard hypersingular integrals. *Uzbek Mathematical Journal* 67(3). (2023), pp. 5-12.

[18] Akhmedov D., Hayotova S., Hayotov M. An optimal approximation formula for reconstruction of tomographic images of radial symmetric functions in the Sobolev space *Uzbek Mathematical Journal* 68(1). (2024), pp. 5-10.

[19] Shadimetov Kh.M., Akhmedov D.M. An optimal approximate solution of the I kind Fredholm singular integral equations. *Filomat* 38(30) (2024), 10765-10796, https://doi.org/10.2298/FIL2430765S

[20] Shadimetov Kh.M., Akhmedov D.M. Numerical integration formulas for hypersingular integrals. *Numer. Math. Theor. Meth. Appl.* 17(3) (2024), pp. 805-826, doi: 10.4208/nmtma.OA-2024-0028

Ismatullaev G.P.
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
email: axmedovdilshod@mail.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.100-113
DOI: 10.29229/uzmj.2025-3-10

Marcinkiewicz multiplier and Littlewood-Paley type theorems in Smirnov-Orlicz classes

Jafarov S.Z.

Abstract. Let G be finite Jordan domain bounded a Dini smooth curve Γ in the complex plane. Marcinkiewicz multiplierand Littlewood-Paley type theorems in Smirnov-Orlicz classes, defined in domain G, are proved.

Keywords: Orlicz spaces, Smirnov spaces, Dini-smooth curve, Faber series, Faber-Laurent series, Marcinkiewicz multiplier therem, Littlewood-Paley theorem.

MSC (2020): 41A10, 41A17, 41A25, 41A27, 41A28, 41A81, 42A10, 46E30.

1. Introduction and main results

Let $\mathbb{T}$ denote the interval $[0,\ 2\pi]$. Let $L_p(\mathbb{T})$, $1 \leq p < \infty$ be the Lebesgue space of all measurable $2\pi-$periodic functions defined on $\mathbb{T}$ such that

$$\|f\|_p := \left(\int\limits_{\mathbb{T}} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Let $\mathbb{T}$ denote the interval $[-\pi,\ \pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$. A convex and continuous function $M : [0, \infty) \to [0, \infty)$ which satisfies the conditions

$$\begin{aligned} M(0) &= 0, \ M(x) > 0 \ \text{ for } x > 0, \\ \lim_{x \to 0} (M(x)/x) &= 0; \quad \lim_{x \to \infty} (M(x)/x) = \infty, \end{aligned}$$

is called a *Young function.* We will say that M satisfies the Δ_2-condition if $M(2u) \leq cM(u)$ for any $u \geq u_0 \geq 0$with some constant c, independent of$u$.

We can consider a right continuous, monotone increasing function $\rho : [0, \infty) \to [0, \infty)$ with

$$\rho(0) = 0; \ \lim_{t \to \infty} \rho(t) = \infty \ \text{ and } \rho(t) > 0 \text{ for } t > 0,$$

then the function defined by

$$N(x) = \int\limits_0^{|x|} \rho(t) \, dt$$

is called $N-$*function.* For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$for which

$$\int\limits_{\mathbb{T}} M(|f(x)|) \, dx < \infty.$$

The $N-$*function complementary to* M is defined by

$$N(y) := \max_{x \geq 0} (xy - M(x)), \text{ for } y \geq 0.$$

Let N be the complementary Young function of M. It is well-known [23, p. 69], [37, pp. 52-68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup \left\{ \int\limits_{\mathbb{T}} |f(x)g(x)| \, dx : g \in \widetilde{L}_N(\mathbb{T}), \ \int\limits_{\mathbb{T}} N(|g(x)|) \, dx \leq 1 \right\}, \tag{1.1}$$

or with the *Luxemburg norm*

$$\|f\|^*_{L_M(\mathbb{T})} := \inf\left\{k>0 : \int\limits_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right)dx \le 1\right\}$$

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [23, p. 26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T}),\ 1<p<\infty$. If $M(x)=M(x,p):=x^p$, $1<p<\infty$, then Orlicz spaces $L_M(\mathbb{T})$ coincide with the usual Lebesgue spaces $L_p(\mathbb{T}),\ 1<p<\infty$. Note that the Orlicz spaces play an important role in many areas such as applied mathematics, mechanics, regularity theory, fluid dynamics and statistical physics (e.g., [12, 32, 39]). Therefore, investigation of approximation of functions by means of Fourier trigonometric series in Orlicz spaces is also important in these areas of research.

The Luxemburg norm is equivalent to the Orlicz norm. The inequalities

$$\|f\|^*_{L_M(\mathbb{T})} \le \|f\|_{L_M(\mathbb{T})} \le 2\,\|f\|^*_{L_M(\mathbb{T})}, \qquad f\in L_M(\mathbb{T})$$

hold [33, p. 80].

If we choose $M(u)=u^p/p,\ 1<p<\infty$ then the complementary function is $N(u)=u^q/q$ with $1/p+1/q=1$ and we have the relation

$$p^{-1/p}\,\|u\|_{L_p(\mathbb{T})} = \|u\|^*_{L_M(\mathbb{T})} \le \|u\|_{L_M(\mathbb{T})} \le q^{1/q}\,\|u\|_{L_p(\mathbb{T})},$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int\limits_{\mathbb{T}} |u(x)|^p\,dx\right)^{1/p}$ stands for the usual norm of the $L_p(\mathbb{T})$space.

If N is complementary to M in Young's sense and $f\in L_M(\mathbb{T})$, $g\in L_N(\mathbb{T})$ then the so-called strong Hölder inequalities [23, p. 80]

$$\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx \le \|f\|_{L_M(\mathbb{T})}\,\|g\|^*_{L_N(\mathbb{T})},$$

$$\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx \le \|f\|^*_{L_M(\mathbb{T})}\,\|g\|_{L_N(\mathbb{T})}$$

are satisfied.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the Δ_2-condition [37, p. 113].

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by the rectifiable Jordan curve Γ. Without loss of generality we assume $0\in Int\ \Gamma$. Let $G^-:\ =Ext\ \Gamma$. Let also $\mathbb{T}:=\{w\in\mathbb{C}:\ |w|=1\}$, $\mathbb{D}=Int\ \mathbb{T}$ and $\mathbb{D}^-=Ext\ \mathbb{T}$. We recall that if for a given analytic function f on G, there exists a sequence of rectifiable Jordan curves (Γ_n) in G tending to the boundary Γ in the sense that Γ_n eventually surrounds each compact subdomain of G such that

$$\int\limits_{\Gamma_n} |f(z)|^p\,|dz| \le K < \infty,$$

then we say that f belongs to the *Smirnov class* $E^p(G),\ 1\le p<\infty$. Each function $f\in E^p(G)$ has non-tangential limit almost everywhere $(a.e.)$ on Γ and the boundary function belongs to $L^p(\Gamma)$.

We define also the *Smirnov -Orlicz classes* $E_M(G)$of analytic functions in G as

$$E_M(G) := \left\{f\in E^1(G) : f\in L_M(\Gamma)\right\}.$$

We define the norm of $f\in E_M(G)$by

$$\|f\|_{E_M(G)} := \|f\|_{L_M(\Gamma)}.$$

Note that Smirnov-Orlicz class $E_M(G)$ is a generalization of the Smirnov class $E^p(G)$. In particular, if $M(x) := x^p$, $1 < p < \infty$, then the Smirnov-Orlicz class coincides with the Smirnov class $E^p(G)$ [20, 28].

Let also χ be a continuous function on 2π. Its modulus of continuity is defined by

$$\omega(t,\chi) := \sup_{t_1,t_2\in[0,2\pi],|t_1-t_2|<t} |\chi(t_1)-\chi(t_2)|, \ t\geq 0 .$$

The curve Γ is called *Dini-smooth* curve if it has the parametrization

$$\Gamma : \chi(t), 0\leq t\leq 2\pi ,$$

such that $\chi'(t)$ is Dini-continuous [36, p.48], i.e.

$$\int_0^{\pi} \frac{\omega(t,\chi')}{t}dt < \infty$$

and

$$\chi'(t)\neq 0.$$

Note that the order of polynomial approximation in $E^p(G)$, $p\geq 1$, has been investigated by several authors. In [42] Walsh and Rusel gave results when Γ is an analytic curve. When Γ is a Dini-smooth curve, direct and inverse theorems were proved by S. Y. Alper [5]. These results were later extended to domains with regular boundaries for $p > 1$ by V. M. Kokilashvili [30] and for $p\geq 1$ by J. E. Andersson [6]. The approximation properties of the $p-$Faber series expansions in the $\omega-$weighted Smirnov class $E^p(G,\omega)$ of analytic functions in G whose boundary is a regular Jordan curve are investigated in [21].

We denote by φ the conformal mapping of G^-onto $\mathbb{D}^-$normalized by

$$\varphi(\infty)=\infty , \quad \lim_{z\to\infty}\frac{\varphi(z)}{z}>0.$$

Let ψ be the inverse of φ. The function φ and ψ have continuous extensions to Γ and $\mathbb{T}$, their derivatives φ' and ψ' have definite non-tangential limit values on Γ and $\mathbb{T}$ a.e., and they are integrable with respect to the Lebesgue measure on on Γ and $\mathbb{T}$, respectively. It is known that $\varphi'\in E^1(G^-)$ and $\psi'\in E^1(\mathbb{D}^-)$. Note that the general information about Smirnov classes can be found in [14, pp. 168-185] and [19, pp. 438-453].

We denote also by $w=\varphi_1(z)$ the conformal mapping of G onto the domain $\mathbb{D}^- := \{w\in\mathbb{C} : |w|>1\}$ normalized by the conditions

$$\varphi_1(0)=\infty, \quad \lim_{z\to 0}(z\varphi_1(z))>0,$$

and let ψ_1 be the inverse mapping of φ_1.

The functions ψ and $\psi_1$have in some deleted neighborhood of the point $w=\infty$ the representations

$$\psi(w)=\gamma w+\gamma_0+\frac{\gamma_1}{w}+\frac{\gamma_2}{w^2}+..., \ \gamma>0$$

and

$$\psi_1(w)=\frac{\alpha_1}{w}+\frac{\alpha_2}{w^2}+...+\frac{\alpha_k}{w^k}+..., \ \alpha_1>0.$$

The following expansions hold [11], [14] and [38]:

$$\frac{\psi'(w)}{\psi(w)-z}=\sum_{k=0}^{\infty}\frac{F_k(z)}{w^{k+1}}, \ z\in G \text{ and } w\in\mathbb{D}^-, \tag{1.2}$$

and

$$\frac{\psi_1'(w)}{\psi_1(w)-z}=\sum_{k=0}^{\infty}-\frac{F^*\left(\frac{1}{z}\right)}{w^{k+1}},\ z\in G^- \text{ and } w\in\mathbb{D}^-, \tag{1.3}$$

where $F_k(z)$ and $F_k^*\left(\frac{1}{z}\right)$ are the Faber polynomials of degree k with respect to z and $\frac{1}{z}$ for the continuums $\overline{G}$and $\overline{\mathbb{C}}\backslash G$, respectively. Also, for the Faber polynomials $\Phi_k(z)$ and rational functions $F_k^*\left(\frac{1}{z}\right)$ the integral representations

$$F_k(z) = [\varphi(z)]^k+\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{[\varphi(\zeta)]^k}{\zeta-z}d\zeta,\ k=0,1,2,...,\ z\in G^-,$$

$$F_k^*\left(\frac{1}{z}\right)=[\varphi_1(z)]^k-\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{[\varphi_1(\zeta)]^k}{\zeta-z}d\zeta,\ k=0,1,2,...,\ z\in G\backslash\{0\}$$

hold [11, 38].

Let $f\in L_1(\Gamma)$. Then the functions f^+ and f^- defined by

$$f^+(z)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f(\psi(w))\,\psi'(w)}{\psi(w)-z}dw,\quad z\in G \tag{1.4}$$

and

$$f^-(z)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f(\psi_1(w))\,\psi_1'(w)}{\psi_1(w)-z}dw,\quad z\in G^- \tag{1.5}$$

are analytic in G and G^-, respectively, and $f^-(\infty)=0$. Thus the limit

$$S_\Gamma(f)(z):=\lim_{\varepsilon\to\infty}\frac{1}{2\pi i}\int\limits_{\Gamma\cap\{\zeta:|\zeta-z|>\varepsilon\}}\frac{f(\zeta)}{\zeta-z}d\zeta$$

exists and is finite for almost all $z\in\Gamma$.

The quantity $S_\Gamma(f)(z)$ is called the *Cauchy singular integral* of f at $z\in\Gamma$.

According to the Privalov theorem [19, p.431] if one of the functions f^+ or f^- has the non-tangential limits almost every (a.e.) on Γ, then $S_\Gamma(f)(z)$ exists a.e. on Γ and also the other one has the non-tangential limits a.e. on Γ. Conversely, if $S_\Gamma(f)(z)$ exists a.e. on Γ, then the functions $f^+(z)$ and $f^-(z)$ have non-tangential limits a.e. on Γ. In both cases, the formulae

$$f^+(z)=S_\Gamma(f)(z)+\frac{1}{2}f(z),\quad f^-(z)=S_\Gamma(f)(z)-\frac{1}{2}f(z) \tag{1.6}$$

and hence

$$f(z)=f^+(z)-f^-(z) \tag{1.7}$$

holds a.e. on Γ. From the results in [25] , it follows that if Γ is a Dini-smooth curve S_Γ is bounded on $L_M(\Gamma)$. Note that some properties of the Cauchy singular integral in the different spaces were investigated in [10, 13, 15, 18, 25, 27, 29, 31, 35].

Let $f\in E_M(G)$. Then taking into account $f\in E^1(G)$ and Cauchy's intergral formula we obtain

$$f(z)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{f(\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f(\psi(w))\,\psi'(w)}{\psi(w)-z}dw,\quad z\in G. \tag{1.8}$$

Then using (1.2) and (1.8) we can associate *Faber series*

$$f(z)\backsim\sum_{k=0}^{\infty}a_kF_k(z), \tag{1.9}$$

where

$$a_k := \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f\left[\psi\left(w\right)\right]}{w^{k+1}} d\omega,\ \ k = 0, 1, 2, \ldots.$$

Let $f \in E_M\left(G^-\right)$. In this case, similar to above by Cauchy's intergral formula and (1.3) we can associate series

$$f\left(z\right) \backsim \sum_{k=1}^{\infty} b_k F_k^*\left(\frac{1}{z}\right), \tag{1.10}$$

where

$$b_k := \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f\left[\psi_1\left(w\right)\right]}{w^{k+1}} dw,\ \ k = 1, 2, \ldots$$

Let $f \in L_M(\Gamma)$. Using (1.2), (1.3), (1.4), (1.5), (1.6) and (1.7) we can associate *Faber-Laurent series*

$$f\left(z\right) \backsim \sum_{k=0}^{\infty} a_k F_k\left(z\right) + \sum_{k=1}^{\infty} b_k F_k^*\left(\frac{1}{z}\right),$$

where the coefficients a_k and b_k are defined by

$$a_k :== \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f\left[\psi\left(w\right)\right]}{w^{k+1}} d\omega,\ \ k = 0, 1, 2, \ldots$$

and

$$b_k := \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f\left[\psi_1\left(w\right)\right]}{w^{k+1}} dw,\ \ k = 1, 2, \ldots$$

The coefficients a_k and b_k are said to be the *Faber-Laurent coefficients* of f.

We use the constants $c_1, c_2, \ldots$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest.

If Γ is a Dini-smooth curve, then from the results in [43], it follows that

$$\begin{array}{ll} 0 < c_1 < \left|\varphi'\left(z\right)\right| < c_2 < \infty, & 0 < c_3 < \left|\varphi_1'\left(z\right)\right| < c_4 < \infty, \\ 0 < c_5 < \left|\psi'\left(w\right)\right| < c_6 < \infty, & 0 < c_7 < \left|\psi_1'\left(\omega\right)\right| < c_8 < \infty, \end{array}$$

where the constants c_1, c_2, c_3, c_4 and c_5, c_6, c_7, c_8 are independent of $z \in \bar{G}^-$and $|w| \geq 1$, respectively.

Let Γ be a Dini-smooth curve and let $f_0\left(w\right) := f\left[\psi\left(w\right)\right]$ for $f \in L_M(\Gamma)$ and let $f_1\left(w\right) := f\left[\psi_1\left(w\right)\right]$ for $f \in L_M(\Gamma)$.Then using (1.10) we obtain $f_0 \in L_M(\mathbb{T})$ and $f_1 \in L_M(\mathbb{T})$ for $f \in L_M(\Gamma)$.
Moreover, $f_0^-(\infty) = f_1^-(\infty) = 0$ and by (1.9)

$$\begin{array}{l} f_0(w) = f_0^+(w) - f_0^-(w), \\ f_1(w) = f_1^+(w) - f_1^-(w). \end{array} \tag{1.11}$$

We also introduce the notations

$$\Delta_k\left(f\right)\left(z\right) = \sum_{s=2^{k-1}}^{2^k-1} a_s\left(f\right) F_s\left(z\right),$$

and

$$\Delta_k^*\left(f\right)\left(z\right) = \sum_{s=2^{k-1}}^{2^k-1} a_s^*\left(f\right) F_s^*\left(1/z\right),$$

for $f \in E_M\left(G\right)$ and $f \in E_M^*\left(G^-\right)$, respectively.

Let Γ be a Dini-smooth curve We obtain $f^+ \in E_M\left(G\right)$ and $f^- \in E_M^*\left(G^-\right)$ for $f \in L_M\left(\Gamma\right)$. Then we can write the series

$$f^+\left(z\right) \sim \sum_{k=0}^{\infty} a_k\left(f^+\right) \Phi_k\left(z\right),\ \ z \in G$$

and

$$f^{+}(z) \sim \sum_{k=0}^{\infty} b_k\left(f^{-}\right) F_k^{*}(1/z), \ z \in G^{-}.$$

The equality $f = f^{+} - f^{-}$ is satisfied a.e. on Γ. Then we can associate with f the formal series

$$f(z) \sim \sum_{k=0}^{\infty} a_k\left(f^{+}\right) F_k(z) - \sum_{k=0}^{\infty} b_k\left(f^{-}\right) F_k^{*}(1/z), \tag{1.12}$$

a.e. on Γ.

We consider the sequences $\{\lambda_s\}_0^{\infty}$ of complex numbers which satisfies the following conditions for all natural numbers s and m,

$$|\lambda_s| \le c_9, \ \sum_{s=2^{m-1}}^{2^m-1} |\lambda_s - \lambda_{s+1}| \le c_{10}. \tag{1.13}$$

Note that for Fourier series the multiplier theorem in Lebesgue spaces was proved by Marcinkiewicz [32] (see also, [46], Vol.II, p.232). Later on the multiplier theorems in the different spaces have been investigated by several authors (see [17, 24, 30]). When the weight function satisfies the Muckenhoupt condition, the Littlewood-Paley type theorem in weighted Lebesgue spaces $L^p(\mathbb{T})$, $1 < p < \infty$, obtained by D. S. Kurtz [24]. When the boundary of domain G is a Carleson curve in the Smirnov classes $E_p(G)$, $1 < p < \infty$, the Littlewood-Paley type theorems have been investigated by A. Guven and D. M. Israfilov [17]. Also, these theorems play an important role in the various problems of approximation theory. Using Littlewood-Paley type theorems, direct and inverse theorems of approximation theory in different spaces are obtained (see [30, 40, 41]).

In this work for Faber series the analogs of Marcinkiewicz multiplier theorem and Littlewood-Paley type theorem are proved in Smirnov-Orlicz classes, defined in the domains with Dini-smooth boundary. Similar problems of approximation of the functions by trigonometric polynomials, Faber polynomials and Faber-Laurent rational functions in different spaces have been investigated by several authors (see ([1-8], [21], [22], [42] and [45]).

Note that in the proof of the main results in this work, we use the methods of proof in the studies [9], [17] and [24].

Our main results are as follows.

Theoem 1.1. *Let G be a finite, simply connected domain with a Dini-smooth boundary Γ and let $L_M(\Gamma)$ be a reflexive Orlicz space on Γ. If $f \in E_M(G)$ with the Faber series (1.9) and $\{\lambda_k\}_0^{\infty}$ is a sequence of complex numbers which satisfies the condition (1.14), then there exists a function $F \in E_M(G)$ which has the Faber series*

$$F(z) \sim \sum_{k=-0}^{\infty} \lambda_k a_k(f) F_k(z), \ z \in G$$

and $\|F\|_{L_M(\Gamma)} \le c\|f\|_{L_M(\Gamma)}$.

We can write similar theorem for $f \in E_M^{*}(G^{-})$:

Theorem 1.2. *Let G be a finite, simply connected domain with a Dini-smooth boundary Γ and let $L_M(\Gamma)$ be a reflexive Orlicz space on Γ. If $f \in E_M^{*}(G^{-})$ with the Faber series (1.10) and $\{\lambda_k\}_0^{\infty}$ is a sequence of complex numbers which satisfies the condition (1.14), then there exists a function $f \in E_M^{*}(G^{-})$ which has the Faber series*

$$F(z) \sim \sum_{k=-1}^{\infty} \lambda_k a_k^{*}(f) F_k^{*}(1/z), \ z \in G^{-}$$

and $\|F\|_{L_M(\Gamma)} \le c\|f\|_{L_M(\Gamma).}$

From Theorem 1.1 and Theorem 1.2 the following Corollary is obtained:

Corollary 1.1. *Let Γ be a Dini-smooth curve. If $f \in L_M(\Gamma)$ has the Faber-Laurent seriers (1.13) and $\{\lambda_k\}_0^{\infty}$ is a sequencve of complex numbers which satisfies the condition (1.14), then there exist a function $F \in L_M(\Gamma)$, which has the Faber-Laurent series*

$$F\left(z\right)\sim\sum_{k=0}^{\infty}\lambda_k a_k\left(f^{+}\right)F_k\left(z\right)-\sum_{k=0}^{\infty}\lambda_k b_k\left(f^{-}\right)F_k^{*}\left(1/z\right)$$

and satisfies $\|F\|_{L_M(\Gamma)}\leq c\,\|f\|_{L_M(\Gamma)}$.

When the boundary of domain G is a Dini-smooth curve ın the Smirnov-Orlicz class $E_M\left(G\right)$ the following Littlewood-Paley theorems hold:

Theorem 1.3. *Let* Γ *be a Dini-smooth curve and* $f\in E_M\left(G\right)$. *Then the two-sided estimate*

$$c_{11}\|f\|_{L_M(\Gamma)}\leq\left\|\left[\sum_{k=0}^{\infty}\left|\Delta_k\left(f\right)\left(z\right)\right|^2\right]^{1/2}\right\|_{L_M(\Gamma)}\leq c_{12}\|f\|_{L_M(\Gamma)}. \tag{1.14}$$

holds.

Theorem 1.4. *Let* Γ *be a Dini-smooth curve and* $f\in E_M^{*}\left(G^{-}\right)$.*Then the two-sided estimate*

$$c_{13}\|f\|_{L_M(\Gamma)}\leq\left\|\left[\sum_{k=0}^{\infty}\left|\Delta_k^{*}\left(f\right)\left(z\right)\right|^2\right]^{1/2}\right\|_{L_M(\Gamma)}\leq c_{14}\|f\|_{L_M(\Gamma)}.$$

holds.

2. Auxiliary results

Let $\mathcal{P}$:={all polynomials (with no restriction on the degree)}, and let $\mathcal{P}(D)$ be the set of traces of members of $\mathcal{P}$ on D. We define two operator as follows [10]:

$$T:=\mathcal{P}(D)\longrightarrow E_M\left(G\right),$$
$$T(P)(z):=\frac{1}{2\pi i}\int_T\frac{P(w)\psi'(w)}{\psi(w)-z}dw,\ z\in G.$$

and

$$T:=\mathcal{P}(D)\longrightarrow E_M^{*}\left(G,\omega\right),$$
$$T^{*}(P)(z):=\frac{1}{2\pi i}\int_T\frac{P(w)\psi_1'(w)}{\psi_1(w)-z}dw,\ z\in G^{-}.$$

It is readily seen that

$$T\left(\sum_{k=0}^{n}b_k w^k\right)=\sum_{k=0}^{n}b_k F_k\left(z\right)\quad\text{and}\quad T^{*}\left(\sum_{k=0}^{n}d_k w^k\right)=\sum_{k=0}^{n}b_k F_k\left(1/z\right).$$

Note that if $z'\in G$, then

$$T(P)(z')=\frac{1}{2\pi i}\int_T\frac{P(w)\psi'(w)}{\psi(w)-z}dw=\frac{1}{2\pi i}\int_T\frac{\left(P\circ\phi\right)\left(\zeta\right)}{\zeta-z}d\zeta=\left(P\circ\phi\right)^{+}\left(z'\right),$$

which by (1.6) imples that

$$T(P)(z)=S_\Gamma(P\circ\phi)\left(z\right)+\frac{1}{2}P\circ\phi)\left(z\right),$$

a.e. on Γ.

Similar to above if $z" \in G^{-}$ the relation

$$T^{*}(P)(z')=\frac{1}{2\pi i}\int_T\frac{P(w)\psi_1'(w)}{\psi_1(w)-z}dw=\frac{1}{2\pi i}\int_T\frac{\left(P\circ\phi_1\right)\left(\zeta\right)}{\zeta-z}d\zeta=\left(P\circ\phi_1\right)^{-}\left(z"\right)$$

holds. Then according to (1.6)

$$T^{*}(P)(z)=S_{\Gamma}(P\circ\phi_{1})(z)-\frac{1}{2}P\circ\phi_{1})(z)$$

holds a.e. on Γ.

According to the Hahn-Banach theorem, we can extend the operators T and T^{*} from $\mathcal{P}(D)$ to the space $E_{M}(\mathbb{D})$ as a linear and bounded operator. Then for these extiension $T:=E_{M}(\mathbb{D})\to E_{M}(G)$ and $T^{*}:=E_{M}(\mathbb{D})\to E_{M}^{*}(G^{-})$ we have the representations

$$\begin{aligned} T(g)(z) &= \frac{1}{2\pi i}\int_{T}\frac{g(w)\psi'(w)}{\psi(w)-z}dw\ ,\ z\in G,\ g\in E_{M}(\mathbb{D}),\\ T^{*}(g)(z) &= \frac{1}{2\pi i}\int_{T}\frac{g(w)\psi_{1}'(w)}{\psi_{1}(w)-z}dw\ ,\ z\in G^{-},\ g\in E_{M}(\mathbb{D}). \end{aligned}$$

In the proof of Theorem 1.1 and Theorem 1.2 we use the following Lemmas:

Lemma 2.1. *Let Γ be a Dini-smooth curve. Furter let g be an analytic function in $\mathbb{D}$ which has the Taylor expansion* $g(w)=\sum\limits_{s=0}^{\infty}c_{s}(g)w^{s}$.

1. *If $g\in E_{M}(\mathbb{D})$, then $T(g)$ has the Faber coefficients $c_{s}(g)$, $k=0,1,2,...$*
2. *If $g\in E_{M}(\mathbb{D})$, then $T^{*}(g)$ has the Faber coefficients $c_{s}(g)$, $k=0,1,2,...$*

Proof. Let's prove the second case first. Let $g_{r}(w):=g(rw)$, $0<r<1$. It is clear that $g\in E_{1}(\mathbb{D})$. The function g coincides with the Poisson integral of its boundary function. Then using [34, Th. 10] we obtain

$$\|g_{r}-g\|=\left\|g\left(re^{i\theta}\right)-g\left(e^{i\theta}\right)\right\|_{L_{M}(\mathbb{T})}\to 0,\ r\to 1^{-}.$$

The operator T^{*} is bounded in the Orlicz space $L_{M}(\Gamma)$. Hence we conclude that

$$\left\|T^{*}(g_{r})-T^{*}(g)\right\|_{L_{M}(\Gamma)}\to 0,\ as\ r\to 1^{-}. \tag{2.1}$$

Note that the series $\sum\limits_{s=0}^{\infty}c_{s}(g)w^{s}$ is uniformly convergent for $|w|=r<1$, Therefore, the series $\sum\limits_{s=0}^{\infty}c_{s}(g)r^{s}w^{s}$ converges uniformly on $\mathbb{T}$. Then we can write the following expansion for the operator:

$$\begin{aligned} T^{*}(g_{r})(z) &= -\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{g_{r}(w)\psi_{1}'(w)}{\psi_{1}'(w)-z}dw\\ &= \sum_{s=0}^{\infty}c_{s}(g)r^{s}\left\{-\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{w^{s}\psi_{1}'(w)}{\psi_{1}'(w)-z}dw\right\}\\ &= \sum_{s=0}^{\infty}c_{s}(g)r^{s}F_{s}^{*}(1/z),z\in G^{-}. \end{aligned}$$

Taking the limit as $z^{*}\to z\in\Gamma$ along all non-tangential paths outside Γ, we have

$$T^{*}(g_{r})(z)=\sum_{s=0}^{\infty}c_{s}(g)r^{s}F_{s}^{*}(1/z), \tag{2.2}$$

for $z\in\Gamma$. Consideration of (2.2) and [18 p.43, Lemma 3] gives us

$$\begin{aligned} b_{s}(T^{*}(g_{r})) &= \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{T^{*}(g_{r})\psi_{1}(w)}{w^{s+1}}dw\\ &= \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\sum\limits_{k=0}^{\infty}c_{k}(g)r^{k}F_{k}^{*}(\psi_{1}(w))}{w^{s+1}}dw\\ &= \sum_{k=0}^{\infty}c_{k}r^{k}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{F_{k}^{*}(\psi_{1}(w))}{w^{s+1}}dw=c_{s}r^{s}. \end{aligned}$$

Hence for $r\to 1^{-}$ we have

$$b_s\left(T^{*}\left(g_r\right)\right)\to c_s. \tag{2.3}$$

Using (1.11), Hölder inequality for the space $L_M\left(\Gamma\right)$ and [36, Theorem 2.1] we find that

$$\begin{aligned}\left|b_s\left(T^{*}\left(g_r\right)\right)-b_s\left(T^{*}\left(g\right)\right)\right| &= \left|\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\left[T^{*}\left(g_r\right)-T^{*}\left(g\right)\right]\psi_1\left(w\right)}{w^{k+1}}dw\right| \\ &\le \frac{1}{2\pi}\int_{\mathbb{T}}\left|\left[T^{*}\left(g_r\right)-T^{*}\left(g\right)\right]\right|\psi_1\left(w\right)dw \\ &= \frac{1}{2\pi}\int_{\Gamma}\left|\left[T^{*}\left(g_r\right)-T^{*}\left(g\right)\right]\left(z\right)\right|\left|\varphi_1'\left(z\right)\right|\left|dz\right| \\ &\le \frac{c}{2\pi}\int_{\Gamma}\left|\left[T^{*}\left(g_r\right)-T^{*}\left(g\right)\right]\left(z\right)\right|\left|dz\right| \\ &\le \frac{c}{2\pi}\left\|T^{*}\left(g_r\right)-T^{*}\left(g\right)\right\|_{L_M(\Gamma)}.\end{aligned} \tag{2.4}$$

Use of (2.1) and (2.4) gives us

$$b_s\left(T^{*}\left(g_r\right)\right)\to b_s\left(T^{*}\left(g\right)\right),\ r\to 1^{-}. \tag{2.5}$$

Then from (2.3) and (2.5) we conclude that

$$b_s\left(T^{*}\left(g\right)\right)=c_s\left(g\right),\ s=0,1,2,...$$

The second case of Lemma 2.1 is proved. The proof of the first case of Lemma 2.1 is done similarly to the proof of the second case. □

Lemma 2.2. *Let $\{\lambda_k\}_0^{\infty}$ be a sequence which satisfies the condition (1.14). If the function $g\in E_m\left(\mathbb{D}\right)$ has the Taylor series*

$$g\left(w\right)=\sum_{s=0}^{\infty}c_s\left(g\right)w^s,\ w\in D,$$

then there exists a function $g^{*}\in E_m\left(\mathbb{D}\right)$ which has the Taylor series

$$g^{*}\left(w\right)=\sum_{s=0}^{\infty}\lambda_s c_s\left(g\right)w^s,\ w\in D$$

and satisfies $\left\|g^{*}\right\|_{L_M(\mathbb{T})}\le c\left\|g\right\|_{L_M(\mathbb{T})}$.

Proof. $\beta_s\left(g\right)$ $(s=...-1,0,1,..)$ denote the Fourier coefficients of the boundary function of g. By Theorem 3.4 in [16, p.38] we have

$$\beta_s\left(g\right)=\begin{cases}c_s\left(g\right), & s\ge 0;\\ 0, & s<0.\end{cases}$$

Using the proof method of Theorem 2 in [34], we can show that there is a function $v\in L_M\left(\mathbb{T}\right)$ with Fourier coefficients $\beta_s\left(v\right)=\lambda_s\beta_s\left(g\right)$ and $\left\|v\right\|_{L_M(\mathbb{T})}\le c\left\|g\right\|_{L_M(\mathbb{T})}$. If we write $g^{*}=v^{+}$, then $g^{*}\in E_M\left(\mathbb{D}\right)$. For Taylor coefficients of g^{*}, we have by (1.7)

$$\begin{aligned}c_s\left(g^{*}\right) &= c_s\left(v^{+}\right)=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{v^{+}\left(w\right)}{w^{s+1}}dw=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{v\left(w\right)}{w^{s+1}}dw+\frac{1}{2\pi}\int_{\mathbb{T}}\frac{v^{-}\left(w\right)}{w^{s+1}}dw \\ &= \frac{1}{2\pi}\int_{\mathbb{T}}\frac{v\left(w\right)}{w^{s+1}}dw=\beta_k\left(v\right)=\lambda_s\beta_s\left(g\right)=\lambda_s c_s\left(g\right),\ s=0,1,2,...\end{aligned}$$

Therefore, we have

$$\left\|g^{*}\right\|_{L_M(\mathbb{T})}=\left\|v^{+}\right\|_{L_M(\mathbb{T})}\le c\left\|v\right\|_{L_M(\mathbb{T})}\le\left\|g\right\|_{L_M(\mathbb{T})}.$$

The proof of Lemma 2.2 is completed. □

3. Proof of the main results

Proof of Theorem 1.1. Let $f\in E_M\left(G\right)$. Using (1.12) we have $f_0\left(w\right)=f_0^+\left(w\right)-f_0^-\left(w\right)$. Then by the definitions of the coefficients $a_k\left(f\right)$ we get

$$\begin{aligned} a_k\left(f\right) &= \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0\left(w\right)}{w^{k+1}}dw=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^+\left(w\right)}{w^{k+1}}dw-\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^-\left(w\right)}{w^{k+1}}dw \\ &= \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_0^+\left(w\right)}{w^{k+1}}dw=c_k\left(f_0^+\right),\ k=0,1,2.... \end{aligned}$$

That is the Faber coefficients of f are the Taylor coefficients of $f_0^+\left(w\right)$ at the orgin. Then for the fungtion f_0^+ the Taylor expansion

$$f_0^+\left(w\right):=\sum_{k=0}^{\infty}c_k\left(f\right)w^k,\ w\in\mathbb{D}$$

holds. By virtue of Lemma 2.2, there is a function $F_0\in E_M\left(\mathbb{D}\right)$ which has the Taylor coefficents $c_k\left(F_0\right)=\lambda_k a_k\left(f\right),\ k=0,1,2,...$and the following inequality holds:

$$\left\|F_0\right\|_{L_M(\mathbb{T})}\leq c_{15}\left\|f_0^+\right\|_{L_M(\mathbb{T})}.$$

Then by [20] $T\left(F_0\right)\in E_M\left(G\right)$ for $F_0\in E_M\left(\mathbb{D}\right)$. It is clear that according to Lemma 2.1 the Faber coefficients of $T\left(F_0\right)$ are $c_k\left(F_0\right)=\lambda_k a_k\left(f\right)$. Then we can write the following expansion for $T\left(F_0\right)$:

$$T\left(F_0\right)\left(z\right)\sim\sum_{k=0}^{\infty}\lambda_k a_k\left(f\right)F_k(z),\ z\in G. \tag{3.1}$$

Using the boundedness of the operator T, (3.1) and the boundedness of the Cauchy singular operator in $L_M\left(\mathbb{T}\right)$ we have

$$\begin{aligned} \left\|T\left(F_0\right)\right\|_{L_M(\Gamma)} &\leq \left\|T\right\|\left\|F_0\right\|_{L_M(\mathbb{T})}\leq\left\|f_0^+\right\|_{L_M(\mathbb{T})} \\ &\leq c_{16}\left\|f_0\right\|_{L_M(\mathbb{T})}\leq c_{17}\left\|f\right\|_{L_M(\Gamma)}. \end{aligned}$$

If $F:=T\left(F_0\right)$ is written in the last inequality, the desired result in Theorem 1.1 is obtained. The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2. By considering the formula of the Faber coefficient of $f\in E_M^*\left(G^-\right)$,

$$\begin{aligned} b_k\left(f\right) &= \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_1\left(w\right)}{w^{k+1}}dw=\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_1^+\left(w\right)}{w^{k+1}}dw-\frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_1^-\left(w\right)}{w^{k+1}}dw \\ &= \frac{1}{2\pi i}\int\limits_{\mathbb{T}}\frac{f_1^+\left(w\right)}{w^{k+1}}dw=c_k\left(f_1^+\right),\ k=1,2... \end{aligned}$$

That is the Faber coefficients of f are the Taylor coefficients of $f_0^+\left(w\right)$ at the orgin. Using Lemma 2.2 there exists a function $F_1\in E_M\left(D\right)$ the following expansion and inequality holds:

$$F_1\left(w\right)=\sum_{k=0}^{\infty}\lambda_k b_k\left(f\right)w^k,\ w\in D,$$

and

$$\left\|F_1\right\|_{L_M(\mathbb{T})}\leq c_{18}\left\|f_1^+\right\|_{L_M(\mathbb{T})}.$$

If $F:=T^*\left(F_1\right)$ is written and using Lemma 2.1, we have

$$F(z)=\sum_{k=1}^{\infty}\lambda_k b_k(f)\,F_k^*(1/z)\,,\ \ z\in G^-.$$

Taking into account the boundedness of the operator T^*, boundedness of the singular operator in $L_M(\mathbb{T})$ [35] and the formulas (1.6) we conclude that

$$\begin{aligned}\|F\|_{L_M(\Gamma)} &= \|T^*(F_1)\|_{L_M(\Gamma)}\le\|T^*\|\,\|F_1\|_{L_M(\mathbb{T})}\\ &\le c_{19}\left\|f_1^+\right\|_{L_M(\mathbb{T})}\le c_{20}\|f_1\|_{L_M(\mathbb{T})}\le c_{21}\|f\|_{L_M(\Gamma)}\,.\end{aligned}$$

Thus, the required result was obtained. □

Proof of Theorem 1.3. Let $\{r_k\}_0^\infty$ be the sequence of Rademacher functions and let $t\in[0,1]$ be not dyadic rational number. If we set $\lambda_0:=r_0(t)$ and $\lambda_j=r_k(t)\quad 2^{k-1}\le j\le 2^k$, then the sequence $\{\lambda_j\}_0^\infty$ satisfies the condition (1.14). By Theorem 1.1 there exists a function $F\in E_M(G)$ such that

$$F(z)\sim\sum_{j=0}^{\infty}\lambda_j a_j(f)\,F_j(z)=\sum_{k=0}^{\infty}r_k(t)\,\Delta_k(f)(z) \tag{3.2}$$

and

$$\|F\|_{L_M(\Gamma)}\le c_{22}\|f\|_{L_M(\Gamma)}\,.$$

On the other hand since

$$F(z)\sim\sum_{k=0}^{\infty}r_k(t)\,\Delta_k(f)(z)$$

and $\{\lambda_j\}_0^\infty$ satisfies (1.14), there is $F^*\in E_M(G)$ for which

$$F^*(z)\sim\sum_{k=0}^{\infty}\lambda_k r_k(t)\,\Delta_k(f)(z)=\sum_{k=0}^{\infty}a_k(f)\,F_k(z)$$

and

$$\|F^*\|_{L_M(\Gamma)}\le c_{23}\|F\|_{L_M(\Gamma)}$$

holds. Since there is no two different functions in $E_M(G)$ have the same Faber series we have $F^*=f$. Then we find that

$$c_{24}\|f\|_{L_M(\Gamma)}\le\|F\|_{L_M(\Gamma)}\le c_{25}\|f\|_{L_M(\Gamma)}\,. \tag{3.3}$$

Using Holder inequality for $f\in L_M(\Gamma)$ and $g\in L_N(\Gamma)$, (3.2) and (1.1) we obtain

$$\begin{aligned}&\int_\Gamma|F(z)\,g(z)|\,dz\\ =\ &\int_\Gamma\left|\sum_{k=0}^{\infty}r_k(t)\,\Delta_k(f)(z)\,\Delta_k(g)(z)\right|dz\le\int_\Gamma\sum_{k=0}^{\infty}|r_k(t)\,\Delta_k(f)(z)\,\Delta_k(g)(z)|\,dz\\ \le\ &c\int_\Gamma\left[\sum_{k=0}^{\infty}|\Delta_k(f)(z)|^2\right]^{1/2}\left[\sum_{k=0}^{\infty}|\Delta_k(g)(z)|^2\right]^{1/2}\\ \le\ &c\left\|\left[\sum_{k=0}^{\infty}|\Delta_k(f)(z)|^2\right]^{1/2}\right\|_{L_M(\Gamma)}\left\|\left[\sum_{k=0}^{\infty}|\Delta_k(g)(z)|^2\right]^{1/2}\right\|_{L_N(\Gamma)}\\ \le\ &c\left\|\left[\sum_{k=0}^{\infty}|\Delta_k(f)(z)|^2\right]^{1/2}\right\|_{L_M(\Gamma)}\|g\|_{L_N(\Gamma)}\,.\end{aligned}$$

Now taking supremum in the last inequality for all functions $g\in L_N(\Gamma)$ satisfying $\|g\|_{L_M(\Gamma)}\le 1$, we find that

$$\|F\|_{L_M(\Gamma)} \leq c_{26} \left\| \left[\sum_{k=0}^{\infty} |\Delta_k(f)(z)|^2 \right]^{1/2} \right\|_{L_M(\Gamma)}. \tag{3.4}$$

Using the proof method in study [44] (see. also [46], Vol 1, p.213) inequality,

$$\|F\|_{L_M(\Gamma)} \geq c_{27} \left\| \left[\sum_{k=0}^{\infty} |\Delta_k(f)(z)|^2 \right]^{1/2} \right\|_{L_M(\Gamma)} \tag{3.5}$$

is proven similary to inequality (3.4). The relations (3.3), (3.4) and (3.5) immediately yield (1.15). The proof of Theorem 1.3 is completed. □

Note that the proof of Theorem 1.4 is similar to proof of Theorem 1.3.

Acknowledgement. The author thanks the referees for all their valuable comments and suggestions and also for his precious remarks, which significantly improved the paper.

References

[1] F. G. Abdullayev, I. A. Shevchuk, *Uniform estimates for polynomial approximation in domains with cornes*, J. Approximation Theory, 137 (2) (2005), 143-165.

[2] F. G. Abdullayev, P. Özkartepe, V. V. Savchuk, A. I. Shidlich, *Exact constants in direct and inverse approximation theorems for functions of several variables in the spaces S^p*, Filomat 33 (5) (2019), 1471-1484.

[3] F. G. Abdullayev, S. Chaichenko, M. I. Kyzy, A. Shidlich, *Direct nnd inverse approximation theorems in the weighted Orlicz-type with variable exponent*, Turkish J. of Math. 44 (1) (2020), 284-299.

[4] F. G. Abdullayev, S. Chaichenko, M. Imashkyzy, A. Shidlich, *Jackson-type inequalities and widths of functional classes in the Musielak-Orlicz type spaces*, Rocky Mountain J. Math. 51 (4) (2021), 1143-1155.

[5] S. Y. Alper, *Approximation in the mean of analytic functions of class E^p*, In: Investigations on the Modern Problems of the Function Theory of a Complex Variable, Gos. Izdat. Fiz.-Mat. Lit., Moscow, 1960, pp. 272-286. (In Russian.)

[6] J. E. Andersson, *On the degree of poynomial approximation in $E^p(D)$. J. Approximation Theory* **19** (1977), 61-68.

[7] V. V. Andrievskii, H. Blatt, Discrepancy of Signed Measures and Polynomial Approximation, Springer Verlag, New York, NY, USA, 2002.

[8] V. V. Andrievskii, V. I. Belyi and V. K. Dzyadyk, Conformal Invariants in Constructive Theory of Functions of Complex Variable, Adv. Series in Math. Sciences and Engineer, WEP Co., Atlanta, Georgia, 1995.

[9] R. Akgün, D.M. Israfilov, *Approximation and moduli of fractional orders in Smirnov-Orlicz classes,* Glas. Mat. **43**(63) (2008), 121-136.

[10] A. Böttcher, Y. I. Karlovich, Carleson Curves, Muckenhoupt Weights and Teoplitz Operators, Birkhauser-Verlag, 1997.

[11] A. Cavus, D. M. Israfilov, *Approximation by Faber-Laurent retional functions in the mean of functions of the class $L_p(\Gamma)$ with $1 < p < \infty$*, Approx. Theory Appl. **11** (1) (1995), 105-118.

[12] M. Colombo and G. Mingione, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215** (2015), no. 2, 443-496.

[13] G. David, *Operateurs integraux singulers sur certains courbes du plan complexe*, Ann. Sci. Ecol. Norm. Super. **4** (1984), 157-189.

[14] P. L. Duren, Theory of H^p spaces, Academic Press, 1970.

[15] E. M. Dyn'kin and B.P. Osilenker, *Weighted estimates for singular integrals and their appllications*, In: Mathematical Analysis, Vol. 21. Akad. Nauk. SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983, pp.42-129.

[16] D. Gaier, Lectures on Complex Approximation, Birkhäuser, 1987.

[17] A. Guven and D. M. Israfilov, *Multiplier theorems in weighted Smirnov spaces,* J. Korean Math. Soc. **45** (6) (2008), 1535-1548.

[18] V. S. Guliyev, J. Hasanov , S. Samko, *Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces*, Math. Scand. **107** (2010), 285-304.

[19] G. M. Goluzin , *Geometric Theory of Functions of a Complex Variable,* Traslation of Mathematical Monographs, 26, Providence, RI: AMS, 1968

[20] D. M. Israfilov, R. Akgün, *Approximation in weighted Smirnov-Orlicz classes*, J. Math. Kyoto University (JMKYAZ), **46** (4) (2006), 755-770.

[21] D. M. Israfilov, *Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G,w)$ and the Bieberbach polynomials,* Constr. Approx. **17** (2001), 335-351.

[22] D. M. Israfilov, N. P. Tozman, *Approximation in Morrey-Smirnov classes*, Azerbaijan J. Math. **1** (2) (2011), 99-113.

[23] M. A. Krasnoselskii and Ya.B.Rutickii, *Convex Functions and Orlicz Spaces,* P. No

[24] D. S. Kurtz, *Littlewood-Paley and multiplier theorems on weighted L^p spaces,* Trans. Amer. Math. Soc. **259** ((1) (1980), 235-254.

[25] A. Yu. Karlovich, *Algebras of singular integral operators with piecewise continuous coefficients on relexive Orlicz spaces*, Math. Nachr. **179** (1996), 187-222.

[26] A. Yu. Karlovich, *Algebras of singular integral operators with PC coefficients in rearrangement -invariant spaces with Muckenhoupt weights,* J. Operator Theory, **47** (2002), 303-323.

[27] V. M. Kokilashvili, V. Paatasvili, S. Samko, *Boundary value problems for analytic functions in the class of Cauchy type integrals with density in* $L^{p(\cdot)}(\Gamma)$,Boundary Value Problems, Vol. 2005, Hindawi Publ. Cor., 2005, pp.43-71.

[28] V. M. Kokilashvili, *On analytic functions of Smirnov-Orlicz classes*, Stud. Math. 31 (1968), 43-59.

[29] V. M. Kokilashvili , S. Samko, *Weighted boundedness in Lebesgue spaces with variable exponentsof classical operators on Carleson curves,* Proc. A. Razmadze Math. Inst. **138** (2005) 106-110.

[30] V. M. Kokilashvili, A *direct theorem for the approximation in the mean of analytic functions by polynomials,* Dokl. Akad. Nauk. SSSR **185** (1969), 749–752.

[31] V. M. Kokilashvili and S. Samko, *Singular integrals and potentials in some Banach function spaces with variable exponent,* J. Funct. Spaces Appl. **1** (1) (2003), 45-59.

[32] J. Marcinkiewicz, *Sur les Multiplicateurs des Series de Fourier*, Studia Math. 8 (1939), 78-91.

[33] W. A. Majewski and L. E. Labuschange, *On application of Orlicz spaces to statistical physics*, Ann. Henri Pioncare **15** (2014), 1197-1221.

[34] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 167 (1972), 207-226.

[35] A. Meskhi, *Maximal functions and singular integrals in Morrey spaces associated with grand Lebesgue spaces,* Proc. A. Razmadze Math. Inst. **151** (2009), 130-143.

[36] Ch. Pommerenke, Boundary Behavior of Conformal Maps, Berlin, SprimgerVerlag, 1992.

[37] M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces,* Marcel Dekker, New York, 1991

[38] P. K. Suetin, *Series of Faber polynomials*, Gordon and Breach Science Publishers, 1998.

[39] A. Swierczewska-Gwiazda, *Nonlinear parabolic problems in Musielak- Orlicz spaces,* Nonlinear Anal. **98** (2014), 48-65.

[40] M. F. Timan, *Inverse theorems of the constructive theory of functions in L_p spaces (*$1 \leq p \leq \infty$), Mat. Sb. N.S. **46** (88) (1958), 125-132.

[41] M. F. Timan, *On Jordan's theorems in L_p spaces*, Ukrain. Mat. Z. **18** (1) (1966), 134-137.

[42] J. L. Walsh, H.G. Russel, *Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions.* Trans. Amer. Math. Soc. **92** (1959), 355-370.

[43] S. E. Warschawskii, *Über das Randverhalten der Ableitung der Abbildungsfunktionen bei Konformer Abbildung*, Math. Z., **35** (1932), 321-456

[44] Y. E. Yıldırır and D. M. Isafilov, *The ğroperties of convolution type transforms in weighted Orlicz spaces,* Glas. Mat. **45**(65) (2010), 461-474.

[45] H. Yurt, A. Guven, *Approximation by Faber-Laurent rational functions on doubly connected domains,* New Zealand Journal of Math. **44** (2014), 113-124.

[46] A. Zygmund, Trigonometric series, Vol. I-II, Cambridge Univ. Press, 2nd-edition, 1959.

Jafarov S.Z.,
Department of Mathematics and Science Education,
MuS Alparslan University, 49250, MuS, Turkey;
Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan,
9 B. Vahabzadeh str., AZ1141, Baku, Azerbaijan
email: s.jafarov@alparslan.edu.tr

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.114-120
DOI: 10.29229/uzmj.2025-3-11

Solution of the Monge-Ampere equation in a ring domain

Kholmurodova G.N.

Abstract. The problem of recovering surfaces by the total or extrinsic curvature is related to the solution of the nonlinear elliptic equation of the Monge-Ampere type. Using the geometric method, the existence and uniqueness of a solution to the Monge-Ampere equation is shown in the problem of recovering a surface by its total curvature in isotropic space. In this article, an exact solution to the Dirichlet problem for a ring domain is found if the total curvature function is given exact form. In this, isotropic space geometry is used.

Keywords: Total curvature, Monge-Ampere equation, ring domain, isotropic space, extrinsic curvature, boundary conditions.

MSC (2020): 35J96, 35J25

1. Introduction

Initially, in the Euclidean space, A.D. Alexandrov posed the problem of recovering of convex polyhedra by extrinsic curvature and proved the existence and uniqueness of a solution[1]. Then, he generalized this problem for convex surfaces [2]. That is, this problem was solved if the extrinsic curvature is a non-negative, complete additive set function defined on the Borel set. A.V. Pogorelov, using the property of monotonicity of the extrinsic curvature of the convex polyhedra, showed that a convex polyhedron exists and is unique by a given monotonic function [17]. I. Ya. Bakelman studied the connection between the extrinsic curvature of convex surfaces and the second-order nonlinear Monge-Ampere equation [10]. In this case, I. Ya. Bakelman showed that the solution of the generalized Dirichlet problem for the Monge-Ampere equation exists and is unique by estimating the area of the normal image of the surface [11]. The listed problems were solved only if the domain $D \subset R_2$ is convex where the function is defined. By applying the geometry of the Galilean space, A. Artykbaev solved the problem of the existence and uniqueness of the convex surface according to the given extrinsic curvature if the domain $D \subset R_2$ is non-convex [3]. Also, in the article [4], the concept of generalized extrinsic curvature is given, and the existence and uniqueness of the solution of the Monge-Ampere equation in the multi-connected domain is proved. In addition, Sh.Sh. Ismoilov found a solution for the family of dual translation surfaces, using the geometry of isotropic space, which is the total curvature is the product of two functions with separate variables [5]. If the total curvature in the isotropic space is zero, in [8, 12, 15], translation surfaces are classified according to their analytical equation. M.E. Aydin and other co-authors studied the class of different surfaces which is the total curvature is equal to a non-zero constant [7, 9, 20]. In this paper, we consider the problem of recovering surfaces in isotropic space where the total curvature is a function defined in the ring domain, and find an exact solution by solving the Dirichlet problem for the Monge-Ampere equation.

2. Preliminaries

It is known that the Monge-Ampere equation is generally as follows:

$$z_{xx}z_{yy} - z_{xy}^2 = \phi(x, y, z, z_x, z_y). \tag{2.1}$$

In this case, if $\phi(x, y, z, z_x, z_y) > 0$, the equation is an elliptic and its solution is a convex surface equation.

Now, if we consider this equation in the semi-Euclidean space, that is, in the isotropic space, it will be as follows [14]:

$$z_{xx}z_{yy} - z_{xy}^2 = K(x, y). \tag{2.2}$$

2.1. **Isotropic space geometry.** Let there be given an affine space A_3 with the coordinate system $O\{e_1, e_2, e_3\}$. If the inner product of two vectors $\overrightarrow{X}\{x_1, x_2, x_3\}$ and $\overrightarrow{Y}\{y_1, y_2, y_3\}$ is determined in this space as follows:

$$(X,Y) = \begin{cases} (X,Y)_1 = x_1y_1 + x_2y_2 & (X,Y)_1 \neq 0, \\ (X,Y)_2 = x_3y_3 & (X,Y)_1 = 0. \end{cases} \tag{2.3}$$

Then this space is called an isotropic space[6].

In the isotropic space the norm of the vector $\overrightarrow{X}$ is determined by the $\overrightarrow{X} = \sqrt{(\overrightarrow{X}, \overrightarrow{X})}$. From this, the distance between two points $A(x_1, x_2, x_3)$ and $B(y_1, y_2, y_3)$ is calculated by the following formula:

$$d = \begin{cases} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} & \text{if } \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \neq 0, \\ |y_3 - x_3| & \text{if } \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 0. \end{cases} \tag{2.4}$$

In the isotropic space, the motion is given by the following and this preserves the distance (2.4):

$$\begin{cases} x' = x\cos\alpha - y\sin\alpha + a, \\ y' = x\sin\alpha + y\cos\alpha + b, \\ z' = h_1x + h_2y + z + c. \end{cases} \tag{2.5}$$

Let the regular surface be given by the following vector equation in this space:

$$\vec{r}(x,y) = r_1(x,y)\vec{e}_1 + r_2(x,y)\vec{e}_2 + r_3(x,y)\vec{e}_3. \tag{2.6}$$

Where, $r_i(x,y)$ are the parametric functions of the surface, and $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are the basis vectors. The first and second fundamental forms of the surface are given by the following formulas:

$$I = ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

$$II = Ldu^2 + 2Mdudv + Ndv^2.$$

The coefficients E, F, G and L, M, N of the first and second fundamental forms of the surface in isotropic space are calculated as follows:

$$\begin{cases} E = \vec{r}_x^2 = r_{1x}^2 + r_{2x}^2, \\ F = \vec{r}_x\vec{r}_y = r_{1x}r_{1y} + r_{2x}r_{2y}, \\ G = \vec{r}_y^2 = r_{1y}^2 + r_{2y}^2, \end{cases} \qquad \begin{cases} L = (\vec{r}_{xx}, \vec{n}), \\ M = (\vec{r}_{xy}, \vec{n}), \\ N = (\vec{r}_{yy}, \vec{n}). \end{cases} \tag{2.7}$$

And from this, as an analogue of Euclidean space, total curvature in isotropic space is determined by the following formula and differs only in the calculation of the coefficients of the first and second fundamental forms:

$$K = \frac{LN - M^2}{EG - F^2}. \tag{2.8}$$

If the surface is one-valued projected onto the plane Oxy, then the total curvature is as follows[13]:

$$K = LN - M^2, \tag{2.9}$$

where, $L = z_{xx}$, $M = z_{xy}$, $N = z_{yy}$. So, we determine equation (2.2) by these coefficients.

3. Main result

Let there be given a regular surface F by explicit form $z = z(x,y)$ in the domain $D \subset R_2$. Then its total curvature is determined by the (2.9) in the isotropic space [16].

In this space, we consider the domain $D \subset R_2$ which is bounded by the curves $L_1 : x^2 + y^2 = b^2$ and $L_2 : x^2 + y^2 = a^2$ on the Oxy plane. Let a spatial closed curve H be given and projected one-valued onto the L_2. The domain $D\subset R_2$ is a doubly-connected domain in the form of the ring and it is given by the following equation:

$$D=\left\{(x,y):b^2\le x^2+y^2\le a^2,0<b<a\right\}. \tag{3.1}$$

We consider the problem of recovering the surface by the total curvature which is being a function defined in the ring domain. That is, we find the analytical equation of the surface by solving the Monge-Ampere equation. For this, we transfer equation (2) to the polar coordinate system. It is known that the connection between polar and Cartesian coordinates is as follows:

$$\begin{cases} x=\rho\cos\varphi, \\ y=\rho\sin\varphi. \end{cases} \tag{3.2}$$

If we determine the first- and second-order and mixed partial derivatives of the equation $z=z\,(x,y)$ with respect to ρ and φ, we obtain the following expressions:

$$z_\rho=z_x\cos\varphi+z_y\sin\varphi\ ,\quad z_\varphi=\rho\left(-z_x\sin\varphi+z_y\cos\varphi\right),$$

$$z_{\rho\rho}=z_{xx}\cos^2\varphi+z_{yy}\sin^2\varphi+z_{xy}\sin 2\varphi,$$

$$z_{\varphi\varphi}=\rho^2\left(z_{xx}\sin^2\varphi+z_{yy}\cos^2\varphi-z_{xy}\sin 2\varphi\right)-\rho\left(z_x\cos\varphi+z_y\sin\varphi\right),$$

$$z_{\rho\varphi}=-\rho\tfrac{\sin 2\varphi}{2}\left(z_{xx}-z_{yy}\right)+z_{xy}\rho\cos 2\varphi+\left(-z_x\sin\varphi+z_y\cos\varphi\right).$$

From this,

$$z_{\rho\rho}=z_{xx}\cos^2\varphi+z_{yy}\sin^2\varphi+z_{xy}\sin 2\varphi, \tag{3.3}$$

$$\frac{z_{\varphi\varphi}}{\rho^2}+\frac{z_\rho}{\rho}=z_{xx}\sin^2\varphi+z_{yy}\cos^2\varphi-z_{xy}\sin 2\varphi, \tag{3.4}$$

$$\frac{z_{\rho\varphi}}{\rho}-\frac{z_\varphi}{\rho^2}=-\frac{\sin 2\varphi}{2}\left(z_{xx}-z_{yy}\right)+z_{xy}\cos 2\varphi. \tag{3.5}$$

By simplifying the expressions (3.3),(3.4),(3.5) we get the following:

$$\begin{cases} z_{xx}+z_{yy}=z_{\rho\rho}+\frac{z_\rho}{\rho}+\frac{z_{\varphi\varphi}}{\rho^2}, \\ z_{xx}-z_{yy}=\left(z_{\rho\rho}-\frac{z_\rho}{\rho}-\frac{z_{\varphi\varphi}}{\rho^2}\right)\cos 2\varphi-2\left(\frac{z_{\rho\varphi}}{\rho}-\frac{z_\varphi}{\rho^2}\right)\sin 2\varphi. \end{cases} \tag{3.6}$$

From equation (3.6), we find the following for the second-order partial derivatives:

$$z_{xx}=\frac{1}{2}\left(z_{\rho\rho}+\frac{z_\rho}{\rho}+\frac{z_{\varphi\varphi}}{\rho^2}\right)+\frac{1}{2}\left(\left(z_{\rho\rho}-\frac{z_\rho}{\rho}-\frac{z_{\varphi\varphi}}{\rho^2}\right)\cos 2\varphi-\left(\frac{z_{\rho\varphi}}{\rho}-\frac{z_\varphi}{\rho^2}\right)\sin 2\varphi\right), \tag{3.7}$$

$$z_{yy}=\frac{1}{2}\left(z_{\rho\rho}+\frac{z_\rho}{\rho}+\frac{z_{\varphi\varphi}}{\rho^2}\right)-\frac{1}{2}\left(\left(z_{\rho\rho}-\frac{z_\rho}{\rho}-\frac{z_{\varphi\varphi}}{\rho^2}\right)\cos 2\varphi-\left(\frac{z_{\rho\varphi}}{\rho}-\frac{z_\varphi}{\rho^2}\right)\sin 2\varphi\right). \tag{3.8}$$

By putting the second-order partial derivatives (3.7), (3.8) into the formula (3.5), we find the mixed derivative:

$$z_{xy}=\frac{1}{2}\left(z_{\rho\rho}-\frac{z_\rho}{\rho}-\frac{z_{\varphi\varphi}}{\rho^2}\right)\sin 2\varphi+\left(\frac{z_{\rho\varphi}}{\rho}-\frac{z_\varphi}{\rho^2}\right)\cos 2\varphi. \tag{3.9}$$

If we put the partial derivatives (3.7), (3.8) and (3.9) into the Monge-Ampere equation, we get its form in the polar coordinates:

$$z_{xx}z_{yy}-z_{xy}^2=\frac{1}{\rho^2}\left[z_{\rho\rho}z_{\varphi\varphi}-z_{\rho\varphi}^2+\rho z_\rho z_{\rho\rho}+\frac{2z_\varphi z_{\rho\varphi}}{\rho}-\frac{z_\varphi^2}{\rho^2}\right]. \tag{3.10}$$

The following main theorem is holds:

Theorem 3.1. *If the total curvature is given by the form*

$$K\left(x,y\right)=\frac{a^2+b^2-2\left(x^2+y^2\right)}{\sqrt{a^2-x^2-y^2}\sqrt{x^2+y^2-b^2}} \tag{3.11}$$

and the Dirichlet problem for equation (2.2) satisfies the following boundary conditions:

$$z\left|_{L_1}=0,\quad z\right|_{L_2}=H\quad H>>0. \tag{3.12}$$

Here, $H-$ is a sufficiently large positive number.
Then the solution of the Monge-Ampere equation in the ring domain D is as follows:

$$z\left(x,y\right)=\frac{\sqrt{x^2+y^2}}{2}\left(\sqrt{a^2-x^2-y^2}+\sqrt{x^2+y^2-b^2}\right)+\frac{a^2}{2}\arcsin\frac{\sqrt{x^2+y^2}}{a}- \tag{3.13}$$

$$-\frac{b^2}{2}\ln\left(\sqrt{x^2+y^2}+\sqrt{x^2+y^2-b^2}\right)+A\sqrt{x^2+y^2}+B,$$

where,

$$A=\frac{1}{a-b}(H-\frac{(a-b)\sqrt{a^2-b^2}}{2}-\frac{\pi a^2}{4}+\frac{a^2}{2}\arcsin\frac{b}{a}+\frac{b^2}{2}\ln(a+\sqrt{a^2-b^2})-\frac{b^2}{2}\ln b),$$

$$B=\frac{1}{a-b}(\frac{\pi a^2 b}{4}-bH-\frac{a^3}{2}\arcsin\frac{b}{a}-\frac{b^3}{2}\ln(a+\sqrt{a^2-b^2})+\frac{ab^2}{2}\ln b).$$

Proof. Taking into account the total curvature, if we write equation (2.2) in polar coordinates, it will be the following form:

$$\frac{1}{\rho^2}\left[z_{\rho\rho}z_{\varphi\varphi}-z_{\rho\varphi}^2+\rho z_\rho z_{\rho\rho}+\frac{2z_\varphi z_{\rho\varphi}}{\rho}-\frac{z_\varphi^2}{\rho^2}\right]=\frac{a^2+b^2-2\rho^2}{\sqrt{a^2-\rho^2}\sqrt{\rho^2-b^2}}. \tag{3.14}$$

We seek the solution in the following special form [18, 19]:

$$z\left(\rho,\varphi\right)=f\left(\rho\right)+\rho g\left(\varphi\right). \tag{3.15}$$

By finding the first and second order partial derivatives of the expression (3.15)

$$z_\rho=f'+g,\quad z_\varphi=\rho g',\ \ z_{\rho\rho}=f'',\ \ z_{\varphi\varphi}=\rho g'',\ \ z_{\rho\varphi}=z_{\varphi\rho}=g'.$$

If we put these expressions to (3.14), then we have the following equality:

$$\frac{1}{\rho^2}\left[f''\cdot\rho g''-g'^2+\rho\left(f'+g\right)f''+\frac{2\rho g'^2}{\rho}-\frac{\rho^2 g'^2}{\rho^2}\right]=\frac{a^2+b^2-2\rho^2}{\sqrt{a^2-\rho^2}\sqrt{\rho^2-b^2}}\Rightarrow$$

$$\Rightarrow f''\cdot\left(g''+g\right)+f'\cdot f''=\rho\left(\frac{a^2+b^2-2\rho^2}{\sqrt{a^2-\rho^2}\sqrt{\rho^2-b^2}}\right). \tag{3.16}$$

In the formula (3.16), we can write the total curvature as:

$$f''(g''+g)+f'\cdot f''=\rho\left(\frac{\sqrt{a^2-\rho^2}}{\sqrt{\rho^2-b^2}}-\frac{\sqrt{\rho^2-b^2}}{\sqrt{a^2-\rho^2}}\right).$$

From this,

$$g''+g=\frac{\rho\left(\frac{\sqrt{a^2-\rho^2}}{\sqrt{\rho^2-b^2}}-\frac{\sqrt{\rho^2-b^2}}{\sqrt{a^2-\rho^2}}\right)-f'\cdot f''}{f''}=a_0, \tag{3.17}$$

where a_0 is a constant.

Thus, from the expression (3.17), we obtain two ordinary differential equations with separate variables. Here, $f'' \neq 0 \quad \Rightarrow \quad f(\rho) \neq c_0\rho + c_1$.
The general solution of the left side of equation (3.17) is as follows:

$$g(\varphi) = C_1 \cos\varphi + C_2 \sin\varphi + a_0, \tag{3.18}$$

where, $C_1, \ C_2$ are constants.

Now, we write the expression on the right side of (3.17) as follows:

$$f' \cdot f'' + a_0 \cdot f'' = \rho\left(\frac{\sqrt{a^2-\rho^2}}{\sqrt{\rho^2-b^2}} - \frac{\sqrt{\rho^2-b^2}}{\sqrt{a^2-\rho^2}}\right).$$

Thus,

$$\left(\frac{f'^2}{2} + a_0 f'\right)' = \rho\left(\frac{\sqrt{a^2-\rho^2}}{\sqrt{\rho^2-b^2}} - \frac{\sqrt{\rho^2-b^2}}{\sqrt{a^2-\rho^2}}\right). \tag{3.19}$$

By integrating the above expression (3.19), we will write as follows:

$$\frac{f'^2 + 2a_0 f'}{2} = I = \int \rho\frac{\sqrt{a^2-\rho^2}}{\sqrt{\rho^2-b^2}}d\rho - \int \rho\frac{\sqrt{\rho^2-b^2}}{\sqrt{a^2-\rho^2}}d\rho. \tag{3.20}$$

By using the method of integration by parts, we obtain:

$$I = \int \sqrt{a^2-\rho^2}d\left(\sqrt{\rho^2-b^2}\right) + \int \sqrt{\rho^2-b^2}d\left(\sqrt{a^2-\rho^2}\right) = \sqrt{a^2-\rho^2}\sqrt{\rho^2-b^2} + \int \rho\frac{\sqrt{\rho^2-b^2}}{\sqrt{a^2-\rho^2}}d\rho +$$

$$+\sqrt{\rho^2-b^2}\sqrt{a^2-\rho^2} - \int \rho\frac{\sqrt{a^2-\rho^2}}{\sqrt{\rho^2-b^2}}d\rho = 2\sqrt{a^2-\rho^2}\sqrt{\rho^2-b^2} - I.$$

From this, $I = \sqrt{a^2-\rho^2}\sqrt{\rho^2-b^2} + D_1$. We will write (3.20) as follows:

$$f'^2 + 2a_0 f' + a_0^2 - a_0^2 = 2\sqrt{a^2-\rho^2}\sqrt{\rho^2-b^2} + 2D_1 \quad \Rightarrow$$

$$(f' + a_0)^2 = 2\sqrt{a^2-\rho^2}\sqrt{\rho^2-b^2} + \left(2D_1 + a_0^2\right) = 2\sqrt{a^2-\rho^2}\sqrt{\rho^2-b^2} + \tilde{D}_1, \tag{3.21}$$

where $\tilde{D}_1 = 2D_1 + a_0^2$ is a constant.

On the right side of (3.21), to use the completing the square method, the following expression must be valid, that is $\tilde{D}_1 = a^2 - b^2$. That is,

$$(f' + a_0)^2 = a^2 - b^2 + 2\sqrt{a^2-\rho^2}\sqrt{\rho^2-b^2} = a^2 - \rho^2 + 2\sqrt{a^2-\rho^2}\sqrt{\rho^2-b^2} + \rho^2 - b^2 =$$

$$= \left(\sqrt{a^2-\rho^2} + \sqrt{\rho^2-b^2}\right)^2,$$

$$f' + a_0 = \pm\left(\sqrt{a^2-\rho^2} + \sqrt{\rho^2-b^2}\right).$$

Here we consider surfaces with one-sided convexity. Therefore, we consider only one equation:

$$f' + a_0 = \sqrt{a^2-\rho^2} + \sqrt{\rho^2-b^2}.$$

From here, integrating again, we find the following:

$$f = -a_0\rho + \int \sqrt{a^2-\rho^2}d\rho + \int \sqrt{\rho^2-b^2}d\rho,$$

$$f(\rho) = -a_0\rho + \tfrac{\rho}{2}\sqrt{a^2-\rho^2} + \tfrac{a^2}{2}\arcsin\tfrac{\rho}{a} + \tfrac{\rho}{2}\sqrt{\rho^2-b^2} - \tfrac{b^2}{2}\ln\left|\rho + \sqrt{\rho^2-b^2}\right| + D_2.$$

Now, putting the found expressions into (3.15), we get the following:

$$z(\rho,\varphi) = \frac{\rho}{2}\left(\sqrt{a^2-\rho^2} + \sqrt{\rho^2-b^2}\right) + \frac{a^2}{2}\arcsin\frac{\rho}{a} - \frac{b^2}{2}\ln\left|\rho + \sqrt{\rho^2-b^2}\right| -$$

$$-a_0\rho + D_2 + C_1\rho\cos\varphi + C_2\rho\sin\varphi + a_0\rho =$$

$$= \frac{\rho}{2}\left(\sqrt{a^2-\rho^2} + \sqrt{\rho^2-b^2}\right) + \frac{a^2}{2}\arcsin\frac{\rho}{a} - \frac{b^2}{2}\ln\left|\rho + \sqrt{\rho^2-b^2}\right| + C_1\rho\cos\varphi + C_2\rho\sin\varphi + D_2. \tag{3.22}$$

Using the boundary conditions (3.12), we find the constants:

$$\tfrac{b}{2}\sqrt{a^2-b^2}+\tfrac{a^2}{2}\arcsin\tfrac{b}{a}-\tfrac{b^2}{2}\ln b+b\left(C_1\cos\varphi+C_2\sin\varphi\right)+D_2=0,$$

$$\tfrac{a}{2}\sqrt{a^2-b^2}+\tfrac{a^2}{2}\cdot\tfrac{\pi}{2}-\tfrac{a^2}{2}\ln\left(a+\sqrt{a^2-b^2}\right)+a\left(C_1\cos\varphi+C_2\sin\varphi\right)+D_2=H.$$

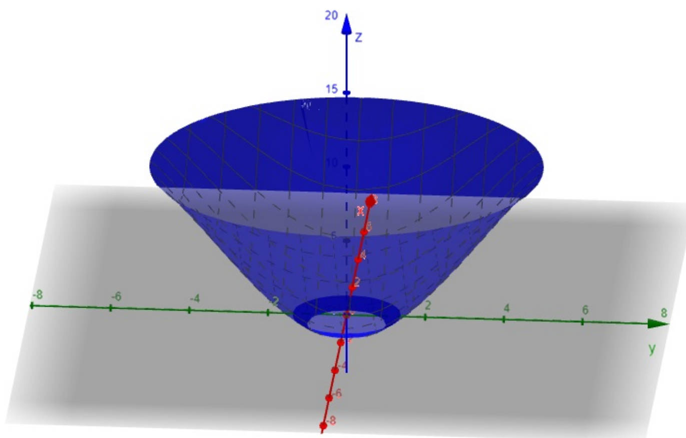
Thus,

$$D_2=\frac{1}{a-b}\left(\frac{\pi a^2 b}{4}-bH-\frac{a^3}{2}\arcsin\frac{b}{a}-\frac{b^3}{2}\ln(a+\sqrt{a^2-b^2})+\frac{ab^2}{2}\ln(b)\right), \tag{3.23}$$

$$C_1\cos\varphi+C_2\sin\varphi=\frac{1}{a-b}\left(H-\frac{(a-b)\sqrt{a^2-b^2}}{2}-\frac{\pi a^2}{4}+\frac{a^2}{2}\arcsin\frac{b}{a}+\frac{b^2}{2}\ln(a+\sqrt{a^2-b^2})-\frac{b^2}{2}\ln b\right). \tag{3.24}$$

By putting the (3.23)-(3.24) expressions into equation (3.22), we find the surface equation given in the theorem. Also, taking into account that the interior boundary is $L_1 : x^2+y^2=b^2$ and the exterior boundary is $L_2 : x^2+y^2=a^2$, it can be shown that the solution obtained in the theorem satisfies the boundary conditions. Theorem is completely proved. □

Example : If we consider $L_1 : x^2+y^2=1$ and $L_2 : x^2+y^2=25$ and $H=10$, then the graph of the surface is the following:



References

[1] Alexandrov A.D.; Convex polyhedra. Springer Monographs in Mathematics.–2005.

[2] Alexandrov A.D.; Intrinsic geometry of convex surfaces. Classics of Soviet Mathematics. –2006.

[3] Artykbayev A.; Recovering convex surfaces from the extrinsic curvature in Galilean space. Math. USSR Sb.–1984.–V.47.–No.1. –pp. 195–214.

[4] Artykbayev A, Ibodullayeva N.M.; The problem of recovering a surface from extrinsic curvature and solving the Monge-Ampere equation. Advances in Science and Technology. Series: Contemporary Mathematics and Its Applications –2021. –201. –pp. 123–131. (in Russian)

[5] Artykbayev A., Ismoilov Sh.Sh.; Surface Recovering by a given total and mean curvature in isotropic space R_3^2 . Palestine J. Math. –2022.–V.11.–No.3. –pp. 351–361.

[6] Artykbayev A., Ismoilov Sh.Sh.; Special mean and total curvature of a dual surface in isotropic spaces. Int. Elect. J. Geom.–2022.–V.15.–No.1.–pp.1–10.

[7] Aydin M.E.; Classifications of translation surfaces in isotropic geometry with constant curvature. Ukrainian Math. J.–2020.–V.72.–No.3.–pp. 329–347.

[8] Aydin M.E., Mihai A.; Ruled surfaces generated by elliptic cylindrical curves in the isotropic space. Georgian Math. J.–2019.–V.26–No.3.–pp.331–340.

[9] Aydin M.E., Kulahci M.A, Ogrenmis A.O. Constant curvature Translation surfaces in Galilean 3-space. Int. Elect. J. Geom.–2019.–V.12.–No.1.–pp. 9–19.

[10] Bakelman I.Ya.; Geometric methods for solving elliptic equations. Nauka, Moscow,–1965.

[11] Bakelman I.Ya.; Generalized solutions of the Monge-Ampere equation. DAN, USSR. –1957.–V.114.–No.6.–pp.1143–1145.

[12] Dede M., Ekici C., Goemans W.; Surfaces of revolution with vanishing curvature in Galilean 3-space. J. Math. Physics, Analysis, Geometry.–2018.–V.14.–No.2.–pp.141–152.

[13] Ismoilov Sh.Sh.; Geometry of the Monge-Ampere equation in an isotropic space. Uzbek Math.J.–2022.–V.65.–No.2.–pp.66–77.

[14] Ismoilov Sh.Sh. Kholmurodova G.N.; Equality of surfaces in Euclidian and semi-Euclidian spaces according to geometric characteristics, Monge-Ampere equation. Bull. Inst. Math.–2023.–V.6.–No.4.–pp.39–46.

[15] Karacan M.K., Cakmak A., Kiziltug S., Es H.; Surfaces generated by Translation surfaces of Type 1 in I_3^1 . Iranian J. Math. Sciences. Informatics.–2021.–V.16.–No.1.–pp.123–135.

[16] Kholmurodova G.N. Surface at a constant distance from the edge of regression on a transfer surface of type 1 in isotropic space. Monge-Ampere equation. Sci. Bull. Namangan. State Uni. –2024.–V.5.–pp.13–16.

[17] Pogorelov A.V.; Extrinsic geometry of convex surfaces. Science, Moscow, –1991.

[18] Polyanin A.D., Zhurov A.I.;Methods for separation of variables and exact solutions of nonlinear equations of mathematical physics. Moskva Ipmex. RAN.–2020.

[19] Polyanin A.D.; Lectures on nonlinear equations of mathematical physics. Moskva Ipmex. RAN.–2023.

[20] Sipus Z.M.; Translation surfaces of constant curvature in a simply isotropic space. Period Math. Hungar.–2014.–V.68.–pp.160–175.

Kholmurodova G.N.,
Tashkent State Transport University,
Tashkent, Uzbekistan.
email: xolmurodovagulnoza3@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.121-125
DOI: 10.29229/uzmj.2025-3-12

Quasitraces on real C*- and AW*-algebras

Kim D.I., Rakhimov A.A.

Abstract. In this paper, a real analogue of a quasitrace is given, and its connection with the quasitrace of the enveloping C*-algebra is found. Similar to the complex case, some interesting properties of a quasitrace and the corresponding metric for real C*-algebras are obtained

Keywords: quasitrace, AW*-algebra, C*-algebra, d_τ-metric, quasinorm.
MSC (2020): 46L35, 46L10, 47L30, 47L40

1. Introduction

AW*-algebras are a generalization of von Neumann algebras (W*-algebras), and naturally the question of generalizing the results obtained for W*-algebras to AW*-algebras arises, which is quite relevant. It is known that in the study and classification of von Neumann algebras, along with projections, the concept of a trace on an algebra plays an important role. For example, [1], it was proved that a von Neumann algebra is finite if and only if there exists a separating family of finite normal traces on it. Actually, for this reason, C*-algebras have been studied relatively poorly, some of them do not even have nontrivial projections, not to mention traces. On the other hand, AW*-algebras have been studied relatively better, since these algebras have a sufficient number of projections, but these algebras also have problems with the trace. There are papers (see, for example, [30]) where, for convenience, the existence of a trace on the AW*-algebra is assumed. And so, in 1982, in the paper of Blackadar and Handelman [3] an analogue of a trace, called a quasitrace, was introduced. Despite the fact that this concept does not completely replace a trace, they and some researchers managed to obtain an analogue of the results available for traces. This paper presents Kaplansky's question: "Is every quasitrace on a C-algebra linear, i.e. a trace?" This problem remains open. The last attempt to solve this problem was presented in [4], where it was proved that in exact C*-algebras every quasitrace is linear, i.e. in this case Kaplansky's problem has a positive solution.

In this paper, a real analogue of a quasitrace is given, and its connection with the quasitrace of the enveloping C*-algebra is found. Similar to the complex case, some interesting properties of a quasitrace and the corresponding metric for real C*-algebras are obtained.

2. Preliminaries

A Banach *-algebra A over a field $\mathbb{C}$ is called a *C*-algebra* if $\|x^*x\| = \|x\|^2$, for any $x \in A$. Let $B(H)$ be the algebra of all bounded linear operators, acting in the complex Hilbert space H. A weakly closed *-subalgebra $M \subset B(H)$ with identity is called W*-algebra. The center $Z(M)$ of an algebra M is the set of elements of M that commute with each element of M. A W*-algebra M is called a factor if $Z(M)$ consists of complex multiples of $\mathbb{1}$, i.e. $Z(M) = \{\lambda\mathbb{1} : \lambda \in \mathbb{C}\}$.

Let A be a ring and S a non-empty subset of A. Assume that $R(S) = \{x \in A \mid sx = 0$ for all $s \in S\}$ and call $R(S)$ the right annihilator of S. Similarly, $L(S) = \{x \in A \mid xs = 0$ for all $s \in S\}$ denotes the left annihilator of S. A *Baer *-ring* is a ring A such that for every non-empty subset S of A, $R(S) = gA$ for a suitable projection g. The equality $L(S) = ((R(S^*)))^* = ((hA))^* = Ah$ (for some projection h) shows that this definition can also be given through the left annihilator. AW*-algebra is a C*-algebra, which is also a Baer *-ring. It is known [5] that every W*-algebra is an AW*-algebra, but the converse is not true.

3. Main results

Definition 3.1. [3]. Let A be a C*-algebra with unity. A quasitrace τ on A is a function $\tau : A \to \mathbb{C}$ that satisfies the following conditions *(i)* $\tau(x^*x) = \tau(xx^*) \geq 0$, $x \in A$;

(ii) $\tau(a + ib) = \tau(a) + i\tau(b)$, for $a, b \in A_h$;

(iii) τ is linear on an abelian C*-subalgebra B of A.

We give a definition of a quasitrace in the real case.

Definition 3.2. Let R be a unital real C*-algebra. A quasitrace τ on R is a function $\tau : R \to \mathbb{R}$ that satisfies the following conditions

(i′) $\tau(x^*x) = \tau(xx^*) \geq 0$, $x \in R$;

(ii′) $\tau(a + b) = \tau(a)$, for $a \in R_h, b \in R_k$;

(iii′) τ is linear on an abelian C*-subalgebra B of R.

We can see that definitions of quasitrace in real and complex cases are slightly different. In the next two theorems we naturally consider *the restriction* of a quasitrace from A to R, and conversely, *the extension* of a quasitrace from R to A.

Theorem 3.3. *Let R be a unital real C*-algebra. If $\overline{\tau}$ is a quasitrace on the C*-algebra $A = R + iR$, then its restriction to the real C*-algebra R, defined as*

$$\tau(a + b) = \overline{\tau}(a), \qquad a \in R_h, b \in R_k \tag{3.1}$$

is a quasitrace on R.

Proof. (i) Let $x \in R$ and $x = a + b$, where $a \in R_h, b \in R_k$. Then by (3.1) we obtain $\tau(x^*x) = \tau(a^2 - b^2 + ab - ba) = \overline{\tau}(a^2 - b^2)$, since $a^2 - b^2 \in R_h$ and $ab - ba \in R_k$. Similarly, $\tau(xx^*) = \tau(a^2 - b^2 + ba - ab) = \overline{\tau}(a^2 - b^2)$. Thus, we obtain the equality $\tau(x^*x) = \tau(xx^*)$. Since $a^2 - b^2 \geq 0$, then $\tau(x^*x) = \overline{\tau}(a^2 - b^2) \geq 0$.

(ii) By (3.1) we have $\tau(a + b) = \overline{\tau}(a) = \tau(a + 0) = \tau(a)$.

(iii) Let B be an arbitrary abelian real C*–subalgebra of R. Consider the complexification $B_c = B + iB$, which is an abelian C*-subalgebra of A. By Definition 3.1, the quasitrace $\overline{\tau}$ is linear on B_c . We show that the restriction of τ to B is linear. Let $\lambda \in \mathbb{R}$ and let $x, y \in B$ be such that $x = a + b, y = c + d$, $a, c \in R_h$ and $b, d \in R_k$. Since $\lambda a + c \in R_h$ and $\lambda b + d \in R_k$, then

$$\tau(\lambda x + y) = \tau(\lambda a + c + \lambda b + d) = \overline{\tau}(\lambda a + c).$$

Since $a, c \in R_h \subset B_c$, then by the linearity of $\overline{\tau}$ on B_c we get

$$\overline{\tau}(\lambda a + c) = \lambda\overline{\tau}(a) + \overline{\tau}(c) = \lambda\tau(a + b) + \tau(c + d) = \lambda\tau(x) + \tau(y).$$

Therefore, τ is linear on B. Thus τ is a quasitrace on R. The theorem is proved. □

Theorem 3.4. *Let R be a unital real C*-algebra. If τ is a quasitrace on R, then its extension $\overline{\tau}$ to $A = R + iR$, defined as $\overline{\tau}(x + iy) = \tau(x) + i\tau(y)$, is a quasitrace on A, where $x, y \in R$.*

Proof. Recall that the algebra $A = R + iR$ can be embedded in $M_2(A)$ as $x \to e_{11} \otimes x$ and the mapping $\pi : M_2(A) \to M_2(C) \otimes A$ defined as $\pi([a_{ij}]) = \sum_{i,j=1}^{2} e_{ij} \otimes a_{ij}$ is an *-isomorphism, that is,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \to e_{11} \otimes a_{11} + e_{12} \otimes a_{12} + e_{21} \otimes a_{21} + e_{22} \otimes a_{22}.$$

Let $x = c + id$, where $c, d \in R$. Then

$$\begin{aligned} \overline{\tau}(x^*x) = \overline{\tau}((c + id)^*(c + id)) &= \overline{\tau}(c^*c + d^*d + i(c^*d - d^*c)) \\ &= \tau(c^*c + d^*d) + i\tau(c^*d - d^*c). \end{aligned}$$

Since $c^*d - d^*c \in R_k$, then applying the equality $\tau(a+b) = \tau(a)$ $(a \in R_h,\ b \in R_k)$ we obtain $\tau(c^*c + d^*d) + i\tau(c^*d - d^*c) = \tau(c^*c + d^*d)$. Since $\tau(x) = \tau(x \otimes e_{11})$, then

$$\begin{aligned}
\overline{\tau}(x^*x) &= \tau(c^*c + d^*d) = \tau((c^*c + d^*d) \otimes e_{11}) = \tau \begin{pmatrix} c^*c + d^*d & 0 \\ 0 & 0 \end{pmatrix} \\
&= \tau\left(\begin{pmatrix} c^* & d^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix}\right) = \tau\left(\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix}\begin{pmatrix} c^* & d^* \\ 0 & 0 \end{pmatrix}\right) \\
&= \tau\begin{pmatrix} cc^* & cd^* \\ dc^* & dd^* \end{pmatrix} = \tau(cc^* + dd^*) = \tau\left(cc^* + dd^*\right) + i\tau\left((cd^* - dc^*)\right) \\
&= \overline{\tau}((c + id)(c + id)^*) = \overline{\tau}(xx^*).
\end{aligned}$$

(ii) Let $x, y \in A_h$ and $x = a + ib$, $y = c + id$. Since $x = x^*$, then $a = a^*$, $b^* = -b$, i.e. $a \in R_h$, $b \in R_k$; similarly we have $c \in R_h$, $d \in R_k$. Since $\tau(b) = \tau(d) = 0$, then we get

$$\begin{aligned}
\overline{\tau}(x + iy) &= \overline{\tau}(a - d + i(b + c)) = \tau(a - d) + i\tau(b + c) = \\
&= \tau(a) + i\tau(c) = \tau(a) + i\tau(b) + i(\tau(c) + i\tau(d)) \\
&= \overline{\tau}(a + ib) + i\overline{\tau}(c + id) = \overline{\tau}(x) + i\overline{\tau}(y).
\end{aligned}$$

(iii) Let B_c be an abelian C*-subalgebra of the AW*- algebra A. Since $A = R + iR$, then for $\forall x \in B_c$ there are such $a, b \in R$: $x, = a + ib$, therefore $B_c = B_1 + iB_2, B_i \subset R$, for which $a \in B_1$,$b \in B_2$.

1) Since $B_c \ni 0 = 0 + i0$ and $B_c \ni \mathbb{1} = \mathbb{1} + i0$, $B_c \ni i\mathbb{1}$, then $0, \mathbb{1} \in B_i, i = 1, 2$.

If $B_c = iB_2$, then for $\lambda \in \mathbb{C}$, $\lambda = \lambda_1 + i\lambda_2$, $x = ia, y = ib$, $a, b \in B_2$ we have

$$\begin{aligned}
\overline{\tau}(\lambda x + y) &= \overline{\tau}((\lambda_1 + i\lambda_2)ia + ib) = \overline{\tau}(-\lambda_2 a + i(\lambda_1 a + b)) = \tau(-\lambda_2 a) + i\tau(\lambda_1 a + b) \\
&= -\lambda_2\tau(a) + i\lambda_1\tau(a) + i\tau(b) = i(\lambda_1 + i\lambda_2)\tau(a) + i\tau(b) = \lambda(0 + i\tau(a)) + (0 + i\tau(b)) \\
&= \lambda\overline{\tau}(x) + \overline{\tau}(y),
\end{aligned}$$

i.e. in this case $\overline{\tau}$ is linear.

2) Let $B_c = B_1 + iB_2, B_i \subset R$, $B_1 \neq \{0\}$. Let $x = a + ic, y = b + id$, $a, b \in B_1, c, d \in B_2$. Then $xy = ab - cd + i(cb + ad)$ and therefore, $ab - cd \in B_1$, $cb + ad \in B_2$. For $c = d = 0$ we get $ab \in B_1$, therefore B_1 is an algebra, similarly B_2 is an algebra. For $d = 1, b = 0$ we have $a \in B_2$, hence $B_1 \subset B_2$. Similarly, we can get $B_2 \subset B_1$. Thus $B_1 = B_2$. Hence $B_c = B + iB$. Since $x^* = a^* - ib^* \in B_c$, then $a^*, b^* \in B$. Thus B is a real *-subalgebra.

3) Let's show the linearity of $\overline{\tau}$ on B_c:

$$\begin{aligned}
\overline{\tau}(x + y) &= \overline{\tau}(a + ib + c + id) = \tau(a + c) + i\tau(b + d) \\
&= \tau(a + c) + i\tau(b + d) = \tau(a) + \tau(c) + i\tau(b) + i\tau(d) \\
&= \tau(a) + \tau(c) + i\tau(b) + i\tau(d) = \overline{\tau}(a + ib) + \overline{\tau}(c + id) \\
&= \overline{\tau}(x) + \overline{\tau}(y),
\end{aligned}$$

means $\overline{\tau}$ is additive. Now let's show homogeneity.

$$\begin{aligned}
\overline{\tau}(\lambda x) &= \overline{\tau}((\lambda_1 + i\lambda_2)(a + ib)) = \overline{\tau}(\lambda_1 a - \lambda_2 b + i(\lambda_1 b + \lambda_2 a)) \\
&= \tau(\lambda_1 a - \lambda_2 b) + i\tau(\lambda_1 b + \lambda_2 a) = \lambda_1\tau(a) - \lambda_2\tau(b) + i\lambda_1\tau(b) + i\lambda_2\tau(a) \\
&= (\lambda_1 + i\lambda_2)\tau(a) + i(\lambda_1 + i\lambda_2)\tau(b) = (\lambda_1 + i\lambda_2)(\tau(a) + i\tau(b)) \\
&= \lambda\overline{\tau}(x).
\end{aligned}$$

Thus $\overline{\tau}$ is linear on B_c. Theorem proved. □

Definition 3.5. A quasitrace τ is called

- *finite* if $\tau(\mathbb{1}) < \infty$;
- *semifinite* if the set $D = \{x \;:\; \tau(x^*x) < \infty\}$ is norm-dense in the algebra itself;
- *faithful* if $\tau(a) > 0$, for $a > 0$, i.e. $\{x : \; \tau(x^*x) = 0\} = \{0\}$;
- *extremal* if it cannot be represented as the sum of two other quasitraces.

Theorem 3.6. *A quasitrace $\overline{\tau}$ is faithful, semifinite and extremal if and only if τ is faithful, semifinite and extremal.*

Proof. 1) Let $\overline{\tau}$ be faithful, $\tau(a^*a) = 0$, for $a \in R$. Since $\tau(a^*a) = \overline{\tau}(a^*a)$, then from the faithfulness of $\overline{\tau}$ we obtain that $a = 0$, therefore τ is faithful.

Conversely, let τ be faithful and $\overline{\tau}(x^*x) = 0$, for $x \in A$. Let $x = a + ib$, $a, b \in R$. Similar to the proof of Theorem 3.4, we obtain $0 = \overline{\tau}(x^*x) = \tau(a^*a) + \tau(b^*b)$. Since $\tau(a^*a) \geq 0$ and $\tau(b^*b) \geq 0$, then $\tau(a^*a) = \tau(b^*b) = 0$, and hence $a = b = 0$, i.e. $x = 0$. Therefore $\overline{\tau}$ is faithful.

2). Let $D_1 = \{a \in R \;:\; \tau(a^*a) < \infty\}$. We show that D_1 is dense in R. For $a \in D_1$ we have $\overline{\tau}(a^*a) = \tau(a^*a) < \infty$, therefore $D_1 \subset D$. Also for $y = ib \in iD_1$ $(b \in D_1)$ we have $\overline{\tau}(y^*y) = \overline{\tau}((ib)^*ib) = \tau(b^*b) < \infty$, therefore $iD_1 \subset D$. Hence $D_1 + iD_1 \subset D$.

Conversely, let $x \in A$ and $\overline{\tau}(x^*x) < \infty$, i.e. $x \in D$. Let $x = a + ib$, where $a, b \in R$. Then, as shown above, $\overline{\tau}(x^*x) = \tau(a^*a + b^*b) + i\tau(a^*b - b^*a) = \tau(a^*a + b^*b)$ and for the element $a^*a + b^*b$ we have

$$\begin{aligned}
\tau(a^*a + b^*b) &= \tau\begin{pmatrix} a^*a + b^*b & 0 \\ 0 & 0 \end{pmatrix} = \tau\left(\begin{pmatrix} a^* & b^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right) \\
&= \tau\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} a^* & b^* \\ 0 & 0 \end{pmatrix}\right) = \tau\begin{pmatrix} aa^* & ab^* \\ ba^* & bb^* \end{pmatrix} = \tau\begin{pmatrix} aa^* & 0 \\ 0 & bb^* \end{pmatrix} \\
&= \tau(aa^*) + \tau(bb^*) = \tau(a^*a) + \tau(b^*b).
\end{aligned}$$

Since $\overline{\tau}(x^*x) < \infty$, then $\tau(a^*a) < \infty$ and $\tau(b^*b) < \infty$, i.e. $a, b \in D_1$, therefore $D \subset D_1 + iD_1$.

Thus we get: $D = D_1 + iD_1$. Since $\overline{D} = \overline{D}_1 + i\overline{D}_1$ and $\overline{D} = A = R + iR$, then $\overline{D}_1 = R$. Therefore, τ is semifinite.

3). If $\tau = \tau_1 + \tau_2$, then by Theorem 3.4 we have

$$\overline{\tau}(x + iy) = \tau(x) + i\tau(y) = \tau_1(x) + \tau_2(x) + i\tau_1(y) + i\tau_2(y) = \overline{\tau}_1(x + iy) + \overline{\tau}_2(x + iy),$$

from here we get $\overline{\tau} = \overline{\tau}_1 + \overline{\tau}_2$.

Conversely, if $\overline{\tau} = \varphi_1 + \varphi_2$, then by Theorem 3.3 we have $\tau(a+b) = \overline{\tau}(a) = \varphi_1(a) + \varphi_2(a)$. Therefore $\tau = \tau_1 + \tau_2$, where $\tau_1(a + b) = \varphi_1(a)$ and $\tau_2(a + b) = \varphi_2(a)$. The theorem is proved. □

Let A be a C*-algebra and τ be a quasitrace on A. Put $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in A$. The mapping $\|.\|_2$, called a *quasi-norm*, is not a norm in general. However, the following properties hold (see [4, Lemma 3.5])

(1) $\tau(x + y)^{1/2} \leq \tau(x)^{1/2} + \tau(y)^{1/2}, \;\; a, b \in A_+$;

(2) $\|x + y\|_2^{2/3} \leq \|x\|_2^{2/3} + \|y\|_2^{2/3}, \;\; x, y \in A$;

(3) $\|xy\|_2 \leq \|x\|\|y\|_2$ and $\|xy\|_2 \leq \|x\|_2\|y\|, \;\; x, y \in A$.

Now let's set $d_\tau(x, y) = ||x - y||_2^{2/3}, \;\; x, y \in A$. Then d is a metric on A and has the following properties (see [4, Definition 3.6 and Lemma 3.7]):

– the involution $x \to x^*$ is continuous in the d_τ-metric;

– the sum is continuous in the d_τ-metric on A;

– the product is continuous in the d_τ-metric on bounded sets A;

– $x \to \tau(x)$ is continuous in the d_τ-metric on A_+.

The quasinorm and the corresponding metric are similarly defined in the real case.

Theorem 3.7. *Let τ be an faithful, normal quasitrace on a real C*-algebra R. If τ is extremal, then the AW*-completion $\overline{R}^{d_\tau}$ of R is a real AW*-factor.*

Proof. Let $\overline{\tau}$ be the extension of τ to $A = R + iR$. By Theorem 3.6, the quasitrace $\overline{\tau}$ is also faithful, normal, and extremal. By [4, Proposition 4.6], the AW*-completion $\overline{A}^{d_{\overline{\tau}}}$ of A is an AW*-factor. Since $\overline{A}^{d_{\overline{\tau}}} = \overline{R}^{d_\tau} + i\overline{R}^{d_\tau}$, then by [6, Proposition 4.3.1], the AW*-completion $\overline{R}^{d_\tau}$ of R is a real AW*-factor. The theorem is proved. □

References

[1] Takesaki M. Theory of Operator Algebras. I, Springer-Verlag.,– 1979. VIII. – 415p.

[2] Wright M.J.D. On AW*-algebras of finite type. J. London Math. Soc., 1976, Vol. 2.– 12.– pp.431-439.

[3] B.Blackadar and D. Handelman, Dimension functions and traces on C*-algebra, J. Funct. Anal.– 1982.– 45. –pp.297-340.

[4] Haagerup U. Quasitraces on exact C*-algebras are traces. C. R. Math. Acad. Sci. Soc. R. Can.,– 2014. – 36(2-3). – pp. 6792 .

[5] Dixmier, J.. Von Neumann Algebras,–2011. – Elsevier, North-Holland Mathematical Library., –Vol 27.

[6] Ayupov Sh.A., Rakhimov A.A. Real W*-algebras. Actions of groups and Index theory for real factors. VDM Publishing House Ltd. Beau-Bassin,– 2010.– Germany, Bonn.,– 138 p.

Kim D.I.,
National University of Uzbekistan,
Tashkent, Uzbekistan.
Tashkent Branch of the G.V. Plekhanov
Russian University of Economic,
Tashkent, Uzbekistan.
email: dmitriy.kim.1995.04.23@gmail.com

Rakhimov A.A. ,
National University of Uzbekistan,
Tashkent, Uzbekistan.
email: rakhimov@ktu.edu.tr

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.126-133
DOI: 10.29229/uzmj.2025-3-13

The phase transition for the Hard-Core model on the closed Cayley tree of branching ratio two

Makhammadaliev T. M., Karshiboev Sh. O.

Abstract. In this paper, we consider the two-state Hard-Core model on the closed Cayley tree of branching ratio two. On the symmetric case, we exactly solve the model and show that there is a phase transition.

Keywords: closed Cayley tree, Gibbs measure, Hard-Core model, phase transition
MSC (2020): 82B05, 82B20, 60K35

1. Introduction

The Hard-Core model is significant in statistical physics as a fundamental representation of a gas consisting of particles with non-negligible sizes. In this model, the state $\sigma(i) = 1$ (resp. 0) indicates that vertex i is occupied by a particle (resp. empty). A key constraint of the model is that no two adjacent vertices can both be occupied, effectively preventing particles from overlapping. This model can also be derived as a limiting case of the antiferromagnetic Ising model (see, for instance, [7]). Moreover, the Hard-Core model has gained relevance in the study of communication networks (see, e.g., [10]).

The Hard-Core model has been extensively studied on various types of lattices and graphs due to its broad applications in statistical physics, combinatorics, and operations research. The choice of lattice often depends on the specific phenomena being investigated and the mathematical properties of interest. Commonly studied lattices include: Cayley trees (Bethe lattices) [3, 4, 8, 11, 12, 14, 16, 18], triangular and hexagonal lattices [1, 7], d-dimensional hypercubic lattices [15, 17], random graphs [10], weighted and bipartite lattices [19]. Each of these lattices provides unique insights into the behavior of the Hard-Core model, ranging from exact solvability on trees to the complexity of dense configurations in higher-dimensional or irregular structures.

In 1979 Jelitto [9] introduced the zero-field Ising model on the closed Cayley tree of branching ratio two and proceeded to solve it exactly. After that a great interest has been devoted to the investigation of various properties of the Ising model (see, e.g., [2, 13, 20]), Potts model (see, e.g., [5, 6]) on the closed Cayley trees.

In the present paper, we consider the two-state Hard-Core model on the closed Cayley tree of branching ratio two. On the symmetric case, we exactly solve the model, i.e., we find the critical value of the parameter such that below this value there are three limiting Gibbs measures. On the other hand, as in [18] it is shown that the model on the Cayley tree of order two possesses a unique limiting Gibbs measure for all values of the parameter.

The paper is organized as follows. Section 2 focuses on the preliminary concepts and foundational material. In Section 3, we derive a functional equation for the model by leveraging the structure of the graph. In Section 4, we review several results concerning the open Cayley tree. Section 5 focuses on the analysis of limiting Gibbs measures for the model on the symmetric tree.

2. Preliminaries

An open tree is a graph $G = \{E, K\}$ that is connected and contains no circuits. Thus, G is an open tree if and only if, for any two distinct vertices $x, y \in E$, there exists a unique path $x = z_1, z_2, \ldots, z_m = y$, where $z_1, \ldots, z_m$ are distinct.

A symmetric closed Cayley tree can be constructed recursively as follows (see Fig. 1). At the initial, or first stage, two nodes are connected by a vertical edge, as shown in Fig. 1a. To construct the next stage, each vertical edge in the previous stage is replaced by the elemental cluster illustrated in Fig. 1b. Consequently, Fig. 1b represents the second stage, Fig. 1c depicts the third stage, and this process continues iteratively.

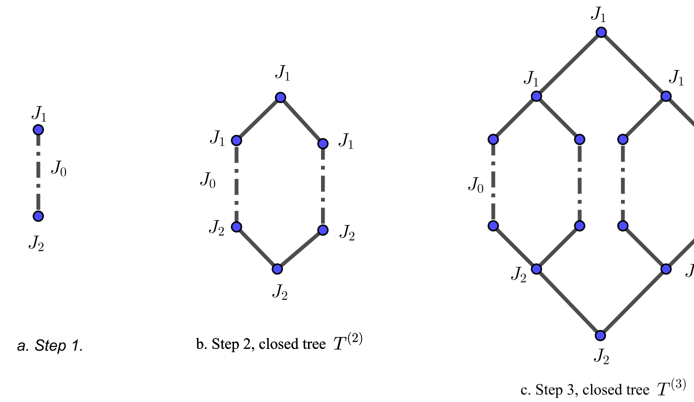


Figure 1. The recursive construction of the closed tree.

To define a random field on an infinite closed tree, it is useful to consider its construction from a different perspective. We begin with a single vertical edge connecting a pair of sites. Next, replicate this structure to the right and connect the two top sites through an additional site with two edges. Similarly, connect the two bottom sites in the same manner, thereby forming the elemental hexagon shown in Fig. 1b.

In the next step, replicate this structure to the right and link the two top sites and two bottom sites as before. This process generates a closed tree with two upper levels and two lower levels, as illustrated in Fig. 1c. Denote the finite closed tree constructed at the nth stage by $T^{(n)}$. The infinite closed Cayley tree is then defined as

$$T = \lim_{n\to\infty} T^{(n)}.$$

The infinite closed Cayley tree T can be naturally decomposed into two distinct parts:

$$T = \Gamma^1 \cup \Gamma^2 \cup S,$$

where S is the set of edges which connects upper and lower tree, $\Gamma^1 = \{V_1, L_1\}$ represents the upper tree, and $\Gamma^2 = \{V_2, L_2\}$ represents the lower tree.

We consider binary random fields on T that can be regarded as the limiting measure of a spin system on a sequence of finite closed Cayley trees. Let $V = V_1 \cup V_2$ and $L = L_1 \cup L_2$. If $l \in L$ an edge with endpoints $x, y \in V$ then we write $l = \langle x, y\rangle$ and the endpoints are called nearest neighbors. We assume that $\Phi = \{0, 1\}$, and $\sigma \in \Omega = \Phi^V$ is a configuration, i.e., $\sigma = \{\sigma(x) \in \Phi : x \in V\}$, where $\sigma(x) = 1$ means that the vertex x on the closed tree $T^{(n)} = \{V^{(n)}, L^{(n)}\}$ is *occupied*, and $\sigma(x) = 0$ means it is *vacant*. A configuration σ on the upper (respectively, lower) tree is said to be *admissible* if $\sigma(x)\sigma(y) = 0$ for every edge $\langle x, y\rangle$ in L_1 (respectively, L_2). If the configuration σ is admissible on both the upper and lower trees, then it is called admissible on V and the set of such configurations is denoted by Ω^a. Clearly, $\Omega^a \subset \Phi^V$. Furthermore, at the first stage, the condition $\sigma(x)\sigma(y) = 1$ may occur. In such a case, the edge $\langle x, y\rangle$ is referred to as an *occupied edge*.

We represent the random field on $T^{(n)}$ in terms of the potential function

$$H(\sigma) = -J_0 \sum_{\langle x,y\rangle : x\in V_1, y\in V_2} \sigma(x)\sigma(y) - J_1 \sum_{i\in V_1} \sigma(i) - J_2 \sum_{j\in V_2} \sigma(j), \tag{2.1}$$

the first summation is carried out over all spins on the surfaces sites (i.e., those for nearest neighbor pairs on upper level 1 and lower level 1), the second (third) summation is taken over the vertices of the upper (lower) tree of $T^{(n)}$ (see Fig. 1).

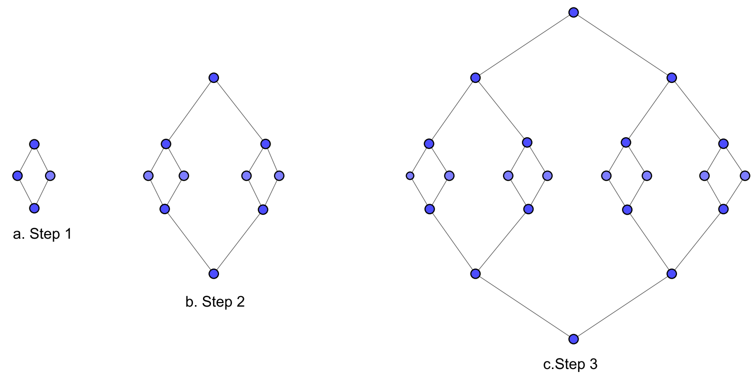


Figure 2. Closed tree model with $J_0 = \infty$.

Let $\mathbf{B}$ be the σ-algebra generated by cylindrical sets with finite base of Ω^a. For any n we let $\mathbf{B}_{V^{(n)}} = \{\omega \in \Omega^a : \omega|_{V^{(n)}} = \omega_n\}$ denote the subalgebra of $\mathbf{B}$, where $\omega|_{V^{(n)}}$ is restriction of ω to $V^{(n)}$ and $\omega_n : x \in V^{(n)} \mapsto \omega_n(x)$ an admissible configuration in $V^{(n)}$.

The resulting for $\sigma \in \mathbf{B}_{V^{(n)}}$ Gibbs distribution is defined by

$$P(\sigma) = \frac{1}{Z^{(n)}} \exp\left(-\frac{1}{kT} H(\sigma)\right),$$

where

$$Z^{(n)} = \sum_{\sigma} \exp\left(-\frac{1}{kT} H(\sigma)\right),$$

is the partition function. We can rewrite this distribution in terms of the number of "occupied" edges $n_0(\omega)$ and number of "occupied" vertices $n_1(\omega)$ and $n_2(\omega)$ on the surface, the upper tree and the lower tree, respectively, as follows:

$$P(\omega) = \frac{1}{Z^{(n)}} \exp\{h_0 n_0(\omega) + h_1 n_1(\omega) + h_2 n_2(\omega)\}, \tag{2.2}$$

where $h_i = \frac{J_i}{kT}$, $i = 0, 1, 2$.

Notice that several special cases of interest can be obtained from the above system as follows:

(a) To recover the system on the open tree, we set $J_0 = 0$. The resulting system is equivalent to two independent hard-core models on open trees.

(b) If $J_0 \neq 0$ and $J_1 = J_2$, we obtain the closed symmetric model.

(c) If we set $J_1 = J_2$ and $J_0 = \infty$, we obtain a spin system on the closed tree (as shown in Fig. 2) in which the upper and lower surface levels (i.e., levels 0) have been fused.

3. Recurrence relations

To study the phase transition, we propose an approach using a recurrence relation for the partition function and subsequently derive some new properties.

The partition function $Z^{(n)}$ can be expressed as the sum of four terms:

$$Z^{(n)} = Z^{(n)}_{0,0} + Z^{(n)}_{0,1} + Z^{(n)}_{1,0} + Z^{(n)}_{1,1}, \tag{3.1}$$

where

$$Z^{(n)}_{\alpha,\beta} = \sum_{\{\omega:\omega_u=\alpha,\omega_l=\beta\}} \exp\{h_0 n_0(\omega) + h_1 n_1(\omega) + h_2 n_2(\omega)\},$$

with α and β taking values 0 or 1, and ω_u and ω_l representing the spins at the uppermost and lowermost sites of Γ^i, $i = 1, 2$, respectively. Each term $Z^{(n)}_{\alpha,\beta}$ can further be decomposed as a combination of different $Z^{(n-1)}_{\alpha,\beta}$. As shown in Figure 3, there are 16 distinct possibilities for each pair (α, β).

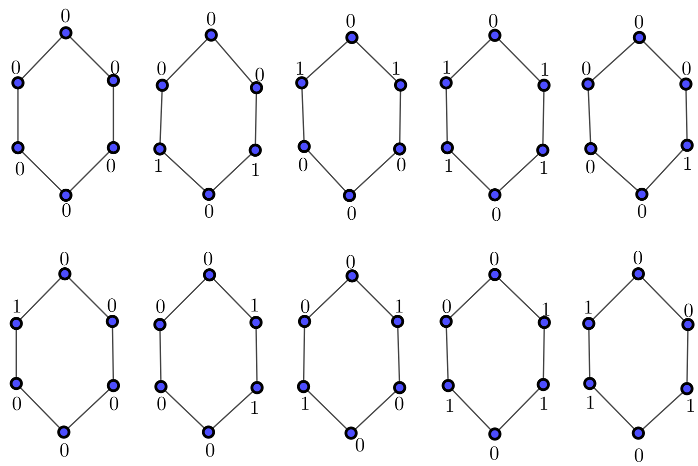


Figure 3. There are 10 fundamental configurations out of the 16 possible configurations with top spin 0 and bottom spin 1. The remaining 6 configurations can be obtained by exchanging the left and right subtrees in equation (3.2).

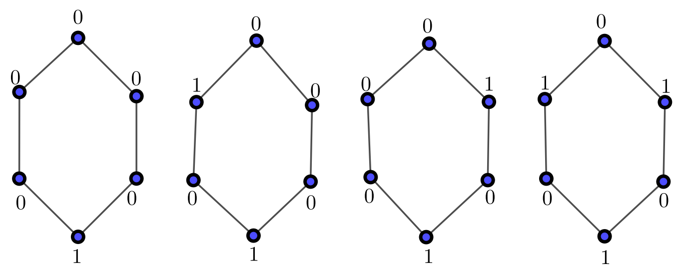


Figure 4. The four fundamental possible configurations for eq.(3.3) and eq.(3.4)

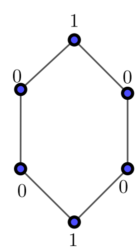


Figure 5. The fundamental possible configurations for eq.(3.5)

Then

$$\begin{aligned} Z_{0,0}^{(n)} = \left(Z_{0,0}^{(n-1)}\right)^2 + \left(Z_{0,1}^{(n-1)}\right)^2 + \left(Z_{1,0}^{(n-1)}\right)^2 + \left(Z_{1,1}^{(n-1)}\right)^2 + \\ + 2Z_{0,0}^{(n-1)}Z_{0,1}^{(n-1)} + 2Z_{0,0}^{(n-1)}Z_{1,0}^{(n-1)} + 2Z_{0,0}^{(n-1)}Z_{1,1}^{(n-1)} + \\ + 2Z_{0,1}^{(n-1)}Z_{1,0}^{(n-1)} + 2Z_{0,1}^{(n-1)}Z_{1,1}^{(n-1)} + 2Z_{1,0}^{(n-1)}Z_{1,1}^{(n-1)} = \\ = \left(Z_{0,0}^{(n-1)} + Z_{0,1}^{(n-1)} + Z_{1,0}^{(n-1)} + Z_{1,1}^{(n-1)}\right)^2. \quad (3.2) \end{aligned}$$

Similarly (see Fig. 4 and Fig. 5),

$$Z_{0,1}^{(n)} = e^{h_2}\left(Z_{0,0}^{(n-1)} + Z_{1,0}^{(n-1)}\right)^2, \tag{3.3}$$

$$Z_{1,0}^{(n)} = e^{h_1}\left(Z_{0,0}^{(n-1)} + Z_{0,1}^{(n-1)}\right)^2, \tag{3.4}$$

and

$$Z_{1,1}^{(n)} = e^{h_1+h_2}\left(Z_{0,0}^{(n-1)}\right)^2. \tag{3.5}$$

The initial values are given by (see Fig. 6)

$$Z_{0,0}^{(0)} = 1, \quad Z_{0,1}^{(0)} = e^{h_2}, \quad Z_{1,0}^{(0)} = e^{h_1}, \quad Z_{1,1}^{(0)} = e^{h_0+h_1+h_2}.$$

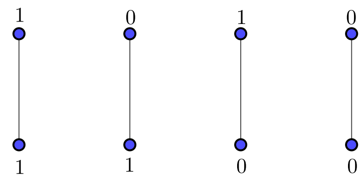


Figure 6. The initial configurations.

Therefore,

$$Z_{0,0}^{(1)} = \left(1 + e^{h_2} + e^{h_1} + e^{h_0+h_1+h_2}\right)^2,\ Z_{1,1}^{(1)} = e^{h_1+h_2},\ Z_{0,1}^{(1)} = e^{h_2}\left(1+e^{h_1}\right)^2,\ Z_{1,0}^{(1)} = e^{h_1}\left(1+e^{h_2}\right)^2.$$

Let

$$x_n = \frac{Z_{0,1}^{(n)}}{Z_{1,0}^{(n)}}, \quad y_n = \frac{Z_{0,0}^{(n)}}{Z_{1,0}^{(n)}}, \quad z_n = \frac{Z_{1,1}^{(n)}}{Z_{1,0}^{(n)}}, \tag{3.6}$$

and

$$a = e^{h_0},\ b = e^{h_1},\ c = e^{h_2}. \tag{3.7}$$

Then

$$\begin{cases} x_n = \frac{c(y_{n-1}+1)^2}{b(x_{n-1}+y_{n-1})^2}, \\ y_n = \frac{(x_{n-1}+y_{n-1}+z_{n-1}+1)^2}{b(x_{n-1}+y_{n-1})^2}, \\ z_n = \frac{cy_{n-1}^2}{(x_{n-1}+y_{n-1})^2}. \end{cases} \tag{3.8}$$

If $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} z_n = z$ exist, then

$$\begin{cases} x = \frac{c(1+y)^2}{b(x+y)^2}, \\ y = \frac{(x+y+z+1)^2}{b(x+y)^2}, \\ z = \frac{cy^2}{(x+y)^2}. \end{cases} \tag{3.9}$$

If the system of equations (3.9) has a unique solution, the corresponding limiting measure is also unique, indicating the absence of a phase transition. On the other hand, if the system has multiple solutions, this implies there is more than one (actually, infinitely many) limiting measures, which is characteristic of a phase transition.

4. Review some known results

When $J_0 = 0$, i.e., $a = 1$, the model is reduced to two identical and independent open tree models each of which may be represented by

$$P(\omega) = \frac{1}{z^{(n)}}\exp\{h_1 n_1(\omega)\}, \tag{4.1}$$

where $h_1 = \frac{J_1}{kT}$. The partition function

$$z^{(n)} = \sum_{\sigma}\exp\left(-\frac{1}{kT}H(\sigma)\right),$$

where

$$H(\sigma) = -J_1 \sum_{x\in V_1} \sigma(x),$$

can be divided into two parts,

$$z^{(n)} = z_0^{(n)} + z_1^{(n)}, \tag{4.2}$$

where

$$z_0^{(n)} = \left(z_0^{(n-1)} + z_1^{(n-1)}\right)^2, \quad z_1^{(n)} = e^h \left(z_0^{(n-1)}\right)^2.$$

The quantities in (3.2)-(3.5) can be expressed as the products of $z_0^{(n)}$ and $z_1^{(n)}$ as follows:

$$Z_{0,1}^{(n)} = Z_{1,0}^{(n)} = z_0^{(n)} \cdot z_1^{(n)}, \quad Z_{0,0}^{(n)} = \left(z_0^{(n)}\right)^2, \quad Z_{1,1}^{(n)} = \left(z_1^{(n)}\right)^2. \tag{4.3}$$

By defining

$$u_n = \frac{z_1^{(n)}}{z_0^{(n)}},$$

we reduce the system of equations (4.3) to a single equation

$$u_n = \frac{\lambda}{(1+u_{n-1})^2} := \phi(u_{n-1}),$$

where $\lambda = e^h$ and $\phi(x) = \frac{\lambda}{(1+x)^2}$. If u_n has a limit, say u, we must have $u = \phi(u)$.

Remark 4.1. Yu. Suhov and U. Rozikov studied the equation $u = \phi(u)$ in [18] and it is shown that it has unique positive solution, which implies that the model on the open Cayley tree of order two does not exhibit a phase transition.

5. Main results

For $J_0 \neq 0$, we obtain a Hard-Core model on the closed tree shown in Figure 1. We consider only the symmetric case, i.e., $J_1 = J_2 = J$ (thus, $h_1 = h_2 = h$ and $b = c$). Since $h = 0$ corresponds to a trivial i.i.d. model, we always assume that $h \neq 0$, i.e., $b \neq 1$.

In the symmetric case $b = c$ under consideration, the system (3.9) becomes

$$\begin{cases} x = \frac{(1+y)^2}{(x+y)^2}, \\ y = \frac{(x+y+z+1)^2}{c(x+y)^2}, \\ z = \frac{cy^2}{(x+y)^2}. \end{cases} \tag{5.1}$$

Denote $c_{cr} = \frac{22\sqrt{33}}{9} + 14 \approx 28.0422$. The following is true

Proposition 5.1. *The system of equations (5.1) has:*

- *three solutions when $c > c_{cr}$;*
- *two solutions when $c = c_{cr}$;*
- *one solution when $0 < c < c_{cr}$.*

Proof. For simplicity, we define $u = \sqrt{x}$, $v = \sqrt{y}$, $t = \sqrt{z}$ and $\gamma = \sqrt{c}$. Then (5.1) is reduced to

$$\begin{cases} u = \frac{1+v^2}{u^2+v^2}, \\ v = \frac{u^2+v^2+t^2+1}{\gamma(u^2+v^2)}, \\ t = \frac{\gamma v^2}{u^2+v^2}. \end{cases} \tag{5.2}$$

From the first equation of (5.2) we get

$$(u-1)(u^2+u+1+v^2)=0.$$

Therefore, $u=1$. Then from the third equation, we obtain

$$t=\frac{\gamma v^2}{1+v^2}.$$

Setting it to the second equation of (5.2), we have

$$\big(\gamma v^3-v^2-1\big)\big(v^4+3v^2-\gamma v+2\big)=0. \tag{5.3}$$

We consider the first factor of equation (5.3). Using the Cardano formula, we obtain that equation

$$\gamma v^3-v^2-1=0$$

has one positive solution of the form

$$v_0=\frac{\rho^2(\gamma)+2\rho(\gamma)+4}{6\gamma\rho(\gamma)},$$

where

$$\rho(\gamma)=\sqrt[3]{108\gamma^2+12\gamma\sqrt{81\gamma^2+12}+8}.$$

Now we consider the second factor of equation (5.3). Using the Ferrari formula, we find the solutions of the equation

$$v^4+3v^2-\gamma v+2=0.$$

After some operations, we get

$$v^4+3v^2-\gamma v+2=\left(v^2+\frac{3}{2}+c_0\right)^2-\left(2c_0v^2+\gamma v+c_0^2+3c_0+\frac{1}{4}\right).$$

Denote

$$c_0(\gamma)=\frac{\beta^2(\gamma)-12\beta(\gamma)+132}{12\beta(\gamma)},$$

where

$$\beta(\gamma)=\sqrt[3]{108\gamma^2+12\sqrt{81\gamma^4-2268\gamma^2-96}-1512}$$

and $\gamma\geq\gamma_{cr}=\frac{\sqrt{22\sqrt{33}+126}}{3}\approx 5.295494$. After some operations, one gets

$$v^4+3v^2-\gamma v+2=$$
$$=\left(v^2+\sqrt{2c_0}v+\frac{3}{2}+c_0+\frac{\gamma}{2\sqrt{2c_0}}\right)\cdot\left(v^2-\sqrt{2c_0}v+\frac{3}{2}+c_0-\frac{\gamma}{2\sqrt{2c_0}}\right)=0.$$

Due to the Vieta formulas, the first factor does not have a positive solution for any $\gamma>0$. The second factor has one positive solution if $\gamma=\gamma_{cr}$ and two positive solutions if $\gamma>\gamma_{cr}$. Therefore, the system of equations (5.1) has three solutions when $c>c_{cr}$, has two solutions when $c=c_{cr}$ and has one solution when $0<c<c_{cr}$. □

Summarising, we obtain

Theorem 5.2. *For the Hard-Core model on the symmetric closed Cayley tree with a branching ratio of two:*

- *if* $h\geq\ln\left(\frac{22\sqrt{33}}{9}+14\right)$, *a phase transition occurs,*
- *if* $h<\ln\left(\frac{22\sqrt{33}}{9}+14\right)$, *a phase transition does not occur.*

Acknowledgments. We are grateful to the reviewer for the careful reading of the manuscript and for the constructive remarks. We also thank Prof. M.M. Rahmatullaev for many stimulating conversations. This work was supported by the Fundamental Project (No FL-8824063343) of the Agency for Innovative Development under the Ministry of Higher Education, Science and Innovation of the Republic of Uzbekistan.

References

[1] Baxter R.J.; Hard Hexagons: Exact Solution. Journal of Physics A: Mathematical and General.,–1980. –13.–No3.–P.61–70.

[2] Berger T., Zhongxing Y.; Cardinality of phase transition of Ising models on closed cayley trees. Physica A: Statistical mechanics and its applications.,–1990.–166.–No3.–P.549–574.

[3] Brightwell G., Häggström O. and Winkler P.; Non monotonic behavior in hard-core and Widom-Rowlinson models, Jour. Stat. Phys.,–1999.–94.–P.415–435.

[4] Brightwell G., Winkler P.; Graph homomorphisms and phase transitions, J.Combin. Theory Ser.B.–1999.–77.–P.221–262.

[5] Christiano P.L., Goulart R.S.; The Potts model on a closed Cayley tree. Physics Letters A.,–1984.–101.–No 5–6.–P.275–278.

[6] De'Bell K., Geldart D.J.W. and Glasser M.L.; Recursion relations for the q-state Potts model on a closed Cayley tree. Physica A: Statistical mechanics and its applications.,–1984.–125.–No.2–3.–P.625–630.

[7] Dobrushin R.L., Kolafa J. and Shlosman S.B.; Phase diagram of the two-dimensional Ising antiferromagnet (computer-assisted proof). Commun.Math. Phys.,–1985.–102.–P.89–103.

[8] Galvin D., Martinelli F., Ramanan K. and Tetali P.: The multi-state Hard Core model on a regular tree, SIAM Journal on Discrete Mathematics., –2011.–25.–No.2.–P.894–915.

[9] Jelitto R. J.; The Ising model on a Closed cayley tree. Physica A: Statistical mechanics and its applications.,–1979.–99.–No.1.–P.268–280.

[10] Kelly F.P.; Loss Networks. Annals of applied probability.,–1991.–1.–No.3–P.319–378.

[11] Khakimov R.M., Makhammadaliev M.T. and Haydarov F.H.; New class of Gibbs measures for two-state hard-core model on a Cayley tree, Infin. Dimens. Anal. Quantum Probab. Relat. Top.,–2023.–26.–No.4. Article ID 2350024.

[12] Khakimov R.M., Makhammadaliev M.T.; Uniqueness and nonuniqueness conditions for weakly periodic Gibbs measures for the Hard-Core model. Theor. Math. Phys.–2020.–204.–No.2.–P.1059–1078.

[13] Krizan J.E., Peter F.B. and Glasser M.L.; Phase transitions for the Ising model on the closed Cayley tree. Physica A: Statistical mechanics and its applications.–1983.–119.–No.1–2.–P.230–242.

[14] Martin J.B.; Reconstruction thresholds on regular trees. Discrete random walks., (electronic), Discrete Math. Theor. Comput. Sci. Proc., AC, Assoc. Discrete Math. Theor. Comput. Sci.,–2003.–P.191–204.

[15] Mazel A.E., Suhov Y.M.; Random surfaces with two-sided constraints: An application of the theory of dominant ground states. J Stat. Phys., –1991.–64.–P.111–134.

[16] Rozikov U.A., Khakimov R.M. and Makhammadaliev M.T.; Gibbs Periodic measures for a two-state HC-Model on a Cayley Tree. Journal of Mathematical Sciences.,–2024.–278.–No.4.

[17] Sinclair A., Srivastava P.; Hard-Core Model on Hypercubic Lattices: Phase Transitions and Computational Complexity. Annals of Probability.,–2017.

[18] Suhov Yu.M., Rozikov U.A.; A hard-core model on a Cayley tree: an example of a loss network. Queueing Systems.,–2004.–46.–P.197–212.

[19] Weitz D.; Counting independent sets up to the tree threshold. Proceedings of the ACM Symposium on Theory of Computing (STOC).,–2006.

[20] Zhongxing Y., Berger T.; A bound on the phase transition region for Ising models on closed cayley trees. Physica A: Statistical mechanics and its applications.,–1990.–169.–No.3.–P.430–443.

Makhammadaliev M.T. ,
Namangan state university, Namangan, Uzbekistan
email: mmtmuxtor93@mail.ru
Karshiboev O.Sh.,
Oriental university,
Samarkand, Uzbekistan.
Namangan state university, Namangan, Uzbekistan
email: okarshiboevsher@mail.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.134-141
DOI: 10.29229/uzmj.2025-3-14

About the Conditions for Locally Relative Controllability of a Differential Inclusion

Otakulov S., Rahimov B.SH.

Abstract. The mathematical model of the control system considered in this paper is presented in the form of a linear nonstationary differential inclusion. For this model of the control system, the problem of locally relative controllability is studied. The necessary conditions of locally relative $M-$controllability for compact terminal set M are obtained. Property of locally relative controllability of differential inclusion has been studied using the methods of convex analysis, properties of multi-valued maps, support functions and the fundamental matrix for solving linear systems of differential equations.

Keywords: dynamic system, differential inclusion, terminal set, locally relative controllability, conditions for controllability.

MSC (2020): 34A60, 34H05, 49J21.

1. Introduction

Differential inclusions have wide applications in the theory of optimal control, in differential games, in the theory of differential equations with discontinuous right-hand sides and in other fields. Differential inclusions are a convenient mathematical apparatus in the research of such fundamental issues of the mathematical theory of optimal control as the problems of controllability of dynamic systems, the existence of optimal control, necessary and sufficient optimality conditions [1]-[5]. By now, the scope of research of differential inclusion and applications has significantly expanded.

One of the most important problems for a dynamic system is its controllability property. Along with full controllability, the problems of conditional, relative and local controllability are also of great interest in control theory. The necessary and sufficient conditions of controllability have been studied for various classes of models of dynamic systems [6]-[18]. Many results on controllability problem obtained for ordinary continuous systems are developed for systems with delays and discrete systems [19]-[21].

The issues of controllability of dynamical systems can be studied as controllability problems for models described by various classes of differential inclusions. Some properties of the type of local controllability for systems described by differential inclusions were initially studied by V.I.Blagodatskikh [3],[22]. The property of controllability of differential inclusions are also investigated by F. Clark[2], B. Sh. Mordukhovich [23], E. S. Polovinkin and G. V. Smirnov [24], H. Frankowska [25].

At the present stage of development of the theory of optimal processes, much attention is devote to the issues of building optimal control systems in conditions of inaccuracy and insufficient information. Therefore, the issues of controllability of the ensemble of trajectories of differential inclusions with control parameters are of particular interest. The works [26]-[28] are devoted to the study of the controllability property of an ensemble of trajectories. Some properties of the set of relative controllability of the differential inclusion were studied in [29],[30]. In this paper for one model of a control system in the form of a differential inclusion the problem of necessary conditions of locally relative controllability is studied.

2. Statement of the problem

We will use the designations: $\mathbb{R}^n$ is n - dimensional Euclidean space;(x, y) is the inner product of vectors $x, y \in \mathbb{R}^n$; $||x||$ is the norm of the vector $x \in \mathbb{R}^n$; $c(X, \psi) = \sup\{(x, \psi) : x \in X\}$ is the support function of a limited set X from $\mathbb{R}^n$; $||X|| = \sup\limits_{x\in X} ||x||$ is the norm of compact set X; $L_1(T)$ is the space of Lebesgue integrable (summable) functions defined on the segment $T = [t_0, t_1]$.

Consider a control object whose dynamics in the n dimensional state space $\mathbb{R}^n$ is described by differential inclusion

$$\frac{dx}{dt} \in A(t)x + B(t), t \geq t_0 \tag{2.1}$$

where $A(t)$ is a $n \times n$-matrix, $B(t)$ is a multi-valued mapping. We will assume that the following conditions are met:

1) the elements of the matrix $A(t)$ are summable on any $T = [t_0, t_1] \subset [t_0, +\infty)$; 2) for each $t \geq t_0$ a set $B(t) \subset \mathbb{R}^n$ is compact and multivalued mapping $t \to B(t)$ is measurable on an arbitrary segment $T \subset [t_0, +\infty)$ and $||B(t)|| \leq \beta(t)$,where $\beta(\cdot) \in L_1(T)$.

By the admissible trajectories of the control system we will understand each absolutely continuous n-vector function $x = x(t)$ on a certain segment $T = [t_0, t_1]$, satisfying almost everywhere on $T = [t_0, t_1]$ a given differential inclusion (2.1).

Let $X(t_0, t_1, x_0, A, B)$ be the reachability set of differential inclusion (2.1) from the starting point $x_0 \in \mathbb{R}^n$ at time $t_1 > t_0$, i.e. the set of all possible points $x_1 \in \mathbb{R}^n$ for which there are trajectories $x = x(t)$, $t \in T = [t_0, t_1]$, such that $x(t_0) = x_0$ and $x(t_1) = x_1$.

Let M be a given compact set of the space $\mathbb{R}^n$ which will use as set of terminal states of the control object (2.1); $M^\varepsilon = \{\xi : \xi = m + \nu, m \in M, ||\nu|| \leq \varepsilon\}$ is the ε- neighborhood of the set M.

Definition 2.1. We will say that a differential inclusion (2.1)locally relative M- controllable if there is a number $\varepsilon > 0$ and a time interval $T = [t_0, t_1]$, such that for any starting point $x_0 \in M^\varepsilon$ there exist an admissible trajectory $x(t), t \in T$, satisfying the condition $x(t_1) \in M$, i.e. the relation $X(t_0, t_1, x_0, A, B) \cap M \neq \emptyset \ \forall x_0 \in M^\varepsilon$ holds.

In the case when the set M consists of a single element, i.e. if $M = \{m\}$, then according to this definition we will say about the local $\{m\}$-contractility of the differential inclusion.And in the case of $m = 0$, according to the definition, we obtain a definition of the local zero-contractility of the system under consideration.

From the theory of multivalued maps and differential inclusions and [1], [3] and linear systems of differential equations is known that for the set $X(t_0, t_1, \xi, A, B)$ the following formula is true [14],[18],[26]

$$X(t_0, t_1, \xi, A, B) = \Phi_A(t_1, t_0)\xi + \int_{t_0}^{t_1} \Phi_A(t_1, t)B(t)dt, \tag{2.2}$$

where $\Phi_A(t, \tau)$ is the fundamental matrix of solutions to equation $\dfrac{dx}{dt} = A(t)x$, $t \in T$. From this formula and the properties of the integral of multivalued maps, it easily follows that $X(t_0, t_1, \xi, A, B)$ is a convex compact of space $\mathbb{R}^n$. According to results of the theory of multivalued mappings and the properties of support functions [3], from the formula (2.2) follows that for the support function of the set $X(t_0, t_1, \xi, A, B)$ the formula holds

$$c(X(t_0, t_1, \xi, A, B), \psi) = (\Phi_A(t_1, t_0)\xi, \psi) + \int_{t_0}^{t_1} c(\Phi_A(t_1, t)B(t), \psi)dt. \tag{2.3}$$

Using the methods of convex analysis, the properties of multivalued maps and support functions, and the fundamental matrix for solving linear systems of differential equations [3]-[6] we will study the property of locally relative controllability of differential inclusion (2.1).

3. Necessary conditions for locally relative M-controllability

Theorem 3.1. *For locally relative M-controllability of differential inclusion (2.1), it is necessary that the condition*

$$\inf_{||\psi||=1} \sup_{\xi \in M} \int_{t_0}^{t_1} c(\Phi_A(t_1, \tau)[A(\tau)\xi + B(\tau)], \psi)d\tau > 0, \tag{3.1}$$

was performed at some moment of time $t_1 > t_0$.

Proof. Suppose by contradiction, i.e. let for each $t_1 > t_0$,there exists $\psi^0 = \psi^0(t_1) \in \mathbb{R}^n, ||\psi^0|| = 1$ such that there is an inequality

$$\sup_{\xi \in M} \int_{t_0}^{t_1} c(\Phi_A(t_1, \tau)[A(\tau)\xi + B(\tau)], \psi^0)d\tau \leq 0. \tag{3.2}$$

Since the differential inclusion (2.1) is locally relative M- controllable, then by virtue of definition 2.1, there exists a number $\varepsilon > 0$ and some moment of time $t_1 > t_0$ such that for all $x_0 \in M^\varepsilon$ there is a ratio $X(t_0, t_1, x_0, A, B) \cap M \neq \emptyset$, which is equivalent to inclusion $0 \in X(t_0, t_1, x_0, A, B) - M$. Using of the support functions, the last relation can be written as an inequality

$$\inf_{||\psi||=1} [c(X(t_0, t_1, x_0, A, B), \psi) + c(M, -\psi)] \geq 0.$$

Using the formula (2.3), the resulting inequality can be written as follows:

$$\inf_{||\psi||=1} [(\Phi_A(t_1, t_0)x_0, \psi) + \int_{t_0}^{t_1} c(\Phi_A(t_1, \tau)B(\tau), \psi)d\tau + c(M, -\psi)] \geq 0.$$

Hence, in particular, we obtain that for the vector $\psi^0 = \psi^0(t_1) \in \mathbb{R}^n, ||\psi^0|| = 1$ satisfying the inequality (3.2), the following inequality is also valid:

$$(\Phi_A(t_1, t_0)x_0, \psi^0) + \int_{t_0}^{t_1} c(\Phi_A(t_1, \tau)B(\tau), \psi^0)d\tau + c(M, -\psi^0) \geq 0. \tag{3.3}$$

Due to the compactness of the set M, there exists a point $m^0 \in M$ such that $(m^0, -\psi^0) = c(M, -\psi^0)$, i.e.

$$(m^0, \psi^0) + c(M, -\psi^0) = 0. \tag{3.4}$$

Consider an arbitrary point $\xi = m^0 + \nu, ||\nu|| \leq \varepsilon$. By virtue of the properties of the fundamental matrix $\Phi_A(t_1, t)$, we have:

$$\Phi_A(t_1, t_0)m^0 - m^0 = -\int_{t_0}^{t_1} \frac{\partial \Phi_A(t_1, t)}{\partial t} m^0 dt = \int_{t_0}^{t_1} \Phi_A(t_1, t)A(t)m^0 dt.$$

Therefore,

$$(\Phi_A(t_1, t_0)\xi, \psi^0) = (m^0, \psi^0) + \int_{t_0}^{t_1} (\Phi_A(t_1, t)A(t)m^0, \psi^0)dt + (\Phi_A(t_1, t_0)\nu, \psi^0).$$

Thus, given (3.4), we have:

$$(\Phi_A(t_1, t_0)\xi, \psi^0) + \int_{t_0}^{t_1} c(\Phi_A(t_1, \tau)B(\tau), \psi^0)d\tau + c(M, -\psi^0) = (\Phi_A(t_1, t_0)\nu, \psi^0) +$$

$$+(m^0, \psi^0) + c(M, -\psi^0) + \int_{t_0}^{t_1} (\Phi_A(t_1, \tau)A(\tau)m^0, \psi^0)d\tau + \int_{t_0}^{t_1} c(\Phi_A(t_1, \tau)B(\tau), \psi^0)d\tau =$$

$$= (\Phi_A(t_1, t_0)\nu, \psi^0) + \int_{t_0}^{t_1} c(\Phi_A(t_1, \tau)[A(\tau)m^0 + B(\tau)], \psi^0)d\tau. \tag{3.5}$$

Consider the vector $\nu^0 = -\frac{\varepsilon \Phi_A'(t_1, t_0)\psi^0}{||\Phi_A'(t_1, t_0)\psi^0||}$. It is clear that $||\nu^0|| = \varepsilon$. Therefore, assuming $\nu = \nu^0$, for the vector $\xi \equiv x_0 = m^0 + v^0$ from (3.5) we get:

$$(\Phi_A(t_1, t_0)x_0, \psi^0) + \int_{t_0}^{t_1} c(\Phi_A(t_1, \tau)B(\tau), \psi^0)d\tau + c(M, -\psi^0) =$$

$$= -\varepsilon||\Phi'_A(t_1,t_0)\psi^0|| + \int\limits_{t_0}^{t_1} c(\Phi_A(t_1,\tau)[A(\tau)m^0 + B(\tau)], \psi^0)d\tau. \tag{3.6}$$

Now, considering (3.2), from (3.6) we obtain

$$(\Phi_A(t_1,t_0)x_0, \psi^0) + \int\limits_{t_0}^{t_1} c(\Phi_A(t_1,\tau)B(\tau), \psi^0)d\tau + c(M, -\psi^0) < 0.$$

The latter inequality contradicts condition (3.3), the validity of which, as already stated above, follows from the local-relative M- controllability of the differential inclusion (2.1). The resulting contradiction proves the theorem.

Theorem 3.2. *If the differential inclusion (2.1) is locally-relative M- controllable, then there exists a moment of time $t_1 > t_0$ such that following inclusion takes place:*

$$0 \in int \int\limits_{t_0}^{t_1} \Phi_A(t_1,\tau)\big[A(\tau)M + B(\tau)\big]d\tau. \tag{3.7}$$

Proof. Using the properties of the support functions the necessary condition of local controllability in form (3.1) can be written as an inequality

$$\inf_{||\psi||=1} c(\bigcup_{\xi\in M} \int\limits_{t_0}^{t_1} \Phi_A(t_1,\tau)[A(\tau)\xi + B(\tau)]d\tau, \psi) > 0. \tag{3.8}$$

It is clear that

$$c(\bigcup_{\xi\in M} \int\limits_{t_0}^{t_1} \Phi_A(t_1,\tau)[A(\tau)\xi + B(\tau)]d\tau, \psi) \le c(\int\limits_{t_0}^{t_1} \Phi_A(t_1,\tau)[A(\tau)M + B(\tau)]d\tau, \psi).$$

So, from the necessary condition of locally relative M-controllability in from (3.8), we obtain

$$\inf_{||\psi||=1} c(\int\limits_{t_0}^{t_1} \Phi_A(t_1,\tau)[A(\tau)M + B(\tau)]d\tau, \psi) > 0.$$

Due to the properties of the support functions, the relation (3.8) follows from the last inequality. Theorem 2.1 is proven.

Corollary 3.1. *If differential inclusion (2.1) is locally relative M- controllable, then there exists a moment of time $t_1 > t_0$ such that*

$$0 \in int[X(t_0,t_1,coM,A,B) - coM]. \tag{3.9}$$

Indeed, for an arbitrary point $m \in M$ we have:

$$\int\limits_{t_0}^{t_1} \Phi_A(t_1,\tau)\big[A(\tau)m + B(\tau)\big]d\tau = \int\limits_{t_0}^{t_1} \Phi_A(t_1,\tau)A(\tau)md\tau + \int\limits_{t_0}^{t_1} \Phi_A(t_1,\tau)B(\tau)d\tau =$$

$$= -\int\limits_{t_0}^{t_1} \frac{\partial\Phi_A(t_1,\tau)}{\partial\tau}d\tau m + \int\limits_{t_0}^{t_1} \Phi_A(t_1,\tau)B(\tau)d\tau = (\Phi_A(t_1,t_0) - E)m + \int\limits_{t_0}^{t_1} \Phi_A(t_1,\tau)B(\tau)d\tau.$$

Therefore,

$$\bigcup_{m\in M}\int_{t_0}^{t_1}\Phi_A(t_1,\tau)[A(\tau)m+B(\tau)]d\tau=\bigcup_{m\in M}\big[(\Phi_A(t_1,t_0)-E)m+\int_{t_0}^{t_1}\Phi_A(t_1,\tau)B(\tau)d\tau\big]=$$

$$=\bigcup_{m\in M}\big[(\Phi_A(t_1,t_0)-E)m\big]+\int_{t_0}^{t_1}\Phi_A(t_1,\tau)B(\tau)d\tau=(\Phi_A(t_1,t_0)-E)M+\int_{t_0}^{t_1}\Phi_A(t_1,\tau)B(\tau)d\tau,$$

where E is a single $n\times n$ - matrix. Since the condition (3.1) is equivalent to the ratio (3.8), we obtain that the necessary condition (3.1) takes the form of inclusion

$$0\in int\big[(\Phi_A(t_1,t_0)-E)coM+\int_{t_0}^{t_1}\Phi_A(t_1,\tau)B(\tau)d\tau\big].$$

Therefore, using formula (2.2), we obtain a formula (3.9).

Remark 3.1. From the condition (3.9) follows that if the differential inclusion (2.1) is locally $\{m\}$-controllable, then

$$m\in intX(t_0,t_1,m,A,B)$$

for some moment of time $t_1>t_0$.

Corollary 3.2. *If the differential inclusion (2.1) is locally relative M- controllable, then there exists a moment of time $t_1>t_0$ such that there is an inequality*

$$\inf_{||\psi||=1}\sup_{t\in[t_0,t_1]}c\big(\Phi_A(t_1,t)[A(t)M+B(t)],\psi\big)>0. \tag{3.10}$$

Indeed, if we assume that (3.10) does not hold for any $t_1>t_0$, then for each moment of time $t_1>t_0$ there exists $\psi^*=\psi^*(t_1)\in\mathbb{R}^n, ||\psi^*||=1$ such that

$$\sup_{t\in[t_0,t_1]}c\big(\Phi_A(t_1,t)[A(t)M+B(t)],\psi^*\big)\le 0.$$

Then, from this inequality we get

$$c\big(\int_{t_0}^{t_1}\Phi_A(t_1,\tau)[A(\tau)M+B(\tau)]d\tau,\psi^*\big)=\int_{t_0}^{t_1}c(\Phi_A(t_1,\tau)[A(\tau)M+B(\tau)],\psi^*)d\tau\le 0.$$

But this contradicts to the necessary condition of locally relative M-controllability in the form (3.7).

Theorem 3.3. *If the differential inclusion (2.1) is locally-relative M- controllable, then there exists a moment of time $t_1>t_0$ such that*

$$0\in int\,\overline{co}\bigcup_{t\in[t_0,t_1]}\Phi_A(t_1,t)[A(t)M+B(t)].$$

Proof. According to corollary 3.2, there exist of a moment of time $t_1>t_0$ and the condition (3.10) is fulfilled. The left part of (3.10) is denoted by $\delta>0$. Then it is clear that

$$\delta\le\sup_{t\in[t_0,t_1]}c\big(\Phi_A(t_1,t)[A(t)M+B(t)],\psi\big)=$$

$$=c\big(\bigcup_{t\in[t_0,t_1]}\Phi_A(t_1,t)[A(t)M+B(t)],\psi\big)\ \forall\psi\in\mathbb{R}^n, ||\psi||=1.$$

Therefore, according to the properties of the support functions, we have

$$0\in int\,\overline{co}\bigcup_{t\in[t_0,t_1]}\Phi_A(t_1,t)[A(t)M+B(t)].$$

Theorem 3.3 is proven.

Theorem 3.4. *Let $B(t) = C(t)U(t)$, where $C(t) = (c_{ij}(t))$ is a $n \times m$ -matrix whose elements $c_{ij}(t) \in L_1(T)$ at each $T = [t_0, t_1]$, $t \to U(t)$ is a measurable multivalued map, $U(t)$ is compact set from $\mathbb{R}^m$, $||U(t)|| \le g(t)$, $t \in T = [t_0, t_1]$, $g(\cdot) \in L_1(T)$. Then if the system (2.1) is locally zero-controllable, then there exists $t_1 > t_0$ such that for any $\psi \in \mathbb{R}^n, ||\psi|| = 1$ the relation $\mu\Theta(\psi) > 0$ is valid, where $\mu\Theta(\psi)$ is the Lebesgue measure of the set $\Theta(\psi) = \{t \in T = [t_0, t_1] : C^{'}(t)\Phi^{'}_A(t_1, t)\psi \neq 0\}$.*

Proof. Suppose contrary, i.e. for any $t_1 > t_0$ there exists $\psi^* = \psi(t_1) \in \mathbb{R}^n, ||\psi^*|| = 1$ such that $\mu\{t \in [t_0, t_1] : C^{'}(t)\Phi^{'}_A(t_1, t)\psi^* \neq 0\} = 0$. Then we have:

$$\int_{t_0}^{t_1} c\big(\Phi_A(t_1, t)C(t)U(t), \psi^*\big)dt = \int_{t_0}^{t_1} c\big(U(t), C'(t)\Phi^{'}_A(t_1, t)\psi^*\big)dt =$$

$$= \int_{\mu\Theta(\psi^*)} c\big(U(t), C'(t)\Phi^{'}_A(t_1, t)\psi^*\big)dt = 0.$$

But, on the other hand, since the zero-controllability property takes place, then by virtue of Theorem 3.1 the ratio is valid.

$$\int_{t_0}^{t_1} c\big(\Phi_A(t_1, t)C(t)U(t), \psi^*\big)dt = \int_{t_0}^{t_1} c\big(U(t), C'(t)\Phi^{'}_A(t_1, t)\psi^*\big)dt > 0,$$

which contradicts the equality obtained above. This contradiction shows that our assumption is incorrect, and therefore, the theorem has been proved.

Corollary 3.3. *Let $A(t) \equiv A, B(t) \equiv CU(t)$, where C is a $n \times m$- matrix, $t \to U(t)$ is a measurable multivalued map, $U(t)$ is compact set of space $\mathbb{R}^m$, $||U(t)|| \le g(t)$, $t \in T = [t_0, t_1]$, $g(\cdot) \in L_1(T)$. Then, if the system (2.1) is locally zero-controllable, then $rankK = n$, where $K = \{C, AC, A^2C, ..., A^{n-1}C\}$.*

In fact, if we assume that $rankK < n$, then there exists $\psi \in \mathbb{R}^n, ||\psi|| = 1$ such that

$$\psi^{'}C = \psi^{'}AC = \psi^{'}A^2C = ... = \psi^{'}A^{n-1}C = 0.$$

According to the Cayley-Hamilton theorem from algebra, a square matrix A of size n satisfies the equality

$$A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + ... + \alpha_{n-1}A + \alpha_n E = 0,$$

where $\alpha_1, \alpha_2, ..., \alpha_n$ are the coefficients of the characteristic equation $det(A-\lambda E) = 0$. It follows that all degrees $A^\nu, \nu \ge n$ of the matrix A are expressed as linear combinations of matrices $E, A, A^2, ..., A^{n-1}$. Therefore, using the definition of an exponential matrix, we have:

$$e^{tA} = E + tA + \frac{t^2}{2!}A^2 + ... + \frac{t^{n-1}}{(n-1)!}A^{n-1} + \frac{t^n}{n!}A^n + ... =$$

$$= \beta_1(t)E + \beta_2(t)A + ... + \beta_n(t)A^{n-1},$$

where $\beta_1, \beta_2, ..., \beta_n$ are the analytical functions of the argument t. Therefore, given that $\Phi_A(t_1, t) = e^{A(t_1-t)}$, we have:

$$\psi^{'}\Phi_A(t_1, t)C = \psi^{'}e^{A(t_1-t)}C = \psi^{'}[\beta_1(t)E + \beta_2(t)A + ... + \beta_n(t)A^{n-1}]C =$$

$$= \beta_1(t)\psi^{'}C + \beta_2(t)\psi^{'}AC + ... + \beta_n(t)\psi^{'}A^{n-1}C = 0, \forall t \in [t_0, t_1].$$

Thus, $\psi^{'}\Phi_A(t_1, t)C = 0 \ \forall t \in [t_0, t_1]$. And this contradicts the necessary condition of local zero-controllability $\mu\{t \in T = [t_0, t_1] : C^{'}(t)\Phi^{'}_A(t_1, t)\psi \neq 0\} > 0$ from Theorem 3.4. The resulting contradiction proves the statement of the corollary.

Remark 3.2. *The fulfillment of the inclusion $0 \in int\, \overline{co} \bigcup_{t\in[t_0,t_1]} \Phi_A(t_1, t)B(t)$ for a certain $t_1 > t_0$ is necessary condition of zero-controllability. A necessary condition for local M-controllability with there will be condition $0 \in int\, \overline{co} \bigcup_{t\in[t_0,t_1]} B(t)$, $t_1 > t_0$. If $A(t) \equiv 0$ and $B(t) \equiv B$, then the condition $0 \in int\, \overline{co}B$ is a necessary condition for the local M-controllability of the system (2.1).*

4. Conclusion

The paper researches the problem of local relative controllability for a mathematical model of a control system in the form of a linear nonstationary differential inclusion. Assuming that the terminal set M is compact, the necessary conditions for locally-relative M- controllability are studied. From these conditions, the consequences are derived, which clarify and supplement the results obtained. These results, in particular, are of interest for the question concerning the conditions of openness of the field of zero-controllability of a dynamical system.

References

[1] Aubin J.P., Cellina A. Differential inclusions. Set-valued maps and viability theory. Berlin a.o. : Springer, 1984.

[2] Clark F.H. Optimization and non-smooth analysis. John-Wiley and Sons, New York, 1983.

[3] Blagodatskikh V.I., Filippov A.F. Differential inclusions and optimal control. Proceedings of the Mathematical Institute of Academy of Sciences of USSR. 1985, vol. 169. pp. 194-252.

[4] Borisovich Yu.G., Gelman B.D., Myshkis A.D.,Obukhovsky V.V. Introduction to the theory of multivalued maps and differential inclusions. Moscow: KomKniga, 2005.

[5] Polovinkin E.S. Multivalued analysis and differential inclusions. Fizmatlit, Moskow, 2015.

[6] Lee E.B., Marcus L. Fundamentals of the theory of optimal control. New York-London-Sydney, 1967.

[7] Gabasov R.F., Kirillova F.M. Qualitative theory of optimal processes. Nauka, Moscow, 1971.

[8] Bartoshevich Z. Approximate controllability of neutral systems with control delays. J. Diff. Equat. 1984, volume 51, No. 3. pp. 295-325.

[9] Margaery A. On the 0-local controllability of a linear control system. J. Opt. Theory and Applications, 1990, vol. 66, No. 1. pp. 61-69.

[10] Frankowska H., Local controllability of control systems with feedbacks, Journal of Optimization Theory and Applications, 1989, No 60. pp. 277-296.

[11] Aubin J.-P. , Frankowska H., Controllability and observability of control systems under uncertainty, 1990, Volume dedicated to Opial, Ann. Polonici Mat, LI. pp.37-67.

[12] Kaczorek T., Linear Control systems, Research Studies Press and John Wiley New York, 1993.

[13] Klamka J., Constrained Controllability of Nonlinear Systems, Journal of mathematical analysis and applications, 1996, No 201. pp. 365-374.

[14] Klamka J., Controllability of Dynamical Systems, Matematyka stosowana, 2008, 9. pp. 57- 75.

[15] Agrachev A.A., Sachkov Yu.L., Control theory from the Geometric Viewpoint, Springer-Verlag, Berlin, 2004.

[16] Jurdjevic V., Geometric Control Theory, Cambridge University Press, 1997.

[17] Kaczorek T., Klamka J., Convex linear combination of the controllability pairs for linear systems, Control and Cybernetics , 2021, Vol. 50, No 4. pp. 111.

[18] Blagodatskikh V.I., Introduction to theory of optimal control, Moscow, Visshaya shkola, 2001.

[19] Klamka J, Controllability of fractional discrete-time systems with delay, Scientific Notebooks of Silesian University of Technology, Series of Automatics, 2008, 151. pp. 6772.

[20] Klamka J., Constrained controllability of semilinear systems with delays, Nonlinear Dynamics, 2009, 56 (12). pp. 169177.

[21] Razmyslovich G.P., Krakhotko V. V. Controllability of linear systems with many delays in the control of differential regulators. Bulletin of the Belarusian State University. Mathematics and computer science. 2018, No.3. pp. 82-85.

[22] Blagodatskikh V.I. On the local controllability of differential inclusions. Differential Equations, 1973, vol.9, No. 2. pp. 361-362.

[23] Mordukhovich B. Discrete approximations and refined Euler-Lagrange conditions for nonconvex differential inclusions. SIAM J. Control and optimization. 1995. Vol. 33. pp.882-915.

[24] Polovinkin E.S., Smirnov G.V. On the time optimal control problem for differential inclusions. Differential equations. 1986. Vol.20, No. 2. pp. 1351-1365.

[25] Frankowska H. Local controllability and infinitesimal generators of semigroups of multivalued maps. SIAM J. Control and Optimization. 1987. Volume 25. pp. 412-432.

[26] Otakulov S. Problems of controlling an ensemble of trajectories of differential inclusions. Lambert Academic Publishing House, 2019.

[27] Jr-Shin Li. Ensemble Control of Finite-Dimensional Time-Varying Linear Systems. IEEE , Transactions on automatic control, 2011, vol. 56, No. 2. P. 345357.

[28] Rahimov B. Sh. About conditions of controllability of ensemble trajectories of differential inclusion. Bull. Inst. Math. (Uzbekistan), 2024, vol.7, 2. -p. 83-91(in Russian).

[29] Otakulov S., Rahimov B. Sh. Haydarov T.T. On the property of relative controllability for the model of dynamic system with mobile terminal set. AIP Conference Proceedings, 2022, 2432, 030062. -p. 15.

[30] Rahimov B.Sh. Properties of the Controllability Set of One Class of Differential Inclusions. Russian Mathematics, 2024, Vol, 68, No. 9, pp. 63-69.

Otakulov S.,
University of Innovation Technologies,
Republic of Karakalpakstan, Uzbekistan
email: otakulov52@mail.ru

Rahimov B.SH.,
Jizzakh Polytechnic Institute,
Jizzakh, Uzbekistan.
email: raximovboyxoroz@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.142-152
DOI: 10.29229/uzmj.2025-3-15

Effective Dimension Aware Fractional-Order Stochastic Gradient Descent for Convex Optimization Problems

Partohaghighi M., Marcia R., Chen Y.Q.

Abstract. Fractional-order stochastic gradient descent (FOSGD) leverages fractional exponents to capture long-memory effects in optimization. However, its utility is often limited by the difficulty of tuning and stabilizing these exponents. We propose 2SED Fractional-Order Stochastic Gradient Descent (2SEDFOSGD), which integrates the Two-Scale Effective Dimension (2SED) algorithm with FOSGD to adapt the fractional exponent in a data-driven manner. By tracking model sensitivity and effective dimensionality, 2SEDFOSGD dynamically modulates the exponent to mitigate oscillations and hasten convergence. Theoretically, this approach preserves the advantages of fractional memory without the sluggish or unstable behavior observed in naïve fractional SGD. Empirical evaluations in Gaussian and α-stable noise scenarios using an autoregressive (AR) model, highlight faster convergence and more robust parameter estimates compared to baseline methods, underscoring the potential of dimension-aware fractional techniques for advanced modeling and estimation tasks.

Keywords: Fractional Calculus, Stochastic Gradient Descent, Two Scale Effective Dimension, More Optimal Optimization.

MSC: 26A33, 34A08, 78M50, 68T05

1. Introduction

Machine learning (ML) and scientific computing increasingly rely on sophisticated optimization methods to tackle complex, high-dimensional problems. Classical stochastic gradient descent (SGD) has become a mainstay in training neural networks and large-scale models, owing to its simplicity and practical performance. However, standard SGD exhibits notable limitations: it typically treats updates as short-term corrections, discarding a rich history of past gradients. In contrast, fractional approaches in optimization draw upon the theory of fractional calculus to capture long-memory effects, thereby influencing the trajectory of updates by retaining historical gradient information over extended intervals [2, 3].

We assume the parameter space $\Theta \subseteq \mathbb{R}^d$ represents the parameters of a neural network with L layers, where each layer j has parameters $\theta^{(j)} \in \mathbb{R}^{d_j}$, and $d = \sum_{j=1}^{L} d_j$. The Fisher Information Matrix and 2SED measures are computed for each layer to adapt optimization updates.

Fractional calculus extends traditional calculus to include non-integer orders, offering a powerful tool for modeling and control in various fields, including optimization. It allows for the incorporation of memory and hereditary properties into models, which is particularly beneficial in dynamic systems and optimization, e.g. [4]. By leveraging fractional derivatives, optimization algorithms can potentially achieve better convergence properties and robustness against noise, as they account for the accumulated effect of past gradients rather than relying solely on the most recent updates [6]. This approach has shown promise in enhancing the performance of optimization algorithms in machine learning and other scientific computing applications [14].

By embracing these generalized derivatives, FOSGD modifies the usual gradient step to incorporate a partial summation of past gradients, effectively smoothing updates over a historical window. The method stands especially valuable for scenarios where prior states wield significant impact on the current gradient, as often encountered in dynamic processes or highly non-convex landscape [8]. Nonetheless, the quest for harnessing fractional updates is not without drawbacks. Incorporating fractional operators demands added hyperparameters (particularly the fractional exponent α), which can prove sensitive or unstable to tune. Excessively low or high fractional orders may slow convergence or lead to oscillatory gradients, thus negating the presumed benefits. Bridging the gap between the theoretical elegance of fractional calculus and the pressing computational demands of real-world ML systems remains a formidable challenge.

Although FOSGD helps mitigate short-term memory loss by preserving traces of past gradients, selecting and calibrating the fractional exponents can introduce substantial complexities in real-world settings. For instance, deciding whether $\alpha = 0.5$ or $\alpha = 0.9$ is most appropriate for capturing relevant memory structures is neither straightforward nor reliably robust, with the optimal choice often varying considerably across tasks or even across different stages of training. In practice, if the chosen exponent fails to align with the true dynamics of the loss landscape, updates may drift or stall, resulting in unpredictable or sluggish convergence. Moreover, fractional terms can amplify variance in gradient estimatesespecially under noisy or non-stationary conditions thereby causing oscillatory or chaotic training behaviors that undermine stability.

Beyond these convergence and stability concerns, fractional exponents impose additional burdens on tuning and hyperparameter selection. Even minor changes in α can radically alter the memory effect, forcing practitioners to engage in extensive trial-and-error experiments to achieve consistent results. Such overhead becomes especially prohibitive in large-scale or time-sensitive applications, where iterating over a range of fractional parameters is not feasible. Consequently, despite its theoretical promise as a memory-based learning strategy, FOSGD faces limited adoption in practice, as the algorithm's strong reliance on well-chosen exponents can undercut the potential advantages that long-range gradient retention might otherwise provide.

Studies have highlighted issues such as the need for precise tuning of fractional orders to avoid erratic convergence paths [9, 10]. Additionally, challenge of converging to a real extreme point encountered by the existing fractional gradient algorithms is addressed in [11]. These challenges underscore the need for robust fractional SGD variants that balance computational efficiency with stable convergence. A geometry-aware strategy like Two-Scale Effective Dimension (2SED) can dynamically regulate the fractional exponent in FOSGD. By examining partial diagonal approximations of the Fisher information matrix [15], 2SED identifies regions of high sensitivity and adapts the exponent accordingly. This approach dampens updates in areas prone to instability while exploiting longer memory in flatter regions. Consequently, combining 2SED with FOSGD reduces erratic oscillations, preserves long-term memory benefits, and yields more robust performance across diverse data sets and problem types. We introduce a novel 2SED-driven FOSGD framework that dynamically regulates the fractional exponent using the dimension-aware metrics of 2SED. This adaptive mechanism aligns historical-gradient memory with the sensitivity of the optimization landscape, thereby enhancing stability and data alignment. Under standard smoothness and bounded-gradient assumptions, the method satisfies strong convergence criteria. In practice, geometry-based regularization fosters a more consistent convergence, as evidenced by solving an autoregressive (AR) model under Gaussian and α-stable noise. We organize this paper as follows. In Section 2, we thoroughly examine the 2SED algorithm, illustrating how it approximates second-order geometry to produce dimension-aware updates. Section 3 reviews FOSGD, highlighting its appeal for long-memory processes and the hyperparameter dilemmas that hinder practicality. Also, we detail how to embed 2SEDs dimension metrics into the fractional framework, providing both equations and pseudo-code. Section 4 delves into a convergence analysis, establishing theoretical performance bounds for our method. Section 5 showcases experiments across different tasks, such as an auto-regressive (AR) model and image classification, demonstrating that 2SED-driven exponent adaptation yields measurably stronger results. Finally, Section 6 concludes by summarizing key findings, identifying broader implications for optimization, and suggesting directions for further research in advanced fractional calculus and dimension-based learning techniques. This overarching narrative underscores the growing intersection between fractional approaches and dimension-aware strategies. Aligning memory-based methods with geometry-aware design, we move closer to an optimization paradigm that capitalizes on historical information without succumbing to the pitfalls of unbounded memory effects. Our findings thus underscore the promise of *2SED + FOSGD* as a more stable algorithmic solution poised for wide adoption in deep learning.

2. Two-Scale Effective Dimension (2SED) and Fractional-Order SGD

Classical complexity measures, such as the Vapnik-Chervonenkis (VC) dimension [16] or raw parameter counts, often overestimate the capacity of overparameterized neural networks. Zhang et al. [17] demonstrate that deep networks, such as Inception-style models with millions of parameters, generalize well despite their ability to memorize random labels, undermining naive VC-based bounds.

This discrepancy arises because many directions in the high-dimensional parameter space $\Theta \subseteq \mathbb{R}^d$ are "flat," contributing minimally to model outputs, while a subset of sensitive directions dominates learning [18].

Curvature-aware approaches, leveraging the Fisher Information Matrix (FIM) [19], better capture local sensitivity. We adopt the Two-Scale Effective Dimension (2SED) [15], which integrates global parameter counts with local curvature effects encoded in the FIM, offering a more nuanced complexity measure than Hessian-based metrics [20] or K-FAC approximations [21]. In this section, we define 2SED and its layer-wise variant, Lower 2SED, and propose their use in adapting fractional-order stochastic gradient descent (FOSGD) to improve optimization stability and generalization.

2.1. **Foundational Definitions.** We consider a neural network with L layers, where layer j has parameters $\theta_j \in \mathbb{R}^{d_j}$, and the parameter space is $\Theta = \Theta_1 \times \cdots \times \Theta_L \subseteq \mathbb{R}^d$, with $d = \sum_{j=1}^{L} d_j$. The Fisher Information Matrix (FIM) and 2SED are computed layer-wise to adapt optimization updates, leveraging the Markovian structure of feed-forward networks [15]. Each layers parameter vector $\theta_j \in \Theta_j \subseteq \mathbb{R}^{d_j}$ represents the trainable weights and biases, while Θ_j is the bounded domain of possible parameter values, ensuring regularity in the statistical model.

Definition 2.1 (Fisher Information [15])**.** For a statistical model $p_\theta(x, y)$ with parameters $\theta \in \Theta \subseteq \mathbb{R}^d$, assuming p_θ is differentiable and non-degenerate, define the log-likelihood as

$$\ell_\theta(x, y) = \log p_\theta(x, y).$$

The *Fisher Information Matrix* $F(\theta)$ is given by

$$F(\theta) = \mathbb{E}_{(x,y)\sim p_\theta}\left[(\nabla_\theta \ell_\theta(x, y)) \otimes (\nabla_\theta \ell_\theta(x, y))\right], \tag{2.1}$$

where $\otimes$ denotes the outer product and the expectation is over p_θ. Under regularity conditions, this equals $\mathbb{E}[-\nabla^2_\theta \ell_\theta(x, y)]$ [19].

Definition 2.2 (Empirical Fisher)**.** [15] Given an i.i.d. sample $\{(X_i, Y_i)\}_{i=1}^N$, the *empirical Fisher Information Matrix* is

$$F_N(\theta) = \frac{1}{N}\sum_{i=1}^{N} (\nabla_\theta \ell_\theta(X_i, Y_i)) \otimes (\nabla_\theta \ell_\theta(X_i, Y_i)), \tag{2.2}$$

converging to $F(\theta)$ as $N \to \infty$.

Definition 2.3 (Normalized Fisher Matrix [15])**.** The *normalized Fisher matrix* $\widehat{F}(\theta)$ rescales $F(\theta)$ so that

$$\mathbb{E}_\theta[\mathrm{Tr}\widehat{F}(\theta)] = d,$$

where $d = \dim(\Theta)$. Formally,

$$\widehat{F}(\theta) = \begin{cases} \frac{d}{\mathbb{E}_\theta[\mathrm{Tr}F(\theta)]} F(\theta), & \text{if } \mathbb{E}_\theta[\mathrm{Tr}F(\theta)] > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

2.2. **The 2SED Approach.** Although $d = \dim(\Theta)$ represents the nominal number of parameters, many directions in Θ are flat, contributing minimally to the loss [18]. The Two-Scale Effective Dimension (2SED) integrates a curvature-based term derived from the Fisher Information Matrix with the parameter count d, capturing the effective dimensionality of active directions.

Definition 2.4 (Two-Scale Effective Dimension [15])**.** Let $\widehat{F}(\theta)$ be the normalized Fisher matrix, positive semi-definite under mild conditions. For $0 < \varepsilon < 1$ and $\zeta \in \left[\frac{2}{3}, 1\right)$, the 2SED is:

$$d_\zeta(\varepsilon) = \zeta d + (1 - \zeta) d_{\mathrm{curv}}(\varepsilon), \tag{2.4}$$

where

$$d_{\mathrm{curv}}(\varepsilon) = \frac{\log \mathbb{E}_\theta\left[\det\left(I_d + \varepsilon^{\zeta-1}\widehat{F}(\theta)^{\frac{1}{2}}\right)\right]}{\left|\log\left(\varepsilon^{\zeta-1}\right)\right|}. \tag{2.5}$$

Here, I_d is the $d \times d$ identity matrix, and $\widehat{F}(\theta)^{\frac{1}{2}}$ is the positive semi-definite square root of $\widehat{F}(\theta)$.

For layer-wise optimization, we compute $d_\zeta^{(j)}(\varepsilon)$ for each layer j, using the layer-wise FIM $F_j(\theta_j)$, approximated empirically as in Definition 2.2. The parameter ζ balances the nominal dimension d_j and the curvature term $d_{\text{curv}}(\varepsilon)$. Smaller ε amplifies the contribution of significant eigenvalues, emphasizing high-curvature directions. As $\zeta \to 0$, 2SED prioritizes curvature-based modes, while $\zeta \to 1$ recovers the nominal dimension d_j. The term $\log\det\left(I_d + \varepsilon^{\zeta-1}\widehat{F}(\theta)^{\frac{1}{2}}\right)$ summarizes the spectrum of $\widehat{F}(\theta)^{\frac{1}{2}}$, emphasizing directions with large eigenvalues (high curvature) while suppressing flat directions. Rewriting the determinant as $\prod_i \left(1 + \varepsilon^{\zeta-1}\lambda_i^{1/2}\right)$, we get:

$$d_{\text{curv}}(\varepsilon) = \frac{\sum_{i=1}^{d} \log\left(1 + \varepsilon^{\zeta-1}\lambda_i^{1/2}\right)}{\left|\log\left(\varepsilon^{\zeta-1}\right)\right|}, \tag{2.6}$$

where λ_i are the eigenvalues of $\widehat{F}(\theta)$. This aligns with information geometry, capturing the effective degrees of freedom in parameter space [19].

2.2.1. *Lower 2SED for Layer-wise Complexity.* The Lower 2SED is a critical component of our 2SED-FOSGD algorithm, enabling efficient and adaptive optimization in deep neural networks. Unlike the global 2SED, which requires computing the Fisher Information Matrix (FIM) for all model parame- tersa computationally prohibitive task for deep architectures like ResNet-50 with millions of param- eters Lower 2SED leverages the Markovian structure of feed-forward networks to compute layer-wise complexity measures. This reduces memory requirements from $O(d^2)$ to $O(d_j^2)$ per layer, where d_j is the number of parameters in layer j, and enables scalable computation. By providing a per-layer complexity metric, $\underline{d}_\zeta^j(\varepsilon)$, Lower 2SED allows 2SEDFOSGD to dynamically adjust the fractional-order exponent $\alpha_t^{(j)}$ for each layer, tailoring updates to the local curvature of the loss landscape. This leads to faster convergence and improved generalization compared to standard FOSGD, which uses a uniform fractional order [22]. The Lower 2SED, introduced by Datres et al. [15] for Markovian models like CNNs, is defined iteratively for each layer $j = 1, \ldots, L$ of a model with parameters $\theta = (\theta_1, \ldots, \theta_L)$, where $\theta_j \in \Theta_j$. The FIM for layer j is:

$$\begin{aligned} F_j(\theta_1, \ldots, \theta_j) = \mathbb{E}_{x_0,\ldots,x_{j-1}} \Bigg[\int_{\mathcal{X}_j} & \left(\nabla_{\theta_j} \log p_{\theta_j}(x_j \mid x_{j-1})\right) \\ & \times \left(\nabla_{\theta_j} \log p_{\theta_j}(x_j \mid x_{j-1})\right)^T p_{\theta_j}(dx_j \mid x_{j-1}) \Bigg], \end{aligned} \tag{2.7}$$

where $p_{\theta_j}(x_j \mid x_{j-1})$ is the conditional output distribution of layer j. To model deterministic CNN outputs probabilistically, we assume layer outputs follow a Gaussian distribution with mean equal to the deterministic output and variance $\sigma^2 = 0.01$, as in [15]. The Lower 2SED is computed as:

$$\begin{aligned} \underline{d}_\zeta^j(\varepsilon) &= \underline{d}_\zeta^{j-1}(\varepsilon) \\ &+ \frac{1-\zeta}{|\log \varepsilon|} \oint_{\hat{\Theta}_j} \int_{\Theta_j} \log\det\left(I_j + \varepsilon^{\zeta-1} F_j(\theta_1, \ldots, \theta_j)^{\frac{1}{2}}\right) \\ &\times d\theta_j d\Phi_j, \end{aligned} \tag{2.8}$$

where $\hat{\Theta}_j = \Theta_1 \times \cdots \times \Theta_{j-1}$, $d\Phi_j$ is a normalized measure over previous layers parameters, and $\underline{d}_\zeta^1(\varepsilon)$ is computed for the first layer (see [15] for details). In practice, F_j is approximated empirically using Monte Carlo integration. The Lower 2SED, $\underline{d}_\zeta^j(\varepsilon)$, replaces 2SED in our 2SEDFOSGD algorithm (Algorithm 1), scaling fractional-order gradients layer-wise to enhance convergence and generalization. The selection of the parameter ζ in the two-scale effective dimension (2SED) is pivotal for balancing theoretical rigor and practical applicability in deep learning model complexity analysis. As specified in Theorem 5.1 in [15], $\zeta \in \left[\frac{2}{3}, 1\right)$ ensures the validity of the generalization bound.

3. Fractional-Order SGD and 2SED Adaptation

Classical stochastic gradient descent (SGD) updates parameters using instantaneous gradients. However, optimization in deep learning often exhibits memory effects, suggesting that incorporating past gradients could improve convergence. Fractional calculus, via the Caputo derivative, provides a principled way to encode gradient history, with the fractional order α controlling the memory effect [23].

3.1. **Caputo Fractional Derivative and Fractional Updates.**

Definition 3.1 (Caputo Derivative [23])**.** For $n-1<\alpha<n$, the *Caputo fractional derivative* of a function f is:

$$D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\,d\tau. \tag{3.1}$$

The Caputo form is preferred in optimization as it handles initial conditions naturally and yields zero for constant functions [22].

In discrete optimization, the classical gradient $\nabla f(\theta_t)$ is replaced by the fractional gradient $D_t^\alpha f(\theta_t)$. The fractional-order SGD update is [22] $\theta_{t+1}=\theta_t-\eta D_t^\alpha f(\theta_t)$. For $\alpha\in(0,1)$, $\delta>0$, and using a Taylor series approximation [22]:

$$\theta_{t+2}=\theta_{t+1}-\mu_t\frac{\nabla f(\theta_{t+1})}{\Gamma(2-\alpha)}\left(|\theta_{t+1}-\theta_t|+\delta\right)^{1-\alpha}. \tag{3.2}$$

The offset $\delta>0$ prevents stalling when consecutive iterates are similar.

3.2. **Adapting the Fractional Exponent via Lower 2SED.**

To adapt the fractional exponent for each layer j of the neural network, where layer j has parameters $\theta^{(j)}\in\mathbb{R}^{d_j}$, we compute the 2SED $d_\zeta^{(j)}(\varepsilon)|_t$ for layer j at iteration t. A fixed fractional exponent α can lead to instability if the model's curvature changes dramatically during training. Intuitively, high curvature or high 2SED indicates directions of rapid change or "sensitivity," so a smaller $\alpha_t^{(j)}$ (closer to 0) is preferred, as it increases the memory effect by amplifying the fractional term $\left(|\theta_{t+1}^{(j)}-\theta_t^{(j)}|+\delta\right)^{1-\alpha_t^{(j)}}$ in the update rule. This smooths updates, enhancing stability and preventing overshooting in these sensitive directions. Conversely, in regions with low curvature or low 2SED, which correspond to flatter areas of the optimization landscape, a larger $\alpha_t^{(j)}$ (closer to α_0) is preferred, as it reduces the influence of the fractional term, making the update resemble standard SGD. This allows for faster convergence by relying more on the current gradient in regions where large steps are safer. Hence, we propose a 2SED-based FOSGD that dynamically adjusts α using 2SED of each layer. Suppose we compute the 2SED, $d_\zeta^{(j)}(\varepsilon)$, for layer j and let $\alpha_t^{(j)}=\alpha_0-\beta\times\frac{d_\zeta^{(j)}(\varepsilon)\big|_t}{d_{\max}}$, where α_0 is a base fractional order, $\beta>0$ is a tuning parameter and $d_{\max}$ is the maximum observed 2SED among all layers. The fraction $\frac{d_\zeta^{(j)}(\varepsilon)\big|_t}{d_{\max}}$ scales the current 2SED to the range $[0,1]$.

Algorithm 1 2SED-Based Fractional-Order SGD (2SEDFOSGD)

Input: Neural network with L layers; parameters $\theta^0 \in \mathbb{R}^d$; loss function $f(\theta)$; base fractional order $\alpha_0 \in (0, 1]$; tuning parameter $\beta > 0$; singularity offset $\delta > 0$; base learning rate $\mu_0 > 0$; 2SED balance parameter ζ; curvature sensitivity ε; maximum iterations $t_{\max} \in \mathbb{N}$.

Initialize: $\theta^1 \leftarrow \theta^0 - \mu_0 \nabla f(\theta^0)$ (classical SGD step).

for $t = 1, 2, \ldots, t_{\max} - 1$ **do**
  Compute gradient: $g(\theta^t) \leftarrow \nabla f(\theta^t)$
  Compute Fisher matrices: $F_j(\theta^t)$ for $j = 1, \ldots, L$
  for $j = 1, \ldots, L$ **do**
    Compute $\underline{d}_\zeta^{(j)}(\varepsilon)\big|_t$
    Compute $d_{\max} \leftarrow \max_{j,k} \underline{d}_\zeta^{(j)}(\varepsilon)\big|_k$
    Compute $\alpha_t^{(j)} \leftarrow \alpha_0 - \beta \times \frac{\underline{d}_\zeta^{(j)}(\varepsilon)\big|_t}{d_{\max}}$
  end for
  Update learning rate: $\mu_t \leftarrow \frac{\mu_0}{\sqrt{t}}$
  for $j = 1, \ldots, L$ **do**
    Update parameters: $\theta_{t+1}^{(j)} \leftarrow \theta_t^{(j)} - \frac{\mu_t}{\Gamma(2-\alpha_t^{(j)})} \times \left(\left|\theta_t^{(j)} - \theta_{t-1}^{(j)}\right| + \delta\right)^{1-\alpha_t^{(j)}} g_j(\theta^t)$
  end for
end for
Output: $\theta^{t_{\max}} \in \mathbb{R}^d$ ▷ Final optimized parameters

3.3. **2SEDFOSGD Algorithm.**

4. Convergence Analysis for Convex Objectives

This section provides a detailed convergence proof for the 2SEDFOSGD algorithm under convex objectives, where the fractional order $\alpha_j \in (0, 1]$ for each layer j is dynamically adjusted based on the Two-Scale Effective Dimension (2SED). The 2SED quantifies the effective number of parameters by combining the nominal parameter count with curvature information derived from the Fisher Information Matrix. We prove convergence in terms of the expected function value gap, ensuring $\min_{1\leq s\leq T} \mathbb{E}[f(\theta^s) - f(\theta^\star)] = \mathcal{O}(1/\sqrt{T})$. The analysis contains an explicit fractional factor bounds, precise descent lemma constants, and corrected step-size summations.

4.1. **Foundational Definitions and Assumptions.**

Assumption 1 (Convex Objective)**.** Let $f(\theta) : \mathbb{R}^d \to \mathbb{R}$ be convex, with $\theta = (\theta^1, \ldots, \theta^L)$, $\theta^j \in \mathbb{R}^{d_j}$, and $\sum_j d_j = d$. For $\forall \lambda \in [0, 1], \theta, \theta' \in \mathbb{R}^d$, convexity implies:

$$f(\lambda\theta + (1-\lambda)\theta') \leq \lambda f(\theta) + (1-\lambda) f(\theta'). \tag{4.1}$$

We assume f is differentiable, ensuring $\nabla f(\theta)$ exists everywhere, and let $\theta^\star = \operatorname{argmin}_\theta f(\theta)$.

Assumption 2 (Smoothness and Lipschitz Continuity)**.** The function f is L-smooth, meaning for any $\theta, \theta' \in \mathbb{R}^d$:

$$\|\nabla f(\theta) - \nabla f(\theta')\| \leq L\|\theta - \theta'\|.$$

The gradients are bounded, i.e., $\|\nabla f(\theta)\| \leq G$ for all $\theta \in \mathbb{R}^d$, where $G > 0$.

Assumption 3 (Bounded Iterates)**.** We assume the iterates are bounded, with $\|\theta_t^j - \theta_{t-1}^j\| \leq R_\Delta$ for some $R_\Delta > 0$, ensured by the step-size schedule and gradient bounds (Proposition 4.1).

Assumption 4 (Fractional Derivative Parameters)**.** The base fractional order is defined as $\alpha_0 \in (0, 1]$, serving as the starting point for the adaptive fractional order for each layer j. Specifically, the fractional order for layer j is given by $\alpha_j = \alpha_0 - \beta \frac{d_\zeta^j(\varepsilon)}{d_{\max}}$, where $d_\zeta^j(\varepsilon)$ is the Two-Scale Effective Dimension (2SED) for layer j, and $d_{\max} = \max_{k,t} d_\zeta^k(\varepsilon)|_t$ represents the maximum 2SED across all layers k and iterations t. The parameter $\beta > 0$ is chosen to ensure that $\alpha_j \in (0, 1]$, as established by Lemma 4.2, thereby maintaining the validity of the fractional order within the required range.

Assumption 5 (Fractional Factor Boundedness)**.** The update for layer j is $\theta_{t+1}^j = \theta_t^j - \frac{\mu_t}{\Gamma(2-\alpha_j)}(\|\theta_t^j - \theta_{t-1}^j\| + \delta)^{1-\alpha_j} g^j(\theta^t)$, where $\mu_t = \frac{\mu_0}{\sqrt{t}}$, $\mu_0 > 0$, $\delta > 0$ is a small constant to prevent singularities, and $g^j(\theta^t)$ is the stochastic gradient for layer j.
To bound the effective step size η_t^j, we analyze the fractional factor in the update rule. Since the fractional order $\alpha_j \in (0, 1]$, we have $2 - \alpha_j \in [1, 2]$, and the gamma function $\Gamma(x)$, being positive and continuous, satisfies $1 \leq \Gamma(2 - \alpha_j) \leq 1.6$. We define $c_\Gamma = \Gamma(2) = 1$ and $C_\Gamma = \Gamma(1) = 1$ as the lower and upper bounds, respectively, noting that $\Gamma(2 - \alpha_j)$ is typically close to 1 but may reach up to 1.6 for small α_j. Additionally, the term $(\|\theta_t^j - \theta_{t-1}^j\| + \delta)^{1-\alpha_j}$ is bounded given $\|\theta_t^j - \theta_{t-1}^j\| \leq R_\Delta$. With $\alpha_{j,\max} = \max_{j,t} \alpha_j$ and $\alpha_{j,\min} = \min_{j,t} \alpha_j$, we set $c_\Delta = \delta^{1-\alpha_{j,\min}}$ and $C_\Delta = (\delta + R_\Delta)^{1-\alpha_{j,\max}}$, ensuring $0 < c_\Delta \leq (\|\theta_t^j - \theta_{t-1}^j\| + \delta)^{1-\alpha_j} \leq C_\Delta < \infty$. Consequently, the effective step size satisfies $\eta_t^j \in \left[\mu_t \frac{c_\Delta}{C_\Gamma}, \mu_t \frac{C_\Delta}{c_\Gamma}\right]$.

Assumption 6 (Stochastic Gradient Bounds)**.** For the stochastic gradients used in the optimization, we assume that the stochastic gradient $g^j(\theta^t)$ for layer j at iteration t is an unbiased estimate of the true gradient, satisfying $\mathbb{E}[g^j(\theta^t)] = \nabla^j f(\theta^t)$. Additionally, the variance of the stochastic gradient is bounded, with $\mathbb{E}[\|g^j(\theta^t) - \nabla^j f(\theta^t)\|^2] \leq \sigma^2$, where $\sigma^2 \geq 0$ is a positive constant. Furthermore, the norm of the stochastic gradient is bounded such that $\|g^j(\theta^t)\| \leq G + \sigma$, where $G > 0$ represents the bound on the true gradient norm $\|\nabla f(\theta)\|$.

Assumption 7 (Step-Size Schedule)**.** The step-size schedule is defined as $\mu_t = \frac{\mu_0}{\sqrt{t}}$, where $\mu_0 > 0$ is a positive constant, and this schedule satisfies specific bounds on its sums. Specifically, the sum of the step sizes over T iterations is bounded by $\sum_{t=1}^T \mu_t \leq \mu_0(2\sqrt{T} - 1)$, ensuring controlled growth proportional to $\sqrt{T}$. Additionally, the sum of the squared step sizes is bounded by $\sum_{t=1}^T \mu_t^2 \leq \mu_0^2(1 + \ln T)$, reflecting a logarithmic growth that maintains stability in the optimization process.

4.2. **Propositions and Lemmas.**

Proposition 4.1 (Bounded Iterates [1])**.** *For $\mu_t = \frac{\mu_0}{\sqrt{t}}$ (and μ_0 for $t = 0$), $\|g^j(\theta^t)\| \leq G + \sigma$, the iterates satisfy:*

$$\|\theta_t^j - \theta_{t-1}^j\| \leq R_\Delta = \mu_0 \frac{C_\Delta}{c_\Gamma}(G + \sigma).$$

Lemma 4.2 (Bounding the 2SED Measure [1])**.** *Let $d_\zeta^j(\varepsilon)$ be the 2SED for layer j, updated via exponential moving averages of Fisher blocks. Assume the gradients satisfy $\mathbb{E}[\|g^j(\theta^t)\|^2] \leq G^2 + \sigma^2$, where G^2 and σ^2 are positive constants. There exists a finite constant $d_{max,finite} > 0$ such that:*

$$d_\zeta^j(\varepsilon) \leq d_{max,finite}, \quad \forall t, j.$$

Lemma 4.3 (Descent Lemma [1])**.** *For convex f, with layerwise updates $\theta_{t+1}^j = \theta_t^j - \eta_t^j g^j(\theta^t)$:*

$$\mathbb{E}[f(\theta^{t+1}) \mid \theta^t] \leq f(\theta^t) - \sum_j \eta_t^j \frac{c_\Gamma}{2C_\Delta} \|\nabla^j f(\theta^t)\|^2 + \sum_j (\eta_t^j)^2 \frac{C_\Delta^2}{c_\Gamma^2}(G^2 + \sigma^2).$$

4.3. **Main Convergence Theorem.**

Theorem 4.4 (Convergence in Convex Setting)**.** *Under the above assumptions, the iterates $\{\theta^t\}$ satisfy $\min_{1 \leq s \leq T} \mathbb{E}[f(\theta^s) - f(\theta^\star)] = \mathcal{O}(1/\sqrt{T})$ as $T \to \infty$.*

Proof. From Lemma 4.3:

$$\mathbb{E}[f(\theta^{t+1}) - f(\theta^\star) \mid \theta^t] \le f(\theta^t) - f(\theta^\star) - C_1 \sum_j \eta_t^j \|\nabla^j f(\theta^t)\|^2 + C_2 \sum_j (\eta_t^j)^2.$$

where $C_1 = \frac{c_\Gamma}{2C_\Delta}$, $C_2 = \frac{C_\Delta^2}{c_\Gamma^2}(G^2 + \sigma^2)$. Taking expectations:

$$\mathbb{E}[f(\theta^{t+1}) - f(\theta^\star)] \le \mathbb{E}[f(\theta^t) - f(\theta^\star)] - C_1 \mathbb{E}\left[\sum_j \eta_t^j \|\nabla^j f(\theta^t)\|^2\right] + C_2 \mathbb{E}\left[\sum_j (\eta_t^j)^2\right].$$

Summing from $t = 1$ to T:

$$\mathbb{E}[f(\theta^{T+1}) - f(\theta^\star)] \le f(\theta^1) - f(\theta^\star) - C_1 \sum_{t=1}^{T} \mathbb{E}\left[\sum_j \eta_t^j \|\nabla^j f(\theta^t)\|^2\right] + C_2 \sum_{t=1}^{T} \mathbb{E}\left[\sum_j (\eta_t^j)^2\right].$$

Since $f(\theta^{T+1}) \ge f(\theta^\star)$, we have:

$$C_1 \sum_{t=1}^{T} \mathbb{E}\left[\sum_j \eta_t^j \|\nabla^j f(\theta^t)\|^2\right] \le f(\theta^1) - f(\theta^\star) + C_2 \sum_{t=1}^{T} \mathbb{E}\left[\sum_j (\eta_t^j)^2\right].$$

Bound the error term:

$$\sum_j (\eta_t^j)^2 \le L\mu_t^2 \frac{C_\Delta^2}{c_\Gamma^2}, \quad \sum_{t=1}^{T} \mu_t^2 \le \mu_0^2(1 + \ln T).$$

Thus:

$$\sum_{t=1}^{T} \mathbb{E}\left[\sum_j (\eta_t^j)^2\right] \le L\mu_0^2(1 + \ln T)\frac{C_\Delta^2}{c_\Gamma^2}.$$

Bound the gradient term:

$$\sum_j \eta_t^j \ge L\mu_t \frac{c_\Delta}{C_\Gamma}, \quad \sum_{t=1}^{T} \mu_t \ge \sum_{t=1}^{T} \frac{\mu_0}{\sqrt{t}} \ge \int_1^T \frac{\mu_0}{\sqrt{x}}\, dx = 2\mu_0(\sqrt{T} - 1).$$

So:

$$\sum_{t=1}^{T} \sum_j \eta_t^j \ge L\mu_0 \frac{c_\Delta}{C_\Gamma} \cdot 2(\sqrt{T} - 1).$$

The expected function value gap is:

$$\min_{s \le T} \mathbb{E}[f(\theta^s) - f(\theta^\star)] \le \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[f(\theta^t) - f(\theta^\star)].$$

Using Jensens inequality for convex f, $\mathbb{E}[f(\theta^t)] \ge f(\mathbb{E}[\theta^t])$, and assuming $f(\theta^t) - f(\theta^\star) \le f_{\max} - f_{\min}$, we bound:

$$\sum_{t=1}^{T} \mathbb{E}[f(\theta^t) - f(\theta^\star)] \le C_1 \sum_{t=1}^{T} \mathbb{E}\left[\sum_j \eta_t^j \|\nabla^j f(\theta^t)\|^2\right].$$

Thus:

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[f(\theta^t)-f(\theta^\star)] \le \frac{f(\theta^1)-f(\theta^\star)+C_2L\mu_0^2(1+\ln T)}{C_1L\mu_0\frac{c_\Delta}{C_\Gamma}2(\sqrt{T}-1)}.$$

For large T, the dominant term in the denominator is $2\sqrt{T}$, so:

$$\min_{s\le T}\mathbb{E}[f(\theta^s)-f(\theta^\star)] \le \frac{f(\theta^1)-f(\theta^\star)+C_2L\mu_0^2(1+\ln T)}{C_1L\mu_0\frac{c_\Delta}{C_\Gamma}\cdot 2\sqrt{T}}$$
$$= \mathcal{O}(1/\sqrt{T}).$$

since $\ln T/\sqrt{T}\to 0$. Hence, the convergence rate is $\mathcal{O}(1/\sqrt{T})$. □

5. Illustrative Examples

To illustrate the effectiveness of the proposed algorithm, we first consider a system identification task based on an auto-regressive (AR) model of order p. The system output is given by [22] $y(k) = \sum_{i=1}^{p} a_i y(k-i)+\xi(k)$, where $y(k-i)$ denotes the output at time $k-i$, $\xi(k)$ is a stochastic noise sequence, and a_i are the parameters to be estimated. Our objective is to determine these unknown coefficients. The corresponding regret function is $J_k(\hat{\theta}) = \frac{1}{2}\left[y(k)-\phi^T(k)\hat{\theta}(k)\right]^2$, with $\hat{\theta}(k) = [\hat{a}_1(k),\ldots,\hat{a}_p(k)]^T$ and $\phi(k) = [y(k-1),\ldots,y(k-p)]^T$. We consider an AR model: $y(k) = 1.5y(k-1)-0.7y(k-2)+\xi(k)$, where $\xi(k)$ is α-stable noise with zero mean and variance 0.5. The goal is to estimate the true coefficients $a_1 = 1.5$ and $a_2 = -0.7$ under $\alpha_0 = 0.98$ and $\beta = 0.01$.

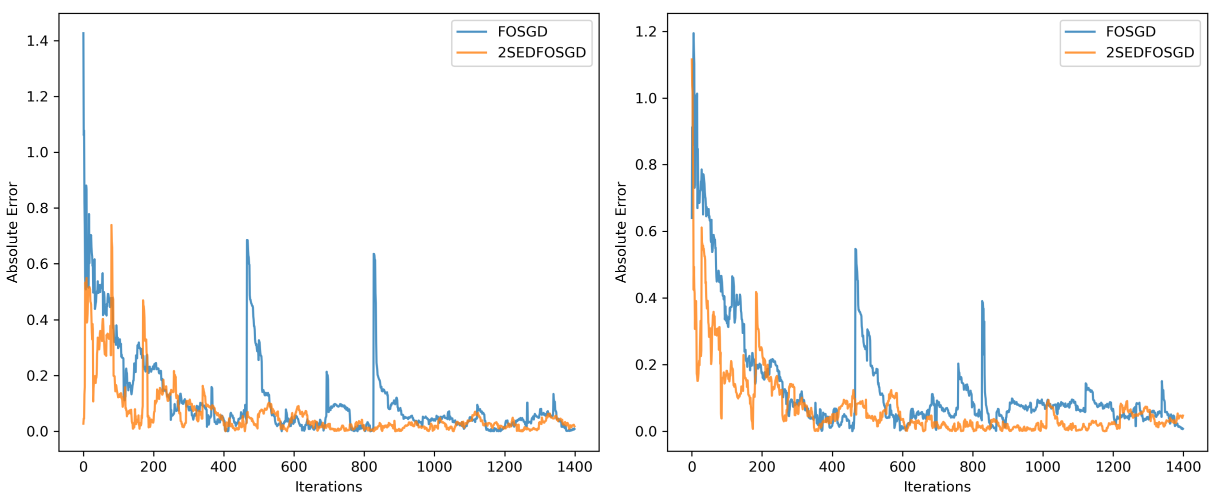


Figure 4. Convergence of a_1 and a_2 under α-stable noise.

Figure 4 illustrates the convergence of absolute errors in a_1 (left) and a_2 (right) under α-stable noise ($\alpha = 1.8$), comparing FOSG and 2SEDFOSGD. 2SEDFOSGD achieves smoother, lower error trajectories by adapting to heavy-tailed fluctuations, while FOSGD exhibits spikes due to its sensitivity to outliers.

6. Conclusion

In this paper, we proposed the 2SED Fractional-Order Stochastic Gradient Descent (2SEDFOSGD) algorithm, which augments fractional-order SGD (FOSGD) with a Two-Scale Effective Dimension (2SED) framework to dynamically adapt the fractional exponent. By continuously monitoring model sensitivity and effective dimensionality, 2SEDFOSGD mitigates oscillatory or sluggish convergence behaviors commonly encountered with naive fractional approaches. We evaluated the performance of 2SEDFOSGD through a system identification task using an autoregressive (AR) model under both Gaussian and α-stable noise.

Declaration of AI Use: During the preparation of this work, the authors used Copilot to check grammar and improve readability.

References

[1] Mohammad Partohaghighi and Roummel Marcia and YangQuan Chen, Effective Dimension Aware Fractional-Order Stochastic Gradient Descent for Convex Optimization Problems, arXiv https://arxiv.org/abs/2503.13764, pp. 1–8, 2025

[2] X. Zhou, Z. You, W. Sun, D. Zhao, and S. Yan, Fractional-order stochastic gradient descent method with momentum and energy for deep neural networks, *Neural Networks*, vol. 181, pp. 106810, Jan. 2025. doi: 10.1016/j.neunet.2024.11.019.

[3] D. Sheng, Y. Wei, Y. Chen, and Y. Wang, Convolutional neural networks with fractional order gradient method, *Neurocomputing*, vol. 408, pp. 42–50, Sep. 2020. doi: 10.1016/j.neucom.2020.05.041.

[4] Hu Sheng, YangQuan Chen, and TianShuang Qiu, Fractional Processes and Fractional-Order Signal Processing: Techniques and Applications, Springer London, 2012. doi: 10.1007/978-1-4471-2233-3.

[5] Z. Yu, G. Sun, and J. Lv, A fractional-order momentum optimization approach of deep neural networks, *Neural Computing and Applications*, vol. 34, no. 9, pp. 7091–7111, 2022. doi: 10.1007/s00521-021-06765-2.

[6] Y. Chen, Y. Wei, Y. Wang, and Y. Chen, Fractional order gradient methods for a general class of convex functions, in *Proc. 2018 American Control Conf. (ACC)*, Milwaukee, WI, USA, Jun. 27–29, 2018, pp. 3763–3767. doi: 10.23919/ACC.2018.8431083.

[7] J. Liu, R. Zhai, Y. Liu, W. Li, B. Wang, and L. Huang, A quasi fractional order gradient descent method with adaptive stepsize and its application in system identification, *Applied Mathematics and Computation*, vol. 393, pp. 125797, 2021. doi: 10.1016/j.amc.2020.125797.

[8] Y. Shin, J. Darbon, and G. E. Karniadakis, Accelerating gradient descent and Adam via fractional gradients, *Neural Networks*, vol. 161, pp. 185–201, 2023. doi: 10.1016/j.neunet.2023.01.004.

[9] S. M. Elnady, M. El-Beltagy, A. G. Radwan, and M. E. Fouda, A comprehensive survey of fractional gradient descent methods and their convergence analysis, *Journal of Computational and Applied Mathematics*, vol. 453, pp. 116147, 2024.

[10] P. Harjule, R. Sharma, and R. Kumar, Fractional-order gradient approach for optimizing neural networks: A theoretical and empirical analysis, *Neural Networks*, vol. 178, pp. 106446, 2024.

[11] G. Chen and Z. Xu, λ-FAdaMax: A novel fractional-order gradient descent method with decaying second moment for neural network training, *Expert Systems with Applications*, vol. 279, 2025.

[12] O. Herrera-Alcntara, Fractional Derivative Gradient-Based Optimizers for Neural Networks and Human Activity Recognition, *Applied Sciences*, vol. 12, no. 18, pp. 9264, 2022. doi: 10.3390/app12189264.

[13] G. Chen, Y. Liang, S. Li, and Z. Xu, A Novel Gradient Descent Optimizer Based on Fractional Order Scheduler and Its Application in Deep Neural Networks, *Applied Mathematical Modelling*, vol. 128, pp. 26–57, 2024.

[14] Y. Chen, Q. Gao, Y. Wei, and Y. Wang, Study on fractional order gradient methods, *Applied Mathematics and Computation*, vol. 314, pp. 310–321, 2017. doi: 10.1016/j.amc.2017.07.006.

[15] M. Datres, G. P. Leonardi, A. Figalli, and D. Sutter, A Two-Scale Complexity Measure for Deep Learning Models, *arXiv preprint*, arXiv:2401.09184, 2024.

[16] V. Vapnik, The Nature of Statistical Learning Theory, Springer Science & Business Media, 1999.

[17] C. Zhang, S. Bengio, M. Hardt, B. Recht, and O. Vinyals, Understanding Deep Learning Requires Rethinking Generalization, *International Conf. on Learning Representations (ICLR)*, 2017.

[18] R. Karakida, S. Akaho, and S. Amari, Universal Statistics of Fisher Information in Deep Neural Networks, *Neural Computation*, 2019.

[19] S. Amari, Natural Gradient Works Efficiently in Learning, *Neural Computation*, 1998.

[20] T. Liang, T. Poggio, A. Rakhlin, and J. Stokes, Fisher-Rao Metric, Geometry, and Complexity of Neural Networks, in *Proc. 22nd Int. Conf. on Artificial Intelligence and Statistics (AISTATS)*, 2019.

[21] J. Martens and R. Grosse, Optimizing Neural Networks with Kronecker-factored Approximate Curvature, in *Proc. 32nd Int. Conf. on Machine Learning (ICML)*, 2015.

[22] Y. Yang, L. Mo, Y. Hu, and F. Long, The Improved Stochastic Fractional Order Gradient Descent Algorithm, *Fractal Fract*, vol. 7, pp. 631, 2023.

[23] C. Monje, Y. Chen, B. M. Vinagre, D. Xue, and V. Feliu, Fractional-Order Systems and Controls, Springer, 2010.

Mohammad Partohaghighi,
University of California at Merced,
Merced, CA 95343, USA
e-mail: mpartohaghighi@ucmerced.edu

Roummel Marcia,
Department of Applied Mathematics,
University of California at Merced,
Merced, CA 95343, USA
e-mail: rmarcia@ucmerced.edu

YangQuan Chen,
Department of Mechanical and
Aerospace Engineering,
School of Engineering, University of California,
Merced, CA 95343, USA
e-mail: ychen53@ucmerced.edu

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.153-160
DOI: 10.29229/uzmj.2025-3-16

Global Stability of Polynomial Stochastic Operators Associated with Higher-Order Diagonally Primitive Doubly Stochastic Hyper-Matrices

Saburov M., Saburov Kh.

Abstract. In this paper we study a global stability problem for polynomial stochastic operators associated with higher-order diagonally primitive doubly stochastic hyper-matrices.

Keywords: Primitive matrix; stochastic hyper-matrix; global stability; polynomial stochastic operator.

MSC (2020): 37C75; 47H60; 60J10

1. Introduction

Let $\{\mathbf{e}_k\}_{k=1}^m$ be the standard basis of the space $\mathbb{R}^m$. Suppose that $\mathbb{R}^m$ is equipped with the l_1-norm $\|\mathbf{x}\|_1 := \sum_{k=1}^m |x_k|$ where $\mathbf{x} = (x_1, \cdots, x_m)^T \in \mathbb{R}^m$. We say that $\mathbf{x} \geq 0$ (respectively, $\mathbf{x} > 0$) if $x_k \geq 0$ (respectively, $x_k > 0$) for all $k \in \mathbf{I}_m := \{1, 2, 3, \cdots, m\}$. Let $\mathbb{S}^{m-1} = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geq 0,\ \|\mathbf{x}\|_1 = 1\}$ be the $(m-1)-$dimensional standard simplex. An element of the simplex $\mathbb{S}^{m-1}$ is called *a stochastic vector*. Let $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ be the center of the simplex $\mathbb{S}^{m-1}$. Let $\mathrm{int}\mathbb{S}^{m-1} = \{\mathbf{x} \in \mathbb{S}^{m-1} : \mathbf{x} > 0\}$ and $\partial\mathbb{S}^{m-1} = \mathbb{S}^{m-1} \setminus \mathrm{int}\mathbb{S}^{m-1}$ be, respectively, an interior and boundary of the simplex $\mathbb{S}^{m-1}$.

1.1. Higher-Order Singly and Doubly Stochastic Hyper-Matrices.

Let us first recall the definitions of higher-order *singly* and *doubly* stochastic hyper-matrices.

Definition 1.1 (**Higher-Order Singly Stochastic Hyper-Matrix**). A $(k+1)-$order $m-$dimensional hyper-matrix $\mathcal{P}_{k+1} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is called *stochastic* if one has that

$$\sum_{j=1}^m p_{i_1\cdots i_k j} = 1, \quad p_{i_1\cdots i_k j} \geq 0, \quad \forall\ i_1, \cdots, i_k, j \in \mathbf{I}_m.$$

Definition 1.2 (**Higher-Order Doubly Stochastic Hyper-Matrix**). A $(k+1)-$order $m-$dimensional hyper-matrix $\mathcal{P}_{k+1} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is called *doubly stochastic* if one has that

$$\sum_{i_k=1}^m p_{i_1\cdots i_k j} = \sum_{j=1}^m p_{i_1\cdots i_k j} = 1, \quad p_{i_1\cdots i_k j} \geq 0, \quad \forall\ i_1, \cdots, i_k, j \in \mathbf{I}_m.$$

Let $\mathfrak{P} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ be a polynomial stochastic operator

$$\mathfrak{P}(\mathbf{x}) := \sum_{i_1=1}^m \cdots \sum_{i_k=1}^m x_{i_1} \cdots x_{i_k} \mathbf{p}_{i_1\cdots i_k \bullet}, \qquad \forall\ \mathbf{x} \in \mathbb{S}^{m-1}, \tag{1.1}$$

where $\mathcal{P}_{k+1} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is a $(k+1)-$order $m-$dimensional stochastic hyper-matrix such that

$$\mathbf{p}_{i_1\cdots i_k \bullet} := (p_{i_1\cdots i_k 1}, \cdots, p_{i_1\cdots i_k m}) \in \mathbb{S}^{m-1}, \quad \forall\ i_\ell \in \mathbf{I}_m,\ \ell \in \mathbf{I}_k.$$

It is worth mentioning that if $k = 2$ then we derive a *quadratic stochastic operator* and if $k = 3$ then we derive a *cubic stochastic operator*.

Historically, a quadratic stochastic operator was first introduced by S. Bernstein [1], back in 1942. A quadratic stochastic process (see [2, 22]) is the simplest *nonlinear Markov chain*. The analytic theory of the quadratic stochastic process generated by cubic stochastic matrices was established in the papers [2, 22]. The quadratic stochastic operator was considered an important source of analysis for the study of dynamical properties and modeling in various fields such as biology, physics, control system. The fixed point sets, omega limiting sets, ergodicity and chaotic dynamics of quadratic stochastic operators defined on a finite-dimensional simplex were studied in the references [3, 4, 5, 6]. A long, self-contained exposition of recent achievements and open problems in the theory of quadratic stochastic operators and processes were presented in the references [7, 16]. The analytic theory of the cubic stochastic processes was established in the paper [19]. Accordingly, the fixed point sets, omega limiting sets, ergodicity and chaotic dynamics of cubic stochastic operators defined on a finite-dimensional simplex were studied in the references [8, 9, 10, 11, 12, 14, 15, 17, 18, 20, 21].

In this paper we are aiming to study a global stability problem for polynomial stochastic operators (1.1) associated with higher-order diagonally primitive doubly stochastic hyper-matrices.

1.2. **Global Stability.** Let us now recall a notion of global stability of polynomial operators associated with higher-order stochastic hyper-matrices.

Definition 1.3 (**Global Stability**, [13])**.** Let $\mathfrak{P}:\mathbb{S}^{m-1}\to\mathbb{S}^{m-1}$ be a polynomial stochastic operator given by (1.1) and let $\mathbf{q}\in\mathbb{S}^{m-1}$ be a fixed point, i.e., $\mathfrak{P}(\mathbf{q})=\mathbf{q}$. A fixed point $\mathbf{q}\in\mathbb{S}^{m-1}$ is called *globally stable* within the simplex $\mathbb{S}^{m-1}$ if one has that $\lim\limits_{n\to\infty}\mathfrak{P}^{(n)}(\mathbf{x})=\mathbf{q}$ for any initial point $\mathbf{x}\in\mathbb{S}^{m-1}$, where $\mathfrak{P}^{(n+1)}(\mathbf{x})=\mathfrak{P}(\mathfrak{P}^{(n)}(\mathbf{x}))$ for all $n\in\mathbb{N}$. In this case, a polynomial stochastic operator $\mathfrak{P}:\mathbb{S}^{m-1}\to\mathbb{S}^{m-1}$ given by (1.1) is also called *globally stable* within the simplex $\mathbb{S}^{m-1}$.

1.3. **The Matrix Form.** Throughout this section, we always assume that a $(k+1)-$order $m-$dimensional hyper-matrix $\mathcal{P}_{k+1}=\left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is doubly stochastic unless explicitly specified otherwise. Furthermore, it is important to emphasize that we do not impose the condition

$$p_{i_1 i_2\cdots i_k,j}=p_{i_{\pi(1)}i_{\pi(2)}\cdots i_{\pi(k)},j}$$

where π is a permutation of the set $\mathbf{I}_k$.

Let $\mathcal{P}_{k+1}=\left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ be the $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix and $\mathcal{P}_{\bullet\cdots\bullet l|k}=(p_{i_1\cdots i_k l})_{i_1,\cdots,i_k=1}^{m,\cdots,m}$ be its $k-$order $m-$dimensional l^{th} subhyper-matrix for fixed $l\in\mathbf{I}_m$. It is clear that $\mathcal{P}_{\bullet\cdots\bullet l|k}=(p_{i_1\cdots i_k l})_{i_1,\cdots,i_k=1}^{m,\cdots,m}$ is also stochastic hyper-matrix.

We define a polynomial stochastic operator $\mathfrak{P}:\mathbb{S}^{m-1}\to\mathbb{S}^{m-1}$ associated with $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix $\mathcal{P}_{k+1}=\left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ as follows

$$\left(\mathfrak{P}(\mathbf{x})\right)_l=\sum_{i_1=1}^{m}\cdots\sum_{i_k=1}^{m}p_{i_1\cdots i_k l}\;x_{i_1}\cdots x_{i_k},\qquad\forall l\in\mathbf{I}_m. \tag{1.2}$$

We also define a polynomial stochastic operator $\mathfrak{P}_l:\mathbb{S}^{m-1}\to\mathbb{S}^{m-1}$ associated with the $k-$order $m-$dimensional stochastic hyper-matrix $\mathcal{P}_{\bullet\cdots\bullet l|k}=(p_{i_1\cdots i_k l})_{i_1,\cdots,i_k=1}^{m,\cdots,m}$ as

$$\left(\mathfrak{P}_l(\mathbf{x})\right)_j=\sum_{i_1=1}^{m}\cdots\sum_{i_{k-1}=1}^{m}p_{i_1\cdots i_{k-1}jl}\;x_{i_1}\cdots x_{i_{k-1}},\qquad\forall j\in\mathbf{I}_m. \tag{1.3}$$

for all $l\in\mathbf{I}_m$. It follows from (1.2) and (1.3) that

$$\left(\mathfrak{P}(\mathbf{x})\right)_l=\sum_{j=1}^{m}\left(\mathfrak{P}_l(\mathbf{x})\right)_j x_j=\big(\mathfrak{P}_l(\mathbf{x}),\mathbf{x}\big),\quad\forall l\in\mathbf{I}_m.$$

where $(\cdot,\cdot)$ stands for the standard inner product of two vectors.

Therefore, the polynomial stochastic operator $\mathfrak{P}:\mathbb{S}^{m-1}\to\mathbb{S}^{m-1}$ given by (1.2) can be written as follows

$$\mathfrak{P}(\mathbf{x})=\Big(\big(\mathfrak{P}_1(\mathbf{x}),\mathbf{x}\big),\cdots,\big(\mathfrak{P}_m(\mathbf{x}),\mathbf{x}\big)\Big)^T \tag{1.4}$$

where $\mathfrak{P}_l:\mathbb{S}^{m-1}\to\mathbb{S}^{m-1}$ is defined by (1.3) for all $l\in\mathbf{I}_m$.

We now define an $m \times m$ matrix as follows

$$\mathbb{P}(\mathbf{x}) \quad = \quad \begin{pmatrix} \big(\mathfrak{P}_1(\mathbf{x})\big)_1 & \big(\mathfrak{P}_1(\mathbf{x})\big)_2 & \cdots & \big(\mathfrak{P}_1(\mathbf{x})\big)_m \\ \big(\mathfrak{P}_2(\mathbf{x})\big)_1 & \big(\mathfrak{P}_2(\mathbf{x})\big)_2 & \cdots & \big(\mathfrak{P}_2(\mathbf{x})\big)_m \\ \vdots & \vdots & \ddots & \vdots \\ \big(\mathfrak{P}_m(\mathbf{x})\big)_1 & \big(\mathfrak{P}_m(\mathbf{x})\big)_2 & \cdots & \big(\mathfrak{P}_m(\mathbf{x})\big)_m \end{pmatrix}. \tag{1.5}$$

We show that $\mathbb{P}(\mathbf{x})$ is doubly stochastic matrix for every $\mathbf{x} \in \mathbb{S}^{m-1}$. In fact we know that $\mathbb{P}(\mathbf{x}) = \big(p_{lj}(\mathbf{x})\big)_{l,j=1}^{m}$ where

$$p_{lj}(\mathbf{x}) = \big(\mathfrak{P}_l(\mathbf{x})\big)_j = \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} p_{i_1\cdots i_{k-1}jl}\ x_{i_1}\cdots x_{i_{k-1}}. \tag{1.6}$$

Therefore, it follows from (1.6) that

$$\begin{aligned} \sum_{l=1}^{m} p_{lj}(\mathbf{x}) &= \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} \left(\sum_{l=1}^{m} p_{i_1\cdots i_{k-1}jl} \right) x_{i_1}\cdots x_{i_{k-1}} \\ &= \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} x_{i_1}\cdots x_{i_{k-1}} = (x_1 + \cdots + x_m)^{k-1} = 1, \\ \sum_{j=1}^{m} p_{lj}(\mathbf{x}) &= \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} \left(\sum_{j=1}^{m} p_{i_1\cdots i_{k-1}jl} \right) x_{i_1}\cdots x_{i_{k-1}} \\ &= \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} x_{i_1}\cdots x_{i_{k-1}} = (x_1 + \cdots + x_m)^{k-1} = 1. \end{aligned}$$

Hence, it follows from (1.4) and (1.5) that

$$\mathfrak{P}(\mathbf{x}) = \mathbb{P}(\mathbf{x})\mathbf{x} \tag{1.7}$$

and it is called *matrix form* of the polynomial stochastic operator $\mathfrak{P} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ (1.2) associated with the $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix $\mathcal{P}_{k+1} = \big(p_{i_1\cdots i_k j}\big)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$.

2. The Main Result

Let us first introduce a notion of higher-order *diagonally primitive* doubly stochastic hyper-matrices.

Definition 2.1 (Higher-Order Diagonally Primitive Doubly Stochastic Hyper-Matrices). A $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix $\mathcal{P}_{k+1} = \big(p_{i_1\cdots i_k j}\big)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is called *diagonally primitive* if its diagonal matrix $\operatorname{diag}(\mathcal{P}_{k+1}) := \big(p_{i\cdots ij}\big)_{i,j=1}^{m,m}$ is a *primitive* square stochastic matrix, i.e., there exists $s \in \mathbb{N}$ such that the $s^{\text{th}}-$power of the square stochastic matrix $\operatorname{diag}(\mathcal{P}_{k+1}) := \big(p_{i\cdots ij}\big)_{i,j=1}^{m,m}$ is positive, i.e., $\big[\operatorname{diag}(\mathcal{P}_{k+1})\big]^s > 0$ where

$$\operatorname{diag}(\mathcal{P}_{k+1}) := \begin{pmatrix} p_{1\cdots 11} & p_{1\cdots 12} & \cdots & p_{1\cdots 1m} \\ p_{2\cdots 21} & p_{2\cdots 22} & \cdots & p_{2\cdots 2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m\cdots m1} & p_{m\cdots m2} & \cdots & p_{m\cdots mm} \end{pmatrix}.$$

We are now ready to state the main result of this section.

Let $\mathbf{e}_1 = (1,0,0,\cdots,0)^T$, $\mathbf{e}_2 = (0,1,0,\cdots,0)^T$, $\mathbf{e}_m = (0,0,0,\cdots,1)^T$ be vertices of the simplex $\mathbb{S}^{m-1}$ and $\mathbf{e}_l^{(n+1)} := \mathfrak{P}\left(\mathbf{e}_l^{(n)}\right)$, where $\mathbf{e}_l^{(1)} := \mathbf{e}_l$ for all $l \in \mathbf{I}_m$ and $n \in \mathbb{N}$. Let $\mathbf{x}^{(n+1)} := \mathfrak{P}\left(\mathbf{x}^{(n)}\right)$ for all $n \in \mathbb{N}$ be a trajectory starting from an initial point $\mathbf{x}^{(1)}$.

Theorem 2.2 (**Global Stability Criterion**). *Let $\mathcal{P}_{k+1} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ be a $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix and let $\mathfrak{P} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ be the associated polynomial stochastic operator. Assume that a $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix $\mathcal{P}_{k+1} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is diagonally primitive. Then the trajectory $\{\mathbf{x}^{(n)}\}_{n=1}^{\infty}$ starting from any initial point $\mathbf{x}^{(1)} \in \mathbb{S}^{m-1}$ of the simplex $\mathbb{S}^{m-1}$ converges to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex $\mathbb{S}^{m-1}$ if and only if for each $l \in \mathbf{I}_m$ there exists $n_l \in \mathbb{N}$ such that $\mathbf{e}_l^{(n_l)} \in \mathrm{int}\mathbb{S}^{m-1}$.*

Proof. **The "only if" part.** Let us assume that the trajectory $\{\mathbf{x}^{(n)}\}_{n=1}^{\infty}$ starting from any initial point $\mathbf{x}^{(1)} \in \mathbb{S}^{m-1}$ of the simplex $\mathbb{S}^{m-1}$ converges to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex $\mathbb{S}^{m-1}$. Particularly, the trajectory $\{\mathbf{e}^{(n)}\}_{n=1}^{\infty}$ starting from any vertex $\mathbf{e}_k^{(1)} := \mathbf{e}_k$, $k \in \mathbf{I}_m$ of the simplex $\mathbb{S}^{m-1}$ also converges to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex $\mathbb{S}^{m-1}$. Consequently, since $\mathbf{c} \in \mathrm{int}\mathbb{S}^{m-1}$, it is evident that for every $l \in \mathbf{I}_m$ there exists $n_l \in \mathbb{N}$ such that $\mathbf{e}_l^{(n_l)} \in \mathrm{int}\mathbb{S}^{m-1}$.

The "if" part. Let $\left\{\mathbf{x}^{(n)}\right\}_{n=1}^{\infty}$ where $\mathbf{x}^{(n+1)} = \mathfrak{P}(\mathbf{x}^{(n)})$ be a trajectory of the polynomial stochastic operator $\mathfrak{P} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ starting from an initial point $\mathbf{x}^{(1)} \in \mathbb{S}^{m-1}$. Particularly, let $\left\{\mathbf{e}_l^{(n)}\right\}_{n=1}^{\infty}$ be a trajectory of the polynomial stochastic operator $\mathfrak{P} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ starting from a vertex $\mathbf{e}_l$ of the simplex $\mathbb{S}^{m-1}$ for all $l \in \mathbf{I}_m$. According to the definition, the multi-agent system eventually reaches a consensus if $\{\mathbf{x}^{(n)}\}_{n=1}^{\infty}$ converges to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex $\mathbb{S}^{m-1}$ for any initial point $\mathbf{x}^{(1)} \in \mathbb{S}^{m-1}$. We accomplish it under two hypotheses:

(i) For each $l \in \mathbf{I}_m$ one has $\mathbf{e}_l^{(n_l)} \in \mathrm{int}\mathbb{S}^{m-1}$ for some $n_l \in \mathbb{N}$;

(ii) A doubly stochastic hyper-matrix $\mathcal{P}_{k+1} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is *diagonally primitive.*

Step-1. We first show that $\mathfrak{P}(\mathrm{int}\mathbb{S}^{m-1}) \subset \mathrm{int}\mathbb{S}^{m-1}$. Indeed, let $\mathbf{x} \in \mathrm{int}\mathbb{S}^{m-1}$. This means that $x_i > 0$ for all $i \in \mathbf{I}_m$. Since $\mathbb{P}(\mathbf{x}) = \left(p_{lj}(\mathbf{x})\right)_{l,j=1}^{m}$ is a square doubly stochastic matrix and $\mathfrak{P}(\mathbf{x}) = \mathbb{P}(\mathbf{x})\mathbf{x}$, we derive that

$$0 < \min_{j\in\mathbf{I}_m} x_j \leq \sum_{j=1}^{m} p_{lj}(\mathbf{x})x_j = (\mathfrak{P}(\mathbf{x}))_l\,, \qquad \forall\ l \in \mathbf{I}_m.$$

This means that $\mathfrak{P}(\mathbf{x}) \in \mathrm{int}\mathbb{S}^{m-1}$.

Step-2. We now show that there exists $n_0 \in \mathbb{N}$ such that for any initial point $\mathbf{x}^{(1)} \in \mathbb{S}^{m-1}$ one has $\mathbf{x}^{(n_0)} \in \mathrm{int}\mathbb{S}^{m-1}$. It has been noted that n_0 does not depend on an initial point $\mathbf{x}^{(1)} \in \mathbb{S}^{m-1}$. Indeed, since for each $l \in \mathbf{I}_m$ one has $\mathbf{e}_l^{(n_l)} \in \mathrm{int}\mathbb{S}^{m-1}$ for some $n_l \in \mathbb{N}$, it then follows from the previous step that for each $l \in \mathbf{I}_m$ one has $\mathbf{e}_l^{(n)} \in \mathrm{int}\mathbb{S}^{m-1}$ for any $n > n_l$.

Let $n_0 := \max\limits_{l\in\mathbf{I}_m} n_l$. Then $\mathbf{e}_l^{(n_0)} \in \mathrm{int}\mathbb{S}^{m-1}$ for all $l \in \mathbf{I}_m$. We now show that $\mathbf{x}^{(n_0+1)} = \mathfrak{P}(\mathbf{x}^{(n_0)}) \in \mathrm{int}\mathbb{S}^{m-1}$ for any initial point $\mathbf{x}^{(1)} \in \mathbb{S}^{m-1}$. In order to prove it, we first prove the following inequality for any initial point $\mathbf{x}^{(1)} \in \mathbb{S}^{m-1}$

$$\mathbf{x}^{(n+1)} \geq x_1^{\mathrm{K(n)}}\mathbf{e}_1^{(n)} + x_2^{\mathrm{K(n)}}\mathbf{e}_2^{(n)} + \cdots + x_m^{\mathrm{K(n)}}\mathbf{e}_m^{(n)}, \quad n \in \mathbb{N} \tag{2.1}$$

where $\mathrm{K(n)} = k^n$ for any $n \in \mathbb{N}$. Let us first introduce some necessary notations. Let $\mathcal{M}_{\mathcal{P}_{k+1}} : (\mathbb{R}^m)^{\times k} \to \mathbb{R}^m$ be a multi-linear operator associated with $(k+1)-$order $m-$dimensional stochastic hyper-matrix $\mathcal{P}_{k+1} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ as follows

$$\mathcal{M}_{\mathcal{P}_{k+1}}\left(\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \cdots, \mathbf{y}^{(k)}\right) = \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} y_{i_1}^{(1)} y_{i_2}^{(2)} \cdots y_{i_k}^{(k)} \mathbf{p}_{i_1\cdots i_k \bullet}$$

where $\mathbf{p}_{i_1\cdots i_k\bullet} = (p_{i_1\cdots i_k 1}, \cdots, p_{i_1\cdots i_k m}) \in \mathbb{S}^{m-1}$ for any $i_1, \cdots, i_k \in \mathbf{I}_m$. It is clear that $\mathfrak{P}(\mathbf{x}) = \mathcal{M}_{\mathcal{P}_{k+1}}(\mathbf{x}, \mathbf{x}, \cdots, \mathbf{x})$ for any $\mathbf{x} \in \mathbb{S}^{m-1}$. Moreover, if $\mathbf{x} = \lambda_1\mathbf{v}_1 + \cdots + \lambda_q\mathbf{v}_q \in \mathbb{S}^{m-1}$ with $\mathbf{v}_1, \cdots, \mathbf{v}_q \in$

$\mathbb{S}^{m-1}$, $\lambda_1+\cdots+\lambda_q=1$, and $\lambda_1,\cdots,\lambda_q\geq 0$ then

$$\begin{aligned}\mathfrak{P}(\mathbf{x}) &= \sum_{i_1=1}^{q}\cdots\sum_{i_k=1}^{q}\lambda_{i_1}\cdots\lambda_{i_k}\mathcal{M}_{\mathcal{P}_{k+1}}\left(\mathbf{v}_{i_1},\cdots,\mathbf{v}_{i_k}\right)\\ &= \lambda_1^k\mathfrak{P}(\mathbf{v}_1)+\cdots+\lambda_q^k\mathfrak{P}(\mathbf{v}_q) \qquad + \sum_{\substack{\text{at least for two}\\ i_\mu,i_\nu:\ i_\mu\neq i_\nu}}\lambda_{i_1}\cdots\lambda_{i_k}\mathcal{M}_{\mathcal{P}_{k+1}}\left(\mathbf{v}_{i_1},\cdots,\mathbf{v}_{i_k}\right) \quad (2.2)\end{aligned}$$

Hence, it follows form (2.2) that

$$\begin{aligned}\mathbf{x}^{(2)} &= \mathfrak{P}(\mathbf{x}^{(1)}) = x_1^{\mathrm{K}(1)}\mathbf{e}_1^{(1)}+x_2^{\mathrm{K}(1)}\mathbf{e}_2^{(1)}+\cdots+x_m^{\mathrm{K}(1)}\mathbf{e}_m^{(1)}+\text{remaining terms},\\ \mathbf{x}^{(3)} &= \mathfrak{P}(\mathbf{x}^{(2)}) = x_1^{\mathrm{K}(2)}\mathbf{e}_1^{(2)}+x_2^{\mathrm{K}(2)}\mathbf{e}_2^{(2)}+\cdots+x_m^{\mathrm{K}(2)}\mathbf{e}_m^{(2)}+\text{remaining terms},\\ &\ \vdots\\ \mathbf{x}^{(n+1)} &= \mathfrak{P}(\mathbf{x}^{(n)}) = x_1^{\mathrm{K(n)}}\mathbf{e}_1^{(n)}+x_2^{\mathrm{K(n)}}\mathbf{e}_2^{(n)}+\cdots+x_m^{\mathrm{K(n)}}\mathbf{e}_m^{(n)}+\text{remaining terms}.\end{aligned}$$

Consequently, the last equality yields the inequality (2.1).

Moreover, it follows from the inequality (2.1) and $\mathbf{e}_l^{(n)}>0$ for any $n>n_0,\ l\in\mathbf{I}_m$ that $\mathbf{x}^{(n+1)}>0$ for any $n>n_0$ and for any $\mathbf{x}^{(1)}\in\mathbb{S}^{m-1}$. This shows that $\mathbf{x}^{(n+1)}\in\mathrm{int}\mathbb{S}^{m-1}$ for any $n>n_0$.

Step-3. We now show that for any $\mathbf{x}^{(1)}\in\mathbb{S}^{m-1}$ the omega limit set $\omega\left(\left\{\mathbf{x}^{(n)}\right\}\right)$ of the sequence $\{\mathbf{x}^{(n)}\}_{n=1}^{\infty}$ is a subset of the interior $\mathrm{int}\mathbb{S}^{m-1}$ of the simplex $\mathbb{S}^{m-1}$ i.e., $\omega\left(\left\{\mathbf{x}^{(n)}\right\}\right)\Subset\mathrm{int}\mathbb{S}^{m-1}$. This indeed follows from the previous step that $\mathfrak{P}^{(n_0+1)}(\mathbb{S}^{m-1})\Subset\mathrm{int}\mathbb{S}^{m-1}$. Since the image of simplex under the polynomial stochastic operator is a compact set, there exists $\alpha>0$ such that

$$\mathfrak{P}^{(n_0+1)}(\mathbb{S}^{m-1})\geq\alpha\mathbf{e}:=(\alpha,\alpha,\cdots,\alpha)^T \qquad \forall\ \mathbf{x}\in\mathbb{S}^{m-1}.$$

On the other hand, since the interior $\mathrm{int}\mathbb{S}^{m-1}$ of the simplex $\mathbb{S}^{m-1}$ is an invariant set (see **Step-1**) and

$$\mathfrak{P}^{(n+1)}(\mathbb{S}^{m-1})\subset\mathfrak{P}^{(n_0+1)}(\mathbb{S}^{m-1})\subset\mathbb{S}_\alpha$$

for any $n>n_0$, we have that $\{\mathbf{x}^{(n)}\}_{n=n_0+1}^{\infty}\subset\mathbb{S}_\alpha$, i.e., $\mathbf{x}^{(n)}\geq\alpha\mathbf{e}$ for any $n>n_0$ where

$$\mathbb{S}_\alpha:=\{\mathbf{x}\in\mathbb{S}^{m-1}:\mathbf{x}\geq\alpha\mathbf{e}\}.$$

Consequently, the omega limit set $\omega\left(\left\{\mathbf{x}^{(n)}\right\}\right)$ of the sequence $\{\mathbf{x}^{(n)}\}_{n=1}^{\infty}$ is a subset of the set $\mathbb{S}_\alpha$, i.e., $\omega\left(\left\{\mathbf{x}^{(n)}\right\}\right)\subset\mathbb{S}_\alpha\subset\mathrm{int}\mathbb{S}^{m-1}$ for any $\mathbf{x}^{(1)}\in\mathbb{S}^{m-1}$.

Step-4. As we showed in the previous step $\mathfrak{P}^{(n)}(\mathbb{S}^{m-1})\subset\mathbb{S}_\alpha$ for any $n>n_0$. It is therefore enough to study the dynamics of the polynomial stochastic operator over the set $\mathbb{S}_\alpha$ which is an invariant set. Let $\mathbf{x}^{(1)}\in\mathbb{S}_\alpha$. Then $\mathbf{x}^{(n)}\in\mathbb{S}_\alpha$, i.e., $\mathbf{x}^{(n)}\geq\alpha\mathbf{e}$ for any $n\in\mathbb{N}$. It follows from the matrix form (1.7) of the polynomial stochastic operator that

$$\mathbf{x}^{(n+1)}=\mathfrak{P}(\mathbf{x}^{(n)})=\mathbb{P}\left(\mathbf{x}^{(n)}\right)\mathbf{x}^{(n)}=\mathbb{P}\left(\mathbf{x}^{(n)}\right)\cdots\mathbb{P}\left(\mathbf{x}^{(2)}\right)\mathbb{P}\left(\mathbf{x}^{(1)}\right)\mathbf{x}^{(1)}$$

where $\mathbb{P}(\mathbf{x})$ is the square doubly stochastic matrix defined by (1.6). Let us set for any two integer numbers $n>r$

$$\mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(r)}\right]}:=\mathbb{P}\left(\mathbf{x}^{(n)}\right)\mathbb{P}\left(\mathbf{x}^{(n-1)}\right)\cdots\mathbb{P}\left(\mathbf{x}^{(r+1)}\right)\mathbb{P}\left(\mathbf{x}^{(r)}\right).$$

Then for any $n\geq r\geq 0$, we obtain

$$\mathbf{x}^{(n+1)}=\mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(1)}\right]}\mathbf{x}^{(1)}=\mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(r)}\right]}\mathbf{x}^{(r)}.$$

Then from (1.6), for a stochastic matrix $\mathbb{P}(\mathbf{x}^{(n)})=\left(p_{lj}(\mathbf{x}^{(n)})\right)_{l,j=1}^{m}$ we have

$$p_{lj}(\mathbf{x}^{(n)})=\sum_{i_1=1}^{m}\cdots\sum_{i_{k-1}=1}^{m}p_{i_1\cdots i_{k-1}jl}\ x_{i_1}^{(n)}\cdots x_{i_{k-1}}^{(n)}\geq p_{j\cdots jjl}\alpha^k>0$$

for any $l, j \in \mathbf{I}_m$ and $n \in \mathbb{N}$. Hence, we obtain that

$$\mathbb{P}(\mathbf{x}^{(n)}) \geq \alpha^k \big[\mathrm{diag}(\mathcal{P}_{k+1})\big]^T, \qquad \forall\, n \in \mathbb{N}.$$

Consequently, since $\big[\mathrm{diag}(\mathcal{P}_{k+1})\big]^s > 0$ for some $s \in \mathbb{N}$, the last inequality yields that

$$\mathbb{P}^{\left[\mathbf{x}^{(n+s)}, \mathbf{x}^{(n)}\right]} = \mathbb{P}\big(\mathbf{x}^{(n+s)}\big) \cdots \mathbb{P}\big(\mathbf{x}^{(n+1)}\big)\mathbb{P}\big(\mathbf{x}^{(n)}\big) \geq \alpha^{ks}\Big[\big[\mathrm{diag}(\mathcal{P}_{k+1})\big]^s\Big]^T > 0$$

for any $n \in \mathbb{N}$.

Step-5. Let $\delta(\mathbb{P}) = \frac{1}{2}\max\limits_{i_1,i_2}\sum\limits_{j=1}^{m}|p_{i_1j} - p_{i_2j}|$ be Dobrushin's ergodicity coefficient of a square stochastic matrix $\mathbb{P} = (p_{ij})_{i,j=1}^m$. We first recall some properties of Dobrushin's ergodicity coefficient for the reader's convenience. The following statements are true for any two square stochastic matrices $\mathbb{P}$ and $\mathbb{Q}$ (see [23]):

(*i*) $0 \leq \delta(\mathbb{P}) \leq 1$;

(*ii*) $\delta(\mathbb{P}) = 0$ if and only if $\mathrm{rank}(\mathbb{P}) = 1$, i.e., $\mathbb{P}$ is a stable stochastic matrix;

(*iii*) $\delta(\mathbb{P}) < 1$ if and only if $\mathbb{P}$ is scrambling. If $\mathbb{P} > 0$, then $\delta(\mathbb{P}) < 1$;

(*iv*) $\delta(\mathbb{P}\mathbb{Q}) \leq \delta(\mathbb{P})\delta(\mathbb{Q})$ and $|\delta(\mathbb{P}) - \delta(\mathbb{Q})| \leq \|\mathbb{P} - \mathbb{Q}\|_\infty$.

As it was shown at the end of **Step-4**, the square doubly stochastic matrix $\mathbb{P}^{\left[\mathbf{x}^{(n+s)}, \mathbf{x}^{(n)}\right]}$ is positive and its entries are uniformly bounded away from zero for any $n \in \mathbb{N}$. It is worthy noting that, by using the same idea, we can also show that not only $\mathbb{P}^{\left[\mathbf{x}^{(n+s)}, \mathbf{x}^{(n)}\right]}$ but also $\mathbb{P}^{\left[\mathbf{y}^{(s+1)}, \mathbf{y}^{(1)}\right]}$ is positive and its entries are uniformly bounded away from zero for all $\mathbf{y}^{(1)} \geq \alpha\mathbf{e}$. Since $\delta(\cdot)$ is continuous, we then obtain that

$$\lambda := \max_{\mathbf{y}^{(1)} \in \mathbb{S}_\alpha} \delta\left(\mathbb{P}^{\left[\mathbf{y}^{(s+1)}, \mathbf{y}^{(1)}\right]}\right) = \delta\left(\mathbb{P}^{\left[\mathbf{y}_*^{(s+1)}, \mathbf{y}_*^{(1)}\right]}\right) < 1$$

for some $\mathbf{y}_*^{(1)} \in \mathbb{S}_\alpha$. Hence, for any $n \geq s+1$, we have that

$$\delta\left(\mathbb{P}^{\left[\mathbf{x}^{(n)}, \mathbf{x}^{(1)}\right]}\right) \leq \prod_{t=1}^{\lfloor \frac{n}{s+1} \rfloor} \delta\left(\mathbb{P}^{\left[\mathbf{x}^{t(s+1)}, \mathbf{x}^{(t-1)s+t}\right]}\right) \leq \lambda^{\lfloor \frac{n}{s+1} \rfloor} \quad \text{and} \quad \lim_{n\to\infty} \delta\big(\mathbb{P}^{\left[\mathbf{x}^{(n)}, \mathbf{x}^{(1)}\right]}\big) = 0,$$

here $f(t) = \lfloor t \rfloor$ is a floor function. Therefore, due to Lemma 4.1, page 136, [23], the backwards products (which are the transpose of forwards products) of doubly stochastic matrices $\big\{\mathbb{P}(\mathbf{x}^{(n)})\big\}_{n=1}^\infty$ are weakly ergodic (see Definition 4.5, page 136, [23]). Moreover, weak and strong ergodicity (see Definitions 4.6, page 136, [23]) are equivalent for the backwards products of doubly stochastic matrices (see Theorem 4.17, page 154, [23]). Due to the definition of strong ergodicity (see Definitions 4.6, page 136, [23]), this means that the backwards products $\left\{\mathbb{P}^{\left[\mathbf{x}^{(n)}, \mathbf{x}^{(1)}\right]}\right\}_{n=1}^\infty$ of doubly stochastic matrices $\big\{\mathbb{P}(\mathbf{x}^{(n)})\big\}_{n=1}^\infty$ must converge to the rank-1 doubly stochastic matrix. Since the only rank-1 doubly stochastic matrix is $m\mathbf{c}^T\mathbf{c}$, we obtain that

$$\lim_{n\to\infty} \mathbb{P}^{\left[\mathbf{x}^{(n)}, \mathbf{x}^{(1)}\right]} = m\mathbf{c}^T\mathbf{c}, \quad \lim_{n\to\infty} \mathbf{x}^{(n+1)} = \lim_{n\to\infty} \mathbb{P}^{\left[\mathbf{x}^{(n)}, \mathbf{x}^{(1)}\right]}\mathbf{x}^{(1)} = \mathbf{c}, \quad \forall\, \mathbf{x}^{(1)} \in \mathbb{S}_\alpha,$$

where $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$. This completes the proof. □

Corollary 2.3. *Let $\mathcal{P}_{k+1} = \big(p_{i_1\cdots i_kj}\big)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ be a $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix and $\mathfrak{P} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ be a polynomial stochastic operator associated with a $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix $\mathcal{P}_{k+1} = \big(p_{i_1\cdots i_kj}\big)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$. If a $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix $\mathcal{P}_{k+1} = \big(p_{i_1\cdots i_kj}\big)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is positive (or only diagonally positive) then the trajectory $\{\mathbf{x}^{(n)}\}_{n=1}^\infty$ starting from any initial point $\mathbf{x}^{(1)} \in \mathbb{S}^{m-1}$ of the simplex $\mathbb{S}^{m-1}$ converges to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex $\mathbb{S}^{m-1}$.*

Example 2.4. Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{S}^2$ be a positive stochastic vector, i.e., $a_1, a_2, a_3 > 0$ and $a_1 + a_2 + a_3 = 1$. Let $\mathbf{e} = (1, 1, 1)$. The following cubic stochastic operator $\mathfrak{P}_{\mathbf{a}} : \mathbb{S}^2 \to \mathbb{S}^2$ is globally stable to the center $\mathbf{c} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ of the simplex where

$$\mathfrak{P}_{\mathbf{a}}(\mathbf{x}) = \mathbf{a}\left(x_1^3 + x_2^3 + x_3^3 + 6x_1x_2x_3\right) + \frac{3}{2}(\mathbf{e} - \mathbf{a})(x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2).$$

Example 2.5. Let $\mathbf{a} = (a_1, \cdots, a_m) \in \mathbb{S}^{m-1}$ be a positive stochastic vector, i.e., $a_1, \cdots, a_m > 0$ and $a_1 + \cdots + a_m = 1$. Let $\mathbf{e} = (1, \cdots, 1)$ and $m > 3$. The following cubic stochastic operator $\mathfrak{P}_{\mathbf{a}} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ is globally stable to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex where

$$\mathfrak{P}_{\mathbf{a}}(\mathbf{x}) = \mathbf{a}\sum_{i=1}^{m} x_i^3 + 3\frac{\mathbf{e} - \mathbf{a}}{m-1}\sum_{i<j}(x_i^2x_j + x_ix_j^2) + 6\frac{(m-3)\mathbf{e} + 2\mathbf{a}}{(m-1)(m-2)}\sum_{i<j<k} x_ix_jx_k.$$

References

[1] Bernstein S.; Solution of a mathematical problem connected with the theory of heredity. Annals of Mathematical Statistics.–1942.–13.–P.53–61.

[2] Ganihodzhaev N.; On stochastic processes generated by quadratic operators. J. Theoretical Prob.–1991.–4.–P.639–653.

[3] Ganikhodjaev N., Ganikhodjaev R. Jamilov U.; Quadratic stochastic operators and zero-sum game dynamics. Ergod. Th. & Dynam. Sys.–2015.–35.–5.–P.1443–1473.

[4] Ganikhodzhaev R.; Quadratic stochastic operators, Lyapunov functions and tournaments. Russian Acad.Sci. Sbornik. Math.– 1993.–76.–P.489–506.

[5] Ganikhodzhaev R.; A chart of fixed points and Lyapunov functions for a class of discrete dynamical systems. Math Notes.–1994.–56.–(5-6).–P.1125–1131.

[6] Ganikhodzhaev R., Eshmamatova D.; Quadratic automorphisms of the simplex and asymptotic behavior of their trajectory. Vladikavkaz Math. Jour.–2006.–8.–P.2–13.

[7] Ganikhodzhaev R., Mukhamedov F., Rozikov U.; Quadratic stochastic operators and processes: Results and Open Problems. Inf. Dim. Anal. Quan. Prob. Rel. Top.–2011.–14.–2.–P.279–335.

[8] Ganikhodzhaev R., Saburov M.; A Generalized model of the nonlinear operators of Volterra type and Lyapunov functions. Jour. Sib. Fed. Univ. Math and Phys.–2008.–1.–2.–P.188–196.

[9] Jamilov U.; Certain Polynomial Stochastic Operators. Math Notes.–2021.–109.–P.828–831.

[10] Jamilov U., Khamraev A., Ladra M.; On a Volterra Cubic Stochastic Operator. Bull Math. Biol.–2018.–80.–P.319–334.

[11] Jamilov U., Khamraev A.; On dynamics of Volterra and non-Volterra cubic stochastic operators. An International Journal Dynamical Systems.–2022.–37.–1.–P.66–82.

[12] Jamilov U., Reinfelds U.; On constrained Volterra cubic stochastic operators. Jour. Diff. Eq. Appl.–2020.–26–2.–P.261–274.

[13] Katok A., Hasselblatt B.; Introduction to the Modern Theory of Dynamical Systems.–1995.–Cambridge University Press.

[14] Khamraev A. Yu.; On cubic operators of Volterra type. Uzbek. Math. Zh.–2004.–2.–P.79–84.

[15] Khamraev A. Yu.; On a Volterra type cubic operators. Uzbek. Math. Zh.–2009.–3.–P.65–71.

[16] Mukhamedov F., Ganikhodjaev N.; Quantum quadratic operators and processes.–2015–Springer.

[17] Mukhamedov F., Saburov M.; On dynamics of Lotka–Volterra type operators. Bull. Malays. Math. Sci. Soc.–2014.–37.–P.59–64.

[18] Mukhamedov F., Saburov M.; Stability and Monotonicity of Lotka–Volterra Type Operators. Qual. Theory Dyn. Syst.–2017.–16.–P.249–267.

[19] Mamurov B., Rozikov U.; On cubic stochastic operators and processes. J. Phys.: Conf. Ser.–2016.–697.—P.012017.

[20] Rozikov U., Khamraev A.; On a cubic operator defined in finite dimensional simplex. Ukr.Math.Jour.–2004.–56.–P.1418–1427.

[21] Rozikov U., Khamraev A.; On construction and a class of non-Volterra cubic stochastic operators. Nonlinear Dynamics and Systems Theory.–2014.–14.–1.–P.92–100.

[22] Sarymsakov T., Ganikhodjaev N.; Analytic methods in the theory of quadratic stochastic processes. J. Theoretical Prob.–1990.–3.–P.51–70.

[23] Seneta E.; Nonnegative Matrices and Markov Chains.–1981.–Springer-Verlag.–New York.

Saburov M.
Department of Mathematics and Natural Science
College of Arts and Sciences (CAS)
Center for Applied Mathematics and Bioinformatics (CAMB)
Gulf University for Science and Technology (GUST), Kuwait;
V.I.Romanovsky Institute of Mathematics, Uzbekistan
email: Saburov.M@gust.edu.kw

Saburov Kh.
Kimyo International University in Tashkent, Uzbekistan
V.I.Romanovsky Institute of Mathematics, Uzbekistan
email: h.saburov@nuu.uz

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.161-167
DOI: 10.29229/uzmj.2025-3-17

A discrete analogue of the second-order differential operator with variable coefficient

Shadimetov Kh.M., Esanov Sh.E.

Abstract. This paper constructs a discrete analogue of the second-order differential operator with variable coefficient. The construction is based on harrow-shaped functions and the application of direct and inverse Fourier transforms.

Keywords: Discrete analogue, differential operator, functional space, fundamental solution, optimal quadrature formulas.

MSC: 65N06, 65D30

1. Introduction

Discrete analogues of differential operators play a crucial role in the construction of optimal cubature, quadrature, interpolation, and finite-difference formulas in various functional spaces.

The discrete analogue of the polyharmonic operator was first studied by S.L.Sobolev [1].

Subsequently, the construction of discrete analogues of various differential operators and their applications have been considered in works [2, 3, 4, 5, 6, 7, 8, 9, 10].

It is not difficult to verify that the function

$$\mu(x) = \frac{e^{-|x|}}{4}(|x| - 1)$$

is the fundamental solution of the following differential operator L:

$$L \equiv \frac{d^2}{dx^2} + 2\,\mathrm{sign}\,x\frac{d}{dx} + 1,$$

i.e., it satisfies the equation

$$L\mu = \delta,$$

where $\delta(x)$ is the Dirac delta-function.

In this work, we aim to construct a discrete analogue of the operator L.

Definition 1.1. The function $\varphi(h\beta)$ is a function of discrete argument if it is given on some set of integer values of β.

This discrete operator $L_h[\beta]$ is the solution of the equation

$$L_h[\beta] * \mu_h[\beta] = \delta[\beta]. \tag{1.1}$$

Here, convolution is defined as

$$L_h[\beta] * \mu_h[\beta] = \sum_{\gamma=-\infty}^{\infty} L_h[\gamma] \cdot \mu_h[\beta - \gamma],$$

$$\mu_h[\beta] = \frac{e^{-|h\beta|}}{4}(|h\beta| - 1), \tag{1.2}$$

$\delta[\beta]$ is the discrete delta-function and it is equal to one when $\beta = 0$, and zero for all other integers of β.

2. Algorithm for constructing the discrete operator

To proceed, we use the concept of of harrow-shaped functions [1].
In the space of harrow-shaped functions, equation (1.1) takes the following form

$$\overline{L}\,(x) * \overline{\mu}\,(x) = \overline{\delta}\,(x). \tag{2.1}$$

Here, these harrow-shaped functions are defined as follows:

$$\overline{L}\,(x) = \sum_{\beta=-\infty}^{\infty} L_h[\beta]\delta(x-h\beta), \tag{2.2}$$

$$\overline{\mu}\,(x) = \sum_{\beta=-\infty}^{\infty} \mu_h[\beta]\delta(x-h\beta), \quad \overline{\delta}\,(x) = \sum_{\beta=-\infty}^{\infty} \delta[\beta]\delta(x-h\beta) = \delta(x).$$

It is known that there exists an isomorphism between the class of harrow-shaped functions and the class of functions with discrete arguments [1].
To construct the discrete operator$L_h[\beta]$, we apply the Fourier transform to both sides of equation (2.1), consequently having

$$F\left[\overline{L}\,(x) * \overline{\mu}\,(x)\right] = F\left[\delta(x)\right]. \tag{2.3}$$

Using the known formulas

$$F\left[\psi(x) * \varphi(x)\right] = F\left[\psi(x)\right] \cdot F\left[\varphi(x)\right],$$

and

$$F\left[\delta(x)\right] = 1,$$

equation (2.3) takes the following form

$$F\left[\overline{L}\,(x)\right] \cdot F\left[\overline{\mu}\,(x)\right] = 1.$$

Hence, the Fourier transform of the harrow-shaped function $F\left[\overline{L}\,(x)\right]$ is given by the equation

$$F\left[\overline{L}\,(x)\right] = \frac{1}{F\left[\overline{\mu}\,(x)\right]}. \tag{2.4}$$

Applying the inverse Fourier transform to both sides of (2.4), we obtain

$$\overline{L}\,(x) = F^{-1}\left[\frac{1}{F\left[\overline{\mu}\,(x)\right]}\right]. \tag{2.5}$$

From this and equation (2.2), we derive the desired operator $\overline{L}_h[\beta]$.
In the following sections, we implement the given algorithm.

3. Fourier transform of the harrow-shaped function $\overline{\mu}\,(x)$

Now we calculate the Fourier transform of the function $\overline{\mu}\,(x)$.
By definition, the transform takes the form

$$F\left[\overline{\mu}\,(x)\right] = F\left[\sum_{\beta=-\infty}^{\infty} \mu_h[\beta]\delta(x-h\beta)\right] = \sum_{\beta=-\infty}^{\infty} \mu_h[\beta] F\left[\delta(x-h\beta)\right] = \sum_{\beta=-\infty}^{\infty} \mu_h[\beta]\exp(2\pi i p h\beta).$$

Using expression (1.2), we obtain

$$F\left[\overline{\mu}\,(x)\right] = \sum_{\beta=-\infty}^{\infty} \frac{e^{-|h\beta|}}{4}(|h\beta|-1)\exp(2\pi i p h\beta). \tag{3.1}$$

Taking into account the evenness of the function $\mu_h[\beta]$, i.e., $\mu_h[\beta] = \mu_h[-\beta]$ and denoting $\lambda = \exp(2\pi i p h)$ expression (3.1) takes the following form

$$F\left[\overline{\mu}\ (x)\right] = \sum_{\beta=-\infty}^{\infty} \frac{e^{-|h\beta|}}{4}(|h\beta|-1)\lambda^{\beta} = \sum_{\beta=1}^{\infty} \frac{e^{-(h\beta)}}{4}(h\beta-1)\left(\lambda^{\beta}+\lambda^{-\beta}\right) - \frac{1}{4}. \tag{3.2}$$

From this, expanding the brackets, we obtain

$$F\left[\overline{\mu}\ (x)\right] = -\frac{1}{4} - \frac{1}{4}\sum_{\beta=1}^{\infty} e^{-h\beta}\lambda^{\beta} - \frac{1}{4}\sum_{\beta=1}^{\infty} e^{-h\beta}\lambda^{-\beta} + +\frac{h}{4}\sum_{\beta=1}^{\infty}\beta e^{-h\beta}\lambda^{\beta} + \frac{h}{4}\sum_{\beta=1}^{\infty}\beta e^{-h\beta}\lambda^{-\beta}. \tag{3.3}$$

Since $|\lambda| = |\exp(2\pi i p h)| = 1$, the series in (3.3) converges. To compute the series, we use the following formulas

$$\sum_{\beta=1}^{\infty}\left(e^{-h}\lambda\right)^{\beta} = \frac{e^{-h}\lambda}{1-e^{-h}\lambda}, \tag{3.4}$$

$$\sum_{\beta=1}^{\infty}\left(e^{-h}\lambda^{-1}\right)^{\beta} = \frac{e^{-h}\lambda^{-1}}{1-e^{-h}\lambda^{-1}} = \frac{1}{e^{h\lambda}-1}, \tag{3.5}$$

$$\sum_{\beta=1}^{\infty} h\beta\left(e^{-h}\lambda\right)^{\beta} = h\frac{e^{-h}\lambda}{\left(1-e^{-h}\lambda\right)^{2}}, \tag{3.6}$$

$$\sum_{\beta=1}^{\infty} h\beta\left(e^{-h}\lambda^{-1}\right)^{\beta} = h\frac{e^{h}\lambda}{\left(e^{h\lambda}-1\right)^{2}}. \tag{3.7}$$

Substituting expressions (3.4)-(3.7) into (3.3) after some simplifications, we obtain

$$\begin{aligned}
F\left[\overline{\mu}\ (x)\right] &= -\frac{1}{4}\left[1 + \frac{e^{-h}\lambda}{1-e^{-h}\lambda} + \frac{e^{-h}\lambda^{-1}}{1-e^{-h}\lambda^{-1}} - h\frac{e^{-h}\lambda}{\left(1-e^{-h}\lambda\right)^{2}} - h\frac{e^{-h}\lambda^{-1}}{\left(1-e^{-h}\lambda^{-1}\right)^{2}}\right] \\
&= -\frac{1}{4}\left[1 + \frac{e^{-h}\lambda}{1-e^{-h}\lambda} + \frac{1}{e^{h}\lambda-1} - h\frac{e^{-h}\lambda}{\left(1-e^{-h}\lambda\right)^{2}} - h\frac{e^{h}\lambda}{\left(e^{h}\lambda-1\right)^{2}}\right] \\
&= -\frac{1}{4}\left[\frac{-\lambda^{2}+\lambda\left(e^{h}+e^{-h}\right)-1+\lambda^{2}-e^{-h}\lambda+1-e^{-h}\lambda}{-\lambda^{2}+\lambda\left(e^{h}+e^{-h}\right)-1}\right. \\
&\quad \left. - h\left[\frac{e^{-h}\lambda\left(e^{h}\lambda-1\right)^{2}+e^{h}\lambda\left(1-e^{-h}\lambda\right)^{2}}{\left(-\lambda^{2}+\lambda\left(e^{h}+e^{-h}\right)-1\right)^{2}}\right]\right] \\
&= -\frac{1}{4}\left[\frac{\lambda\left(e^{h}+e^{-h}\right)}{-\lambda^{2}+\lambda\left(e^{h}+e^{-h}\right)-1} - h\frac{e^{-h}\lambda\left(e^{2h}\lambda^{2}-2e^{h}\lambda+1\right)+e^{h}\lambda\left(1-2e^{-h}\lambda+e^{-2h}\lambda^{2}\right)}{\left(-\lambda^{2}+\lambda\left(e^{h}+e^{-h}\right)-1\right)^{2}}\right] \\
&= -\frac{1}{4}\left[\frac{\lambda\left(e^{h}+e^{-h}\right)}{-\lambda^{2}+\lambda\left(e^{h}+e^{-h}\right)-1} - h\frac{e^{h}\lambda^{3}-2\lambda^{2}+e^{-h}\lambda+e^{h}\lambda-2\lambda^{2}+e^{-h}\lambda^{3}}{\left(-\lambda^{2}+\lambda\left(e^{h}+e^{-h}\right)-1\right)^{2}}\right] \\
&= -\frac{1}{4}\left[\frac{\lambda\left(e^{h}+e^{-h}\right)\left(-\lambda^{2}+\lambda\left(e^{h}+e^{-h}\right)-1\right)}{\left(-\lambda^{2}+\lambda\left(e^{h}+e^{-h}\right)-1\right)^{2}} - \frac{h\left[\left(e^{h}+e^{-h}\right)\lambda^{3}-4\lambda^{2}+\left(e^{h}+e^{-h}\right)\lambda\right]}{\left(-\lambda^{2}+\lambda\left(e^{h}+e^{-h}\right)-1\right)^{2}}\right] \\
&= -\frac{\lambda}{4}\cdot\frac{2\sinh(h)\left[-\lambda^{2}+2\cosh(h)\lambda-1\right]-h\left[2\cosh(h)\lambda^{2}-4\lambda+2\cosh(h)\right]}{\left(\lambda^{2}-2\cosh(h)\lambda+1\right)^{2}} \\
&= -\frac{\lambda}{4}\cdot\left[\frac{\lambda^{2}\left[-2\sinh(h)-2h\cosh(h)\right]+\lambda\left[4h+4\cosh(h)\,\mathrm{sh}(h)\right]}{\left(\lambda^{2}-2\cosh(h)\lambda+1\right)^{2}} + \frac{\left[-2h\cosh(h)-2\sinh(h)\right]}{\left(\lambda^{2}-2\cosh(h)\lambda+1\right)^{2}}\right] \\
&= \frac{\lambda}{2}\cdot\frac{\lambda^{2}\left[h\cosh(h)+\sinh(h)\right]-2\lambda\left[h+\cosh(h)\sinh(h)\right]+h\cosh(h)+\sinh(h)}{\left(\lambda^{2}-2\cosh(h)\lambda+1\right)^{2}}.
\end{aligned}$$

Thus, we have ultimately obtained the expression for the Fourier transform of the harrow-shaped function $\overset{\frown}{\mu}(x)$:

$$F\left[\overset{\frown}{\mu}(x)\right] = \frac{\lambda}{2}\cdot\left[\frac{\lambda^2\left[h\cosh(h)+\sinh(h)\right]-2\lambda\left[h+\cosh(h)\sinh(h)\right]}{\left(\lambda^2-2\cosh(h)\lambda+1\right)^2}+\frac{h\cosh(h)+\sinh(h)}{\left(\lambda^2-2\cosh(h)\lambda+1\right)^2}\right].$$

Let us denote $a_1 = h\cosh(h)+\sinh(h)$, $a_2 = h+\cosh(h)\sinh(h)$, then $F\left[\overset{\frown}{\mu}(x)\right]$ can be written in the following form

$$F\left[\overset{\frown}{\mu}(x)\right] = \frac{\lambda}{2}\cdot\frac{a_1\lambda^2-2a_2\lambda+a_1}{\left(\lambda^2-2\cosh(h)\lambda+1\right)^2}.$$

4. The discrete operator $L_h[\beta]$

We now compute

$$\frac{1}{F\left[\overset{\frown}{\mu}(x)\right]} = \frac{2\left(\lambda^2-2\cosh(h)\lambda+1\right)^2}{\lambda\left(a_1\lambda^2-2a_2\lambda+a_1\right)} = $$
$$= \frac{2\left(\lambda^4-4\cosh(h)\lambda^3+2\lambda^2+4\cosh^2(h)\lambda^2-4\cosh(h)\lambda+1\right)}{a_1\lambda^3-2a_2\lambda+a_1\lambda} \quad (4.1)$$

Thus,

$$\frac{2\left(\lambda^2-2\cosh(h)\lambda+1\right)^2}{\lambda\left(a_1\lambda^2-2a_2\lambda+a_1\right)} = \frac{2}{a_1}\lambda+\frac{4}{a_1{}^2}\left(a_2-2a_1{}^2\cosh(h)\right)+\frac{R_2(\lambda)}{a_1\lambda^3-2a_2\lambda^2+a_1\lambda}, \quad (4.2)$$

where

$$R_2(\lambda) = \left[2+\frac{8}{a_1{}^2}\left(a_1{}^2\cosh^2(h)+a_2{}^2-2a_2a_1\cosh(h)\right)\right]\lambda^2-\frac{4}{a_1}\left(2a_1\cosh(h)+a_2-2a_1\cosh(h)\right)\lambda+2$$
$$= \left[2+\frac{8}{a_1{}^2}\left(a_1\cosh(h)-a_2\right)^2\right]\lambda^2-4\frac{a_2}{a_1}\lambda+2 = \left[2+8\left(\cosh(h)-\frac{a_2}{a_1}\right)^2\right]\lambda^2-4\frac{a_2}{a_1}\lambda+2. \quad (4.3)$$

Next, we decompose $\frac{R_2(\lambda)}{\lambda(a_1\lambda^2-2a_2\lambda+a_1)}$ into elementary fractions. For this, we obtain

$$\frac{R_2(\lambda)}{\lambda\left(a_1\lambda^2-2a_2\lambda+a_1\right)} = \frac{B_0}{\lambda}+\frac{B_{1,1}}{\lambda-\lambda_1}+\frac{B_{1,2}}{\lambda-\lambda_2}. \quad (4.4)$$

Here, λ_1 and λ_2 are roots of the polynomial $a_1\lambda^2-2a_2\lambda+a_1$.

Using (4.4), expression (4.2) becomes

$$\frac{2\left(\lambda^2-2\cosh(h)\lambda+1\right)^2}{\lambda\left(a_1\lambda^2-2a_2\lambda+a_1\right)} = \frac{2}{a_1}\lambda+\frac{4}{a_1{}^2}\left(a_2-2a_1{}^2\cosh(h)\right)+\frac{B_0}{\lambda}+\frac{B_{1,1}}{\lambda-\lambda_1}+\frac{B_{1,2}}{\lambda-\lambda_2}.$$

Hence, we have

$$2\left(\lambda^2 2\cosh(h)\lambda+1\right)^2 = \frac{2}{a_1}\lambda^2\left(a_1\lambda^2-2a_2\lambda+a_1\right)+\frac{4}{a_1{}^2}\left(a_2-2a_1\cosh(h)\right)\lambda\left(a_1\lambda^2-2a_2\lambda+a_1\right)$$
$$+B_0\left(a_1\lambda^2-2a_2\lambda+a_1\right)+\frac{B_{1,1}\lambda a_1\left(\lambda-\lambda_1\right)\left(\lambda-\lambda_2\right)}{\lambda-\lambda_1}+\frac{B_{1,2}\lambda a_1\left(\lambda-\lambda_1\right)\left(\lambda-\lambda_2\right)}{\lambda-\lambda_2}.$$

Hence, by substituting $\lambda=0,\ \lambda=\lambda_1,\ \lambda=\lambda_2$ we obtain $2=B_0a_1$, or

$$B_0 = \frac{2}{a_1}, \quad (4.5)$$

$$2\left(\lambda_1{}^2 - 2\cosh(h)\lambda_1 + 1\right)^2 = B_{1,1}\lambda_1 a_1\left(\lambda_1 - \lambda_2\right),$$
$$2\left(\lambda_2{}^2 - 2\cosh(h)\lambda_2 + 1\right)^2 = B_{2,2}\lambda_2 a_1\left(\lambda_2 - \lambda_1\right),$$

$$B_{1,1} = \frac{2\left(\lambda_1{}^2 - 2\cosh(h)\lambda_1 + 1\right)^2}{\lambda_1 a_1\left(\lambda_1 - \lambda_2\right)}, \tag{4.6}$$

$$B_{1,2} = \frac{2\left(\lambda_2{}^2 - 2\cosh(h)\lambda_2 + 1\right)^2}{\lambda_2 a_1\left(\lambda_2 - \lambda_1\right)}. \tag{4.7}$$

Considering $\lambda_1 = \frac{1}{\lambda_2}$, we obtain

$$\lambda_2{}^2 - 2\cosh(h)\lambda_2 + 1 = \frac{1}{\lambda_1{}^2} - 2\cosh(h)\frac{1}{\lambda_1} + 1 = \frac{\lambda_1{}^2 - 2\cosh(h)\lambda_1 + 1}{\lambda_1{}^2},$$

$$B_{1,2} = \frac{2\left(\lambda_2{}^2 - 2\cosh(h)\lambda_2 + 1\right)^2}{\lambda_2 a_1\left(\lambda_2 - \lambda_1\right)} = \frac{2\left(\lambda_1{}^2 - 2\cosh(h)\lambda_1 + 1\right)^2}{\lambda_1{}^4\lambda_2 a_1\left(\lambda_2 - \lambda_1\right)} = \frac{2\left(\lambda_1{}^2 - 2\cosh(h)\lambda_1 + 1\right)^2}{\lambda_1{}^4\frac{1}{\lambda_1}a_1\left(\lambda_2 - \lambda_1\right)}$$
$$= \frac{2\left(\lambda_1{}^2 - 2\cosh(h)\lambda_1 + 1\right)^2}{\lambda_1{}^3 a_1\left(\lambda_2 - \lambda_1\right)} = \frac{2\left(\lambda_1{}^2 - 2\cosh(h)\lambda_1 + 1\right)^2}{\lambda_1{}^2\lambda_1 a_1\left(\lambda_2 - \lambda_1\right)} = -\frac{B_{1,1}}{\lambda_1{}^2}.$$

Hence

$$B_{1,2} = \frac{-B_{1,1}}{\lambda_1^2}. \tag{4.8}$$

Then we have $B_{1,1} = -B_{1,2}\lambda_1^2 = -\frac{B_{1,2}}{\lambda_2^2}$ and $B_{1,1} = -\frac{B_{1,2}}{\lambda_2^2}$. Let $0 < \lambda_1 < 1$, $\lambda_2 > 1$, then in the expression

$$\frac{R_2(\lambda)}{\lambda\left(a_1\lambda^2 - 2a_2\lambda + 1\right)} = \frac{B_0}{\lambda} + \frac{B_{1,1}}{\lambda - \lambda_1} + \frac{B_{1,2}}{\lambda - \lambda_2},$$

the second and third terms can be rewritten as

$$\frac{B_{1,1}}{\lambda - \lambda_1} = \frac{1}{\lambda}\cdot\frac{B_{1,1}}{1 - \frac{\lambda_1}{\lambda}} = \frac{1}{\lambda}B_{1,1}\sum_{\beta=0}^{\infty}\left(\frac{\lambda_1}{\lambda}\right)^{\beta}, \tag{4.9}$$

$$\frac{B_{1,2}}{\lambda - \lambda_2} = -\frac{B_{1,2}}{\lambda_2\left(1 - \frac{\lambda}{\lambda_2}\right)} = -\frac{B_{1,2}}{\lambda_2}\sum_{\beta=0}^{\infty}\left(\frac{\lambda}{\lambda_2}\right)^{\beta}. \tag{4.10}$$

From (4.1), (4.2), and (4.4), we have

$$\frac{1}{F\left[\overleftarrow{\mu}\ (x)\right]} = \frac{2}{a_1}\lambda + \frac{4}{a_1^2}\left(a_2 - 2a_1^2\cosh(h)\right) + \frac{B_0}{\lambda} + \frac{B_{1,1}}{\lambda - \lambda_1} + \frac{B_{1,2}}{\lambda - \lambda_2}. \tag{4.11}$$

Here, the coefficients were determined in (4.5)-(4.7). From (4.9) and (4.10), expression (4.11) becomes

$$\frac{1}{F\left[\overleftarrow{\mu}\ (x)\right]} = \frac{2}{a_1}\lambda + \frac{4}{a_1^2}\left(a_2 - 2a_1^2\cosh(h)\right) + \frac{B_0}{\lambda} + \frac{1}{\lambda}B_{1,1}\sum_{\beta=0}^{\infty}\left(\frac{\lambda_1}{\lambda}\right)^{\beta} - \frac{B_{1,2}}{\lambda_2}\sum_{\beta=0}^{\infty}\left(\frac{\lambda}{\lambda_2}\right)^{\beta}.$$

From this, using formula (4.8), we obtain

$$\frac{1}{F\left[\overleftarrow{\mu}\ (x)\right]} = \frac{2}{a_1}\lambda + \frac{4}{a_1^2}\left(a_2 - 2a_1^2\cosh(h)\right) - \frac{B_0}{\lambda} + \frac{1}{\lambda}B_{1,1}\sum_{\beta=0}^{\infty}\left(\frac{\lambda_1}{\lambda}\right)^{\beta} + \frac{B_{1,1}}{\lambda_1^2\lambda_2}\sum_{\beta=0}^{\infty}\left(\frac{\lambda}{\lambda_2}\right)^{\beta}$$
$$= \frac{2}{a_1}\lambda + \frac{4}{a_1^2}\left(a_2 - 2a_1^2\cosh(h)\right) + \frac{B_0}{\lambda} + \frac{B_{1,1}}{\lambda}\sum_{\beta=0}^{\infty}\left(\frac{\lambda_1}{\lambda}\right)^{\beta} + \frac{B_{1,1}}{\lambda_1}\sum_{\beta=0}^{\infty}\left(\frac{\lambda}{\lambda_2}\right)^{\beta}. \tag{4.12}$$

Now, substituting into (4.12) the value of $\exp(2\pi iph)$ in place of λ, we obtain

$$\frac{1}{F\left[\overline{\mu}\ (x)\right]} = \frac{2}{a_1}\exp(2\pi iph) + \frac{4}{a_1^2}\left(a_2 - 2a_1^2\cosh(h)\right) + B_0\exp(-2\pi iph)$$
$$+\frac{B_{1,1}}{\lambda_1}\sum_{\beta=0}^{\infty}\left(\frac{\lambda_1}{\exp(2\pi iph)}\right)^{\beta+1} + \frac{B_{1,1}}{\lambda_1}\sum_{\beta=0}^{\infty}\left(\lambda_1\exp(2\pi iph)\right)^{\beta} = \frac{2}{a_1}\exp(2\pi iph)$$
$$+\frac{4}{a_1^2}\left(a_2 - 2a_1^2\cosh(h)\right) + \frac{B_{1,1}}{\lambda_1}\sum_{\beta=0}^{\infty}\lambda_1^{\beta+1}(\exp(-2\pi iph))^{\beta+1}$$
$$+\frac{B_{1,1}}{\lambda_1}\sum_{\beta=0}^{\infty}\lambda_1^{\beta}(\exp(2\pi iph))^{\beta}. \tag{4.13}$$

Expression (4.13) can be written as

$$\frac{1}{F\left[\overline{\mu}\ (x)\right]} = \sum_{\beta=0}^{\infty} D_h\left[\beta\right]\exp(2\pi iph\beta).$$

Here,

$$D_h[0] = \frac{4}{a_1^2}\left(a_2 - 2a_1^2\cosh(h)\right) + \frac{B_{1,1}}{\lambda_1},$$
$$D_h[-1] = B_0 + \frac{B_{1,1}\lambda_1}{\lambda_1},$$
$$D_h[1] = \frac{2}{a_1} + \frac{B_{1,1}}{\lambda_1}\lambda_1,$$
$$D_h[\beta] = \frac{B_{1,1}}{\lambda_1}\lambda_1^{|\beta|}\ ,\quad |\beta| \geq 2,$$

from (2.4) and (2.5), it follows that

$$F\left[\overline{L}\ (x)\right] = \sum_{\beta=-\infty}^{\infty} D_h\left[\beta\right]\exp(2\pi iph\beta).$$

From this and from equations (2.4) and (2.5), it follows that the desired operator $L_h\left[\beta\right]$ is equal to $D_h[\beta]$, .. $L_h\left[\beta\right] = D_h\left[\beta\right]$ and it has the form

$$L_h[0] = \frac{4a_2}{a_1^2} - 8\cosh(h) + \frac{B_{1,1}}{\lambda_1}, \tag{4.14}$$

$$L_h[1] = L_h[-1] = \frac{2}{a_1} + B_{1,1}, \tag{4.15}$$

$$L_h[\beta] = B_{1,1}\lambda_1^{|\beta|-1},\ |\beta| \geq 2. \tag{4.16}$$

Here,

$$a_1 = h\cosh(h) + \sinh(h),$$
$$a_2 = h + \cosh(h)\sinh(h),$$
$$B_{1,1} = \frac{2(\lambda_1^2 - 2\cosh(h) + 1)^2}{a_1\lambda_1(\lambda_1 - \lambda_2)},$$
$$\lambda_1 = \frac{h + \sinh(h)\left(\cosh(h) - \sqrt{\sinh^2(h) - h^2}\right)}{h\cosh(h) - \sinh(h)}.$$

Thus, we have proven the following theorem.

Theorem 4.1. *The discrete analogue $L_h[\beta]$ of the second-order differential operator*

$$L = \frac{d^2}{dx^2} + 2signx\frac{d}{dx} + 1$$

is defined by formulas (4.14)-(4.16).

5. Conclusion

In this paper, using functions of a discrete argument and harrow-shaped functions, a discrete analogue of the second-order differential operator with a variable coefficient has been constructed.

References

[1] Sobolev S.L.; Introduction to the Theory of Cubature Formulas, Nauka, Moscow, 1974, -808 p.

[2] Shadimetov Kh.M.; Discrete analogue of an operator and its construction, Problems of Computational and Applied Mathematics 1985, (79), pp. 22-35.

[3] Hayotov A.R.; The discrete analogue of the differential operator and its applications. Lithuanian Mathematical Journal., 2014, 54(3), pp. 290-307.

[4] Shadimetov Kh.M., Davronov J.R.; The discrete analogue of high-order differential operator and its application to finding coefficients of optimal quadrature formulas. Journal of Inequalities and Applications. 2024, 2024:46 https://doi.org/10.1186/s13660-024-03111-7.

[5] Shadimetov Kh.M., Jalolov I.I.; Weighted Optimal Formulas for Approximate Integration. Mathematics, 2024, 12, 738. https://doi.org/10.3390/math12050738

[6] Ahmadaliev G.N., Hayotov A.R.; A Discrete Analogue of the Differential Operator $\frac{d^{2m}}{dx^{2m}} - 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}}$, Uzbek Mathematical Journal, 2017, Volume 3, pp. 10-22.

[7] Shadimetov Kh. M., Shonazarov S.K.; On an implicit optimal difference formula. Uzbek Mathematical Journal, 2024, Volume 68, Issue 4, pp. 128-136, DOI: 10.29229/uzmj.2024-4-15

[8] Akhmedov D.M., Abdikayumova G.A.; Construction of optimal quadrature formulas with derivatives for Cauchy type singular integrals in the Sobolev space. Uzbek Mathematical Journal, 2020, Volume 64, Issue 1, pp.4-9, DOI: 10.29229/uzmj.2020-1-1

[9] AkhmedovD.M., AtamuradovaB.M.; Construction of optimal quadrature formulas for Cauchy type singular integrals in the $W_2^{(1,0)}(01)$ space. Uzbek Mathematical Journal, 2022, Volume 66, Issue 2, pp. 5-9, DOI:10.29229/uzmj.2022-2-1

[10] Akhmedov D.M.; Optimal approximation of the Hadamard hypersingular integrals. Uzbek Mathematical Journal, 2023, Volume 67, Issue 3, pp. 5-12, DOI: 10.29229/uzmj.2023-3-1

Shadimetov Kh.M.
Tashkent State Transport University,
1, Odilkhodjaev street, Tashkent, 100167, Uzbekistan;
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
9,University Street, Tashkent 100174, Uzbekistan
e-mail: kholmatshadimetov@mail.ru
Esanov Sh.E.
Termiz State University,
43, Barkamol Avlod str., Termiz, 190111, Uzbekistan
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
9,University Street, Tashkent 100174, Uzbekistan
e-mail: esanovsher@mail.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.168-175
DOI: 10.29229/uzmj.2025-3-18

The Dirichlet problem in the class of m-convex functions

Sharipov R.A.

Abstract. The well-known classical Dirichlet problem states that if $D \subset \mathbb{R}^n$ is a regular domain, then for any continuous function $\varphi(\xi) \in C(\partial D)$, there exists a unique harmonic function $\omega(x) \in C(\overline{D})$, such that $\omega|_{\partial D} = \varphi$. In the work of Sadullaev-Sharipov [24], under an additional condition of strict m-convexity of the domain $D \subset \mathbb{R}^n$, an analogous result for m-convex $(m - cv)$ functions has been proven.

In this paper, a stronger result on the existence of the solution to the Dirichlet problem in regular domains $D \subset \mathbb{R}^n$ is proved under one necessary condition on the boundary function $\varphi(\xi)$.

Keywords: strongly m-subharmonic functions, m-convex functions, Borel measures, Hessians, Dirichlet problem

MSC (2020): 26B25, 39B62, 52A41

1. Classical Dirichlet problem

Let a bounded domain $D \subset \mathbb{R}^n$ and a function $\varphi(\xi) \in C(\partial D)$ be given. The classical Dirichlet problem claims that there exists a function $\omega(x) \in h(D) \cap C(\overline{D})$ such that $\omega|_{\partial D} = \varphi$. It immediately follows from the maximum principle for harmonic functions that, if a solution to the Dirichlet problem exists, then it is unique.

To solve the Dirichlet problem in the domain $D \subset \mathbb{R}^n$, we use the well-known Perron method. We consider it as a very convenient tool in potential theory and in the theory of harmonic functions. Moreover, it may be useful in other boundary problems of elliptic equations. For a given function $\varphi(\xi) \in C(\partial D)$, we set

$$\mathcal{U}(\varphi, D) = \left\{ u \in sh(D) : \overline{\lim_{x \to \xi \in D}} u(x) \leq \varphi(\xi) \right\}, \quad \omega(x) = \sup_{u \in \mathcal{U}(\varphi, D)} u(x).$$

Here, $h(D)$ class refers to the class of harmonic functions in D, for which the Laplace equation holds: $\Delta u = 0$. The $sh(D)$ class refers to the class of subharmonic functions in D, for which the inequality holds: $\Delta u \geq 0$.

To ensure that the extremal function $\omega(x)$ is a solution to the Dirichlet problem $\omega(x) \in h(D) \cap C(\overline{D})$ with $\omega|_{\partial D} = \varphi$ in the domain $D \subset \mathbb{R}^n$, an additional condition requiring the existence of a barrier is imposed (see [6]).

Definition 1.1. We say that the domain $D \subset \mathbb{R}^n$ has a barrier at the point $\xi \in \partial D$, if there exists a function $b(x) \in sh(D) \cap C(\overline{D})$ such that:

1) $b(\xi) = 0$,
2) $\sup\limits_{|x-\xi| \geq \varepsilon,\, x \in \bar{D}} b(x) < 0$ for any $\varepsilon > 0$.

In this case, the function $b(x)$ is called a barrier at the point $\xi \in \partial D$. Note that if any Dirichlet problem is solvable in the domain $D \subset \mathbb{R}^n$, then the domain $D \subset \mathbb{R}^n$ has a barrier at any point $\xi \in \partial D$.

Theorem 1.2. *If a bounded domain $D \subset \mathbb{R}^n$ has a barrier at all boundary points $\xi \in \partial D$, then the Dirichlet problem for the Laplace equation*

$$\Delta \omega = 0, \quad \omega|_{\partial D} = \varphi, \quad \varphi(\xi) \in C(\partial D)$$

always (for any function $\varphi(\xi) \in C(\partial D)$) has a solution $\omega \in h(D) \cap C(\overline{D})$, and this solution is unique.

Definition 1.3. A bounded domain $D \subset \mathbb{R}^n$ is called a regular domain, if there exists a strictly negative function $\rho(x) \in sh(D)$ such that $\rho(x) < 0$, $\lim\limits_{x \to \partial D} \rho(x) = 0$. The latter condition means that for any number $c < 0$, the set $\{x \in D : \rho(x) < c\}$ is a compact subset in D.

Theorem 1.4. *(Regularity Criterion). The following conditions are equivalent:*
1) D has a barrier at every point $\xi \in \partial D$;
2) $D \subset \mathbb{R}^n$ is regular domain.

2. m-CONVEX FUNCTIONS

The Potential theory in the class of strongly m-subharmonic functions is based on differential forms and currents $(dd^c u)^k \wedge \beta^{n-k} \geq 0$, $k = 1, 2, \dots, n-m+1$, where $\beta = dd^c \|z\|^2$ is the standard volume form in $\mathbb{C}^n$. Then the potential theory in the class of m-convex ($m - cv$) functions, in particular, maximal $m - cv$ functions and the Dirichlet problem, are related to Hessians $H^k(u)$, $k = 1, 2, ..., n - m + 1$. The main method for studying maximal m-convex functions, which, in general, are not smooth, is to connect $m - cv$ functions with strongly m-subharmonic (sh_m) functions (see [9], [28]). The theory of sh_m functions is well-studied and is currently the subject of research by many mathematicians (see Z.Błocki [9], [8], S.Dinew and S.Kolodziej [14], S. Li [16], H.C. Lu [17], [18], H. Bremermann [7], A.Sadullaev, B.Abdullaev [20], [1]).

We recall that a twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is called strongly m-subharmonic, $u \in sh_m(D)$, if at each point of the domain D the followings holds:

$$sh_m(D) = \left\{u \in C^2 : (dd^c u)^k \wedge \beta^{n-k} \geq 0, \quad k = 1, 2, \dots, n-m+1\right\} =$$

$$= \left\{u \in C^2 : dd^c u \wedge \beta^{n-1} \geq 0, \ (dd^c u)^2 \wedge \beta^{n-2} \geq 0, \dots, (dd^c u)^{n-m+1} \wedge \beta^{m-1} \geq 0\right\}, \tag{2.1}$$

where $\beta = dd^c \|z\|^2$ is the standard volume form in $\mathbb{C}^n$.

Operators $(dd^c u)^k \wedge \beta^{n-k}$ are closely related to the Hessians. For a twice continuously differentiable function $u \in C^2(D)$, the second-order differential $dd^c u = \frac{i}{2} \sum\limits_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\,\bar{z}_k$ (at the fixed point $o \in D$) is a Hermitian quadratic form. After a unitary transformation of coordinates, this form can be reduced to the diagonal form $dd^c u = \frac{i}{2}\left[\lambda_1 dz_1 \wedge d\bar{z}_1 + \dots + \lambda_n dz_n \wedge d\bar{z}_n\right]$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the Hermitian matrix $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)$, which are real: $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. Note that, the unitary transformation does not change the differential form $\beta = dd^c \|z\|^2$. It is easy to see that

$$(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)! H^k(u) \beta^n, \tag{2.2}$$

where $H^k(u) = \sum\limits_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1} \dots \lambda_{j_k}$ is the Hessian of dimension k of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$.

Consequently, a function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is strongly m-subharmonic if at each point $o \in D$, the following inequalities hold:

$$H^k(u) = H^k_0(u) \geq 0, \quad k = 1, 2, ..., n - m + 1. \tag{2.3}$$

Note that the concept of a strongly m-subharmonic function is defined, in general, in the distribution sense

Definition 2.1. A function $u \in L^1_{loc}(D)$ is called sh_m in the domain $D \subset \mathbb{C}^n$, if it is upper semicontinuous and for any twice continuously differentiable function sh_m functions $v_1, \dots, v_{n-m}$, the current $dd^c u \wedge dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ defined as

$$\left[dd^c u \wedge dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1}\right](\omega) =$$

$$= \int u dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \quad \omega \in F^{0,0} \tag{2.4}$$

is positive,

$$\int u dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \quad \forall \omega \in F^{0,0}, \ \omega \geq 0.$$

In the work of Błocki [8], it was proven that this definition is correct, in the sense that, for functions $u \in C^2(D)$, this definition coincides with the original definition of sh_m. Moreover, the operators $(dd^c u)^k \wedge \beta^{n-k} \geq 0$, for $k = 1, 2, ..., n-m+1$ are defined in the class of bounded sh_m functions as Borel measures in the domain D (see [20], [8]).

Let now $D \subset \mathbb{R}^n$ and $u(x) \in C^2(D)$. Similar to (2.1), we want to define $m-cv$ functions in the domain $D \subset \mathbb{R}^n$. The matrix $\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right)$ is symmetric, $\frac{\partial^2 u}{\partial x_k \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_k}$. Therefore, after a suitable orthonormal transformation, it is transformed into a diagonal form

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right) \to \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ are the eigenvalues of the matrix $\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right)$. Let

$$H^k(u) = H^k(\lambda) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} \lambda_{j_1} ... \lambda_{j_k}$$

be the Hessian of dimension k of the $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Definition 2.2. A function $u \in C^2(D)$ is called m-convex in $D \subset \mathbb{R}^n$, $u \in m-cv(D)$, if its $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x))$ satisfies the conditions

$$H^k(u) = H^k(\lambda(x)) \geq 0, \quad \forall x \in D, \ k = 1, ..., n-m+1.$$

When $m = n$, the class $n - cv \cap C^2(D) = \{\lambda_1 + \lambda_2 + \ldots + \lambda_n \geq 0\}$ coincides with the class of subharmonic functions, and when $m = 1$, this class $1 - cv \cap C^2(D) = \{\lambda_1 \geq 0, \lambda_2 \geq 0, \ldots, \lambda_n \geq 0\}$ coincides with functions that are convex in $\mathbb{R}^n$. The class of convex functions is well studied (A. Alexandrov, I. Bakelman, A.Pogorelov, see [2, 3, 4, 5, 19]). For $m > 1$, the class m-convex functions has been studied in series of works by N. Ivochkina, N. Trudinger, X. Wang, etc. (see [12, 13, 15, 27, 28, 29, 30]).

The key point in studying $m-cv \cap L^1_{loc}$ functions is the relationship between $m-cv$ and sh_m functions (see for instanse [8, 12, 25], [26]). We embed $\mathbb{R}^n_x$ into $\mathbb{C}^n_z$ by $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y$ $(z = x + iy)$, as a real $n-$dimensional subspace of the complex space $\mathbb{C}^n$.

Theorem 2.3. *A function $u(x) \in C^2(D)$, $D \subset \mathbb{R}^n$, is m-convex in D, if and only if a function $u^c(z) = u^c(x + iy) = u(x)$ does not depend on variables $y \in \mathbb{R}^n_y$ and is sh_m in the domain $D \times i\mathbb{R}^n_y$.*

Definition 2.4. An upper semicontinuous function $u(x)$ in a domain $D \subset \mathbb{R}^n_x$ is called m-convex in D, if the function $u^c(z)$ is strongly m-subharmonic, $u^c(z) \in sh_m(D \times i\mathbb{R}^n_y)$.

We can now define Hessians H^k, for $k = 1, 2, ..., n-m+1$, in the class of locally bounded, m-convex functions in the domain $D \subset \mathbb{R}^n_x$. Let $u(x)$ be a locally bounded, m-convex function in the domain $D \subset \mathbb{R}^n_x$. According to definition 2.4 we constract $u^c(z) \in sh_m(D \times i\mathbb{R}^n_y)$ and define Borel measures μ_k in the domain $D \times i\mathbb{R}^n_y \subset \mathbb{C}^n_z$:

$$\mu_k = (dd^c u^c)^k \wedge \beta^{n-k}, \quad k = 1, 2, \ldots, n-m+1.$$

Since $u^c \in sh_m\left(D \times i\mathbb{R}^n_y\right)$ does not depend on $y \in \mathbb{R}^n_y$, then for any Borel sets $E_x \subset D$ and $E_y \subset\subset \mathbb{R}^n_y$, the measures $\frac{1}{mesE_y}\mu_k\left(E_x \times E_y\right)$ do not depend on the set $E_y \subset\subset \mathbb{R}^n_y$, i.e., $\frac{1}{mesE_y}\mu_k\left(E_x \times E_y\right) = \nu_k\left(E_x\right)$. Borel measures are defined as

$$\nu_k : \ \nu_k\left(E_x\right) = \frac{1}{mesE_y}\mu_k\left(E_x \times E_y\right), \quad k = 1, 2, \ldots, n-m+1, \tag{2.5}$$

where we call by $H^k = H^k\left(E_x\right)$, for $k = 1, 2, \ldots, n-m+1$, as Hessians for a locally bounded, m-convex function $u(x) \in m-cv(D)$. For a twice continuously differentiable function, $u(x) \in m-cv(D) \cap C^2(D)$, the Hessians are ordinary functions. However, for a non-twice continuously differentiable function but bounded upper semicontinuous function $u(x) \in m-cv(D) \cap L^\infty(D)$, the Hessians H^k, $k = 1, 2, ..., n-m+1$, are positive Borel measures (see [23]).

3. Maximal functions and the Dirichlet problem

Definition 3.1. A function $u(x) \in m - cv(D)$ is called maximal in the domain $D \subset \mathbb{R}^n$ if for this function the maximum principle holds in the class of $m - cv(D)$, i.e., if $v \in m - cv(D)$ and $\varliminf_{x \to \partial D} (u(x) - v(x))$, then $u(x) \geq v(x)$, $\forall x \in D$.

Below we will use the following more convenient criterion of maximality: a function $u(x) \in m-cv(D)$ is maximal in the domain $D \subset \mathbb{R}^n$ if and only if for any domain $G \subset\subset D$ the inequality $u(x) \geq v(x)$, $\forall x \in G$ holds for all functions $v \in m - cv(D): \ u|_{\partial G} \geq v|_{\partial G}$ (see [24]).

Maximal functions are closely related to the Dirichlet problem.

Theorem 3.2. *(see [24]). Let $D = \{\rho(x) < 0\}$ be a strictly $m - cv$ convex domain in $\mathbb{R}^n$ and $\varphi(\xi)$ be a continuous function defined on the boundary ∂D. Put*

$$\mathcal{U}(\varphi, D) = \{u \in m - cv(D) \cap C(\overline{D}) : u|_{\partial D} \leq \varphi\}$$

and

$$\omega(x) = \sup\{u(x) : u \in \mathcal{U}(\varphi, D)\}. \tag{3.1}$$

Then, $\omega(x) \in m - cv(D) \cap C(\overline{D})$, $\omega|_{\partial D} = \varphi$ and in addition, $\omega(x)$ is the maximal $m - cv$ function in D, i.e. $H^{n-m+1}(\omega(x)) = 0$.

We recall that, the domain $D = \{\rho(x) < 0\}$ is called strictly $m - cv$, if the function $\rho(x)$ is strictly $m - cv$ in a neighborhood $D^+ \supset \overline{D}$, $\rho(x) \in m - cv(D^+)$. Moreover $\rho(x) - \delta|x|^2 \in m - cv(D^+)$ for some $\delta > 0$. We note, that the ball $B(0, r) = \{x \in \mathbb{R}^n : \ |x| < r\}$ is strictly $m - cv$ domain, but the parallelepiped $\Pi = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n : \ \ |x_1| < 1, |x_2| < 1, ..., |x_n| < 1\}$ is not strictly $m - cv$ in $\mathbb{R}^n$, although Π is a regular domain in the sense of classical potential theory.

It is natural to call the function $\omega(x)$ as a solution to the Dirichlet problem that is $\omega(x)$ is maximal and $\omega|_{\partial D} = \varphi$. For regularization ω^* which is $m - cv$ function in the domain D condition of continuity on the boundary is also satisfied: $\lim_{x \to \xi} \omega^*(x) = \varphi(\xi)$, $\forall \xi \in \partial D$. From $\omega^*(x) \in m - cv(D)$, $\lim_{x \to \partial D} \omega^* = \varphi$ follows that $\omega^*(x) \leq \omega(x)$, i.e. $\omega^*(x) \equiv \omega(x)$ and $\omega(x)$ is $m - cv$ function. Let us show that $\omega^*(x) \equiv \omega(x)$ is maximal.

Assume the contrary, let there be a domain $G \subset\subset D$ and a function $\phi(x) \in m - cv(D): \ \phi|_{\partial G} \leq \omega|_{\partial G}$, but $\phi(x^0) > \omega(x^0)$ at some point x^0.

Function

$$w(x) = \begin{cases} \max\{\omega(x), \phi(x)\}, & if \ \ x \in \overline{G} \\ \omega(x), & if \ \ x \in D \backslash G \end{cases}$$

is m-convex, $w(x) \in m - cv(D)$, $w|_{\partial D} = \omega|_{\partial D} = \varphi$. Therefore, $w(x) \leq \omega(x)$ and $\phi(x^0) \leq \omega(x^0)$. This is contradiction.

4. The Dirichlet problem for non strictly $m - cv$ regular domain $D \subset \mathbb{R}^n$

As we saw in Section 3, some condition is imposed on the domain $D \subset \mathbb{R}^n$ for the existence of a solution to the Dirichlet problem, which is related to its strictly m-convexity.

Let's provide an example where the domain $D \subset \mathbb{R}^n$ is simply m-convex or regular in the sense of classical potential theory, and Theorem 3.2 does not hold. First, let us note that if the Dirichlet problem is solvable in the domain $D \subset \mathbb{R}^n$, then

$$H^{n-m+1}(\omega(x)) = 0, \ \omega(x) \in m - cv(D) \cap C(\overline{D}), \quad \omega|_{\partial D} = \varphi, \quad \varphi(\xi) \in C(\partial D),$$

the given boundary function $\varphi(\xi)$ necessarily continues inside D as m-convex function. However, not every function defined on the boundary ∂D can be m- convexly continued inside D.

Example 4.1. $n = 2$, $m = 1$. Let $D = l^2 = \{|x_1| < 1, \ |x_2| < 1\}$ be a square. Then the class $1 - cv$ coincides with the class of convex functions. For any convex function $\omega(x) \in 1 - cv(l^2) \cap C(\overline{l^2})$, its boundary values $\omega^*(\xi)$ consist of two convex functions $\omega^*(\xi_1, 0)$ and $\omega^*(0, \xi_2)$. Thus, if the given boundary function $\varphi(\xi)$ is not convex on the interval $\{|\xi_1| < 1, \xi_2 = 0\}$ or $\{\xi_1 = 0, \ |\xi_2| < 1\}$, then such a function cannot be continued in $D = l^2$ as an $1 - cv$ (convex) function. For example, the function $\varphi(\xi_1, 0) = -\xi_1^2$ is not convex on the interval $(-1, +1)$.

Nevertheless, it is true

Theorem 4.2. *Let $D \subset \mathbb{R}^n$ be a regular domain in the sense of classical potential theory, and let $\varphi \in C(\partial D)$ be a function such that it is the trace of some function $w \in m-cv(D)$ with $\lim\limits_{x\to\xi} w(x) = \varphi(\xi)$ for $\xi \in \partial D$. Then, the function $\omega(x)$ is maximal function in $D \subset \mathbb{R}^n$. Moreover, $\omega \in m-cv(D) \cap C(\overline{D})$ and $\omega|_{\partial D} = \varphi$. This function is called the solution to the Dirichlet problem: $H^{n-m+1}(\omega(x)) = 0,\ \omega|_{\partial D} = \varphi$.*

We note that if in Theorem 3.2 the condition for the existence of a solution to the Dirichlet problem pertains to the domain $D \subset \mathbb{R}^n$, in this theorem the condition is imposed on the boundary function $\varphi \in C(\partial D)$.

Proof. (Proof of Theorem 4.2) According to the property of $m - cv$ functions for the envelope:

$$\omega(x) = \sup\left\{u(x) \in m - cv(D) \cap C(\overline{D}) :\ u|_{\partial D} \le \varphi\right\},$$

the regularization $\omega^*(x)$ will also be an $m - cv$ function.

Along with the family $\{u(x) \in m - cv(D) \cap C(\overline{D}),\ u|_{\partial D} \le \varphi\}$ we also take the family $\{v(x) \in sh(D) \cap C(\overline{D}), v|_{\partial D} \le \varphi\}$, which is involved in solving the classical Dirichlet problem. Since any m-convex function is subharmonic, we have:

$$\{u(x) \in m - cv(D) \cap C(\overline{D}),\ u|_{\partial D} \le \varphi\} \subset \{v(x) \in sh(D) \cap C(\overline{D}), v|_{\partial D} \le \varphi\}. \tag{4.1}$$

Since D is a regular domain in the sense of classical potential theory, the function

$$q(x) = \sup\{v(x) \in sh(D) \cap C(\overline{D}),\ v|_{\partial D} \le \varphi\}$$

represents a harmonic function in D, $q(x) \in h(D) \cap C(\overline{D})$ and $q|_{\partial D} = \varphi$. Furthermore, from (4.1), it follows that:

$$\omega(x) = \sup\{u(x) \in m - cv(D) \cap C(\overline{D}) :\ u|_{\partial D} \le \varphi\} \le q(x).$$

From this, we get

$$\overline{\lim_{x\to\xi}}\omega(x) \le \overline{\lim_{x\to\xi}} q(x) = \varphi(\xi), \quad \xi \in \partial D. \tag{4.2}$$

To prove the reverse inequality, we use the function $w \in m - cv(D)$ with $\lim\limits_{x\to\xi} w(x) = \varphi(\xi)$, $\xi \in \partial D$. From the definition of the class $\mathcal{U}(\varphi, D)$, it is clear that $w \in \mathcal{U}(\varphi, D)$. Thus $w(x) \le \omega(x)$, i.e., $\lim\limits_{x\to\xi} \omega(x) \ge \lim\limits_{x\to\xi} w(x) = \varphi(\xi),\ \xi \in \partial D$, and this together with (4.2) gives us $\lim\limits_{x\to\xi} \omega^*(x) = \varphi(\xi),\ \xi \in \partial D$.

From the proof, it follows that $\omega^* \in \mathcal{U}(\varphi, D)$ and, consequently, $\omega^* = \omega$. We prove, that the solution $\omega^* = \omega$ is maximal. Regularization ω^* is $m - cv$ function in the domain D, for which the continuity condition on the boundary is satisfied: $\lim\limits_{x\to\xi} \omega^*(x) = \varphi(\xi),\ \ \forall\xi \in \partial D$. It follows from $\omega^*(x) \in m - cv(D),\ \ \lim\limits_{x\to\partial D} \omega^* = \varphi$, that $\omega^*(x) \le \omega(x)$, i.e. $\omega^*(x) \equiv \omega(x)$ and $\omega(x)$ is $m - cv$ function. Let us show that it is maximal.

By contradiction, let there exist a domain $G \subset\subset D$ and a function $\phi(x) \in m-cv(D) :\ \ \phi|_{\partial G} \le \omega|_{\partial G}$, but $\phi(x^0) > \omega(x^0)$ at some point $x^0 \in G$.

The function

$$g(x) = \begin{cases} \max\{\omega(x), \phi(x)\} & if\ \ x \in \bar{G} \\ \omega(x) & if\ \ x \in D \backslash G \end{cases}$$

is m-convex, $g(x) \in m - cv(D),\ g|_{\partial D} = \omega|_{\partial D} = \varphi$. Therefore, $g(x) \le \omega(x)$ and $\phi(x^0) \le \omega(x^0)$. Contradiction.

Now we can prove, that ω is continuous in $\overline{D}$. Let's build an approximation $\omega_\delta(x) = \omega \circ K_\delta(x - y) \in m - cv(D_\delta) \cap C^\infty(D_\delta),\ D_\delta = \{x \in D :\ \rho(x) < \delta\},\ \omega_\delta(x) \downarrow \omega(x)$, as $\delta \downarrow 0$, where $D_\delta = \{x \in D :\ \ dist\,(x, \partial D) > \delta\}$. For small enough $\delta > 0$ each interior normal $n_\xi,\ \xi \in \partial D$ intersects ∂D_δ at a single point $\eta(\xi) \in \partial D_\delta$, so that a homeomorphism n_δ is defined $n_\delta :\ \ \partial D \to \partial D_\delta$. Let us put $\varphi_\delta(\eta) = \varphi(n_\delta(\xi)), \eta \in \partial D_\delta,\ \xi \in D$. Since $\lim\limits_{x\to\xi} \omega(x) = \varphi(\xi),\ \ \ \forall\xi \in \partial D$, then for any fixed $\varepsilon > 0$ there is a $\delta_0 > 0$ such that $|\omega(x) - \varphi_{\delta_0}(x)| < \varepsilon,\ \ \forall x \in \partial D_{\delta_0}$. For a fixed $\delta_0 > 0$ the domain $D_{\delta_0} \subset\subset D$ and the approximation $\omega_\delta(x) \downarrow \omega(x)$, for $\delta \downarrow 0$ covers the domain D_{δ_0}.

Applying Hartogs' lemma to a compact set ∂D_{δ_0} and a function $\varphi_{\delta_0}(x) \in C(\partial D_{\delta_0})$ we find $0 < \delta' < \delta_0$ such that

$$\omega_\delta(x) < \omega_{\delta_0}(x) + 3\varepsilon, \quad \forall x \in \partial D_{\delta_0}, \quad \delta < \delta'. \tag{4.3}$$

Since the solution to the Dirichlet problem $\omega(x)$ is maximal in D, from $\omega_\delta(x) < \varphi_{\delta_0}(x) + 3\varepsilon$, $\forall x \in \partial D_{\delta_0}$, $\delta < \delta'$ follows that $\omega_\delta(x) < \omega(x) + 4\varepsilon$, $\forall x \in D_{\delta_0}$, $\delta < \delta'$ because $\omega(x) > \varphi_{\delta_0}(x) - 3\varepsilon$, $\forall x \in \partial D_{\delta_0}$. From here, $\omega(x) < \omega_\delta(x) < \omega(x) + 4\varepsilon$, $\forall x \in \partial D_{\delta_0}$, $\delta < \delta'$, i.e. $|\omega_\delta(x) - \omega(x)| < 4\varepsilon$, $\forall x \in D_{\delta_0}$, $\delta < \delta'(\delta_0)$. Since $\varepsilon > 0$ arbitrary, then the convergence $\omega_\delta(x) \downarrow \omega(x)$ will be uniform inside D and $\omega(x) \in C(D)$, because $\omega_\delta(x) \in C^\infty(D_\delta)$.

It remains to prove that, $H^{n-m+1}(\omega(x)) = 0$. This statement is directly followed by the following theorem

Theorem 4.3. *(see. [24]). A continuous $m - cv$ function $u(x) \in m - cv(D) \cap C(D)$ is maximal if and only if the Borel measure is $H^{n-m+1}(u) = 0$.*

□

Given the importance and need for work, we will formulate the following comparison principle

Theorem 4.4. *(Comparison principle), (see. [24]). Let $u(x)$, $v(x) \in m - cv\,(D) \cap C\,(D)$ and $F = \{u(x) < v(x)\} \subset\subset D$ be open set. Then*

$$\int_F H^{n-m+1}(u) \geq \int_F H^{n-m+1}(v). \tag{4.4}$$

The following maximum principle in a class of $m - cv$ functions can be found in the work [16].

Theorem 4.5. *Let $u(x), v(x) \in m - cv\,(D) \cap C^2\left(\bar{D}\right) : \; H^{n-m+1}(u) \leq H^{n-m+1}(v)$, $x \in D$. Then, if $u|_{\partial D} \geq v|_{\partial D}$, then $u\,(x) \geq v\,(x) \;\; \forall x \in D$.*

Proof. (Proof of theorem 4.3) Let $u(x) \in m - cv\,(D) \cap C\,(D)$ is maximal. Let's take a ball $B \subset\subset D$ and consider the Dirichlet problem in Hessians:

$$\begin{aligned} &H^{n-m+1}(u) = \psi\,(x)\,, \; u\,(x) \in m - cv, \\ &u|_{\partial B} = \varphi, \end{aligned} \tag{4.5}$$

where $\psi(x) \in C\left(\bar{B}\right), \;\; \psi\,(x) \geq 0, \;\; \varphi\,(\xi) \in C\,(\partial B)$.

Many works are devoted to solutions of the Dirichlet problem (4.5). Thus, in the non-degenerate smooth case $\psi(x) \in C^\infty\left(\bar{B}\right), \;\; \psi\,(x) > 0, \;\; \varphi\,(\xi) \in C^\infty\,(\partial B)$, equation (4.5) has a unique solution $u(x) \in m - cv\,(B) \cap C^\infty\left(\bar{B}\right), \;\; u|_{\partial B} = \varphi$ (see. [10]-[12], [16]).

For the degenerate case, to solve the equation

$$\begin{aligned} &H^{n-m+1}(u) = 0, \quad u\,(x) \in m - cv, \\ &u|_{\partial B} = \varphi, \;\; \varphi\,(\xi) \in C\,(\partial D) \end{aligned}$$

we will use above statement. Fix a ball $B \subset\subset D$. We approximate the function $\varphi\,(\xi)$ by infinitely smooth functions: $\varphi_j\,(\xi) \downarrow \varphi\,(\xi)\,, \;\; \varphi_j\,(\xi) \in C^\infty\,(\partial B)$. According to the above equation

$$\begin{aligned} &H^{n-m+1}(u) = \tfrac{1}{j}, \; u\,(x) \in m - cv, \\ &u|_{\partial B} = \varphi_j \end{aligned}$$

has a unique solution $u_j\,(x) \in m - cv\,(B) \cap C^\infty\left(\bar{B}\right), \;\; u_j|_{\partial B} = \varphi_j$. According to Theorem 4.5 the sequence $u_j\,(x)$ is decreasing, $u_j\,(x) \geq u_{j+1}\,(x)$. Due to the Hessian property, the Hessian sequence $H^{n-m+1}(u_j\,(x))$ weakly converges: $H^{n-m+1}(u_j\,(x)) \mapsto H^{n-m+1}(u\,(x))$. Since $H^{n-m+1}(u_j\,(x)) = \frac{1}{j}$, then Borel measure $H^{n-m+1}(u) = 0$ in $B \subset\subset D$ and since the ball $B \subset\subset D$ is arbitrary, then $H^{n-m+1}(u) = 0$ in D.

Vice versa, let $u\,(x) \in m - cv\,(D) \cap C\,(D) : \; H^{n-m+1}(u) = 0$. We will show that u is maximal. Let's assume the opposite, that u is not maximal. Then for some domain $G \subset\subset D$ there exists a function $v \in m - cv\,(D) : \;\; u|_{\partial G} \geq v|_{\partial G}$, but $v\,(z^0) - u\,(z^0) = \varepsilon > 0$ for some point $z^0 \in G$.

Approximating v infinitely smooth $m-cv$ functions $v_j \downarrow v$. Then by Hartogs' lemma we find $j_0 \in \mathbb{R}$ such that, $v_{j_0}|_{\partial G} < u|_{\partial G} + \frac{\varepsilon}{2}$.

Let's compare the function $u(z)$ with function $v_{j_0}(z) + \delta \|z\|^2$, where $\delta = \frac{\varepsilon}{3\cdot\max\left\{\|z\|^2 : z\in\overline{G}\right\}}$. For such $\delta > 0$, the set $F = \left\{u(z) + \frac{\varepsilon}{2} < v_{j_0}(z) + \rho(z)\right\}$ is not empty and lies compactly in G.

Then according to the comparison principle (Theorem 4.4)

$$\delta^n \int\limits_F \left(dd^c \|z\|^2\right)^n \leq \int\limits_F \left(dd^c v + \delta dd^c \|z\|^2\right)^n \leq \int\limits_F (dd^c u)^n = 0,$$

which contradicts, what $\int\limits_F \left(dd^c \|z\|^2\right)^n > 0$. Theorem 4.3 and with this the main Theorem 4.2 are proved. □

References

[1] Abdullaev B.; On the Dirichlet problem for $m - sh$ functions, Dokl. Akad. Nauk Resp. Uzb., –2013. –4. –P. 8-10. (in Russian)

[2] Aleksandrov A.D.; Intrinsic geometry of convex surfaces, OGIZ, Moscow, –1948; German transl., AkademieVerlag, Berlin, –1955.

[3] Aleksandrov A.D.; Dirichlet's problem for the equation $\det(z_{ij}) = \varphi$, Vestnic Leningrad University, –1958. –13. –P. 5-24.

[4] Bakelman I.J.; Variational problems and elliptic Monge-Ampere equations, Journal of differential geometry, –1983. –18. –P. 669-999.

[5] Bakelman I.J.; Convex Analysis and Nonlinear Geometric Elliptic Equations. Springer-Verlag: Berlin-Heidelberg, –1994.

[6] Brelot M.; lments de la thorie classique du potentiel. Les Cours de Sorbonne. 3e cycle, Centre de Documentation Universitaire, Paris, –1965.

[7] Bremermann H.J.; On a generalized Dirichlet problem for plurisubharmonic functions and pseudoconvex domains. Characterization of Silov boundaries. Trans. Amer. Math.Soc., –1959. –91:2. –P. 246-276.

[8] Błocki Z.; The domain of definition of the complex Monge-Ampere operator. Amer.J.Math., –2006. –128:2. –P. 519-530.

[9] Błocki Z.; Weak solution to the complex Hessian equation, Ann. Inst. Fourier, Grenoble. – 2005. –55:5. –P. 1735-1756

[10] Caffarelli L., Nirenberg L., Spruck J.; The Dirichlet problem for nonlinear second-order elliptic equations I. Monge-Ampere equation. Comm. on pure and Appl. Math., V. XXXVII, –1984. –P. 369-402.

[11] Caffarelli L., Kohn J., Nirenberg L., Spruck J.; The Dirichlet problem for nonlinear second-order elliptic equations II. Complex Monge-Ampere and uniform elliptic equations equations. Comm. on pure and Appl. Math., V. XXXVIII, –1985. –P. 209-252.

[12] Caffarelli L., Nirenberg L., Spruck J.; The Dirichlet problem for nonlinear second-order elliptic equations III. Functions of the eigenvalues of the Hessian. Acta. Math., –1985. –155. –P. 261- 301.

[13] Chou K.S., Wang X.J.; Variational theory for Hessian equations, Communications on Pure and Applied Mathematics, –2001. –54(9). –P. 1029-1064.

[14] Dinew S., Kolodziej S.; A priori estimates for the complex Hessian equation. Anal. PDE, –2014. –7. –P. 227-244.

[15] Ivochkina N., Trudinger N.S., Wang X.-J.; The Dirichlet problem for degenerate Hessian equations, Comm. Partial Diff. Equations, –2004. –29. –P. 219-235.

[16] Li S.Y.; On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian., Asian J.Math., –2004. –8. –P. 87-106.

[17] Lu H.C.; Solutions to degenerate Hessian equations, Jornal de Mathematique Pures et Appliques. –2013. –100(6). –P. 785-805.

[18] Lu H.C., Nguyen V. D.; Degenerate complex Hessian equations on compact Kehler manifolds. Indiana University Mathematics Journal, –2015. –64(2), –P. 1721-1745.

[19] Pogorelov A.V.; Extrinsic geometry of convex surfaces, "Nauka", Moscow, –1969; English transl., Amer. Math. Soc, Providence, R. I., –1973.

[20] Sadullaev A., Abdullaev B.; Potential theory in the class of m-subharmonic functions. Proc. Steklov Inst. Math., –2012. –279. –P. 155-180.

[21] Sadullaev A.; Potential theory, Tashkent, – 2022, 163 p. (in Russian).

[22] Sadullaev A.; Pluripotential Theory. Applications. Monograph, Palmarium academic publishing, germany, Germany. –2012. (in Russian)

[23] Sadullaev A.; Definition of Hessians for m-convex functions as Borel measures, Analysis and Applied Mathematics. AAM 2022. Trends in Mathematics, Birkhauser, Cham. –2024. –6. –P. 13-19.

[24] Sadullaev A., Sharipov R.; Maximal Functions and the Dirichlet Problem in the Class of m-convex Functions. Journal of Siberian Federal University. Mathematics & Physics. –2024. –17(4). –P. 519-527.

[25] Sharipov R.A., Ismoilov M.B.; m-convex $(m-cv)$ functions, Azerbaijan Journal of Mathematics, –2023. –13(2). –P. 237-247.

[26] Sharipov R.A., Ismoilov M.B.; Hessian measures in the class of m-convex $(m-cv)$ functions. Bulletin of the Karaganda University. Mathematics Series, –2024. –3(15). –P. 93-100.

[27] Trudinger N.S., Wang X.J.; Hessian measures I, Topological Methods in Nonlinear Analysis, –1997. –10(2). –P. 225-239.

[28] Trudinger N.S., Wang X.J.; Hessian measures II, Annals of Mathematics, –1999. –150(2). –P. 579-604.

[29] Trudinger N.S., Wang X.J.; Hessian measures III, Journal of Functional Analysis, –2002. –192(1). –P. 1-23.

[30] Wang X.J.; The k-Hessian equations. In: Chang, SY., Ambrosetti, A., Malchiodi, A. (eds) Geometric Analysis and PDEs. Lecture Notes in Mathematics, –2009. –1977. –P. 177-252.

Sharipov Rasulbek Axmedovich,
Department of Mathematical analysis,
Urgench State University, Urgench, Uzbekistan
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences
email: r.sharipov@urdu.uz

Uzbek Mathematical Journal
2025, Volume 69, Issue 3, pp.176-180
DOI: 10.29229/uzmj.2025-3-19

Limit theorems for auto regression process with random parameter $v,\ 0\ <\ v<1$

Zuparov T. M., Jovliev A.I.

Abstract. In this paper we obtain the criterion of weak convergence of the sequence of the sum of the first n terms of the linear process $\{X_{kn},\ k=1,2,...,n;\ n=1,2,...\}$ with random coefficients $\{v^k, k\in\mathbb{N}\}$, generated by the innovation sequence $\{\xi_{kn}, k\in Z\}$ satisfying the condition of infinite smallness to the limit distribution and as a consequence of this result we obtain the analog of the Lindeberg-Feller theorem for the auto regression process with random parameter $v,\ 0\ <\ v<1$. In addition, the strong law of large numbers and the law of iterated logarithm are proved.

Keywords: Auto regression process, linear process, central limit theorem, strong law of large numbers, law of iterated logarithm.

MSC (2020): 60F05

1. Introduction

$\{\xi_{kn}, k\in\mathbb{Z}, k\leq n, n\geq 1\}-$ a sequence of series of random variables satisfy the following condition (A): for any $\varepsilon>0$

$$P\left\{\sup_{k\in\mathbb{Z}}|\xi_{kn}|>\varepsilon\right\}\xrightarrow[n\to\infty]{}0.$$

Definition 1.1. If a series $X_{kn}=\sum_{i=0}^{\infty}a_{in}\xi_{k-i,n}$ converges with probability 1, then a sequence of random variables $\{X_{kn}, k\in\mathbb{Z}\}$ is called a linear process with coefficients $\{a_{kn}, k\in Z, n\geq 1\}$ generated by an innovation sequence $\{\xi_{kn}, k\in Z\}$.

Remark 1.1. If the conditions (A), $E\xi_{kn}=0,\ k\in\mathbb{Z}$ are satisfied, then the series $\sum_{i=0}^{\infty}v^i\xi_{k-i,n}$ converges with probability 1, and therefore the linear process is defined quite correctly in this case.

The paper is devoted to the asymptotic analysis of the sum of the first n terms of a linear process with random coefficients $a_{kn}=v^k$, generated by an innovative sequence $\{\xi_{kn}\}$. By means of BN-decomposition (Beveridge-Nilson distribution) we prove the criterion of weak convergence of the randomly normalized sum of the linear process to the iterated limits and as a consequence of this result we obtain the analog of the Lindeberg-Feller theorem for the auto regression process with a random parameter $v,\ 0\ <\ v<1$. In addition, we prove the strong law of large numbers (SLLN) and the law of iterated logarithm for a linear process associated with a 1 - order auto regression process with random parameter v. Many known results obtained about the asymptotics of the distribution of the sum distribution of a linear process can be found in the articles [4], [6 - 12]. In [2] a class of auto regression processes of 1 -order with random coefficients with the form $X_n-\mu=\rho_n(X_{n-1}-\mu)+\varepsilon_n,\quad n\in\mathbb{Z}$, where, $\mu=EX_n$, $\{\varepsilon_n, n\in\mathbb{Z}\}$ white noise: a sequence of independent, identically distributed random variables with mathematical expectations equal to zero and unit variance and $\{\rho_n\}$ - a sequence of independent, identically distributed random variables satisfying the condition $\sup\limits_n|\rho_n|<1$(with probability 1). Obviously, the auto regression process considered in this paper does not belong to the class of processes considered in [2].

In this paper we consider a 1st-order auto regression process with a random parameter v:

$$X_n=\ vX_{n-1}+\xi_n,\quad n\in\mathbb{Z},\tag{1.1}$$

where $v, 0\ <\ v<1$ is a random variable (r.v.) independent of sequence $\{\xi_n, n\in\mathbb{Z}\}$, $\{\xi_n, n\in\mathbb{Z}\}$ is a sequence independent and identically distributed random variables with $E\xi_0=0, E\xi_0^2=\sigma^2<\infty$. There exists only one solution to equation (1.1) such that $EX_k=0$. This solution is of the form see [2, 3]

$$X_k = \sum_{j=0}^{\infty} v^j \xi_{k-j}, \quad k \in \mathbb{N}. \tag{1.2}$$

Many problems related to the asymptotics of the sum of n first terms of a linear process $X_k = \sum_{j=0}^{\infty} a_{jn}\xi_{k-j}$ are solved using the following decomposition :

Lemma 1.1. If a series $\sum_{k=-\infty}^{\infty} a_{kn}$ is absolutely convergent, then there is a decomposition of

$$X_t = \left(\sum_{k=0}^{\infty} a_{kn}\right)\xi_{tn} + \sum_{j=1}^{\infty}\gamma_j\xi_{t-j,n} - \sum_{j=1}^{\infty}\gamma_j\xi_{t-j+1,n}, \tag{1.3}$$

where $\gamma_j = \sum_{k=j}^{\infty} a_{kn}$.

Equality (1.3) can be proved directly by comparing the coefficients ahead of the random variables ξ_l; $l = 0, \pm1, \pm2, ...$ in the expression of the random variable X_n. In particular, from the expansion (1.3) we obtain the following expansion for the sum of the first n terms of the linear process:

$$S_n = \sum_{t=1}^{n} X_t = \left(\sum_{j=0}^{\infty} a_{jn}\right)\sum_{t=1}^{n}\xi_{tn} + \sum_{j=0}^{\infty}\gamma_j\left(\xi_{1-j,n} - \xi_{n-j+1,n}\right). \tag{1.4}$$

Decompositions (1.3) and (1.4) are commonly referred to in the literature as BN-decomposition. The BN-decomposition was formally applied by Beveridge and Nelson (1981) in the study of cycle fluctuations in commercial activities. Phillips and Solo (1992) [4] gave a general treatment of the BN-decomposition and applied it to prove the CLT the SLLN and the invariance principle of the law of iterated logarithm for linear processes generated by independent and identically distributed innovations. Similar results, in the more general unequally distributed case are obtained in the work of one of the authors (see [7]). In [8], an asymptotic analysis of the distribution of the sum of linear processes with non-random coefficients satisfying the condition of infinite smallness (the transfer theorem) to the limit distribution was carried out and, as a consequence of this result, an analog of the Lindeberg-Feller theorem for linear processes generated by $\varphi-$ mixing innovation sequence was obtained. Almost all known results about asymptotics of linear processes follow from the main result of this paper.

BN- decomposition in the special case when the linear process has the form (1.2) is defined by the formula

$$S_n = \left(\frac{1}{1-v}\right)\sum_{t=1}^{n}\xi_{tn} + \sum_{j=0}^{\infty}\gamma_j\left(\xi_{1-j,n} - \xi_{n-j+1,n}\right), \tag{1.5}$$

where $\gamma_j = \sum_{k=j}^{\infty} v^k$.

2. Main Results

Before we formulate the main results of the paper, we prove the following main lemma.

Lemma 2.1. If condition (A) holds, then the following relations can be written:

$$\sum_{j=0}^{\infty}\gamma_j\left(\xi_{1-j,n} - \xi_{n-j+1,n}\right) \longrightarrow 0 \text{ with probability 1.} \tag{2.1}$$

Proof. Using the formula of the sum of infinitely decreasing geometric progression, we obtain

$$\sum_{j=0}^{\infty}\gamma_j\left(\xi_{1-j,n} - \xi_{n-j+1n}\right) = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} v^k(\xi_{1-j,n} - \xi_{n-j+1,n}) = \sum_{j=1}^{\infty}\frac{v^j}{1-v}(\xi_{1-j,n} - \xi_{n-j+1}).$$

It follows that, relation (2.1) is equivalent (see [5], p.269) to the statement: for any $\varepsilon > 0$, at $n \to \infty$

$$P_n(\varepsilon) := P\left(\sup_{k\geq n}\left|\sum_{j=1}^{\infty}\frac{v^j}{1-v}(\xi_{1-j,k}-\xi_{k-j+1})\right|\geq\varepsilon\right)\to 0, n\to\infty.$$

There is inequality:

$$P_n(\varepsilon)\leq 2P\left(\sum_{j=1}^{\infty}\frac{v^j}{1-v}\sup_{k\in\mathbb{Z}}|\xi_{kn}|>\frac{\varepsilon}{2}\right)=2P\left(\frac{v}{(1-v)^2}\sup_{k}|\xi_{kn}|>\frac{\varepsilon}{2}\right).$$

Hence, since the value of $\frac{v}{(1-v)^2}$ with probability 1 is bounded and independent of n, based on condition (A), it follows that the relation in (2.1) of the lemma holds. The lemma is proved.

As a first application of the lemma 2.1, we prove the SLLN for an auto regression process with a random parameter.

Theorem 2.1. (SLLN) The auto regression process of 1st-order with a random parameter $v,\ 0 < v < 1$ obeys the SLLN, i.e. the following relation is true

$$\frac{S_n}{n}\xrightarrow[n\to\infty]{} 0 \text{ with probability 1.}$$

Proof. The auto regression process is a linear process$X_k=\sum_{j=0}^{\infty}v^j\xi_{k-j}$ where$\{\xi_j\}$—is a sequence of independent identically distributed random variables with mathematical expectation equal to zero, which satisfies the strong law of large numbers. Hence, and from lemma 2.1, by virtue of the decomposition (1.5), the proof of the theorem follows.

We now formulate and prove the following main theorem.

Theorem 2.2 (transfer theorem). If the innovation process $\{\xi_{kn}, k\in Z\}$ satisfies some limit relation (in the sense of weak convergence, almost sure convergence, or convergence in probability), then the random sum $(1-v)S_n$ follows the same limit relation, and conversely.

Proof. According to decomposition (1.5), we have

$$(1-v)S_n=\sum_{t=1}^{n}\xi_{tn}+(1-v)\sum_{j=0}^{\infty}\gamma_j\left(\xi_{1-j,n}-\xi_{n-j+1,n}\right) \tag{2.2}$$

since the second summand in the right-hand side of equality (2.2) tends to zero with probability 1 according to lemma 2.1, the sequence $(1-v)S_n$ has the same limiting relation as $\sum_{k=1}^{n}\xi_{kn}$ and vice versa.

It follows from theorem 2.2 that almost all asymptotic statements valid for the sum of the first n terms of the sequence $\{\xi_{kn}\}$ are also valid for the linear process $X_k=\sum_{j=0}^{\infty}v^j\xi_{k-j,n}$. Below we give some of these statements.

Corollary 2.1 (analog of Kolmogorov's theorem). Let $X_k=\sum_{j=0}^{\infty}v^j\xi_{k-j}$— be a linear process generated by a sequence of independent and identically distributed random variables, and assume that $E|\xi_1|\ <\infty$. Then the following relation holds:

$$\frac{(1-v)S_n}{n}\xrightarrow[n\to\infty]{} m \text{ with probability 1.}$$

Here, $m=E\xi_1.$ is the mathematical expectation of the random variable.

The proof follows directly from theorem 2.2, by virtue of the well-known Kolmogorov theorem valid for a sequence of independent identically distributed random variables with finite mathematical expectations.

To prove the next result, the following result established in [13] will be required.

Lemma 2.3. Let $\{\xi_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with $E\xi_1 = 0,\ E\xi_1^2 = 1$. Then following relation hold

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^n \xi_n|}{\sqrt{2n \ln\ln n}} = 1 \text{ with probability 1.}$$

Corollary 2.2. Let auto regression process of order 1 with random parameter $v,\ 0 < v < 1$ generated by a sequence of independent and identically distributed random variables with $E\xi_1 = 0,\ E\xi_1^2 = 1$. Then the is true

$$\limsup_{n\to\infty} \frac{(1-v)S_n}{\sqrt{2n \ln\ln n}} = 1 \text{ with probability 1.}$$

The proof follows directly from Theorem 2.2 and Lemma 2.3.

Theorem 2.3. Let a linear process $X_k = \sum_{j=0}^{\infty} v^j \xi_{k-j,n}$ is generated by a sequence of independent random variables satisfying the condition (A). Then, for the following relation to take place

$$P\left((1-v)\sum_{k=1}^{n} X_k \leq x\right) \Rightarrow F(x),$$

where $F(x)$ — is an arbitrary distribution function, it is necessary and sufficient to fulfill the condition (B) $F_n(x) \Rightarrow F(x)$, where $F_n(x) = P\left(\sum_{k=1}^n \xi_{kn} \leq x\right)$.

Proof. Since the summand in the right-hand side of the expansion (2.2), according to lemma 2.1 converges to zero with probability 1 (hence also in probability), then, according to (2.2) the random variables $(1-v)S_n$ and $\sum_{k=1}^n \xi_{kn}$ are asymtotic identically distributed.The theorem has been proved. From theorem 3, by virtue of the Lindeberg-Feller theorem, the following statement follows.

Corollary 2.3. Let a linear process $X_k = \sum_{j=0}^{\infty} v^j \xi_{k-j,n}$ be generated by a sequence of independent random variables with $E\xi_{kn} = 0,\ E\xi_{kn}^2 = \sigma_{kn}^2 < \infty$, satisfying condition (A). Then the sequence $(1-v)S_n/B_n$, satisfies CLT if and only if Lindeberg's condition is satisfied:

$$\text{(L) } \tfrac{1}{B_n^2} \sum_{k=1}^n E\xi_{kn}^2 I\left\{|\xi_{kn}| \geq \varepsilon(B_n)\right\} \xrightarrow[n\to\infty]{} 0,$$

for any positive ε, where $B_n^2 = \sum_{k=1}^n \sigma_{kn}^2$.

References

[1] Beveridge S. andNelson C.R. A new approach to decomposition o economic time series into permanent and transitory components with particular attention to measurement of the "business cycle". Journal of monetary Economics, 1981, 7, 151-174

[2] Bonkhar S., Mourid T. Limit theorems for hilbertian avtoregressive processes with random coefficients. Annales de PISUP, 2018, 62(3), pp.59-74.

[3] Bosq D. Linear Processes in Function Spaces. Theory and Applications, Springer-Verlag New York, Inc, 2000.

[4] Phillips, P.C.B and Solo, V. Asymptotes for linear processes. Annals of statistics, 1992, 20, 971-1001.

[5] Shiryaev A.N. Probability - M: Nauka, 1980, pp576.

[6] Zuparov T.M., Mukhamedov A.K. Invariance principle for processes with uniformly strong mixing. In Proceedings: Functionals from random processes and statistical conclusions.Tashkent, : Fan, 1989, p.27 - 36.

[7] Zuparov T.M. Limit theorems for linear processes. Proceedings of Respub. Scientific and Practical Conference "Statistics and its Applications", Tashkent, 2012, pp. 112 -123.

[8] Zuparov T.M. Asymptotic analysis of the distribution of the sum of a linear process, International collection of scientific papers. Statistical methods of estimation and hypothesis testing. Perm, Izd. of Perm University, 2019, issue 29, pp.19 - 29.

[9] Peligrad M. and Utev S. Central limit theorem for linear processes. Ann Probab., 1997, 25(1), 443 - 456.

[10] Peligrad M. and Utev S. Central limit theorem for stationary linear processes. Ann Probab., 2006, 34(4), 1608 - 1626.

[11] Wu W.B. Central limit theorem for functional of linear processes and their applications. Statist, Sinica, 2002, 12, p.635 - 649.

[12] Wu W.B. and Min W. On linear processes with dependent innovations. Stochastic Process. Appl, 115, p.939 - 958.

[13] Hartman P., Vintner A. On the law of the iterated logarithm, Amer. J. Math., (1941), 63, no. 1b 169 – 176.

T.M.Zuparov,
Chirchiq State Pedagogical University,
Tashkent, Uzbekistan
email: Tm_zuparov@rambler.ru

A.I.Jovliev
Denov Institute of entrepreneurship and pedagogy
Surkhandaryo, Uzbekistan
email: jovliyevaziz91@gmail.com

Contents

Computer imposition: *A.F. Aliyev*

The journal was registered by the Press and Information Agency of the Republic of Uzbekistan on December 22, 2006. Register. No 0044.

Handed over to the set on 11/03/2022. Signed for printing on 12/04/2023
Format 60×84 1/16. Literary typeface. Offset printing.

V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences 9 University st. 100174

Printed in a printing house "MERIT-PRINT"
Tashkent city, Yakkasaray district, Sh. Rustaveli street, 91