UZBEKISTAN ACADEMY OF SCIENCES
V.I.ROMANOVSKIY INSTITUTE OF MATHEMATICS

UZBEK MATHEMATICAL JOURNAL

Journal was founded in 1957. Until 1991 it was named by "Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk". Since 1991 it is known as "Uzbek Mathematical Journal". It has 4 issues annually.

Volume 69 Issue 4 2025

Uzbek Mathematical Journal is abstracting and indexing by

SCOPUS **EBSCO** **CrossRef**
MathSciNet **Zentralblatt** **Math VINITI**

TASHKENT - 2025

Address: University str., 9, Tashkent 100174, Uzbekistan

Rasul Nabiyevich Ganikhodzhaev

(On the Occasion of His 80th Birthday)

On November 19, 2025, the distinguished mathematician, Doctor of Physical and Mathematical Sciences, and Professor at the National University of Uzbekistan, Rasul Nabiyevich Ganikhodzhaev, celebrates his 80th birthday. He is one of the leading representatives of the Uzbek school of functional analysis and the theory of discrete dynamical systems.

R.N. Ganikhodzhaev was born in 1945 in Tashkent into a family of scholars. His father, Candidate of Technical Sciences N.G. Ganikhodzhaev, worked at the Institute of Power Engineering and Automation of the Uzbekistan Academy of Sciences and greatly influenced the formation of his sons scientific interests. From an early age, Rasul Nabiyevich demonstrated a deep aptitude for the exact sciences, actively participating in physics and mathematics Olympiads. After completing secondary school in 1963, he entered the Faculty of Mechanics and Mathematics at Tashkent State University (now the National University of Uzbekistan).

From 1964 to 1967, he served in the armed forces and, upon returning, resumed his studies, graduating with distinction in 1971. Influenced by the seminars of Professor (now Academician) J.Kh. Khadjiev, he chose functional analysis as his primary field of research – a direction that would define his entire scientific career.

In 1971, R.N. Ganikhodzhaev began his postgraduate studies at the Department of Functional Analysis under Academicians T.A. Sarymsakov and J.Kh. Khadjiev. He defended his Candidates Dissertation in 1978 on "Quadratic Operator Equations," where he classified such equations in Hilbert spaces and established solvability conditions for elliptic-type cases. These results laid the foundation for his later work on uniformly convex operators.

Inspired by Academician Sarymsakov, who introduced quadratic operators to model biological evolution, Ganikhodzhaev made fundamental contributions to the theory of quadratic stochastic operators (QSOs). Using combinatorial and graph-theoretic methods, he described quadratic homeomorphisms on simplices, formulated conditions for QSO regularity, proved the ergodic principle, and proposed an analog of the Kolmogorov-Chapman equation.

He also advanced the study of discrete dynamical systems, introducing concepts such as fixed-point cards and trajectory routes. His work achieved wide international recognition, culminating in his 1995 Doctoral Dissertation "Investigations on the Theory of Quadratic Stochastic Operators." Professor Ganikhodzhaev has authored over 100 scientific and educational works, including textbooks and manuals, and has supervised numerous researchersfour Candidates of Sciences, six PhDs, and one Doctor of Science.

Since 1975, Professor Ganikhodzhaev has been deeply involved in teaching and mentoring. He has delivered a wide range of courses, including Functional Analysis, General Topology, Calculus of Variations, and Mathematical Analysis, as well as specialized lectures on Operator Theory and Convex Analysis. With tireless enthusiasm, he has conducted research seminars and mathematical circles, inspiring generations of young mathematicians.

He serves on the editorial boards of several leading journals, including the Uzbek Mathematical Journal and the Bulletin of the National University of Uzbekistan. He is also a member of specialized dissertation councils.

Rasul Nabiyevich has played a vital role in developing mathematical education and promoting scientific culture in Uzbekistan. He actively supports young researchers, contributes to the formation of new scientific schools, and organizes student Olympiads. Thanks to his dedication and mentorship, many talented students have chosen careers in scientific research.

At present, Professor Ganikhodzhaev continues to be actively engaged in research and teaching. He regularly participates in conferences and seminars and pursues new directions in the theory of discrete dynamical systems and quadratic mappings.

The Editorial Board of the Uzbek Mathematical Journal, together with his colleagues and students, extends its heartfelt congratulations to Professor Rasul Nabiyevich Ganikhodzhaev on the occasion of his 80th birthday, wishing him good health, long life, creative energy, and continued outstanding contributions to Uzbek mathematics.

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.7-14
DOI: 10.29229/uzmj.2025-4-1

Exponential stability of a numerical solution of a hyperbolic system with negative nonlocal characteristic velocities and measurement error

Aloev R., Berdyshev A., Alimova V.

Abstract. In this work, the problem of stabilizing the equilibrium state for a hyperbolic system with negative nonlocal characteristic velocities and measurement error is investigated. A mixed problem is considered for such systems, when a limited perturbation of measurement errors is taken into account in the boundary conditions. The study is based on the use of the adequacy between the stability for a mixed problem for the original hyperbolic system of linear differential equations and the stability of the initial-boundary difference problem for it. When analyzing the initial-boundary difference problems constructed in this way, the properties of logarithmic norms are used. Algorithms are proposed that make it possible to obtain sufficient conditions for the exponential stability of a numerical solution of an initial-boundary difference problem with nonlocal coefficients and limited perturbation of measurement errors in boundary conditions. Sufficient conditions are presented in the form of matrix inequalities, which involve matrices of boundary conditions. The results are presented in the form of an a priori estimate of the numerical solution in the norm through the norms of the functions of the initial data and the norms of perturbation of measurement errors.

Keywords: Lyapunov's function, hyperbolic system, nonlocal characteristic velocity, Lyapunov stability.

MSC (2020): 65M12, 65M06, 35L40

1. Introduction

In [1], the Lyapunov stability of the equilibrium position of a mixed problem for a one-dimensional single hyperbolic equation with a positive nonlocal coefficient of the derivative with respect to x is studied. Exponential stability is tested using spectral analysis of the linearized hyperbolic equation. The Lyapunov function is constructed. Using the Lyapunov function, the exponential stability of the steady-state solution of the hyperbolic equation is proved. The case of positive characteristic velocities for the scalar case is considered.

In [2], a mixed problem is studied for a nonlinear equation with positive coefficients of the derivative with respect to x. The Lyapunov stability of an equilibrium state is investigated based on the Lyapunov stability theory. The Lyapunov exponential stability of the solution to the mixed problem is proven. Based on the construction of a discrete Lyapunov function, the authors were able to transfer the results obtained to the case of a discrete problem. The case of positive characteristic velocities for the scalar case is considered.

The work [3] is devoted to the analysis of exponential stability in the Lyapunov sense of steady-state numerical solutions of a hyperbolic differential equation with nonlocal coefficients in front of the derivatives with respect to x. They obtain sufficient and necessary conditions for the exponential stability of a numerical solution of an initial-boundary difference problem with nonlocal coefficients and limited perturbation of measurement errors in boundary conditions. The case of positive characteristic velocities for the scalar case is considered.

The Articles [4-11], is devoted to the analysis of exponential stability in the Lyapunov sense of steady-state numerical solutions of a classical hyperbolic system of linear differential equations in canonical form. A mixed problem is considered for such systems. The study is based on the use of the adequacy between the stability for a mixed problem for the original hyperbolic system of linear differential equations and the stability of the initial-boundary difference problem for it. When analyzing the initial-boundary difference problems constructed in this way, the properties of logarithmic norms are used. Algorithms are proposed that make it possible to obtain sufficient conditions for the exponential stability of a numerical solution of an initial-boundary difference problem for a classical hyperbolic system of linear differential equations in canonical form. Sufficient conditions are presented in the form algebraic inequalities, which involve coefficients of boundary conditions. The results are presented in the form of an a priori estimate of the numerical solution in the norm through the norms of the functions of the initial data. Nonlocal characteristic velocities are not considered.

2. DIFFERENTIAL MIXED PROBLEM

Consider the following symmetric t-hyperbolic system:

$$\frac{\partial U}{\partial t} - \mathbf{M}(\mathcal{A}(t))\frac{\partial U}{\partial x} = 0, \quad t \in [0, +\infty),\ x \in [0, 1], \tag{2.1}$$

where

$$\begin{aligned}\mathbf{M}(\mathcal{A}(t)) &\triangleq \operatorname{diag}\left({}_1\mu({}_1a(t)), {}_2\mu({}_2a(t)), \ldots, {}_n\mu({}_na(t))\right),\\ U &\triangleq ({}_1u, {}_2u, \ldots, {}_nu)^T,\\ \mathcal{A}(t) &\triangleq ({}_1a(t), {}_2a(t), \ldots, {}_na(t))^T,\\ &{}_i\mu(s) \text{ are some specified functions.}\end{aligned}$$

Here the characteristic speeds $\mathbf{M}(\mathcal{A}(t))$ depend on the integral of the unknown vector function $U(t, x)$ over the entire region [0,1]

$$\mathcal{A}(t) = \int_0^1 U(t, x)dx, \quad t \in (0, +\infty) \tag{2.2}$$

or component by component

$${}_ia(t) = \int_0^1 {}_iu\, dx, \quad i = \overline{1, n}.$$

Initial conditions for system (2.1):

$$U(0, x) = \Phi(x), \quad x \in [0, 1], \tag{2.3}$$

Here $\Phi(x) \triangleq \left({}_1\varphi(x), {}_2\varphi(x), \ldots, {}_n\varphi(x)\right)^T$ - is the given initial vector function.

In this work, we limit ourselves to the case when the characteristic velocity functions are negative, i.e. $\mathbf{M}(\mathcal{A}(t)) > 0$. In this case, it is known from the theory of hyperbolic systems that boundary conditions for system (2.1) are required only on the right boundary, at $x = 1$:

$$-\mathbf{M}(\mathcal{A}(t))U(t, 1) = V(t), \tag{2.4}$$

where $\mathbf{V}(t) \triangleq \left({}_1\mathcal{V}(t), {}_2\mathcal{V}(t), \ldots, {}_n\mathcal{V}(t)\right)^T$ - is the vector function controller.

From works [1] and [2] it follows that with an appropriate choice of $\mathcal{M}(\mathcal{A}(t))$, $U(t, 0)$, $\mathbf{V}(t)$ it is possible to prove the correctness of the formulation of the mixed problem (2.1)-(2.4).

In this work we will consider one special case of specifying boundary conditions.

$$-\mathbf{V}(t) + \mathbf{M}^*\dot{U}^* = \mathbf{R}\left\{-\mathbf{M}(\mathcal{A}(t))\left[U(t, 0) + \Delta(t)\right] + \mathbf{M}^*\dot{U}^*\right\}, \quad t \in (0, +\infty), \tag{2.5}$$

where

$$\mathbf{M}^* \triangleq \mathbf{M}(U^*) = \operatorname{diag}\left({}_1\mu({}_1u^*), {}_2\mu({}_2u^*), \ldots, {}_n\mu({}_nu^*)\right),$$

$$U^* \triangleq ({}_1u^*, {}_2u^*, \ldots, {}_nu^*)^T, \quad \mathbf{R} \triangleq \operatorname{diag}({}_1r, {}_2r, \ldots, {}_nr),$$

$$\Delta(t) \triangleq \left({}_1\delta(t), {}_2\delta(t), \ldots, {}_n\delta(t)\right)^T.$$

and ${}_ir \in [0, 1)$, $i = \overline{1, n}$ are given coefficients, and U^* is quadratic matrix with coefficients ${}_iu^*$, where ${}_iu^* > 0$, $i = \overline{1, n}$ are given equilibrium state and $\Delta(t)$ is constrained disturbance.

Note that for a given equilibrium state U^*, the value of the characteristic vector function is calculated as follows

$$-\mathbf{M}\left(\mathcal{A}(t)\right)\Big|_{\bar{U}=\bar{U}^*} = -\mathbf{M}\left(U^*\right)$$

In this work we limit ourselves to the following family of characteristic velocities of the type

$$_i\mu(s) = \frac{_iP}{_iQ + s}, \quad s \in [0, +\infty), \quad i = \overline{1, n} \tag{2.6}$$

with $_iP > 0,\ _iQ > 0,\ \forall i \in \{m+1, m+2, \ldots, n\}$.

So, consider the following mixed control problem

$$\begin{cases} \dfrac{\partial U}{\partial t} - \mathbf{M}(\mathcal{A}(t))\dfrac{\partial U}{\partial x} = 0, & t \in [0, +\infty),\ x \in (0, 1), \\ U(0, x) = \Phi(x), & x \in (0, 1), \\ -\mathbf{V}(t) + \mathbf{M}^*\dot{U}^* = \mathbf{R}\left\{-\mathbf{M}(\mathcal{A}(t))[U(t, 0) + \Delta(t)] + \mathbf{M}^*\dot{U}^*\right\}, & t \in (0, +\infty), \\ -\mathbf{M}(\mathcal{A}(t))U(t, 1) = \mathbf{V}(t), & t \in [0, +\infty), \\ \mathcal{A}(t) = \displaystyle\int_0^1 U(t, x)dx, & t \in (0, +\infty), \end{cases} \tag{2.7}$$

where U - is the vector function to be determined.

Let's consider transformations regarding equilibrium U^*:

$$\widetilde{U}(t, x) = U(t, x) - U^*, \widetilde{A}(t) = \mathcal{A}(t) - U^*, \quad \widetilde{\Phi}(x) = \Phi(x) - U^*,$$

$$\widetilde{\mathbf{M}}_{\widetilde{A}}(t) = \mathbf{M}(U^* + \widetilde{A}(t)).$$

Then system (2.7) with (2.6) for $t \in (0, +\infty)$ can be rewritten as follows:

$$\begin{cases} \dfrac{\partial \widetilde{U}}{\partial t} - \widetilde{\mathbf{M}}_{\widetilde{A}}(t)\dfrac{\partial \widetilde{U}}{\partial x} = 0, & x \in (0, 1), \\ \widetilde{U}(0, x) = \widetilde{\Phi}(x), & x \in (0, 1), \\ \widetilde{\mathbf{V}}(t) = R\widetilde{\mathbf{M}}_{\widetilde{A}}(t)\left[\widetilde{U}(t, 0) + \Delta(t)\right] + (\mathbf{E} - \mathbf{R})\left\{\mathbf{M}^* - \widetilde{\mathbf{M}}_{\widetilde{A}}(t)\right\}U^*, & \\ \widetilde{\mathbf{M}}_{\widetilde{A}}(t) = \mathbf{M}(U^* + \widetilde{A}(t)), \widetilde{\mathbf{M}}_{\widetilde{A}}(t)\widetilde{U}(t, 1) = \widetilde{\mathbf{V}}(t) & \\ \widetilde{\mathcal{A}}(t) = \displaystyle\int_0^1 \widetilde{U}(t, x)dx \quad \text{where} \quad \int_0^1 {_i\widetilde{u}}(t, x)dx = {_iu^*}, & i = \overline{1, n}, \\ _i\mu(s) = \dfrac{_iP}{_iQ + s}, \quad \text{with } _iP > 0,\ _iQ > 0,\ s \in [0, +\infty), & i = \overline{1, n}. \end{cases} \tag{2.8}$$

Using the expressions given for functions $_i\mu,\ i = \overline{1, n}$ of characteristic velocities (2.6) in equation (2.8), we have

$$\left\{\mathbf{M}^* - \widetilde{\mathbf{M}}_{\mathcal{A}}(t)\right\}U^* = \left[\begin{array}{c} \operatorname{diag}\left(\dfrac{_1P}{_1Q + {_1u^*}}, \ldots, \dfrac{_nP}{_nQ + {_nu^*}}\right) \\ -\operatorname{diag}\left(\dfrac{_1P}{_1Q + {_1u^*} + {_1\tilde{a}(t)}}, \ldots, \dfrac{_nP}{_nQ + {_nu^*} + {_n\tilde{a}(t)}}\right) \end{array}\right]U^* \tag{2.9}$$

$$= \operatorname{diag}\left(\frac{_1P\,{_1\tilde{a}(t)}}{(_1Q + {_1u^*})(_1Q + {_1u^*} + {_1\tilde{a}(t)})}, \ldots, \frac{_nP\,{_n\tilde{a}(t)}}{(_nQ + {_nu^*})(_nQ + {_nu^*} + {_n\tilde{a}(t)})}\right)U^*$$

$$U^* = \Omega\,\widetilde{\mathbf{M}}_{\mathcal{A}}(t)\,\tilde{\mathcal{A}}(t),$$

where

$$\Omega \triangleq \operatorname{diag}\left({}_1\varpi, {}_2\varpi, \ldots, {}_n\varpi\right), \qquad {}_i\varpi = \frac{{}_iu^*}{{}_iQ + {}_iu^*}, \qquad i = \overline{1, n}$$

Note that matrix inequality $\Omega < E$ is true.

For convenience, we omit the "~" symbol. Then for $t \in (0, +\infty)$ the system in equation (2.8) with equation (2.9) can be rewritten as follows:

$$\begin{cases} \dfrac{\partial U}{\partial t} - \mathbf{M}_{\mathcal{A}}(t)\dfrac{\partial U}{\partial x} = 0, & x \in (0,1), \\ U(0,x) = \Phi(x), & x \in (0,1), \\ \mathbf{V}(t) = \mathbf{R}\mathbf{M}_{\mathcal{A}}(t)\left[U(t,0) + \Delta(t)\right] + (\mathbf{E} - \mathbf{R})\Omega\mathbf{M}_{\mathcal{A}}(t)\mathcal{A}(t), & \\ \mathbf{M}_{\mathcal{A}}(t) = \mathbf{M}(U^* + \mathcal{A}(t)), \quad \mathbf{M}_{\mathcal{A}}(t)U(t,1) = \overline{\mathbf{V}}(t), & \\ \mathcal{A}(t) = \displaystyle\int_0^1 U(t,x)\,dx, \quad \text{where} \quad \int_0^1 {}_iu(t,x)\,dx \geq -{}_iu^*, & i = \overline{1,n}, \\ {}_i\mu(s) = \dfrac{{}_iP}{{}_iQ + s}, \quad \text{with } {}_iP > 0,\ {}_iQ > 0,\ s \in [0, +\infty), & i = \overline{1,n}. \end{cases} \tag{2.10}$$

3. EXPONENTIAL STABILITY OF THE NUMERICAL SOLUTION

In this section we establish the exponential stability of the numerical solution of the initial-boundary difference problem.

To obtain the initial-boundary difference problem, we will use the upwind difference scheme for the numerical calculation of system (2.7).

To do this, we cover the spatial region $[0,1]$ using a uniform grid $\Omega_h = \left\{x_j = ih,\ j = \overline{0, J}\right\}$, h is step by x. We calculate the integral $\mathcal{A}(t)$ for each value of $t^k \triangleq k\tau$ (τ is step by time), $k \in \{0, 1, 2, \ldots\}$ using the quadrature formula

$$A^k \triangleq \left({}_1a^k, {}_2a^k, \ldots, {}_na^k\right)^T, \qquad {}_ia^k = h\sum_{j=0}^{J} {}_iu_j^k, \quad k \in \{0, 1, 2, \ldots\}. \tag{3.1}$$

Next, we define the discrete value $\mathbf{M}^k$:

$$\mathbf{M}^k \triangleq \mathbf{M}(A^k) \equiv \operatorname{diag}({}_1\mu^k, {}_2\mu^k, \ldots, {}_n\mu^k),$$

$${}_i\mu^k \triangleq \mu({}_ia^k) = \frac{{}_iP}{{}_iQ + {}_ia^k}, \qquad {}_iP > 0, \quad {}_iQ > 0, \quad i = \overline{1,n}, \quad k \in \{0, 1, 2, \ldots\}. \tag{3.2}$$

Let us assume that the Courant-Friedrichs-Levy condition is satisfied

$$0 < \Lambda^k \triangleq \frac{\tau}{h}\mathbf{M}^k \leq E, \quad k \in \{0, 1, 2, \ldots\} \tag{3.3}$$

where

$$\Lambda^k = \operatorname{diag}\left({}_1\lambda^k, {}_2\lambda^k, \ldots, {}_n\lambda^k\right), \quad {}_i\lambda^k = \frac{\tau}{h}{}_i\mu^k, \quad i = \overline{1,n}, \quad k \in \{1, 2, \ldots, K\}.$$

To numerically solve system (2.7), we propose an upwind difference scheme

$$\begin{cases} U_j^{k+1} = \left(1 - \Lambda^k\right)U_j^k + \Lambda^k U_{j+1}^k, & j = 0, \ldots, J-1; \quad k \in \{0, 1, \ldots\}, \\ U_J^{k+1} = RU_0^{k+1} + (E - R)\left(\mathbf{M}^k\right)^{-1}\mathbf{M}^*U^* + R\Delta^{k+1}, & k \in \{0, 1, \ldots\}, \\ U_j^0 = \Phi(x_j), & j = 0, \ldots, J. \end{cases} \tag{3.4}$$

$$U_j^k = \left({}_1u_j^k, {}_2u_j^k, \ldots, {}_nu_j^k\right)^T, \quad \Delta^k \triangleq \left({}_1\delta^k, {}_2\delta^k, \ldots, {}_n\delta^k\right)^T$$

Let us introduce the following matrices:

$$U^k \triangleq \operatorname{diag}\left({}_1u_0^k, {}_2u_0^k, \ldots, {}_nu_0^k, {}_1u_1^k, {}_2u_1^k, \ldots, {}_nu_1^k, \ldots\ldots, {}_1u_{J-1}^k, {}_2u_{J-1}^k \ldots, {}_nu_{J-1}^k\right),$$

$$U^0 \triangleq \operatorname{diag}\left({}_1\varphi_0, {}_2\varphi_0, \dots, {}_n\varphi_0, {}_1\varphi_1, {}_2\varphi_1, \dots, {}_n\varphi_1, \dots\dots, {}_1\varphi_{J-1}, {}_2\varphi_{J-1}, \dots, {}_n\varphi_{J-1}\right),$$

$$U^* \triangleq \operatorname{diag}\left(\begin{pmatrix} {}_1u^*, {}_2u^*, \dots, {}_nu^* \\ {}_1u^*, {}_2u^*, \dots, {}_nu^* \\ \vdots \\ {}_1u^*, {}_2u^*, \dots, {}_nu^* \end{pmatrix}_{n\times J}\right), \qquad \Delta^k \triangleq \operatorname{diag}\left(\begin{pmatrix} {}_1\delta^k, 0, 0, \dots, 0 \\ 0, {}_2\delta^k, 0, \dots, 0 \\ \vdots \\ 0, 0, 0 \dots, {}_n\delta^k \end{pmatrix}_{n\times J}\right).$$

Definition 3.1. Let $\Xi > 0$. Equilibrium state U^* of the initial-boundary difference problem (3.4) is stable in the l^2-norm with respect to discrete perturbations that satisfy matrix inequalities $\Delta^k \le \Xi$, $k \in \{1, 2, \dots\}$, if there exist positive real constants $\zeta_1 > 0$, $\zeta_2 > 0$, $\zeta_3 > 0$ such that for any initial condition $\Phi(x_j)$, $j = \overline{0, J}$, solution U_j^k, $k \in \{1, 2, \dots\}$, $j = \overline{0, J}$ of the initial-boundary difference problem (3.4) satisfies the inequality

$$\|U^k - U^*\|_{l^2} \le \zeta_2 e^{-\zeta_1 t^k} \|\Phi - U^*\|_{l^2} + \zeta_3 \max_{0\le s<k} (|\Delta^s|), \quad k \in \{1, 2, \dots\}, \tag{3.5}$$

where

$$U^k \triangleq \begin{pmatrix} U_0^k \\ U_1^k \\ \vdots \\ U_{J-1}^k \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Phi(x_0) \\ \Phi(x_1) \\ \vdots \\ \Phi(x_{J-1}) \end{pmatrix}, \quad U^* \triangleq \begin{pmatrix} U_0^* \\ U_1^* \\ \vdots \\ U_{n-1}^* \end{pmatrix}_{n\times J}, \quad |\Delta^s| = \max_{1\le i\le n} |{}_i\delta^s|.$$

and

$$\|U^k - U^*\|_{l^2}^2 \triangleq h \sum_{j=0}^{J-1} \left([U_j^k - U^*], [U_j^k - U^*]\right), \quad k \in \{0, 1, \dots\}.$$

Definition 3.2. (Discrete Lyapunov function). That function $\mathbf{L} : \mathbb{R}^{n\times J} \to \mathbb{R}_0^+$ is a discrete Lyapunov function for the initial-boundary difference problem (3.4) if:

(1) there are positive constants $\chi_1 > 0$ and $\chi_2 > 0$ such that for all $k \in \{0, 1, \dots\}$

$$\chi_1 \|U^k - U^*\|_{l^2}^2 \le \mathbf{L}(U^k) \le \chi_2 \|U^k - U^*\|_{l^2}^2, \tag{3.6}$$

(2) there are positive constants $\eta > 0$ and $\nu > 0$ such that for all $k \in \{0, 1, \dots\}$

$$\frac{\mathbf{L}(U^{k+1}) - \mathbf{L}(U^k)}{\Delta t} \le -\eta \mathbf{L}(U^k) + \nu\left(\Delta^k, \Delta^k\right).$$

To simplify the notation, in what follows we will define the sequence of discrete values $\mathcal{L}^k$ as

$$\mathcal{L}^k = \mathcal{L}(U^k), \quad k \in \{0, 1, \dots\}$$

and where U^k is a given solution of the initial-boundary difference problem (3.4).

Theorem 3.1. (Discrete stability for the case $U^* \ge 0$). Assume that the CFL condition (3.3) is satisfied. Let $\Xi \ge 0$. For each U^* satisfying the matrix inequality $U^* \ge 0$, each R satisfying the matrix inequality $0 \le R < E$, each $u > 0$, and for any initial vector function Φ satisfying the matrix inequality with $U^0 \ge 0$, and

$$\|\Phi - U^*\|_{l^2} < u \tag{3.7}$$

the solution U^k of the initial-boundary value difference problem (3.4) satisfies the matrix inequalities $U^k \ge 0, \quad k \in \{0, 1, \dots\}$, and the stationary state U^* of the initial-boundary value difference problem (3.4) is stable in the norm is l^2 with respect to any discrete perturbation function Δ^k, $k \in \{0, 1, \dots\}$, such that the matrix inequality $\Delta^k \le \Xi$ holds.

To analyze the stability of the initial-boundary difference problem (3.4) using the discrete Lyapunov method, we use the following transformation:

$$\widetilde{U}_j^k = U_j^k - U^*, \qquad \widetilde{A}^k = h \sum_{j=0}^{J-1} \widetilde{U}_j^k, \qquad \widetilde{\mathbf{M}}_{\widetilde{A}^k}^k = \mathbf{M}(U^* + \widetilde{A}^k), \qquad \widetilde{\Lambda}^k = \frac{\tau}{h} \widetilde{\mathbf{M}}_{\widetilde{A}^k}^k, \qquad k \in \{0, 1, \dots\},$$

$$\mathbf{M}(U^* + \widetilde{A}^k) \equiv \operatorname{diag}\left(\frac{{}_1P}{{}_1Q + {}_1u^* + {}_1a^k}, \frac{{}_2P}{{}_2Q + {}_2u^* + {}_2a^k}, \ldots, \frac{{}_nP}{{}_nQ + {}_nu^* + {}_na^k}\right) \tag{3.8}$$

For simplicity, we omit the symbol "~" in notation (3.8) and discretize system (2.10) as follows

$$\begin{cases} U_j^{k+1} = \left(1 - \Lambda^k\right) U_j^k + \Lambda^k U_{j+1}^k, j = \overline{0, J-1}; \quad k \in \{0, 1, \ldots\}; \\ U_J^{k+1} = R U_0^{k+1} + (E - R)\,\Theta A^{k+1} + R\Delta^{k+1}, k \in \{0, 1, \ldots\}; \\ c\,\Theta = \operatorname{diag}\left(\frac{{}_1u^*}{{}_1Q + {}_1u^*}, \frac{{}_2u^*}{{}_2Q + {}_2u^*}, \ldots, \frac{{}_nu^*}{{}_nQ + {}_nu^*}\right); \\ A^k = h\sum_{j=0}^{J-1} U_j^k, \quad M_{A^k}^k = \mathrm{M}(U^* + A^k), \quad \Lambda^k = \frac{\tau}{h} M_{A^k}^k, k \in \{0, 1, \ldots\}; \\ \mathrm{M}(U^* + A^k) \equiv \operatorname{diag}\left(\frac{{}_1P}{{}_1Q + {}_1u^* + {}_1a^k}, \frac{{}_2P}{{}_2Q + {}_2u^* + {}_2a^k}, \ldots, \frac{{}_nP}{{}_nQ + {}_nu^* + {}_na^k}\right); \\ {}_ia^k = h\sum_{j=0}^{J-1} {}_iu_j^k \geq -{}_iu^*, i = \overline{1, n}, \quad k \in \{0, 1, \ldots\}; \\ U_j^0 = \Phi(x_j), \quad j = \overline{0, J}. \end{cases} \tag{3.9}$$

Thus, the assumption in the form of inequality (3.7) in Theorem 1 is now expressed as

$$\|\Phi\|_{\ell^2} < U. \tag{3.10}$$

Note that it can be rewritten in the form

$$\left\|U^k\right\|_{\ell^2} \leq \zeta_2 e^{-\zeta_1 t^k} \|\Phi\|_{\ell^2} + \zeta_3 \max_{0 \leq s < k} \left(|\Delta^s|\right), \quad k \in \{1, 2, \ldots\}.$$

4. COMPUTATIONAL EXPERIMENT

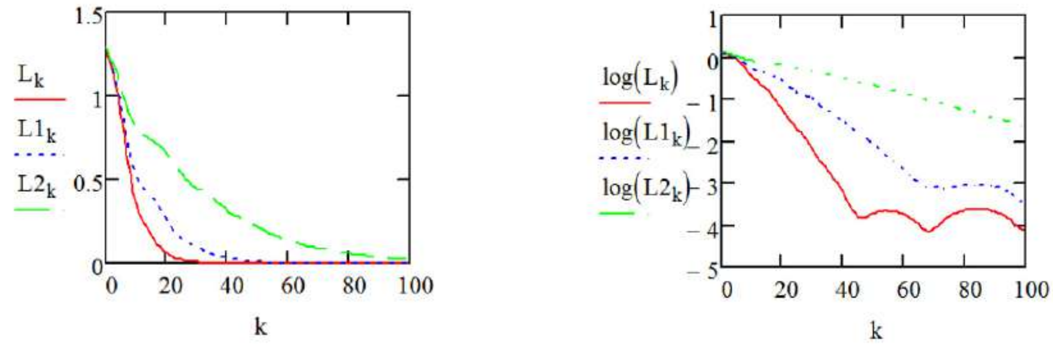
Option. Consider the influence of the parameter r on the numerical solution.

To carry out the computational experiment, the following data were entered: Time $T = 10$ seconds, time step $\tau = 0.1$, spatial domain $[0, 1]$ divided into a grid with step $h = 0.1$. Parameters $P = 1$, $Q = 1$, equilibrium solution $u^* = 0$, limited (known) disturbance $\delta(t) = 2.4 \cdot 10^{-3} \cdot \sin(t)$. Initial condition $u(x) = 1.1 + \sin(2\pi x)$.

Norm of the numerical solution for the case when $r = 0.1$; $r = 0.3$; $r = 0.6$ is shown in Fig. 1. Graph of the numerical solution when $r = 0.1$; $r = 0.3$; $r = 0.6$ is shown in Fig. 2. As can be seen from the graph, the smaller the parameter r, the faster the numerical solution reaches an equilibrium solution.

CONCLUSION

The *Lyapunov* exponential stability of stationary numerical solutions of a mixed problem for a hyperbolic system of linear differential equations with nonlocal matrix coefficients of derivatives with respect to x is studied. In this case, the boundary conditions take into account a limited disturbance of measurement errors. A methodology is proposed that makes it possible to obtain sufficient conditions for the exponential stability of a numerical solution of an initial-boundary difference problem with nonlocal coefficients and limited perturbation of measurement errors in boundary conditions. Sufficient conditions are presented in the form of matrix inequalities, including matrices of boundary conditions. The results are presented in the form of an a priori estimate of the numerical solution in the ℓ^2-norm through the ℓ^2-norm of the functions of the initial data and the norms of perturbation of measurement errors. The proposed method can be used to study the influence of nonlocal coefficients of the original system and perturbation of measurement errors on the exponential stability of the numerical solution of the initial-boundary difference problem for hyperbolic systems. A computational experiment was carried out that confirmed the theoretical results obtained.



(a) *norm of the numerical solution*

(b) *The logarithm of the norm of the numerical solution*

FIGURE 1. Graph of the norm of the numerical solution. Here k is time step, L_k is the norm of the numerical solution at $r = 0.1$; $L1_k$ is the norm of numerical solution at $r = 0.3$; $L2_k$ is the norm of numerical solution at $r = 0.6$.

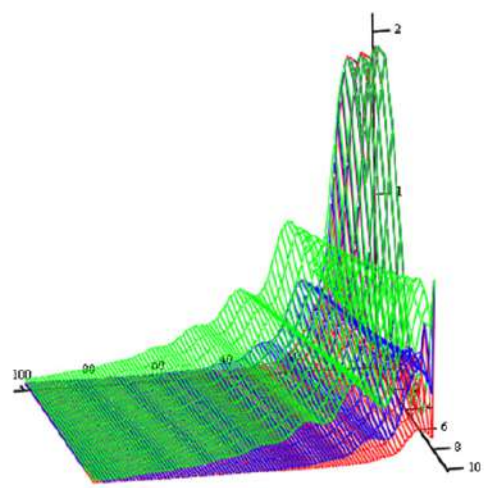


FIGURE 2. Numerical solution graph. Here the graph of the numerical solution at $r = 0.1$ is shown in red; blue shows the graph of the numerical solution at $r = 0.3$; Green shows the graph of the numerical solution at $r = 0.6$

REFERENCES

[1] J. M. Coron; Z. Wang Output Feedback Stabilization for a Scalar Conservation Law with a Nonlocal Velocity. SIAM J. Math. Anal. 2013, 45, 2646-2665, doi:10.1137/120902203.

[2] W. Chen; C. Liu; Z. Wang, Global Feedback Stabilization for a Class of Nonlocal Transport Equations: The Continuous and Discrete Case. SIAM J. Control Optim. 2017, 55, 760-784, doi:10.1137/15m1048914.

[3] Simone Gttlich, Michael Herty and Gediyon Weldegiyorgis. Input-to-State Stability of a Scalar Conservation Law with Nonlocal Velocity. Axioms 2021, 10, 12. https://doi.org/10.3390/axioms10010013.

[4] R. Aloev, A. Berdyshev, D. Bliyeva, S. Dadabayev, Z. Baishemirov, Stability Analysis of an Upwind Difference Splitting Scheme for Two-Dimensional SaintVenant Equations. Symmetry, 2022-09, journal-article DOI: 10.3390/sym14101986. Source: Multidisciplinary Digital Publishing Institute.

[5] R.D.Aloev, S.U. Dadabaev, Stability of the upwind difference splitting scheme for symmetric t-hyperbolic systems with constant coefficients. Results in Applied Mathematics. 2022 | journal-article. DOI: 10.1016/j.rinam.2022.100298. EID: 2-s2.0-85131461551. Part of ISSN: 25900374.

[6] R. D. Aloev, M. U. Hudayberganov, A Discrete Analogue of the Lyapunov Function for Hyperbolic Systems. Journal of Mathematical Sciences (United States).2022, journal-article.DOI: 10.1007/s10958-022-06028-y. EID: 2-s2.0-85135683283. Part of ISSN: 15738795 10723374.

[7] R. D. Aloev, Z. K. Eshkuvatov, M. U. Khudoyberganov, D. E. Nematova, The Difference Splitting Scheme for n-Dimensional Hyperbolic Systems. Malaysian Journal of Mathematical Sciences. 2022 | journal-article. EID: 2-s2.0-85130020938. Part of ISSN: 18238343.

[8] R. Aloev, A. Berdyshev, A. Akbarova, Z. Baishemirov, Development of an algorithm for calculating stable solutions of the saint-venant equation using an upwind implicit difference scheme. Eastern-European Journal of Enterprise Technologies 2021 | journal-article. DOI: 10.15587/1729-4061.2021.239148. EID: 2-s2.0-85116525899. Part of ISSN: 17294061 17293774.

[9] R. D. Aloev, Z. K. Eshkuvatov, M. U. Khudoyberganov, D. E. Nematova, The difference splitting scheme for hyperbolic systems with variable coefficients. Mathematics and Statistics 2019 | journal-article. DOI: 10.13189/ms.2019.070305. EID: 2-s2.0-85071017777.

[10] R.D.Aloev, M.Khudayberganov, A.Blokhin. Construction and research of adequate computational models for quasilinear hyperbolic systems. Numerical Algebra, Control and Optimization 2018 | journal-article. DOI: 10.3934/naco.2018017. EID: 2-s2.0-85056814686.

[11] R. D. Aloev, Z. K. Eshkuvatov, M. U. Khudoyberganov, and D. E. Nematova, The Difference Splitting Scheme for n-Dimensional Hyperbolic Systems. Malaysian Journal of Mathematical Sciences 16(1): 110 (2022) https://doi.org/10.47836/mjms.16.1.0

Rakhmatillo Aloev
Department of Computational Mathematics
and Information Systems,
National University of Uzbekistan,
Tashkent, Uzbekistan
email: aloevr@mail.ru

Abdumauvlen Berdyshev
Department of Mathematics and Mathematical
Modelling, Institute of Mathematics,
Physics and Informatics,
Abai Kazakh National Pedagogical University,
Almaty, Kazakhstan
email: berdyshev@mail.ru

Vasila Alimova,
Department of Computational Mathematics
and Information Systems,
National University of Uzbekistan,
Tashkent, Uzbekistan
email:vasilarobiyaxon@gmail.com.

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.15-21
DOI: 10.29229/uzmj.2025-4-2

Optimal approximation techniques for solving Hadamard type integral equations

Akhmedov D.M., Pirnapasov S.O.

Abstract. This paper discusses the frequent occurrence of the Hadamard type integral equation of the first kind in solving problems in aerodynamics, fluid dynamics, wave propagation theory, and ecology. Since the exact solution of this equation is also expressed by an integral, the problem of constructing an optimal quadrature formula for its approximate calculation is considered. The analytical form of the coefficients of the optimal quadrature formula is found.

Keywords: Hadamard type integral eqaution, Cauchy type singular integral, quadrature formula, error functional, extremal function, optimal coefficients.

Mathematics Subject Classification (2020): 65D32

1. Introduction

Hypersingular integral equations are a type of integral equations where the kernel the function inside the integral is highly "singular," meaning it has an infinitely large value at a certain point. This singularity is of a higher order than what's found in more common singular integral equations. These equations are a powerful tool for analyzing complex, three-dimensional problems in various fields, including:

Aerodynamics and Fluid Dynamics: Studying how air and fluids move around objects; Elasticity: Analyzing how materials deform under stress; Wave Theory: Understanding the diffraction of electromagnetic and acoustic waves; Ecology: Modeling certain environmental systems;

Often, hypersingular integral equations are derived from Neumann boundary value problems. These are mathematical problems for equations like the Laplace or Helmholtz equations, where you're given the values of the function's derivative on the boundary of a region, not the function's value itself. The process of converting these boundary value problems into integral equations involves using a concept called the double-layer potential.

We consider the following hypersingular integral equation of the first kind

$$\int_{-1}^{1} \frac{g(x)}{(x-t)^2} dx = \varphi(t), \tag{1.1}$$

here the functions $g(x)$ and $\varphi(x)$ satisfy the Hölder condition (or belongs to the class H), $-1 < t < 1$. We integrate the left side of integral equation (1.1) by parts and obtain the following

$$\frac{g(-1)}{-1-t} + \frac{g(1)}{1-t} + \int_{-1}^{1} \frac{g'(x)}{(x-t)} dx = \varphi(t). \tag{1.2}$$

For integral equation (1.1) or (1.2) to have a unique solution in the given class, the value of the function $g(x)$ at the boundaries of the segment must be equal to zero, i.e.

$$g(1) = g(-1) = 0$$

or

$$\int_{-1}^{1} g'(x) dx = 0. \tag{1.3}$$

We introduce denotation $g'(x) = \rho(x)$, and get

$$\int_{-1}^{1} \frac{\rho(x)}{x-t} dx = \varphi(t), \quad -1 < t < 1. \tag{1.4}$$

The unique exact analytical solution of integral equation (1.4) that satisfies condition (1.3) is equal to the following [16]

$$\rho(t) = -\frac{1}{\pi^2\sqrt{1-t^2}}\int_{-1}^{1}\frac{\sqrt{1-x^2}\varphi(x)}{x-t}dx \tag{1.5}$$

The function under this integral (1.5) is very complex in practical problems. As a result, it is impossible to find the antiderivative of this function. Therefore, to approximate such singular integral, scientists have used the following methods: the discrete vortex method[15, 10], interpolation methods [14, 11, 12, 5], the piecewise interpolation method [6, 7, 9] and other numerical quadrature methods [8, 13]. These methods are not an optimal approximation techniques. In the present paper, we construct an optimal quadrature formula for approximation of the integral (1.5) in the space $L_2^{(2)}(-1,1)$.

2. Statement of the problem

We consider the following quadrature formula

$$\int\limits_{-1}^{1}\frac{\sqrt{1-x^2}\varphi(x)}{x-t}dx \cong \sum_{\beta=0}^{N}\left(C_0[\beta]\varphi(x_\beta)+C_1[\beta]\varphi'(x_\beta)\right), -1<t<1, \tag{2.1}$$

in the Sobolev space $L_2^{(2)}(-1,1)$. This space is a Hilbert space of classes of all real valued functions φ defined in the interval $[-1,1]$ that differ by a linear polynomial of degree second and square integrable with the second order derivative. Here $C_0[\beta]$, $C_1[\beta]$ are the coefficients, x_β are the nodes of the quadrature formula, N is a natural number.

The following difference is called *the error* of quadrature formula (2.1):

$$(\ell,\varphi) = \int\limits_{-1}^{1}\frac{\sqrt{1-x^2}\varphi(x)}{x-t}dx - \sum_{\beta=0}^{N}C_0[\beta]\varphi(x_\beta) - \sum_{\beta=0}^{N}C_1[\beta]\varphi'(x_\beta) = \int\limits_{-\infty}^{\infty}\ell(x)\varphi(x)dx,$$

where

$$\ell_1(x) = \frac{\sqrt{1-x^2}\varepsilon_{[-1,1]}(x)}{x-t} - \sum_{\beta=0}^{N}C_0[\beta]\delta(x-x_\beta) + \sum_{\beta=0}^{N}C_1[\beta]\delta'(x-x_\beta), \tag{2.2}$$

$\varepsilon_{[-1,1]}(x)$ is the indicator of the interval $[-1,1]$, δ is the Dirac delta-function, $\ell(x)$ is the error functional of quadrature formula (2.1).

It is important to note that the coefficients are determined and are given by the following [14, 15, 16].

Theorem 2.1. *Among all quadrature formulas of the form*

$$\int\limits_{-1}^{1}\frac{\sqrt{1-x^2}\varphi(x)}{x-t}\,\mathrm{d}x \cong \sum_{\beta=0}^{N}C_0[\beta]\varphi(x_\beta), -1<t<1,$$

with the error functional

$$\ell(x) = \frac{\sqrt{1-x^2}\varepsilon_{[-1,1]}(x)}{x-t} - \sum_{\beta=0}^{N}C_0[\beta]\delta(x-x_\beta),$$

in the space $L_2^{(1)}(-1,1)$, there is a unique quadrature formula whose coefficients are defined by the equalities

$$C_0[0] = h^{-1}\left[f_0(h) + \frac{\pi}{4}\left(2t^2+2t-1\right) - \frac{h}{2}\pi t\right],$$

$$C_0[\beta] = h^{-1}\left[f_0(h\beta-h) - 2f_0(h\beta) + f_0(h\beta+h)\right], \beta=\overline{1,N-1},$$

$$C_0[N] = h^{-1}[f_0(1-h) - \frac{\pi}{4}\left(2t^2 - 2t - 1\right) - \frac{h}{2}\pi t],$$

$$f_0(h\beta) = \int_{-1}^{1} \frac{\sqrt{1-x^2}|x-h\beta+1|}{2(x-t)}\,\mathrm{d}x = \left(\frac{h\beta-1}{2} - t\right)\sqrt{1-(h\beta-1)^2} + \left(t^2 - t(h\beta-1) - \frac{1}{2}\right)\mathrm{arcsin(h\beta-1)}$$

$$+ (t-(h\beta-1))\sqrt{1-t^2}\ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|. \tag{2.3}$$

Since functional (2.2) defined in the space $L_2^{(2)}(-1,1)$ in [17], then we have

$$(\ell, x) = 0. \tag{2.4}$$

The main aim of the present paper is to construct optimal quadrature formulas in the sense of Sard of the form (2.1) in the space $L_2^{(2)}(-1,1)$ by the Sobolev method for approximate integration of the Cauchy type singular integral. This means to find the coefficients $C_1[\beta]$ which satisfy the following variation problem

$$\|\mathring{\ell}|L_2^{(2)*}\| := \inf_{C_1[\beta]} \|\ell|L_2^{(2)*}\|. \tag{2.5}$$

Thus, in order to construct optimal quadrature formulas of the form (2.1) in the sense of Sard we have to consequently solve the following problems.

Problem 2.1. Find the norm of error functional (2.2) of quadrature formula (2.1) in the space $L_2^{(2)*}(-1,1)$.

Problem 2.2. Find the coefficients $C_1[\beta]$ which satisfy equality (1.4).

In the works [1, 18, 19] for the norm of the error functional the following form was obtained

$$\begin{aligned}\|\ell|L_2^{(2)*}(-1,1)\|^2 \; = \; & \sum_{k=0}^{1}\sum_{\alpha=0}^{1}\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N}(-1)^k C_k[\gamma]C_\alpha[\beta]\frac{(h\beta-h\gamma)^{3-\alpha-k}\mathtt{sign}(h\beta-h\gamma)}{2(3-\alpha-k)!} - \\ & -2\sum_{\alpha=0}^{1}\sum_{\beta=0}^{N}(-1)^\alpha C_\alpha[\beta]\int_{-1}^{1}\frac{\sqrt{1-x^2}(x-(h\beta-1))^{3-\alpha}\mathtt{sign}(x-(h\beta-1))}{2(3-\alpha)!(x-t)}dx + \\ & + \int_{-1}^{1}\int_{-1}^{1}\frac{\sqrt{1-x^2}\sqrt{1-y^2}(x-y)^3\mathtt{sign}(x-y)}{12(x-t)(y-t)}dxdy\end{aligned} \tag{2.6}$$

Thus, Problem 2.1 is solved for quadrature formulas of the form (2.1) in the space $L_2^{(2)}(-1,1)$.

3. The main results

Now we turn to minimizing the norm (2.5) of the error functional for the quadrature formulas with the orthogonality condition (2.4).

Here, we use the $C_0[\beta]$ coefficients of Theorem 2.1 and substitute them into expression (2.5). We then minimize $\|\ell\|^2$ with respect to the $C_1[\beta]$ based on condition (2.4). In order to do this, we apply the Lagrange method.

We denote $\mathbf{C} = (C_1[0], C_1[1], ..., C_1[N])$ and λ_1. Consider the function

$$\Phi(C,\lambda) = \|\ell|L_2^{(2)*}(-1,1)\|^2 - 2\lambda_1(\ell, x).$$

We obtain

$$\sum_{\gamma=0}^{N} C_1[\gamma]\frac{|h\beta-h\gamma|}{2} - \lambda_1 = F_1(h\beta), \quad \beta = 0,1,2,...,N, \tag{3.1}$$

$$\sum_{\gamma=0}^{N} C_1[\gamma] = g_1 - \sum_{\gamma=0}^{N} C_0[\gamma](h\gamma-1), \tag{3.2}$$

where

$$
\begin{aligned}
F_1(h\beta) &= -f_1(h\beta) + \sum_{\gamma=0}^{N} C_0[\gamma]\frac{|h\beta - h\gamma|^2}{4}, \\
f_1(h\beta) &= \int_{-1}^{1} \frac{\sqrt{1-x^2}|x-h\beta+1|^2}{4(x-t)}dx \\
&= -\frac{1}{2}\Bigg[\frac{1}{6}\left(2(h\beta)^2 - h\beta(4+9t) + 9t + 6t^2\right)\sqrt{1-(h\beta-1)^2} \qquad (3.3)\\
&+\left(\frac{1}{2}(t-2(h\beta-1)) - t(t-(h\beta-1))^2\right)\mathtt{arcsin}(h\beta-1) - \\
&-(t-(h\beta-1))^2\sqrt{1-t^2}ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|\Bigg],
\end{aligned}
$$

$$
g_1 = \int_{-1}^{1} \frac{\sqrt{1-x^2}x}{x-t}dx = \frac{\pi}{2}(1-2t^2). \qquad (3.4)
$$

The uniqueness of the solution of such type of systems was discussed in [17, 20].

We give the algorithm for solution of system (3.1)-(3.2) when $x_\beta = h\beta - 1$, $h = \frac{2}{N}$, N is a natural number. Here we use similar method suggested by S.L. Sobolev [17] for finding the coefficients of optimal quadrature formulas in the Sobolev space $L_2^{(2)}(-1,1)$.

Suppose that $C_1[\beta] = 0$ when $\beta < 0$ and $\beta > N$. Using the definition of convolution, we rewrite system (3.1)-(3.2) in the following form:

$$
G_1(h\beta) * C_1[\beta] + \lambda_1 = F_1(h\beta), \quad \beta = 0, 1, ..., N, \qquad (3.5)
$$

$$
\sum_{\beta=0}^{N} C_1[\beta] = g_1 - \sum_{\gamma=0}^{N} C_0[\gamma](h\gamma - 1), \qquad (3.6)
$$

where

$$
G_1(h\beta) = \frac{(h\beta)\mathtt{sgn}(h\beta)}{2},
$$

λ_1 is an arbitrary constant, $\mathtt{sgn}(h\beta)$ is the signum function.

Thus we have the following problem.

Problem 3.3. Find the discrete function $C_1[\beta]$ and constant λ_1 which satisfy the system (3.5)-(3.6). Further we investigate Problem 3.3. Instead of $C_1[\beta]$ we introduce the following functions

$$
\begin{aligned}
v(h\beta) &= G_1(h\beta) * C_1[\beta], \qquad (3.7)\\
u(h\beta) &= v(h\beta) + \lambda_1. \qquad (3.8)
\end{aligned}
$$

In such statement the coefficients $C_1[\beta]$ are expressed by the function $u(h\beta)$, i.e. taking into account

$$
hD_1(h\beta) * G_1(h\beta) = \delta(h\beta),
$$

where

$$
D_1(h\beta) = \begin{cases} 0, & |\beta| \geq 2, \\ h^{-2}, & |\beta| = 1, \\ -2h^{-2}, & \beta = 0. \end{cases} \qquad (3.9)
$$

There are (3.9) and (3.8) , for the coefficients we have

$$
C_1[\beta] = hD_1(h\beta) * u(h\beta). \qquad (3.10)
$$

Thus, if we find the function $u(h\beta)$, then the coefficients $C_1[\beta]$ can be found from equality (3.10). To calculate the convolution (3.10) it is required to find the representation of the function $u(h\beta)$ for all integer values of β. From equality (3.5) we get that $u(h\beta) = F_1(h\beta)$ when $h\beta - 1 \in [-1, 1]$, i.e. $\beta = 0, 1, ..., N$. Now we need to find the representation of the function $u(h\beta)$ when $\beta < 0$ and $\beta > N$.

Since $C_1[\beta] = 0$ when $h\beta \notin [-1, 1]$ then $C[\beta] = hD_1(h\beta) * u(h\beta) = 0, \;\; h\beta \notin [-1, 1]$.

Now we calculate the convolution $v(h\beta) = G_1(h\beta) * C_1[\beta]$ when $\beta \leq 0$ and $\beta \geq N$. Suppose $\beta \leq 0$, then taking into account that $G_1(h\beta) = \frac{|h\beta|}{2}$ and equality (3.6), we have

$$v(h\beta) = -\frac{1}{2}(h\beta)g_1 - p_1 + \lambda_1, \tag{3.11}$$

here $p_1 = \frac{1}{2}\sum\limits_{\gamma=0}^{N} C_0[\gamma](h\gamma)$.

Similarly, in the case $\beta \geq N$ for the convolution $v(h\beta) = G_1(h\beta) * C[\beta]$ we obtain

$$v(h\beta) = \frac{1}{2}(h\beta)g_1 + p_1 + \lambda_1. \tag{3.12}$$

Then we denote

$$a_1^- = -p_1 + \lambda_1 \tag{3.13}$$

$$a_0^+ = p_1 + \lambda_1. \tag{3.14}$$

Taking into account (3.8), (3.11) and (3.12) we get the following problem

Problem 3.4. Find the solution of the equation

$$hD_1(h\beta) * u(h\beta) = 0, \qquad h\beta \notin [-1, 1] \tag{3.15}$$

having the form:

$$u(h\beta) = \begin{cases} -\frac{1}{2}(h\beta)g_1 + a_0^-, & \beta < 0, \\ F_1(h\beta), & 0 \leq \beta \leq N, \\ \frac{1}{2}(h\beta)g_1 + a_1^+, & \beta > N. \end{cases} \tag{3.16}$$

Here a_1^- and a_1^+ are unknown constants.

If we find a_1^- and a_1^+ then from (3.13), (3.14) we have

$$\lambda_1 = \frac{1}{2}(a_1^- + a_1^+), \tag{3.17}$$

$$p_1 = \frac{1}{2}(a_1^+ - a_1^-). \tag{3.18}$$

Unknowns a_1^- and a_1^+ can be found from equation (3.15), using the function $D_1(h\beta)$ defined by (3.9). Then we obtain explicit form of the function $u(h\beta)$ and from (3.10) we find the coefficients $C_1[\beta]$. Furthermore, from (3.17) we get λ_1.

Thus, Problem 3.4 and respectively Problems 3.3 will be solved.

Then, using the above algorithm, we obtain explicit formulas for coefficients of the optimal quadrature formula (2.1). It should be noted that the quadrature formula (2.1) is exact for linear function.

The following holds

Theorem 3.1. *Coefficients of the optimal quadrature formulas (2.1), with equally spaced nodes in the space $L_2^{(2)}(-1, 1)$, have the following form*

$$\begin{aligned}
C_1[0] &= h^{-1}\Big(f_1(h) + \tfrac{h}{2}f_0(h) - \tfrac{h}{8}\pi(2t^2 + 2t - 1) + \tfrac{\pi}{8}(2t^3 + 4t^2 + t - 2)\Big), \\
C_1[\beta] &= h^{-1}\Big(f_1(h\beta - h) - 2f_1(h\beta) + f_1(h\beta + h) - 2\Big(f_0(h\beta - h) - 2f_0(h\beta) + \\
&\quad + f_0(h\beta + h)\Big) + h^2\sum_{\gamma=0}^{\beta} C_0[\gamma] + \tfrac{h^2}{2}\pi t\Big), \quad \beta = 1, 2, ..., N-1, \\
C_1[N] &= h^{-1}\Big(f_1(1 - h) - \tfrac{h}{2}f_0(1 - h) - \tfrac{h}{8}\pi(2t^2 - 2t - 1) - \tfrac{\pi}{8}(2t^3 - 4t^2 + t + 2)\Big),
\end{aligned}$$

where f_0, f_1 *are defined by* (2.3), (3.3), *respectively.*

Proof. From (3.16) with $\beta = 0$ and $\beta = N$ we immediately obtain (3.16), i.e.

$$a_1^- = f_1(0), \tag{3.19}$$

$$a_1^+ = f_1(2) - g_1. \tag{3.20}$$

This means that we have obtained an explicit form of the function $u(h\beta)$.

Further, using (3.9) and (3.16), from (3.10) calculating the convolution $hD_1(h\beta) * u(h\beta)$ for $\beta = \overline{0, N}$, respectively, we obtain results of the theorem. Theorem (3.1) is proved. □

Remark 3.1. So, the approximate calculation of equality (1.5) is as follows

$$\rho(t) \cong -\frac{1}{\pi^2\sqrt{1-t^2}} \sum_{\beta=0}^{N} \left(C_0[\beta]\varphi(x_\beta) + C_1[\beta]\varphi'(x_\beta)\right).$$

4. Conclusion

In the introduction of the article, several problems in various fields were listed. It was stated that the solutions to these problems lead to first-kind hypersingular integral equations. Since there are no exact analytical methods to solve these types of integral equations, approximate solution methods have been proposed. In this work, we have provided a new method to partially overcome the shortcomings of the proposed methods. That is, an optimal quadrature formula was constructed to approximate singular integrals with high accuracy, and its analytical form was found

References

[1] D. M. Akhmedov, A. R. Hayotov, Kh. M. Shadimetov *Optimal quadrature formulas with derivatives for Cauchy type singular integrals* Applied Mathematics and Computation (2018) vol.317, pp.150-159.

[2] D.M. Akhmedov, G.A. Abdikayumova *Construction of optimal quadrature formulas with derivatives for Cauchy type singular integrals in the Sobolev space.* Uzbek Mathematical Journal, 2020, Volume 64, Issue 1, pp.4-9, DOI: 10.29229/uzmj.2020-1-1

[3] D.M.Akhmedov, B.M. Atamuradova *Construction of optimal quadrature formulas for Cauchy type singular integrals in the* $W_2^{(1,0)}(01)$ *space.* Uzbek Mathematical Journal, 2022, Volume 66, Issue 2, pp. 5-9, DOI:10.29229/uzmj.2022-2-1

[4] D.M. Akhmedov *Optimal approximation of the Hadamard hypersingular integrals. Uzbek Mathematical Journal,* 2023, Volume 67, Issue 3, pp. 5-12, DOI: 10.29229/uzmj.2023-3-1

[5] I.Boykov, V.Roudnev and A.Boykova, *Approximate methods for solving linear and nonlinear hypersingular integral equations*, Axioms. MDPI. **9(74)** (2020), doi:10.3390/axioms9030074.

[6] C. Dagnino, E. Santi, *Spline product quadrature rules for Cauchy singular integrals*, J. Comput. Appl. Math. **33** (1990), 133–140.

[7] C. Dagnino, P. Lamberti, *Numerical evalution of Cauchy principal value integrals based on local spline approximation operators*, J. Comput. Appl. Math. **76** (1996), 231–238.

[8] K. Diethelm, *Gaussian quadrature formulas of the fisrt kind for Cauchy principal value integrals: basic properties an error estimates*, J. Comput. Appl. Math. **65** (1995), 97–114.

[9] K. Diethelm, *Error bounds for spline-based quadrature methods for strongly singular integrals*, J. Comput. Appl. Math. **89** (1998), 257-261.

[10] S.A. Dovgy, I.K. Lifanov and D.I. Cherniy, *Method of singular integral equations and computing technologies*, Yuston, Kyiv, 2016 (in Russian).

[11] Z.K. Eshkuvatov, N.M.A. Nik Long and M. Abdulkawi, *Quadrature formula for approximating the singular integral of Cauchy type with unbounded weight function on the edges*, J. Comput. Appl. Math. **233** (2009), 334–345.

[12] Z.K.Eshkuvatov, N.M.A.Nik Long, M.Abdulkawi, *Numerical evaluation for Caushy type singular integrals on the interval*, J. Comput. Appl. Math. **233(8)** (2010), 1995–2001.

[13] T. Hasegawa, *Numerical integration of functions with poles near the interval of integration*, J. Comput. Appl. Math. **87** (1997), 339–357.

[14] T. Hasegawa, H.Suguira, *Quadrature rule for indefinite integral of algebraic-logarifmic singular integrands*, J. Comput. Appl. Math. **205** (2007) 487–496.

[15] I.K. Lifanov, *The method of singular equations and numerical experiments*, Moscow, TOO "Yanus", 1995 (in Russian).

[16] I.K. Lifanov, *Hypesingular integral equations and their applacations*, Chapman and Hall, 2004.

[17] S.L. Sobolev *Introduction to the Theory of Cubature Formulas.* Nauka, Moscow, (1974).

[18] Kh.M. Shadimetov *On an extremal function of quadrature formulas* Reports of Uzbekistan Academy of Sciences. (1995) no.9-10, pp.3-5.

[19] Kh.M. Shadimetov *Reducing the construction of composite optimal quadrature formulas to the solution of difference schemes* Reports of Uzbekistan Academy of Sciences.(1998) no.7, pp.3-6.

[20] Kh. M.Shadimetov *Optimal lattice quadrature and cubature formulas in Sobolev spaces.* Tashkent, Uzbekistan (2019).

Dilshod M. Akhmedov
V.I.Romanovskiy Institute of mathematics,
Uzbekistan Academy of Sciences,
Tashkent International University
e-mail: axmedovdilshod@mail.ru

Pirnapasov Sardor Olimboevich
Webster university in Tashkent,
Tashkent, Uzbekistan

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.22-34
DOI: 10.29229/uzmj.2025-4-3

Classification of Frobenius algebra structures on two-dimensional vector space over any base field

Asrorov D., Bekbaev U., Rakhimov I.

Abstract. In this paper, we first classify all associative algebra structures on a two-dimensional vector space over an arbitrary base field equipped with a non-degenerate bilinear form. We then determine which of these are Frobenius algebras. We provide lists of canonical representatives of the isomorphism classes of these algebras over an arbitrary base field.

Keywords: Frobenius algebra, Non-degenerate bilinear form, Classification, Automorphism.
MSC (2020): 16W20; 16S34; 20C05.

1. Introduction

Frobenius algebras are a fundamental concept in mathematics, appearing in areas such as algebra, topology, and theoretical physics, particularly in the study of categories, representation theory, and quantum field theory. They are named after the German mathematician Ferdinand Georg Frobenius.

Initially it was introduced as a finite-dimensional associative algebra equipped with a linear functional whose "kernel" contains no nontrivial ideals. Later, a few equivalent definitions of these algebras appear depending their applications in various areas of science. Frobenius algebras were first studied by Frobenius [13] around 1900. Further properties and relations go back to Nakayama [16] in the 1930s. The characterisation of Frobenius algebras in terms of comultiplication goes back to Lawvere [15] (1967), and it was rediscovered by Quinn [22] and Abrams [1] in the 1990s (we refer the reader to [12] as a most fundamental work on Frobenius algebras and their connections).

Frobenius algebras began to be studied in the 1930s by R. Brauer and C. Nesbitt [6]. T. Nakayama discovered the beginnings of a rich duality theory [16, 17]. J. Dieudonné used this to characterize Frobenius algebras [10]. Frobenius algebras were generalized to quasi-Frobenius rings, those Noetherian rings whose right regular representation is injective. There are works on generalization of the concept Frobenius algebra to some specific classes of algebras (see [3, 8, 14, 18]). In recent times, interest has been renewed in Frobenius algebras due to connections to Topological Quantum Field Theory. TQFTs are functors from the category of cobordisms to the category of vector spaces. It has been found that they play an important role in the algebraic treatment and axiomatic foundation of Topological Quantum Field Theory [1, 4, 21]. Frobenius algebras underlie the algebraic structure of 2D Topological Quantum Field Theory's (TQFT's). They provide a bridge between physics and algebraic topology by encoding information about 2-dimensional surfaces and their invariants. Let us mention a few results, illustrating the importance of the concept. In [19] the author introduces foundational concepts related to Frobenius algebras in the context of Hopf algebra theory. A. Atiyah in [2] discussed the role of Frobenius algebras in the development of TQFT's and first described their axiomatic foundation. The authors of [7] present a unified approach to the study of separable and Frobenius algebras.

In the paper, we first classify all associative algebra structures on two-dimensional vector spaces over any base field equipped with a non-degenerate bilinear form (Section 3). Then we identify which of those are Frobenius algebras (Section 4). Section 5 contains a comparison of two lists of two-dimensional associative algebras over any base fields obtained in [11] and [20].

1.1. Preliminaries.

Let $\mathbb{A}$ be a PI-algebra, with a given set of polynomial identities

$$\{P_j[u_1, u_2, ..., u_n] = 0 : \ j \in J\} \text{ over a field } \mathbb{F}$$

and

$$P_j[u_1, u_2, ..., u_n] = \sum_{i=1}^{k_j} Q_j^i[u_1, u_2, ..., u_n] R_j^i[u_1, u_2, ..., u_n], \text{ where } j \in J.$$

In some applications this kind of algebras appear with a non-degenerate bilinear form $\sigma : \mathbb{A} \times \mathbb{A} \to \mathbb{F}$ such that $\sum_{i=1}^{k_j} \sigma(Q_j^i[a_1, a_2, ..., a_n], R_j^i[a_1, a_2, ..., a_n]) = 0$ at all $a_1, a_2, ..., a_n \in \mathbb{A}$. A pair $(\mathbb{A}, \sigma)$ is said to be a Frobenius PI-algebra. The classification of a given class of Frobenius PI-algebras is of great interest. In this paper we consider as a class PI-algebras the class of associative algebras and provide a complete classification of such algebras on two-dimensional vector space over any base field. Therefore, further an algebra $\mathbb{A}$ always is assumed to be associative. The next theorem establishes the equivalence of several important and useful characterizations of Frobenius algebras (see [9]).

Theorem 1.1. *The following statements about a finite-dimensional unital algebra $\mathbb{A}$ are equivalent:*

- *$\mathbb{A}$ is a Frobenius algebra, i.e., there exists a linear functional $\varepsilon : \mathbb{A} \longrightarrow \mathbb{F}$ whose "kernel" contains no nontrivial ideals.*
- *There exists a non-degenerate bilinear form, $\sigma : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{F}$ such that $\sigma(\mathrm{xy}, \mathrm{z}) = \sigma(\mathrm{x}, \mathrm{yz})$ for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathbb{A}$.*
- *For all left ideals L and right ideals R in $\mathbb{A}$ we have*

$$l(r(L)) = L, \text{ and } (r(L) : \mathbb{F}) + (L : \mathbb{F}) = (\mathbb{A} : \mathbb{F});$$

$$r(l(R)) = R, \text{ and } (l(R) : \mathbb{F}) + (R : \mathbb{F}) = (\mathbb{A} : \mathbb{F}),$$

where $r(P) = \{\mathrm{x} \in \mathbb{A} : P\mathrm{x} = 0\}$ and $l(P) = \{\mathrm{x} \in \mathbb{A} : \mathrm{x}P = 0\}$ are right and left annihilators of a subset $P \subset \mathbb{A}$, respectively and $(\_ : \mathbb{F})$ is the dimension over $\mathbb{F}$.

Definition 1.2. Let $(\mathbb{A}, \sigma)$ and $(\mathbb{B}, \tau)$ be Frobenius algebras on a vector space $\mathbb{V}$ over a field $\mathbb{F}$ with non-degenerate bilinear forms $\sigma : \mathbb{A} \times \mathbb{A} \to \mathbb{F}$ and $\tau : \mathbb{B} \times \mathbb{B} \to \mathbb{F}$, respectively. Pairs $(\mathbb{A}, \sigma)$ and $(\mathbb{B}, \tau)$ are said to be isomorphic if there exists isomorphism of algebras $f : \mathbb{A} \longrightarrow \mathbb{B}$ such that $\sigma(\mathrm{xy}) = \tau(f(\mathrm{x})f(\mathrm{y}))$ for all $\mathrm{x}, \mathrm{y} \in \mathbb{A}$ (we denote it by $\mathbb{A} \cong \mathbb{B}$).

Let $\mathbb{A}$ be an n-dimensional algebra and $\mathbf{e} = (\mathrm{e}_1, \mathrm{e}_2, ..., \mathrm{e}_n)$ its basis. Then x, y and z can be presented by their coordinate vectors $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$ and $z = (z_1, z_2, ..., z_n)^T$ as

$$\mathrm{x} = \mathbf{e}x, \mathrm{y} = \mathbf{e}y \text{ and } \mathrm{z} = \mathbf{e}z, \text{ respectevely.}$$

Here and onward we use the notions x and x for a vector and its coordinate vector on the basis $\mathbf{e}$. Therefore, $\mathrm{xy} = \mathbf{e}A(x \otimes y)$, where the entries a_{ij}^k of

$$A = \begin{pmatrix} a_{11}^1 & a_{12}^1 & ... & a_{1n}^1 & a_{21}^1 & a_{22}^1 & ... & a_{2n}^1 & ... & a_{n1}^1 & a_{n2}^1 & ... & a_{nn}^1 \\ a_{11}^2 & a_{12}^2 & ... & a_{1n}^2 & a_{21}^2 & a_{22}^2 & ... & a_{2n}^2 & ... & a_{n1}^2 & a_{n2}^2 & ... & a_{nn}^2 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ a_{11}^n & a_{12}^n & ... & a_{1n}^n & a_{21}^n & a_{22}^n & ... & a_{2n}^n & ... & a_{n1}^n & a_{n2}^n & ... & a_{nn}^n \end{pmatrix}$$

are defined by:

$$\mathrm{e}_i\mathrm{e}_j = \sum_{k=1}^{n} a_{ij}^k \mathrm{e}_k, \text{ where } i, j = 1, 2, ..., n.$$

The matrix A is said to be the matrix of structure constants (MSC) of $\mathbb{A}$ on the basis $\mathbf{e}$.

If $\mathbb{A}$ is a Frobenius algebra then the Frobenius map σ also is presented by its matrix S: $\sigma(\mathrm{x}, \mathrm{y}) = x^T S y$. Then

$$\sigma(\mathrm{xy}, \mathrm{z}) = (x^T \otimes y^T)A^T S z \text{ and } \sigma(\mathrm{x}, \mathrm{yz}) = x^T S A(y \otimes z).$$

Therefore, one has

Lemma 1.3. *An algebra $\mathbb{A}$ is Frobenius if and only if*

$$(x^T \otimes y^T)A^T S z = x^T S A(y \otimes z). \tag{1.1}$$

Recently, in [5] a result on classification of two-dimensional algebras over any base field $\mathbb{F}$ appeared. Using this classification in [20] the author gave the representatives of isomorphism classes of all associative algebra structures on two-dimensional vector space over any base field and their automorphism groups. Below we give results from [20] which we make use in the paper.

Theorem 1.4. *Any non-trivial 2-dimensional associative algebra over a field* $\mathbb{F}$ $(Char(\mathbb{F}) \neq 2)$ *is isomorphic to only one of the following listed, by their matrices of structure constants, algebras:*

(1) $As_{13}^1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$

(2) $As_3^2 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$

(3) $As_3^3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$

(4) $As_3^4 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$

(5) $As_3^5(\alpha_4) := \begin{pmatrix} 1 & 0 & 0 & \alpha_4 \\ 0 & 1 & 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 0 & a^2\alpha_4 \\ 0 & 1 & 1 & 0 \end{pmatrix}$, *where* $\alpha_4 \in \mathbb{F}$, $a \in \mathbb{F}$ *and* $a \neq 0$.

Theorem 1.5. *Any non-trivial 2-dimensional associative algebra over a field* $\mathbb{F}$, $(Char(\mathbb{F}) = 2)$ *is isomorphic to only one of the following listed by their matrices of structure constants, such algebras:*

(1) $As_{12,2}^1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$

(2) $As_{11,2}^2(\beta_1) := \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 & 1 & 0 \\ b^2(\beta_1 + a^2) & 0 & 0 & 1 \end{pmatrix}$, *where* $a, b \in \mathbb{F}$ *and* $b \neq 0$,

(3) $As_{6,2}^3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$

(4) $As_{4,2}^4(\beta_1) := \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 & 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta_1 + a + a^2 & 0 & 0 & 1 \end{pmatrix}$, *where* $a, \beta_1 \in \mathbb{F}$,

(5) $As_{3,2}^5 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$

(6) $As_{3,2}^6 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$

We also need the automorphism groups of the algebras given in the theorems above.

Theorem 1.6. *The automorphism groups of the algebras listed in Theorem 1.4 are given as follows*

(1) $Aut(As_{13}^1) = \left\{ \begin{pmatrix} p & 0 \\ s & p^2 \end{pmatrix} \mid p, s \in \mathbb{F},\ p \neq 0 \right\},$

(2) $Aut(As_3^2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{F},\ t \neq 0 \right\},$

(3) $Aut(As_3^3) = \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \mid s, t \in \mathbb{F},\ t \neq 0 \right\},$

(4) $Aut(As_3^4) = \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \mid s, t \in \mathbb{F},\ t \neq 0 \right\},$

(5) $Aut(As_3^5(0)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{F}, t \neq 0 \right\},$

(6) $Aut(As_3^5(\alpha_4)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\},\ \alpha_4 \neq 0.$

Theorem 1.7. *The automorphism groups of the algebras listed in Theorem 1.5 are given as follows*

(1) $Aut(As^1_{12,2}) = \left\{ \begin{pmatrix} p & 0 \\ s & p^2 \end{pmatrix} \mid p \neq 0, s \in \mathbb{F} \right\}$,

(2) $Aut(As^2_{11,2}) = \left\{ \begin{pmatrix} p & 0 \\ \beta_1(p-1) & 1 \end{pmatrix} \mid p \neq 0 \in \mathbb{F} \right\}$,

(3) $Aut(As^3_{6,2}) = \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \mid t \neq 0, s \in \mathbb{F} \right\}$,

(4) $Aut(As^4_{4,2}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$,

(5) $Aut(As^5_{3,2}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \neq 0 \in \mathbb{F} \right\}$,

(6) $Aut(As^6_{3,2}) = \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \mid s, t \in \mathbb{F} \text{ and } t \neq 0 \right\}$.

Let $\mathbb{A}$ be a two-dimensional algebra over a field $\mathbb{F}$, $\mathbf{e} = (\mathrm{e}_1, \mathrm{e}_2)$ its basis, $A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$ MSC of $\mathbb{A}$ and $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{F})$ the matrix of σ. Then

$$\begin{array}{rcl} (x^T \otimes y^T)A^T S z & = & (\alpha_1 a + \beta_1 c)x_1y_1z_1 + (\alpha_2 a + \beta_2 c)x_1y_2z_1 + (\alpha_3 a + \beta_3 c)x_2y_1z_1 \\ & & +(\alpha_4 a + \beta_4 c)x_2y_2z_1 + (\alpha_1 b + \beta_1 d)x_1y_1z_2 + (\alpha_2 b + \beta_2 d)x_1y_2z_2 \\ & & +(\alpha_3 b + \beta_3 d)x_2y_1z_2 + (\alpha_4 b + \beta_4 d)x_2y_2z_2 \\ x^T SA(y \otimes z) & = & (\alpha_1 a + \beta_1 b)x_1y_1z_1 + (\alpha_3 a + \beta_3 b)x_1y_2z_1 + (\alpha_1 c + \beta_1 d)x_2y_1z_1 \\ & & +(\alpha_3 c + \beta_3 d)x_2y_2z_1 + (\alpha_2 a + \beta_2 b)x_1y_1z_2 \\ & & +(\alpha_4 a + \beta_4 b)x_1y_2z_2 + (\alpha_2 c + \beta_2 d)x_2y_1z_2 + (\alpha_4 c + \beta_4 d)x_2y_2z_2 \end{array} \quad (1.2)$$

and (1.3) can be written as follows

$$\left\{ \begin{array}{rcl} \alpha_1 a + \beta_1 c - \alpha_1 a - \beta_1 b & = & 0 \\ \alpha_2 a + \beta_2 c - \alpha_3 a - \beta_3 b & = & 0 \\ \alpha_3 a + \beta_3 c - \alpha_1 c - \beta_1 d & = & 0 \\ \alpha_4 a + \beta_4 c - \alpha_3 c - \beta_3 d & = & 0 \\ \alpha_1 b + \beta_1 d - \alpha_2 a - \beta_2 b & = & 0 \\ \alpha_2 b + \beta_2 d - \alpha_4 a - \beta_4 b & = & 0 \\ \alpha_3 b + \beta_3 d - \alpha_2 c - \beta_2 d & = & 0 \\ \alpha_4 b + \beta_4 d - \alpha_4 c - \beta_4 d & = & 0 \end{array} \right. \quad (1.3)$$

as far as the system of monomial functions $\{x_1y_1z_1, x_1y_2z_1, x_2y_1z_1, x_1y_1z_2, x_2y_2z_1, x_1y_2z_2, x_2y_1z_2, x_2y_2z_2\}$ is linearly independent over $\mathbb{F}$.

2. Classification of two-dimensional associative algebras equipped by a non-degenerate form σ

In this section first we classify non-degenerate bilinear forms given as $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to the transformations $g^T S g$, $g \in G$, where G is a fixed nontrivial automorphism group from Theorems 1.6 and 1.7 (the transformation $S' = g^T S g$ is denoted by $S' \simeq S$). For the further usage the list of all nontrivial autmorphism groups in Theorems 1.6 and 1.7 we enumerate as follows

- $G_1=\left\{\begin{pmatrix}1&0\\0&t\end{pmatrix}:\ 0\neq t\in\mathbb{F}\right\}$,
- $G_2=\left\{\begin{pmatrix}1&0\\s&t\end{pmatrix}:\ s,t\in\mathbb{F},\ t\neq 0\right\}$,
- $G_3=\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix}\right\}(Char(\mathbb{F})=2)$,
- $G_4=\left\{\begin{pmatrix}p&0\\s&p^2\end{pmatrix}:\ p,s\in\mathbb{F},p\neq 0\right\}$,
- $G_{5,\beta_1}=\left\{\begin{pmatrix}p&0\\\beta_1(p-1)&1\end{pmatrix}:\ p\in\mathbb{F},p\neq 0\right\}$ $(Char(\mathbb{F})=2)$,
- $G_6=\left\{\begin{pmatrix}1&0\\0&\pm1\end{pmatrix}\right\}(Char(\mathbb{F})\neq 2)$.

Now we treat the action $g\cdot S=g^TSg$ for each $G_i, i=1,2,...,6$ $(g\in G_i)$ one by one and find the canonical representatives of the equivalent classes with respect to this action.

Let $g=\begin{pmatrix}1&0\\0&t\end{pmatrix}\in G_1$. Then $g^T\begin{pmatrix}a&b\\c&d\end{pmatrix}g=\begin{pmatrix}a&tb\\tc&t^2d\end{pmatrix}$ and the following canonical forms occur:

- $\begin{pmatrix}a&1\\c&d\end{pmatrix}$, where $ad-c\neq 0$,
- $\begin{pmatrix}a&0\\1&d\end{pmatrix}$, where $ad\neq 0$,
- $\begin{pmatrix}a&0\\0&d\end{pmatrix}\simeq\begin{pmatrix}a&0\\0&t^2d\end{pmatrix}$, where $0\neq t\in\mathbb{F},\ ad\neq 0$.

If $g=\begin{pmatrix}1&0\\s&t\end{pmatrix}\in G_2$ then $g^T\begin{pmatrix}a&b\\c&d\end{pmatrix}g=\begin{pmatrix}a+sc+sb+s^2d&tb+std\\tc+std&t^2d\end{pmatrix}$ and one comes to the following canonical forms:

- $\begin{pmatrix}a&0\\1&d\end{pmatrix}$, where $ad\neq 0$,
- $\begin{pmatrix}a&0\\0&d\end{pmatrix}\simeq\begin{pmatrix}a&0\\0&t^2d\end{pmatrix}$, where $0\neq t\in\mathbb{F},\ ad\neq 0$,
- $\begin{pmatrix}0&1\\c&0\end{pmatrix}$, where $c\neq 0,\ c+1\neq 0$,
- $\begin{pmatrix}a&1\\-1&0\end{pmatrix}$, where $a\in\mathbb{F}$.

Let $g=\begin{pmatrix}1&0\\1&1\end{pmatrix}\in G_3$. Then $g^T\begin{pmatrix}a&b\\c&d\end{pmatrix}g=\begin{pmatrix}a+c+b+d&b+d\\c+d&d\end{pmatrix}$ and we get the following canonical forms:

- $\begin{pmatrix}a&b\\c&d\end{pmatrix}\simeq\begin{pmatrix}a+c+b+d&b+d\\c+d&d\end{pmatrix}$, where $ad-bc\neq 0$.

If $g=\begin{pmatrix}p&0\\s&p^2\end{pmatrix}\in G_4$, where $p,s\in\mathbb{F}$, $p\neq 0$, then $g^T\begin{pmatrix}a&b\\c&d\end{pmatrix}g=\begin{pmatrix}p^2a+psc+psb+s^2d&p^3b+p^2sd\\p^3c+p^2sd&p^4d\end{pmatrix}$ and the following canonical forms occur:

- $\begin{pmatrix}a&0\\c&d\end{pmatrix}\simeq\begin{pmatrix}p^2a&0\\p^3c&p^4d\end{pmatrix}$, where $p\neq 0$,
- $\begin{pmatrix}0&b\\c&0\end{pmatrix}\simeq\begin{pmatrix}0&p^3b\\p^3c&0\end{pmatrix}$, where $p\neq 0,\ bc\neq 0$,

- $\begin{pmatrix} a & b \\ -b & 0 \end{pmatrix} \simeq \begin{pmatrix} p^2a & p^3b \\ -p^3b & 0 \end{pmatrix}$, where $p \neq 0$.

If $g = \left\{ \begin{pmatrix} p & 0 \\ \beta_1(p-1) & 1 \end{pmatrix} : \ p \neq 0 \in \mathbb{F} \right\} \in G_{5,\beta_1}$ then

$$g^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} p^2a + \beta_1(p+1)(pc + pb + \beta_1 d(p+1)) & pb + \beta_1 d(p+1) \\ pc + \beta_1 d(p+1) & d \end{pmatrix}$$

and one gets the following canonical forms:

- $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$, where $ad \neq 0$,
- $\begin{pmatrix} a & \beta_1 d \\ 0 & d \end{pmatrix}$, where $ad \neq 0$,
- $\begin{pmatrix} a & \beta_1 d \\ \beta_1 d & d \end{pmatrix} \simeq \begin{pmatrix} p^2(a - \beta_1^2 d) + \beta_1^2 d & \beta_1 d \\ \beta_1 d & d \end{pmatrix}$, where $ad - \beta_1^2 d^2 \neq 0$ and the polynomial $u^2(a - \beta_1^2 d) + \beta_1^2 d$ has no root in $\mathbb{F}$,
- $\begin{pmatrix} 0 & \beta_1 d \\ \beta_1 d & d \end{pmatrix}$, where $\beta_1 d \neq 0$,
- $\begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix}$, where $ad \neq 0$,
- $\begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}$, where $ad \neq 0$,
- $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \simeq \begin{pmatrix} p^2a & 0 \\ 0 & d \end{pmatrix}$, where $ad \neq 0$.

Finally, taking $g = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \in G_6$ we get $g^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} a & \pm b \\ \pm c & d \end{pmatrix}$ and only the canonical form appears:

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \simeq \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$, where $ad - bc \neq 0$.

Now we turn to pairs $(\mathbb{A}, \sigma)$, where $\mathbb{A}$ is a two-dimensional associative algebra, $\sigma : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{F}$ is a non-degenerate bilinear form, up to isomorphism. Taking into account the canonical forms of σ along with Theorems 1.4 and 1.7 we state the following results.

Lemma 2.1. *The representatives of isomorphism classes of pairs $(\mathbb{A}, \sigma)$, where $\mathbb{A}$ is a two dimensional associative algebra over a field $\mathbb{F}$ $(Char(\mathbb{F}) \neq 2)$, $\sigma : \mathbb{A} \times \mathbb{A} \to \mathbb{F}$ is a non-degenerate form, are given as follows:*

(1) $\left(As_3^2, \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \right)$, *where* $a, c, d \in \mathbb{F}$, $ad - c \neq 0$,

(2) $\left(As_3^2, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix} \right)$, *where* $a, d \in \mathbb{F}$, $ad \neq 0$,

(3) $\left(As_3^2, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) \cong \left(As_3^2, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix} \right)$, *where* $a, d, t \in \mathbb{F}$, $ad \neq 0$, $t \neq 0$,

(4) $\left(As_3^3, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix} \right)$, *where* $a, d \in \mathbb{F}$, $ad \neq 0$,

(5) $\left(As_3^3, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As_3^3, \begin{pmatrix} a & 0 \\ 0 & t^2d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(6) $\left(As_3^3, \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\right)$, *where* $c \in \mathbb{F},\ c \neq 0,\ c+1 \neq 0$,

(7) $\left(As_3^3, \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}\right)$, *where* $a \in \mathbb{F}$,

(8) $\left(As_3^4, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(9) $\left(As_3^4, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As_3^4, \begin{pmatrix} a & 0 \\ 0 & t^2d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(10) $\left(As_3^4, \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\right)$, *where* $c \in \mathbb{F},\ c \neq 0,\ c+1 \neq 0$,

(11) $\left(As_3^4, \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}\right)$, *where* $a \in \mathbb{F}$,

(12) $\left(As_3^5(0), \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}\right)$, *where* $a, c, d \in \mathbb{F},\ ad - c \neq 0$,

(13) $\left(As_3^5(0), \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(14) $\left(As_3^5(0), \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As_3^5(0), \begin{pmatrix} a & 0 \\ 0 & t^2d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(15) $\left(As_3^5(\alpha_4), \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cong \left(As_3^5(\alpha_4), \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\right)$, *where* $\alpha_4, a, b, c, d \in \mathbb{F},\ \alpha_4 \neq 0,\ ad - bc \neq 0$,

(16) $\left(As_{13}^1, \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\right) \cong \left(As_{13}^1, \begin{pmatrix} p^2a & 0 \\ p^3c & p^4d \end{pmatrix}\right)$, *where* $a, c, d, p \in \mathbb{F},\ ad \neq 0,\ p \neq 0$,

(17) $\left(As_{13}^1, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right) \cong \left(As_{13}^1, \begin{pmatrix} 0 & p^3b \\ p^3c & 0 \end{pmatrix}\right)$, *where* $b, c, p \in \mathbb{F},\ bc \neq 0,\ p \neq 0$,

(18) $\left(As_{13}^1, \begin{pmatrix} a & b \\ -b & 0 \end{pmatrix}\right) \cong \left(As_{13}^1, \begin{pmatrix} p^2a & p^3b \\ -p^3b & 0 \end{pmatrix}\right)$, *where* $a, b, p \in \mathbb{F},\ a \neq 0,\ b \neq 0,\ p \neq 0$.

Lemma 2.2. *The representatives of isomorphism classes of pairs* $(\mathbb{A}, \sigma)$, *where* $\mathbb{A}$ *is a two dimensional associative algebra over a field* $\mathbb{F}$ $(Char(\mathbb{F}) = 2)$, $\sigma : \mathbb{A} \times \mathbb{A} \to \mathbb{F}$ *is a non-degenerate form, are given as follows:*

(1) $\left(As_{12,2}^1, \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\right) \cong \left(As_{12,2}^1, \begin{pmatrix} p^2a & 0 \\ p^3c & p^4d \end{pmatrix}\right)$, *where* $a, c, d, p \in \mathbb{F},\ ad \neq 0,\ p \neq 0$,

(2) $\left(As_{12,2}^1, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right) \cong \left(As_{12,2}^1, \begin{pmatrix} 0 & p^3b \\ p^3c & 0 \end{pmatrix}\right)$, *where* $b, c, p \in \mathbb{F},\ bc \neq 0,\ p \neq 0$,

(3) $\left(As_{12,2}^1, \begin{pmatrix} a & b \\ -b & 0 \end{pmatrix}\right) \cong \left(As_{12,2}^1, \begin{pmatrix} p^2a & p^3b \\ -p^3b & 0 \end{pmatrix}\right)$, *where* $a, b, p \in \mathbb{F},\ b^2 \neq 0,\ p \neq 0$,

(4) $\left(As^2_{11,2}(\beta_1), \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\right)$, *where* $\beta_1, a, c, d \in \mathbb{F},\ \beta_1 ad \neq 0$,

(5) $\left(As^2_{11,2}(\beta_1), \begin{pmatrix} a & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right) \cong \left(As^2_{11,2}(\beta_1), \begin{pmatrix} p^2(a-\beta_1^2 d)+\beta_1^2 d & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right)$, *where* $\beta_1(ad-\beta_1^2 d^2) \neq 0$ *and the polynomial* $u^2(a-\beta_1^2 d)+\beta_1^2 d$ *has no root in* $\mathbb{F}$, $\beta_1, a, d, p \in \mathbb{F}$,

(6) $\left(As^2_{11,2}(\beta_1), \begin{pmatrix} a & \beta_1 d \\ 0 & d \end{pmatrix}\right)$, *where* $\beta_1, a, d \in \mathbb{F},\ \beta_1 ad \neq 0$,

(7) $\left(As^2_{11,2}(\beta_1), \begin{pmatrix} 0 & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right)$, *where* $\beta_1, d \in \mathbb{F},\ \beta_1 d \neq 0$,

(8) $\left(As^2_{11,2}(0), \begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(9) $\left(As^2_{11,2}(0), \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(10) $\left(As^2_{11,2}(0), \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^2_{11,2}(0), \begin{pmatrix} p^2 a & 0 \\ 0 & d \end{pmatrix}\right)$, *where* $p, a, d \in \mathbb{F},\ ad \neq 0$,

(11) $\left(As^3_{6,2}, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(12) $\left(As^3_{6,2}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^3_{6,2}, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(13) $\left(As^3_{6,2}, \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\right)$, *where* $c \in \mathbb{F},\ c \neq 0,\ c+1 \neq 0$,

(14) $\left(As^3_{6,2}, \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}\right)$, *where* $a \in \mathbb{F}$,

(15) $\left(As^4_{4,2}(\beta_1), \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cong \left(As^4_{4,2}(\beta_1), \begin{pmatrix} a+c+b+d & b+d \\ c+d & d \end{pmatrix}\right)$, *where* $\beta_1, a, b, c, d \in \mathbb{F}$, $ad-bc \neq 0$,

(16) $\left(As^5_{3,2}, \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}\right)$, *where* $a, c, d \in \mathbb{F},\ ad-c \neq 0$,

(17) $\left(As^5_{3,2}, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *where* $a, d \in \mathbb{F},\ ad \neq 0$,

(18) $\left(As^5_{3,2}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^5_{3,2}, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(19) $\left(As^6_{3,2}, \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix}\right)$, *wherea*, $d \in \mathbb{F},\ ad \neq 0$,

(20) $\left(As^6_{3,2}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^6_{3,2}, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ ad \neq 0,\ t \neq 0$,

(21) $\left(As^6_{3,2}, \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\right)$, *where* $c \in \mathbb{F},\ c \neq 0,\ c+1 \neq 0,$

(22) $\left(As^6_{3,2}, \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}\right)$, *where* $a \in \mathbb{F}$.

3. Classification of two-dimensional Frobenius algebras

Now we determine those pairs from the lemmas above which are Frobenious algebras.

Theorem 3.1.

- *If* $Char(\mathbb{F}) \neq 2$ *then any two-dimensional Frobenius algebra over* $\mathbb{F}$ *is isomorphic to only one of the following such algebras:*
  - $\left(As^2_3, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^2_3, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ t \neq 0,\ ad \neq 0,$
  - $\left(As^5_3\,(0)\,, \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\right)$, *where* $a \in \mathbb{F}$,
  - $\left(As^5_3\,(\alpha_4)\,, \begin{pmatrix} a & b \\ b & 2\alpha_4 a \end{pmatrix}\right) \cong \left(As^5_3\,(\alpha_4)\,, \begin{pmatrix} a & -b \\ -b & 2\alpha_4 a \end{pmatrix}\right)$, *where* $\alpha_4, a, b \in \mathbb{F}$, $2\alpha_4 a^2 - b^2 \neq 0,\ \alpha_4 \neq 0,$
  - $\left(As^1_{13}, \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\right) \cong \left(As^1_{13}, \begin{pmatrix} 0 & p^3 b \\ p^3 b & 0 \end{pmatrix}\right)$, *where* $b, p \in \mathbb{F},\ p \neq 0,\ b \neq 0.$
- *If* $Char(\mathbb{F}) = 2$ *then any two-dimensional Frobenius algebra over* $\mathbb{F}$ *is isomorphic to only one of the following such algebras:*
  - $\left(As^5_{3,2}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^5_{3,2}, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}\right)$, *where* $a, d, t \in \mathbb{F},\ t \neq 0,\ ad \neq 0,$
  - $\left(As^4_{4,2}(\beta_1), \begin{pmatrix} b+\beta_1 d & b \\ b & d \end{pmatrix}\right) \cong \left(As^4_{4,2}(\beta_1), \begin{pmatrix} b+d+\beta_1 d & b+d \\ b+d & d \end{pmatrix}\right)$, *where* $\beta_1, b, d \in \mathbb{F}$, $\beta_1 d^2 + bd - b^2 \neq 0,$
  - $\left(As^2_{11,2}(\beta_1), \begin{pmatrix} \beta_1 d & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right) \cong \left(As^2_{11,2}(\beta_1), \begin{pmatrix} p^2(\beta_1 d - \beta_1^2 d) + \beta_1^2 d & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right)$, *where the polynomial* $u^2(1-\beta_1)+\beta_1$ *has no root in* $\mathbb{F}$, $\beta_1, d, p \in \mathbb{F},\ \beta_1 \neq 0, 1,\ d \neq 0,$
  - $\left(As^1_{12,2}, \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\right) \cong \left(As^1_{12,2}, \begin{pmatrix} 0 & p^3 b \\ p^3 b & 0 \end{pmatrix}\right)$, *where* $b, p \in \mathbb{F},\ p \neq 0,\ b \neq 0.$

Proof. Let now check the condition $\sigma(\mathrm{xy}, \mathrm{z}) = \sigma(\mathrm{x}, \mathrm{yz})$ for the pairs appeared in Lemma 2.1 ($Char\mathbb{F} \neq 2$) and Lemma 2.2 ($Char\mathbb{F} = 2$), i.e., find the solutions to the system of equations (1.3).

• Let $Char\mathbb{F} \neq 2$. In this case the system of equations (1.3) is consistent only for pairs (3), (12), (15) and (17) of Lemma 2.1. The solutions to the system are given as follows. For the pair

$$(3)\ \left(As^2_3, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As^2_3, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}\right),$$

where $a, d, t \in \mathbb{F}, ad \neq 0, t \neq 0$ of Lemma 2.1 the system becomes an identity, therefore, the pair (3) is a Frobenius algebra.

Consider the pair (12) $\left(As^5_3(0), \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}\right)$, where $a, c, d \in \mathbb{F}$, $ad - c \neq 0$ of Lemma 2.1. As a solution to the system (1.3) we get $c = 1$ and $d = 0$ and the corresponding Frobenius algebras are

$$\left(As^5_3\,(0)\,, \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\right), \text{ where } a \in \mathbb{F}.$$

Among the pairs

$$(15)\ \left(As_3^5(\alpha_4), \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cong \left(As_3^5(\alpha_4), \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\right),$$

where $a, b, c, d, \alpha_4 \in \mathbb{F}$, $ad - bc \neq 0$, $\alpha_4 \neq 0$ of Lemma 2.1 those satisfying $c = b$, and $d = 2\alpha_4 a$ are Frobenius. Thus, we get

$$\left(As_3^5(\alpha_4), \begin{pmatrix} a & b \\ b & 2\alpha_4 a \end{pmatrix}\right) \cong \left(As_3^5(\alpha_4), \begin{pmatrix} a & -b \\ -b & 2\alpha_4 a \end{pmatrix}\right),$$

where $a, b, \alpha_4 \in \mathbb{F}$, $2\alpha_4 a^2 - b^2 \neq 0, \alpha_4 \neq 0$.

Considering the system of equations (1.3) for the pair

$$(17)\ \left(As_{13}^1, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right) \cong \left(As_{13}^1, \begin{pmatrix} 0 & p^3 b \\ p^3 c & 0 \end{pmatrix}\right),$$

where $p \neq 0$, of Lemma 2.1 we obtain $c = b$ and hence,

$$\left(As_{13}^1, \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\right) \cong \left(As_{13}^1, \begin{pmatrix} 0 & p^3 b \\ p^3 b & 0 \end{pmatrix}\right) \text{ where } b, p \in \mathbb{F}\ p \neq 0, \text{ and } b \neq 0$$

is a Frobenius algebra.

• Let $Char\mathbb{F} = 2$. Considering the pairs

$$(2)\ \left(As_{12,2}^1, \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right) \cong \left(As_{12,2}^1, \begin{pmatrix} 0 & p^3 b \\ p^3 c & 0 \end{pmatrix}\right),$$

where $b, c, p \in \mathbb{F}$, $bc \neq 0, p \neq 0$ of Lemma 2.2 we generate the following Frobenius algebras

$$\left(As_{12,2}^1, \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\right) \cong \left(As_{12,2}^1, \begin{pmatrix} 0 & p^3 b \\ p^3 b & 0 \end{pmatrix}\right),$$

where $b, p \in \mathbb{F}$, $p \neq 0, b \neq 0$ as far as the solution to the system of equations (1.3) in this case is $c = b = 0$.

From the pair

$$(5)\ \left(As_{11,2}^2(\beta_1), \begin{pmatrix} a & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right) \cong \left(As_{11,2}^2(\beta_1), \begin{pmatrix} p^2(a - \beta_1^2 d) + \beta_1^2 d & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right),$$

where $\beta_1(ad - \beta_1^2 d^2) \neq 0$ and the polynomial $u^2(a - \beta_1^2 d) + \beta_1^2 d$ has no root in $\mathbb{F}$, $\beta_1, a, d, p \in \mathbb{F}$, of Lemma 2.2 subjecting to the system (1.3) we get $a = \beta_1 d$ and obtain the following Frobenius algebra

$$\left(As_{11,2}^2(\beta_1), \begin{pmatrix} \beta_1 d & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right) \cong \left(As_{11,2}^2(\beta_1), \begin{pmatrix} p^2(\beta_1 d - \beta_1^2 d) + \beta_1^2 d & \beta_1 d \\ \beta_1 d & d \end{pmatrix}\right),$$

where the polynomial $u^2(1 - \beta_1) + \beta_1$ has no root in $\mathbb{F}$, $\beta_1, d, p \in \mathbb{F}$ and $\beta_1 d^2 - \beta_1^2 d^2 \neq 0$.

Let us consider the pair (15) $\left(As_{4,2}^4(\beta_1), \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cong \left(As_{4,2}^4(\beta_1), \begin{pmatrix} a + c + b + d & b + d \\ c + d & d \end{pmatrix}\right)$, where $\beta_1, a, b, c, d \in \mathbb{F}$, $ad - bc \neq 0$ of Lemma 2.2. Then the system of equations (1.3) is equivalent to $b = c$ and $a = b + \beta_1 d$. Therefore,

$$\left(As_{4,2}^4(\beta_1), \begin{pmatrix} b + \beta_1 d & b \\ b & d \end{pmatrix}\right) \cong \left(As_{4,2}^4(\beta_1), \begin{pmatrix} b + d + \beta_1 d & b + d \\ b + d & d \end{pmatrix}\right),$$

where $\beta_1, b, d \in \mathbb{F}$, $\beta_1 d^2 + bd - b^2 \neq 0$ are Frobenius algebras.

Finally, considering the pair

$$(18)\ \left(As_{3,2}^5, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \cong \left(As_{3,2}^5, \begin{pmatrix} a & 0 \\ 0 & t^2 d \end{pmatrix}\right),$$

where $a, d, t \in \mathbb{F}, t \neq 0, ad \neq 0$ of Lemma 2.2 we find the system of equations (1.3) to be an identity and hence, they all are Frobenius algebras.

Note that for all the other classes from Lemma 2.2 the system of equations (1.3) is inconsistent.

□

4. Remarks

(1) The authors of the paper are informed by the referee on a classification of two-dimensional associative algebras obtained earlier by M. Gerstenhaber and F. Kubo in [11]. On the way we compare the classification of [11] with that obtained in [20]. Here are the comparisons and some corrections.

• $Char(\mathbb{F}) \neq 2$:

Algebra from [11]	Algebra from [20]	Isomorphism
$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$As_3^5(\frac{1}{4})$	$P_1 = g^{-1} As_3^5(\frac{1}{4}) g^{\otimes 2}$, where $g = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{pmatrix}$
$P_2(d) = \begin{pmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 1 & 0 \end{pmatrix}, d \neq \frac{1}{4}$ $\cong \begin{pmatrix} 1 & 0 & 0 & r^2 d \\ 0 & 1 & 1 & 0 \end{pmatrix}, \ r \neq 0$	$As_3^5(d), d \neq 0, \frac{1}{4}$	$P_2(d) = As_3^5(d)$
$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$	$As_3^5(0)$	$P_3 = g^{-1} As_3^5(0) g^{\otimes 2}$, where $g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	As_3^3	$P_4 = As_3^3$
$P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	As_3^4	$P_5 = g^{-1} As_3^4 g^{\otimes 2}$, where $g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
$P_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	As_3^2	$P_6 = As_3^2$
$P_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	As_{13}^1	$P_7 = As_{13}^1$

• $Char(\mathbb{F}) = 2$:

Algebra from [11]	Algebra from [20]	Isomorphism
$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$	$As_{4,2}^4(1)$	$Q_1 = g As_{4,2}^4(1) (g^{-1})^{\otimes 2}$, where $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$Q_2(d) = \begin{pmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 1 & 0 \end{pmatrix}, d \neq 0$ $\cong \begin{pmatrix} 1 & 0 & 0 & r^2 d \\ 0 & 1 & 1 & 0 \end{pmatrix}, \ r \neq 0$	$As_{11,2}^2(d), d \neq 0$	$Q_2(d) = g^{-1} As_{11,2}^2(d) g^{\otimes 2}$, where $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$	$As_{11,2}^2(0)$	$P_3 = g^{-1} As_{11,2}^2(0) g^{\otimes 2}$, where $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$As_{3,2}^6$	$P_4 = As_{3,2}^6$
$P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$As_{6,2}^3$	$P_5 = As_{6,2}^3$
$P_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$As_{3,2}^5$	$P_6 = As_{3,2}^5$
$P_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$As_{12,2}^1$	$P_7 = As_{12,2}^1$

Conclusion: In the case of $Char(\mathbb{F}) = 2$ in [11] the family of algebras $As^4_{4,2}(\beta_1), \beta_1 \neq 1$ is missing.

(2) There are the following misprintings in the list of automorphism groups in [20]: in the case of $Char(\mathbb{F}) = 3$ the groups

$$Aut(As^1_{13,3}) = \left\{ \begin{pmatrix} p & 0 \\ s & 2p^2 \end{pmatrix} \mid p, s \in \mathbb{F}, p \neq 0 \right\} \text{and } Aut(As^4_{3,3}) = \left\{ \begin{pmatrix} 1 & 0 \\ 1+2t & t \end{pmatrix} \mid t \in \mathbb{F}, t \neq 0 \right\}$$

must be read as follows

$$Aut(As^1_{13,3}) = \left\{ \begin{pmatrix} p & 0 \\ s & p^2 \end{pmatrix} \mid p, s \in \mathbb{F}, p \neq 0 \right\} \text{and } Aut(As^4_{3,3}) = \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \mid s, t \in \mathbb{F}, t \neq 0 \right\},$$

respectively.

(3) In the paper we used Maple software for some computations.

5. Acknowledgements

The authors would like to thank anonymous reviewer for his/her useful and valuable comments.

References

[1] Abrams L.; Two-dimensional Topological Quantum Field Theories and Frobenius algebras. Journal of Knot Theory and Its Ramifications., –1996. –5(5). –P.569-587.

[2] Atiyah M.; Topological Quantum Field Theories. Publications mathématiques de l' I.H.É.S., –1988. –68. –P.175-186.

[3] Bai C.; Double constructions of Frobenius algebras. Connes cocycles and their duality. J. Noncommut. Geom., –2020. –4. –P.475-530. DOI 10.4171/JNCG/64.

[4] Bar-Natan D.; Khovanov's homology for tangles and cobordisms. Geom. Topol., –2005. –9(3). –P1443-1499. arXiv:math/0410495, doi:10.2140/gt.2005.9.1443.

[5] Bekbaev U.; Classification of two-dimensional algebras over any base field, 2023. AIP Conference Proceedings., –2880. –030001. https://doi.org/10.1063/5.0165726.

[6] Brauer R., Nesbitt C.; On the regular representations of algebras. Proc. Natl. Acad. Sci. USA., –1937. –23(4). –P.236–240. doi:10.1073/pnas.23.4.236.

[7] Caenepeel S., Ion B., Militaru G.; The structure of Frobenius algebras and separable algebras. Journal of K-Theory., –2000. –19(4). –P.365–402. DOI: 10.1023/A:1007849203555

[8] Csikós B., Verhóczki L.; Classification of Frobenius Lie algebras of dimension ≤ 6. Publ. Math. Debrecen., –70/3-4. –2007. –P.427–451.

[9] Curtis C.W., Reiner I.; Representation theory of finite groups and associative algebras. John Wiley & Sons, New York · London · Sydney., –1962. –689 pp.

[10] Dieudonne J.; Remarks on quasi-Frobenius rings. Illinois Journal of Mathematics., –1958. –2(3). –P.346-354. https://doi.org/10.1215/ijm/1255454538

[11] Gerstenhaber M., Kubo F.; On the deformation of the two dimensional associative algebras. JP Journal of Algebra, Number Theory and Applications., –2022. –54. –P.1-17.

[12] Kock J.; Frobenius algebras and 2D topological quantum field theories. London Mathematical Society. Cambridge University Press., –2003.

[13] Frobenius G.; Theorie der hyperkomplexen Grossen. Sitzungsberichte der Koniglich Preussischen Akademie der Wissenschaften., –1903. –24. –P.504-537; 634-645.

[14] Kadison L., Nikshych D.; Frobenius extensions and weak Hopf algebras. Journal of Algebra., –2001. –244. –P.312-342. https://doi.org/10.1006/jabr.2001.8911

[15] Lawvere F.W.; Ordinal sums and equational doctrines. In: Eckmann B. (ed.) Seminar on Triples and Categorical Homology Theory, ETH 1966/67. Lecture Notes in Mathematics, no.80. Springer-Verlag, New York., –1967. –P.141-155.

[16] Nakayama T.; On Frobeniusean algebras I. Annals of Mathematics., –1939. –40. –P.611-633.

[17] Nakayama T.; On Frobeniusean algebras II. Annals of Mathematics., –1941. –42. –P.1-21.

[18] Ooms A.; On Frobenius Lie algebras. Communications in Algebra., –1980. –8(1). –P.13-52. https://doi.org/10.1080/00927878008822445

[19] Radford D.E.; The structure of Hopf algebras with a projection. Journal of Algebra., –1985. –92(2). –P.322-347.

[20] Rakhimov I.S.; Algebraic structures on two-dimensional vector space over any basic field. Malaysian Journal of Mathematical Sciences., –2024. –18(2). –P.227-257. https://doi.org/10.47836/mjms.18.2.02

[21] Turner P.; Five lectures on Khovanov homology. arXiv:math/0606464., –2006.

[22] Quinn F.; Lectures on axiomatic topological quantum field theory. In: Geometry and Quantum Field Theory (Park City, UT, 1991). Amer. Math. Soc., Providence, RI., –1995. –P.323-453.

Asrorov D.U.,
Samarkand State University named after Sharof Rashidov, Samarkand, Uzbekistan.
email: 96asrorovdiorjon@mail.ru

Bekbaev U.D.,
Turin Polytechnic University in Tashkent, Tashkent, Uzbekistan
email: uralbekbaev@gmail.com

Rakhimov I.S.,
Universiti Teknologi MARA (UiTM), Shah Alam, Malaysia and V.I. Romanovsky Institute of Mathematics, Tashkent, Uzbekistan
email: isamiddin@uitm.edu.my

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.35-39
DOI: 10.29229/uzmj.2025-4-4

On Finite-Dimensional Approximations of the Time-Optimal Control Problem for the Heat Equation in a Rod

Azamov A. A., Bakhramov J. A.

Abstract. The time-optimal problem is considered for the controllable heat conduction process in a rod. Using the Fourier method, the problem is usually reduced to an infinite system of one-dimensional control equations whose control parameters are connected by complex relationships that make it difficult to solve. This article presents a method for regrouping the terms of the Fourier series of the control function, enabling the reduction of the problem to a finite-dimensional framework. The resulting reduced problem facilitates the more effective construction of a suboptimal control.

Keywords: heat equation, control problem, time optimality, Fourier expansion, regrouping, finite dimensional reduction, suboptimal control.

MSC (2020): 49J20, 93C20.

1. Statement of the problem

The time-optimal problem for the heat equation is formulated as follows. Consider the equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + v(t,x) \tag{1.1}$$

in the domain $t \geq 0$, $0 \leq x \leq \pi$ with boundary conditions $u(t,0) = u(t,\pi) \equiv 0$ and a given initial state $u(0,x) = \varphi(x)$. Here, the function $v(t,x)$ from the class L_2, playing the role of control, should satisfy the constraint

$$\sup |v(t,x)| \leq v_0. \tag{1.2}$$

If $\varphi(\cdot) \in L_2^0(0,\pi)$ then the equation (1.1) has a unique solution $u(t,x)$ from the Sobolev class $H^{1,2}$ [1, 2, 3]. If at this $u(t,x) \equiv 0$ for some $t = T$, then the function $v(t,x)$ is called an admissible control due to the initial state $\varphi(v)$ and the number T is called a transition time. The time-optimal problem for (1.1) demands to find an admissible control $\hat{v}(t,x)$ such that the transition time is minimal. We call the formulated task a Chernousko problem, as in the work [10] F. Chernousko reduced this problem to an infinite system of one-dimensional control problems

$$\frac{du_n}{dt} = -n^2 u_n + v_n, \quad u_n(0) = \varphi_n \tag{1.3}$$

where u_n, v_n, φ_n are the coefficients of the Fourier expansions of the functions $u(t,x)$, $v(t,x)$, and $\varphi(x)$, respectively, more precisely

$$u_n(t) = \frac{1}{\pi}\int_0^\pi u(t,x)\sin x\,dx,$$
$$v_n(t) = \frac{1}{\pi}\int_0^\pi v(t,x)\sin x\,dx,$$
$$\varphi_n = \frac{1}{\pi}\int_0^\pi \varphi(x)\sin x\,dx.$$

The system (1.3) can be easily solved in the Hilbert space l_2 [11]. However, a control sequence $(v_1, v_2, \ldots, v_k, \ldots) \in l_2$ is subject to the constraint.

$$\sup_{0\leq x\leq\pi} |v_1 \sin x + v_2 \sin 2x + \ldots + v_n \sin nx + \ldots| \leq v_0. \tag{1.4}$$

In other words, the region of values of a control sequence for (1.3), (1.4) is given by the formula

$$V = \left\{ (v_n) \in l_2 \mid \sup_x |v_1 \sin x + v_2 \sin 2x + \ldots + v_n \sin nx + ...| \leq v_0 \right\}. \tag{1.5}$$

Obviously (1.5) is a closed and convex set, but it is unknown whether V is compact. In any case, the time-optimal problem for (1.4), (1.5) remains difficult to solve. That is why it is natural to attempt to find suboptimal controls. It seems logical to truncate the partial sums of the Fourier expansions of $u(t,x)$, $v(t,x)$, and $\varphi(x)$, replacing original the infinite-dimensional problem with a finite-dimensional one. However, it would be difficult to estimate the remainder terms of the Fourier expansions for $v(t,x)$ and $\varphi(x)$.

In the paper [10], the region (1.5) is replaced by the set

$$V_1 = \{(v_n) \in l_2 \mid |v_n| \le U_n,\, n = 1, 2, \ldots\} \tag{1.6}$$

where the sequence $\{U_n\}$ should be chosen to satisfy the following condition

$$\sum U_n = v_0 \tag{1.7}$$

due to (1.2). F. Chernousko showed that the sequence U_n can be chosen so that the time-optimal control for the fully separated system of control equations

$$\dot{u}_n = -n^2 u_n + v_n, \quad |v_n| \le U_n, \quad n = 1, 2, \ldots \tag{1.8}$$

has a solution possessing the property $u_n(T_1) = 0$ for all n and the same time T_1. The inclusion $V_1 \subset V$, implies that $T_1 \ge T_{\text{opt}}$, which means the solution of the system (1.3), (1.6) can serve as a suboptimal control. Taking V_1 instead of V can be interpreted as replacing the domain V with a Hilbert brick embedded in it. In [12], a stronger result was obtained by replacing V with the Hilbert "octahedron" (cocube):

$$\{v_n \in l_2 \mid |v_1| + |v_2| + \ldots + |v_n| + \ldots \le v_0\} \tag{1.9}$$

Another approach to construct suboptimal controls was suggested in the papers [13], [15], [16], based on the idea of regrouping the terms of the expansion $\sum_k v_k \sin kx$ in such a way that the infinite system of finite-dimensional problems would be reduced to a single finite-dimensional problem. This is based on the following arithmetic assertion, that has an independent interest. A set of positive integers $\{n_1, n_2, \ldots, n_m\}$ such that $n_1 < n_2 < \ldots < n_m$, will be called a decomposition basis if the partition

$$\mathbb{N}^+ = \bigoplus_{k \in K} \{kn_1, kn_2, \ldots, kn_m\} \tag{1.10}$$

holds for some $K \subset \mathbb{N}^+$. (The symbol $\oplus$ denotes a union of non-intersecting subsets.)

Obviously, the sets $\{1\}$ and $\{1, p\}$ (where $p > 1$) are decomposition bases.

Proposition 1.1. *The set $\{1, 2, 4, \ldots, 2^{m-1}\}$ forms a decomposition basis.*

Proof. One may assume $m \ge 3$. We will construct the appropriate subset $K = \{k_1, k_2, \ldots, k_n, \ldots\}$ possessing the property (1.10). Every number $n \in \mathbb{N}^+$ can be represented in the form $n = 2^{sm+r}p$, where $s \in \mathbb{N}$ and p is odd, $r = 0, 1, 2, \ldots, m-1$. Thus $n \in K$ if and only if $r = 0$.

The list of respective arrays in the decomposition for $m = 3$ is as follows

$$\mathbf{1}, 2, 4; \quad \mathbf{3}, 6, 12; \quad \mathbf{5}, 10, 20; \quad \mathbf{7}, 14, 28; \quad \mathbf{8}, 16, 32;$$

$$\mathbf{9}, 18, 36; \quad \mathbf{11}, 22, 44; \quad \mathbf{13}, 26, 52; \quad \ldots; \quad \mathbf{23}, 46, 92; \quad \mathbf{24}, 48, 96; \quad \ldots$$

It should be verified that the arrays

$$\{k_i, 2k_i, 4k_i, \ldots, 2^{m-1}k_i\},$$

$$\{k_j, 2k_j, 4k_j, \ldots, 2^{m-1}k_j\}$$

do not intersect for $i < j$. Indeed, the relation $2^\alpha k_i = 2^\beta k_j$ where $i < j$, $\alpha, \beta \in \{0, 1, 2, \ldots, m-1\}$ leads to a contradiction because of the representations $k_i = 2^{s_i m} p_i$ and $k_j = 2^{s_j m} p_j$. □

2. Regrouping method for finite-dimensional approximations

Now using the representation (1.10), terms of the Fourier expansion for a control function $v(t,x)$ will be regrouped:

$$v(t,x) = \sum_{i=1}^{\infty}(v_{k_i}\sin k_i x + v_{2k_i}\sin 2k_i x + \ldots + v_{2^{m-1}k_i}\sin 2^{m-1}k_i x). \tag{2.1}$$

Further, instead of the condition 2.1, we consider the more stronger constraint (naturally loosing optimality)

$$\sum_{i=1}^{\infty}|v_{k_i}\sin k_i x + v_{2k_i}\sin 2k_i x + \ldots + v_{2^{m-1}k_i}\sin 2^{m-1}k_i x| \le v_0 \tag{2.2}$$

for all x. Then, following Chernousko's approach, consider the sequence of constraints

$$|v_{k_i}\sin k_i x + v_{2k_i}\sin 2k_i x + \ldots + v_{2^{m-1}k_i}\sin 2^{m-1}k_i x| \le U_i, \tag{2.3}$$

for $i = 1, 2, \ldots$, with $\sum_{i=1}^{\infty} U_i = v_0$.

In this way, we get the sequence of m-dimensional control problems

$$\begin{cases} \dot{u}_{k_i} = -k_i^2 u_{k_i} + v_{k_i}, \\ \dot{u}_{2k_i} = -(2k_i)^2 u_{2k_i} + v_{2k_i}, \\ \ldots \\ \dot{u}_{2^{m-1}k_i} = -2^{2(m-1)}k_i^2 u_{2^{m-1}k_i} + v_{2^{m-1}k_i}, \end{cases} \tag{2.4}$$

for $i = 1, 2, \ldots$, with the constraints (2.3). Note that the collection (2.4) is equivalent to system (1.3), but conditions (2.3) is stronger than (2.2).

Now let us perform the following transformation.

$$y_1 = \frac{k_i^2}{\mu_{k_i}}u_{k_i}, \quad y_2 = \frac{k_i^2}{\mu_{k_i}}u_{2k_i}, \quad \ldots, \quad y_m = \frac{k_i^2}{\mu_{k_i}}u_{2^{m-1}k_i}, \quad \tau = k_i^2 t.$$

$$v_{k_i} = \mu_{k_i}w_1, \quad v_{2k_i} = \mu_{k_i}w_2, \quad \ldots, \quad v_{2^{m-1}k_i} = \mu_{k_i}w_m.$$

Overall, all systems (2.4) will be reformulated to the single m-dimensional control system:

$$\begin{cases} \dot{y}_1 = -y_1 + w_1, \\ \dot{y}_2 = -4y_2 + w_2, \\ \ldots \\ \dot{y}_m = -4^{m-1}y_m + w_m, \end{cases} \tag{2.5}$$

where

$$w_1 = \frac{1}{\mu_{k_i}}v_{k_i}, \quad w_2 = \frac{1}{\mu_{k_i}}v_{2k_i}, \quad \ldots, \quad w_m = \frac{1}{\mu_{k_i}}v_{2^{m-1}k_i}.$$

Now consider the transformation of the constraints (2.3) for the control parameter of the system (2.5). For that, we replace 2.3 by the even more rigid restriction

$$\max_x \left|w_1 \sin k_i x + w_2 \sin 2k_i x + \ldots + w_m \sin 2^{m-1}k_i x\right| \le 1, \tag{2.6}$$

for $k = 1, 2, \ldots$, that implies 2.3 if $\sum_i \mu_{k_i} = v_0$. Obviously 2.6 can be written in the form

$$\max_{\bar{x}} \left|w_1 \sin \bar{x} + w_2 \sin 2\bar{x} + \ldots + w_m \sin 2^{m-1}\bar{x}\right| \le 1, \tag{2.7}$$

The time-optimal problem (2.5), (2.7) is a concrete finite-dimensional linear system with the region W of vectors $\mathbf{w} = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^m$, satisfying 2.7. Obviously, W is a convex compact body (containing the cube $|w_1| + |w_2| + \ldots + |w_m| \le 1$). Therefore, there exists an optimal control $\hat{\mathbf{w}}(t)$ with a bang-bang property, that can be found by the Pontryagin maximum principle [4], [6]. It is clear that the optimal control $\hat{\mathbf{w}}(t)$ of the system (2.5), (2.7) generates optimal controls $(\hat{v}_{k_i}, \hat{v}_{2k_i}, \ldots, \hat{v}_{2^{m-1}k_i})$ for every problem (2.3) - (2.4), $i = 1, 2, \ldots$.

Finally, by selecting the sequence of numbers $U_1, U_2, \ldots$ according to the technique proposed by Chernousko, it is possible to guarantee that the optimal transition time for all systems (2.3) - (2.4) is identical. Furthermore, by combining the control functions

$$(\hat{v}_{k_i},\ \hat{v}_{2k_i},\ \ldots,\ \hat{v}_{2^{m-1}k_i}), \quad i = 1, 2, \ldots,$$

and subsequently applying inverse regrouping, one obtains a suboptimal control for the problem (2.3) – (2.4).

Concluding Remarks

As demonstrated by F. Chernousko, although the Fourier method does not enable the direct construction of an optimal control for the heat conduction equation in a rod, it remains highly effective for developing finite-dimensional approximations. In contrast to the classical approach, when the problem is reduced to a finite-dimensional system by truncating partial sums of the Fourier seriesthe regrouping method proposed in the present work yields an improvement in the quality of suboptimal control.

A separate investigation will be devoted to a detailed comparison of these two approaches to finite-dimensional approximations. It should be noted that, for small values of m, the regrouping method produces systems for which the optimal control can be determined explicitly [13] (see also [14] - [16]).

The authors express their gratitude to D. Ruzimuradova for her valuable assistance in preparing this article for publication.

References

[1] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, 1968.

[2] S. Mizohata, The Theory of Partial Differential Equations, Cambridge University Press, 1973.

[3] N. Dunford, J. T. Schwartz, Linear Operators, Part II: Spectral Theory, Self-Adjoint Operators in Hilbert Space, Wiley, Pure & Applied Mathematics.

[4] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko. The mathematical theory of optimal processes. Interscience Publishers, 1962.

[5] E. B. Lee, L. Markus. Foundation of Optimal Control Theory. Krieger Pub Co, 1986.

[6] V. I. Blagodatskikh. Introduction to Optimal Control. Moscow: Vysshaya Shkola, 2001. (in Russian)

[7] J. L. Lions. Optimal Control of Systems Governed by Partial Differential Equations. Springer-Verlag, 1971.

[8] V. M. Alexeev, V. M. Tikhomirov, S. V. Fomin. Optimal Control. Moscow: Nauka, 2005. (in Russian)

[9] L. C. Evans. Partial Differential Equations. American Mathematical Society, Providence, Rhode Island, 2010.

[10] F. L. Chernousko, Bounded controls in distributed-parameter systems, *Journal of Applied Mathematics and Mechanics*, **56**(5) (1992).

[11] G. I. Ibragimov, A. B. Ja'afaru, On existence and uniqueness of solution to countable number of first-order differential equations in the space ℓ^2, *Journal of Applied Sciences Research*, **7**(12) (2011), 1860–1864.

[12] A. A. Azamov, M. B. Ruziboev, The time-optimal problem for evolutionary partial differential equations, *Journal of Applied Mathematics and Mechanics*, **77** (2013), 220–224.

[13] A. A. Azamov, J. A. Bakhramov, O. S. Akhmedov, On the Chernous'ko time-optimal problem for the equation of heat conductivity in a rod *Ural Mathematical Journal*, **5**(1), (2019), 13–23. DOI: 10.15826/umj.2019.1.002.

[14] A. A. Azamov, J. A. Bakhramov, Synthesis of Suboptimal Control for Time-optimal problem for one dimensional Heat Equation. *Reports of the Academy of Sciences of Uzbekistan*, **3**, (2018), 10–12.

[15] J. A. Bakhramov, Synthesis of suboptimal control in the time-optimal problem. *Uzbek Mathematical Journal*, **65**(1), (2021), 42–47.

[16] A. A. Abdug'aniyev, J. A. Bakhramov, Synthesis of a suboptimal control in three dimensional time-optimal problem for the heat conductivity equation in a rod. *Bulletin of the Institute of Mathematics*, **5**(3), (2022), 29–33.

Azamov A.A.,
V.I.Romanovskiy Institute of mathematics,
Uzbekistan Academy of Sciences, Tashkent, Uzbekistan.
email: abdulla.azamov@gmail.com

Bakhramov J.A.,
V.I.Romanovskiy Institute of mathematics,
Uzbekistan Academy of Sciences, Tashkent, Uzbekistan.
email: bahramovjasurbek@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.40-45
DOI: 10.29229/uzmj.2025-4-5

Symmetric Leibniz algebras whose underlying Lie algebra is almost filiform

Bozorova S.N., Choriyeva I.B.

Abstract. In this paper, we classify the symmetric Leibniz algebras whose underlying Lie algebra is almost filiform. We also describe the symmetric Leibniz algebras associated with Heisenberg and triangular Lie algebras. Moreover, we prove that there is no symmetric Leibniz algebra whose underlying Lie algebra is perfect.

Keywords: Nilpotent Lie algebras, symmetric Leibniz algebras, perfect algebras, almost filiform Lie algebras.

MSC (2020): 17A32; 17B30.

1. Introduction

Leibniz algebras are generalizations of Lie algebras, which are defined with the property that any operator of left (or right) multiplication is a derivation. Leibniz algebras were first introduced in the work of Bloh [7] under the name D-algebras in 1965. Then they were rediscovered by Loday [12], who called them Leibniz algebras. Since the left and right Leibniz algebras have opposite properties, researchers investigated just left (or right) Leibniz algebra as a Leibniz algebra. In the recent years, the theory of Leibniz algebras has been actively examined and many results on Lie algebras have been extended to Leibniz algebras [4, 10, 11, 14].

Symmetric Leibniz algebras are intersections of the left and right Leibniz algebras. The first characterization and theory of symmetric Leibniz algebras are found in the paper of Benayadi and Hidri [6]. They proved that quadratic left (or right) Leibniz algebra which has properties of invariant, non-degenerate and symmetric bilinear forms is a symmetric Leibniz algebras. Recently, the theory of symmetric Leibniz algebras has been intensively studied and many works have been devoted to the investigation of this theory [5, 6, 11, 13]. Symmetric Leibniz algebras are associated to Lie racks [1] and any symmetric Leibniz algebra is flexible, power-associative, and a nilalgebra with nilindex 3. Symmetric Leibniz algebra gives a Poisson algebra under commutator and anticommutator multiplications [2].

It is difficult and fundamental problem that to classify up to isomorphism of any class of algebras. There are several methods for the classification of algebras that were used for the Leibniz algebras. Barreiro and Benayadi in [5] gave a method for the classification of symmetric Leibniz algebras, which is based on the property that a symmetric Leibniz algebra forms a Poisson algebra with respect to the commutator and anticommutator. Using this method, the classification of symmetric Leibniz algebras, whose underlying Lie algebra is filiform was obtained in [8]. The complete classification of complex five-dimensional symmetric Leibniz algebras can be found in [3, 9]. In this work, using this method we give the classification of symmetric Leibniz algebras whose underlying Lie algebra is almost filiform. Moreover, we prove that a non-Lie symmetric Leibniz algebra associated with perfect Lie algebras does not exist.

2. Preliminaries

In this section, we give the basic concepts, definitions and preliminary results which are used in this paper.

Definition 2.1. An algebra $(\mathcal{L}, [-,-])$ over a field $\mathbb{F}$ is called Lie algebra if for any $u, v, w \in \mathcal{L}$ the following identities hold:

$$[u, v] = -[v, u],$$

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0.$$

Definition 2.2. An algebra $(\mathcal{L}, \cdot)$ is said to be a symmetric Leibniz algebra, if for any $u, v, w \in \mathcal{L}$ the following identities hold:

$$u \cdot (v \cdot w) = (u \cdot v) \cdot w + v \cdot (u \cdot w),$$
$$(v \cdot w) \cdot u = (v \cdot u) \cdot w + v \cdot (w \cdot u).$$

Let $(\mathcal{L}, \cdot)$ be an algebra. For all $u, v \in \mathcal{L}$, we define $[-,-]$ and $\circ$ as follows

$$[u, v] = \frac{1}{2}(u \cdot v - v \cdot u), \qquad u \circ v = \frac{1}{2}(u \cdot v + v \cdot u).$$

Proposition 2.3. [5] *Let $(\mathcal{L}, \cdot)$ be an algebra. The following assertions are equivalent:*

1. $(\mathcal{L}, \cdot)$ *is a symmetric Leibniz algebra.*
2. *The following conditions hold:*
   (a) $(\mathcal{L}, [-,-])$ *is a Lie algebra.*
   (b) *For any $x, y \in \mathcal{L}$, $x \circ y$ belongs to the center of $(\mathcal{L}, [-,-])$.*
   (c) *For any $x, y, z \in \mathcal{L}$, $([x, y]) \circ z = 0$ and $(x \circ y) \circ z = 0$.*

According to this Proposition, any symmetric Leibniz algebra is given by a Lie algebra $(\mathcal{L}, [-,-])$ and a symmetric bilinear form $\omega : \mathcal{L} \times \mathcal{L} \to Z(\mathcal{L})$, where $Z(\mathcal{L})$ is the center of the Lie algebra, such that for any $x, y, z \in \mathcal{L}$,

$$\omega([x, y], z) = \omega(\omega(x, y), z) = 0. \tag{2.1}$$

Then the product of the symmetric Leibniz algebra is given by

$$x \cdot_\omega y = [x, y] + \omega(x, y).$$

Proposition 2.4. [1] *Let $(\mathcal{L}, [-,-])$ be a Lie algebra and ω and μ two solutions of* (2.1)*. Then $(\mathcal{L}, \cdot_\omega)$ isomorphic to $(\mathcal{L}, \cdot_\mu)$ if and only if there exists an automorphism A of $(\mathcal{L}, [-,-])$ such that*

$$\mu(x, y) = A^{-1}\big(\omega(A(x), A(y))\big).$$

In the following proposition, we consider symmetric Leibniz algebras whose underlying Lie algebra is perfect.

Proposition 2.5. *Let $\mathcal{L}$ be a complex symmetric Leibniz algebra, whose underlying Lie algebra is perfect, then it is a Lie algebra.*

Proof. Let $(\mathcal{L}, [-,-])$ be a perfect Lie algebra, i.e., $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$. Then, for any $x \in \mathcal{L}$, there exist $z_i, t_i \in \mathcal{L}$, such that $x = \sum_i \alpha_i [z_i, t_i]$. Consider a symmetric bilinear form ω satisfying the condition (2.1). Then, we obtain $\omega(x, y) = \sum_i \alpha_i \omega([z_i, t_i], y) = 0$, for any $y \in \mathcal{L}$. Hence, for any $x, y \in \mathcal{L}$, we have $x \cdot y = [x, y] + \omega(x, y) = [x, y]$. Thus, $(\mathcal{L}, \cdot)$ is a Lie algebra. □

Corollary 2.6. *The symmetric Leibniz algebra structure for the following Lie algebras is trivial:*

- *Schrödinger algebra*

$$\begin{array}{llll} \mathcal{S}_n: & [x_i, y_i] = z, & [h, x_i] = x_i, & [s_{j,k}, x_i] = \delta_{k,i} x_j - \delta_{j,i} x_k, \\ & [e, f] = h, & [h, y_i] = -y_i, & [s_{j,k}, y_i] = \delta_{k,i} y_j - \delta_{j,i} y_k, \\ & [h, e] = 2e, & [e, y_i] = x_i, & [s_{j,k}, s_{l,m}] = \delta_{l,k} s_{j,m} + \delta_{j,m} s_{k,l} + \delta_{m,k} s_{l,j} + \delta_{l,j} s_{m,k}, \\ & [h, f] = -2f, & [f, x_i] = y_i, & 1 \le i \le n, \quad 1 \le j \le k \le n, \quad 1 \le l \le m \le n. \end{array}$$

- *n-th Schrödinger algebra*

$$\begin{array}{lllll} \mathfrak{sch}_n: & [x_i, y_i] = z, & [h, x_i] = x_i, & [e, f] = h, & [h, y_i] = -y_i, \\ & [h, e] = 2e, & [e, y_i] = x_i, & [h, f] = -2f, & [f, x_i] = y_i, \quad 1 \le i \le n. \end{array}$$

- *Virasoro algebra,*

$$Vir: \quad [e_i, e_j] = (i-j)e_{i+j} + \delta_{i,-j}\tfrac{i^3-i}{12}c.$$

- *Not-finitely graded Virasoro algebra with basis* $\{e_{\alpha,i}, c \mid \alpha \in \Gamma, i \in Z_+\}$

$$\widehat{W}(\Gamma): \quad [e_{\alpha,i}, e_{\beta,j}] = (\beta-\alpha)e_{\alpha+\beta,i+j} + (j-i)e_{\alpha+\beta,i+j+1} + \delta_{\alpha+\beta,0}\delta_{i+j,0}\tfrac{\alpha^3-\alpha}{12}c,$$

where $\alpha, \beta \in \Gamma, i, j \in \mathcal{Z}_+$.

- *Virasoro-like algebra with basis* $\{e_{\alpha,i}, c \mid \alpha \in \Gamma, i \in Z\}$

$$\tilde{\mathcal{W}(\Gamma)}: \quad [e_{\alpha,i}, e_{\beta,j}] = (\beta-\alpha)e_{\alpha+\beta,i+j} + (j-i)e_{\alpha+\beta,i+j-1} + \delta_{\alpha+\beta,0}(\delta_{i+j,-1}\alpha^3 + 3i\delta_{i+j,0}\alpha^2 + \\ +3i(i-1)\delta_{i+j,1}\alpha + i(i-1)(i-2)\delta_{i+j,2})c,$$

where $\alpha, \beta \in \Gamma, i, j \in \mathcal{Z}$.

- *The twisted Heisenberg-Virasoro algebra with basis* $\{e_i, f_j, c, c_1, c_2 \mid i, j \in Z\}$

$$H_{Vir}: \quad \begin{array}{l} [e_i, e_j] = (j-i)e_{i+j} + \delta_{i+j,0}\tfrac{i^3-i}{12}c, \\ [f_i, f_j] = i\delta_{i+j,0}c_1, \\ [e_i, f_j] = jf_{i+j} + \delta_{i+j,0}(i^2+i)c_2. \end{array}$$

Where $\delta_{i,j}$ *are Kronecker symbols.*

3. Main result

In this section, we classify symmetric Leibniz algebras associated with the almost filiform algebra $\mathbf{n}_{n,3}$. Furthermore, we also describe the symmetric Leibniz algebras whose underlying Lie algebra is Heisenberg and triangular.

The Lie algebra $\mathbf{n}_{n,3}$ is a nilpotent Lie algebra with the table of multiplications:

$$\mathbf{n}_{n,3} \quad : \quad [e_2, e_n] = e_1, \quad [e_3, e_{n-1}] = e_1, \quad [e_k, e_{n-1}] = e_{k-1}, \quad [e_{n-1}, e_n] = e_2, \quad 4 \le k \le n-2.$$

This algebra is called almost filiform and can be found in the monograph by Snobl and Winternitz [15].

Let $\mathcal{M}$ be a complex symmetric Leibniz algebra whose underlying Lie algebra is $\mathbf{n}_{n,3}$. Since $Z(\mathbf{n}_{n,3}) = \mathrm{span}\{e_1\}$, then by straightforward computations, we get that the corresponding symmetric bilinear form $\omega : \mathbf{n}_{n,3} \times \mathbf{n}_{n,3} \to Z(\mathbf{n}_{n,3})$ satisfying equation (2.1) is

$$\omega(e_{n-2}, e_{n-2}) = A_1 e_1, \quad \omega(e_{n-2}, e_{n-1}) = A_2 e_1, \quad \omega(e_{n-2}, e_n) = A_3 e_1,$$

$$\omega(e_{n-1}, e_n - 1) = A_4 e_1, \quad \omega(e_{n-1}, e_n) = A_5 e_1, \quad \omega(e_n, e_n) = A_6 e_1,$$

where $(A_1, A_2, A_3, A_4, A_5, A_6) \neq (0,0,0,0,0,0)$.

Then, considering the multiplication $x \cdot y = [x, y] + \omega(x, y)$, we obtain that any symmetric Leibniz algebra $\mathcal{M}$ associated with the almost filiform algebra has the following product

$$\mathcal{M}(A_1, A_2, A_3, A_4, A_5, A_6): \quad \begin{array}{lll} e_2 \cdot e_n = e_1, & e_{n-1} \cdot e_n = e_2 + A_5 e_1, & e_{n-2} \cdot e_{n-2} = A_1 e_1, \\ e_n \cdot e_2 = -e_1, & e_n \cdot e_{n-1} = -e_2 + A_5 e_1, & e_{n-2} \cdot e_n = A_3 e_1, \\ e_3 \cdot e_{n-1} = e_1, & e_{n-2} \cdot e_{n-1} = e_{n-3} + A_2 e_1, & e_{n-1} \cdot e_{n-1} = A_4 e_1, \\ e_{n-1} \cdot e_3 = -e_1, & e_{n-1} \cdot e_{n-2} = -e_{n-3} + A_2 e_1, & e_n \cdot e_n = A_6 e_1, \\ e_k \cdot e_{n-1} = e_{k-1}, & e_{n-1} \cdot e_k = -e_{k-1}, & 4 \le k \le n-3. \end{array}$$

It is not difficult to obtain the matrix form of the group of automorphisms of the algebra $\mathbf{n}_{n,3}$ is

$$A = \begin{pmatrix}
a^2b^{2n-8} & a_{2,n-1}ab^{n-5} - a_{2,n}d - a_{3,n}b^2 & a_{n-3,n-2}b^{2n-10} & a_{n-4,n-2}b^{2n-12} & \cdots & a_{3,n-2}b^2 & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\
0 & ab^{n-3} & 0 & 0 & \cdots & 0 & 0 & a_{2,n-1} & a_{2,n} \\
0 & -ab^{n-5}d & a^2b^{2n-10} & a_{n-3,n-2}b^{2n-12} & \cdots & a_{4,n-2}b^2 & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\
0 & 0 & 0 & a^2b^{2n-12} & \cdots & a_{5,n-2}b^2 & a_{4,n-2} & a_{4,n-1} & ab^{n-7}d \\
0 & 0 & 0 & 0 & \cdots & a_{6,n-2}b^2 & a_{5,n-2} & a_{5,n-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & a_{7,n-2}b^2 & a_{6,n-2} & a_{6,n-1} & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & a^2b^2 & a_{n-3,n-2} & a_{n-3,n-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & a^2 & c & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & b^2 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & d & ab^{n-5}
\end{pmatrix},$$

Then, using Proposition (2.4), we obtain the following isomorphism criteria for the algebras of the previous classes.

Proposition 3.1. *Two algebras $\mathcal{M}(A_1, A_2, A_3, A_4, A_5, A_6)$ and $\mathcal{M}(A'_1, A'_2, A'_3, A'_4, A'_5, A'_6)$ are isomorphic, if and only if there exist a, b, c, d such that*

$$\begin{aligned}
A'_1 &= \tfrac{a^2A_1}{b^{2n-8}}, & A'_2 &= \tfrac{cA_1+b^2A_2+dA_3}{b^{2n-8}}, \quad A'_3 = \tfrac{aA_3}{b^{n-3}}, \\
A'_4 &= \tfrac{c^2A_1+2b^2cA_2+2cdA_3+b^4A_4+2b^2dA_5+d^2A_6}{a^2b^{2n-8}}, & A'_5 &= \tfrac{cA_3+b^2A_5+dA_6}{ab^{n-3}}, \quad A'_6 = \tfrac{A_6}{b^2}.
\end{aligned} \tag{3.1}$$

In the following theorem, we give the main result of the work.

Theorem 3.2. *Let $\mathcal{L}$ be an n-dimensional complex symmetric Leibniz algebra, whose underlying Lie algebra is $\boldsymbol{n}_{n,3}$, then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$\mathcal{L}_1^{\alpha,\beta} = \mathcal{M}(1,0,1,\alpha,0,\beta)$ $\mathcal{L}_2^{\alpha,\beta} = \mathcal{M}(1,0,1,0,\alpha,\beta)$ $\mathcal{L}_3^{\alpha} = \mathcal{M}(1,0,0,\alpha,0,1)$ $\mathcal{L}_4 = \mathcal{M}(1,0,0,0,1,0)$
$\mathcal{L}_5 = \mathcal{M}(1,0,0,1,0,0)$ $\mathcal{L}_6 = \mathcal{M}(1,0,0,0,0,0)$ $\mathcal{L}_7^{\alpha} = \mathcal{M}(0,0,1,\alpha,0,1)$ $\mathcal{L}_8 = \mathcal{M}(0,0,1,1,0,0)$
$\mathcal{L}_9 = \mathcal{M}(0,0,1,0,0,0)$ $\mathcal{L}_{10}^{\alpha} = \mathcal{M}(0,\alpha,0,0,0,1)$ $\mathcal{L}_{11} = \mathcal{M}(0,0,0,1,0,1)$ $\mathcal{L}_{12} = \mathcal{M}(0,0,0,0,0,1)$
$\mathcal{L}_{13} = \mathcal{M}(0,1,0,0,1,0)$ $\mathcal{L}_{14} = \mathcal{M}(0,1,0,0,0,0)$ $\mathcal{L}_{15} = \mathcal{M}(0,0,0,0,1,0)$ $\mathcal{L}_{16} = \mathcal{M}(0,0,0,1,0,0)$.

Proof. We make the following denotations:

$$\Delta_1 = A_1, \quad \Delta_2 = A_3, \quad \Delta_3 = A_1A_6 - A_3^2.$$

From this and Proposition 3.1, we get that

$$\Delta'_1 = \frac{a^2}{b^{2n-8}}\Delta_1, \quad \Delta'_2 = \frac{a}{b^{n-3}}\Delta_2, \quad \Delta'_3 = \frac{a^2}{b^{2n-6}}\Delta_3.$$

Therefore, the Δ_i are relative invariants under isomorphisms in the given class of algebras. Now, we consider the following cases:

- If $\Delta_1 \neq 0$, $\Delta_2 \neq 0$, $\Delta_3 \neq 0$, then choosing $a = \frac{\Delta_2^{n-4}}{\sqrt{\Delta_1^{n-3}}}$, $b = \frac{\Delta_2}{\sqrt{\Delta_1}}$, $c = -\frac{\Delta_2(A_2\Delta_2+d\Delta_1)}{\Delta_1^2}$, $d = -\frac{\Delta_2^2(A_5\Delta_1-A_2\Delta_2)}{\Delta_1\Delta_3}$, we obtain the family of algebras $\mathcal{L}_1^{\alpha,\beta}$.
- If $\Delta_1 \neq 0$, $\Delta_2 \neq 0$, $\Delta_3 = 0$, then defining $\Delta_4 = A_1A_5 - A_2A_3$, we get $\Delta'_4 = \frac{a}{b^{3n-13}}\Delta_4$. Next, we consider the following subcases:
  - If $\Delta_4 \neq 0$, then choosing $a = \frac{\Delta_2^{n-4}}{\sqrt{\Delta_1^{n-3}}}$, $b = \frac{\Delta_2}{\sqrt{\Delta_1}}$, $c = -\frac{\Delta_2(A_2\Delta_2+d\Delta_1)}{\Delta_1^2}$, $d = -\frac{\Delta_2^2(A_4\Delta_1-A_2^2)}{2\Delta_1\Delta_4}$, we obtain the family of algebras $\mathcal{L}_2^{\alpha,\beta}$.
- If $\Delta_1 \neq 0$, $\Delta_2 = 0$, $\Delta_3 \neq 0$, then choosing $a = \sqrt{\frac{\Delta_3^{n-4}}{\Delta_1^{n-2}}}$, $b = \sqrt{\frac{\Delta_3}{\Delta_1}}$, $c = -\frac{\Delta_3A_2}{\Delta_1^2}$, $d = -A_5$, we obtain the family of algebras $\mathcal{L}_3^{\alpha}$.
- If $\Delta_1 \neq 0$, $\Delta_2 = 0$, $\Delta_3 = 0$, $A_5 \neq 0$ then choosing $a = \frac{b^{n-4}}{\sqrt{\Delta_1}}$, $b = \sqrt[4n-18]{\Delta_1A_5^2}$, $d = -\frac{\sqrt[2n-9]{\Delta_1A_5^2}(A_1A_4-A_2^2)}{2\Delta_1A_5}$, we obtain the algebra $\mathcal{L}_4$.
- If $\Delta_1 \neq 0$, $\Delta_2 = 0$, $\Delta_3 = 0$, $A_5 = 0$, then defining $\Delta_5 = A_1A_4 - A_2^2$, we get $\Delta'_5 = \frac{1}{b^{4n-20}}\Delta_5$. Next, we consider the following subcases:
  - If $\Delta_5 \neq 0$, then choosing $a = \frac{b^{n-4}}{\sqrt{\Delta_1}}$, $b = \sqrt[4n-20]{\Delta_5}$, we obtain the algebra $\mathcal{L}_5$.

- If $\Delta_5 \neq 0$, then choosing $a = \frac{b^{n-4}}{\sqrt{\Delta_1}}$, we obtain the algebra $\mathcal{L}_6$.

- If $\Delta_1 = 0$, $\Delta_2 \neq 0$, $A_6 \neq 0$, then choosing $a = \frac{\sqrt{A_6^{n-3}}}{\Delta_2}$, $b = \sqrt{A_6}$, we obtain the family of algebras $\mathcal{L}_7^\alpha$.
- If $\Delta_1 = 0$, $\Delta_2 \neq 0$, $A_6 = 0$, then defining $\Delta_6 = A_3A_4 - 2A_2A_5$, we get $\Delta_6' = \frac{1}{ab^{3n-15}}\Delta_6$. Next, we consider the following subcases:
  - If $\Delta_6 \neq 0$, then choosing $a = \frac{b^{n-3}}{\sqrt{\Delta_2}}$, $b = \sqrt[4n-18]{\Delta_2\Delta_6}$, we obtain the algebra $\mathcal{L}_8$.
  - If $\Delta_6 = 0$, then choosing $a = \frac{b^{n-3}}{\sqrt{\Delta_2}}$, we obtain the algebra $\mathcal{L}_9$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 \neq 0$, $A_2 \neq 0$, then choosing $b = \sqrt{A_6}$, $c = -\frac{A_4A_6 - A_5^2}{2A_2}$, $d = -A_5$, we obtain the family of algebras $\mathcal{L}_{10}^\alpha$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 \neq 0$, $A_2 = 0$, then defining $\Delta_7 = A_4A_6 - A_5^2$, we get $\Delta_7' = \frac{1}{a^2b^{2n-10}}\Delta_7$. Next, we consider the following subcases:
  - If $\Delta_7 \neq 0$, then choosing $a = \sqrt{\frac{\Delta_7}{A_6^{n-5}}}$, $b = \sqrt{A_6}$, we obtain the algebra $\mathcal{L}_{11}$.
  - If $\Delta_7 = 0$, then choosing $b = \sqrt{A_6}$, we obtain the algebra $\mathcal{L}_{12}$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 = 0$, $A_2 \neq 0$, $A_5 \neq 0$ then choosing $a = \frac{A_5}{\sqrt{A_2}}$, $b = \sqrt[2n-10]{A_2}$, $c = -\frac{A_4 \sqrt[n-5]{A_2} + 2dA_5}{2A_2}$, we obtain the algebra $\mathcal{L}_{13}$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 = 0$, $A_2 \neq 0$, $A_5 = 0$ then choosing $b = \sqrt[2n-10]{A_2}$, $c = -\frac{A_4}{2\sqrt[n-5]{A_2^{n-6}}}$, we obtain the algebra $\mathcal{L}_{14}$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 = 0$, $A_2 = 0$, $A_5 \neq 0$ then choosing $a = \frac{A_5}{b^{n-5}}$, we obtain the algebra $\mathcal{L}_{15}$.
- $\Delta_1 = 0$, $\Delta_2 = 0$, $A_6 = 0$, $A_2 = 0$, $A_5 = 0$, then $A_4 \neq 0$ and choosing $a = \frac{A_4}{b^{n-6}}$, we obtain the algebra $\mathcal{L}_{16}$. *Box*

Now, we consider a Lie algebra of strictly upper triangular matrices, which is denoted by $\mathbf{n}(n, \mathbb{F})$. It is known that $\mathbf{n}(n, \mathbb{F})$ has a basis $e_{i,j}$ for $i < j$, which the table of multiplication is

$$[e_{i,j}, e_{j,k}] = e_{i,k}, \quad 1 \leq i, j, k \leq n.$$

Consider a symmetric Leibniz algebra $\mathcal{L}$, whose underlying Lie algebra is $\mathbf{n}(n, \mathbb{F})$. Since $Z(\mathbf{n}(n, \mathbb{F})) = \mathrm{span}\{e_{1,n}\}$, then by straightforward computations, we get that the corresponding symmetric bilinear form ω satisfying the condition (2.1) is

$$\omega(e_{i,i+1}, e_{k,k+1}) = A_{i,k}e_{1,n}, \quad 1 \leq i \leq k \leq n-1.$$

Then we obtain the following result.

Proposition 3.3. *Any symmetric Leibniz algebra whose underlying Lie algebra is the algebra of upper triangular matrices has the following multiplications:*

$$\begin{array}{lll}
e_{i,j} \cdot e_{j,k} = -e_{j,k} \cdot e_{i,j} = e_{i,k}, & 1 \leq i, j, k \leq n, & (j-i, k-j) \neq (1,1), \\
e_{i,i+1} \cdot e_{i+1,i+2} = e_{i,i+2} + A_{i,i+1}e_{1,n}, & 1 \leq i \leq n, & \\
e_{i+1,i+2} \cdot e_{i,i+1} = -e_{i,i+2} + A_{i,i+1}e_{1,n}, & 1 \leq i \leq n, & \\
e_{i,i+1} \cdot e_{k,k+1} = e_{k,k+1} \cdot e_{i,i+1} = A_{i,k}e_{1,n}, & i \leq k, & i+1 \neq k.
\end{array}$$

Now, we consider a symmetric Leibniz algebra whose underlying Lie algebra is a Heisenberg algebra. A Heisenberg algebra $\mathfrak{h}$ is a $(2n+1)$-dimensional Lie algebra with basis $\{x_i, y_i, z\}$ and the following multiplication table:

$$[x_i, y_i] = z, \quad 1 \leq i \leq n.$$

Since $Z(\mathfrak{h}) = \mathrm{span}\{z\}$, then by straightforward computations, we get that the corresponding symmetric bilinear form $\omega : \mathfrak{h} \times \mathfrak{h} \to Z(\mathfrak{h})$ satisfying the equation (2.1) is

$$\omega(x_i, x_j) = \alpha_{ij}z, \quad \omega(y_i, y_j) = \beta_{ij}z, \quad \omega(x_i, y_j) = \gamma_{ij}z.$$

Proposition 3.4. *Any symmetric Leibniz algebra whose underlying Lie algebra is the Heisenberg algebra has the following multiplications:*

$$\begin{array}{lll} x_i \cdot y_i = (1+\gamma_{i,i})z, & y_i \cdot x_i = (-1+\gamma_{i,i})z, & 1 \le i \le n, \\ x_i \cdot x_j = x_j \cdot x_i = \alpha_{i,j}z, & y_i \cdot y_j = y_j \cdot y_i = \beta_{i,j}z, & 1 \le i,j \le n, \\ x_i \cdot y_j = y_j \cdot x_i = \gamma_{i,j}z, & 1 \le i \neq j \le n. & \end{array}$$

References

[1] Abchir H., Abid F., Boucetta M.; A class of Lie racks associated to symmetric Leibniz algebras. Journal of Algebra and Its Applications, – 2022. – 21(11). – 2250230.

[2] Albuquerque H., Barreiro E., Benayadi S., Boucetta M., Sánchez J.M.; Poisson algebras and symmetric Leibniz bialgebra structures on oscillator Lie algebras, Journal of Geometry and Physics, – 2021. – 160. – 103939.

[3] Alvarez M.A., Kaygorodov I.; The algebraic and geometric classification of nilpotent weakly associative and symmetric Leibniz algebras. Journal of Algebra, – 2021. – 588. – P. 278-314.

[4] Ayupov Sh.A., Omirov B.A., Rakhimov I.S.; Leibniz Algebras: Structure and Classification. Taylor and Francis Group Publisher, – 2019. – ISBN 0367354810. – 323 pp.

[5] Barreiro E., Benayadi S.; A new approach to Leibniz bialgebras. Algebras and Representation Theory, – 2016. – 19(1). – P. 71-101.

[6] Benayadi S., Hidri S.; Quadratic Leibniz algebras. Journal of Lie Theory, – 2014. – 24(3). – P. 737-759.

[7] Bloh A.,; On a generalization of the concept of Lie algebra. Doklady Akademii Nauk SSSR, – 1965. – 165. – P. 471-473.

[8] Chorieva I.B, Khudoyberdiyev A.Kh.; Classification of symmetric Leibniz algebras associated by naturally graded filiform Lie algebras. AIP Conference Proceedings, – 2023. – 2781. – 020072.

[9] Chorieva I.B, Khudoyberdiyev A.Kh.; Classification of five-dimensional symmetric Leibniz algebras. Bulletin of the Iranian Mathematical Society, – 2024. – 50(33).

[10] Feldvoss J.; Leibniz algebras as non-associative algebras. Contemporary Mathematics, – 2019. – 721. – P. 115-149.

[11] Jibladze M., Pirashvili T.; Lie theory for symmetric Leibniz algebras. Journal of Homotopy and Related Structures, – 2020. – 15 (1). – P. 167-183.

[12] Loday J.L.; Une version non commutative des algèbres de Lie: les algèbres de Leibniz. L'Enseignement Mathématique, – 1993. – 39. – P. 269-293.

[13] Remm E.; Weakly associative and symmetric Leibniz algebras. Journal of Lie Theory, – 2022. – 32(4). – P. 1171-1186.

[14] Siciliano S., Towers D.A.; On the subalgebra lattice of a Leibniz algebra. Communications in Algebra, – 2022. – 50(1). – P. 255-267.

[15] Snobl L., Winternitz P.; Classification and Identification of Lie algebras. CRM Monograph series, – 2017. – P. 366.

Bozorova S.N.,
National University of Uzbekistan, Tashkent, Uzbekistan
email: bozorovasitorabonu5@gmail.com
Choriyeva I.B.,
V.I.Romanovskiy Institute of mathematics,
Uzbekistan Academy of Sciences, Tashkent, Uzbekistan
email: irodachoriyeva1@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.46-52
DOI: 10.29229/uzmj.2025-4-6

On e^*-Semisimple Modules and the Associated Socle

Bouziri A.

Abstract. Let R be a ring, and let M be a right R-module. Recently, Baannon and Khalid introduced and studied the concept of e^*-essential submodules, extending the notion of essential submodules. In this context, we consider e^*-semisimple submodules, and we define the e^*-socle of M as the largest e^*-semisimple submodule of M. It turns out that several properties of the socle can be extended to the e^*-socle. It is shown that the e^*-socle of M is exactly the intersection of all its e^*-essential submodules.

Keywords: e^*-essential submodules, e^*-semisimple modules, socle, e^*-socle.
MSC (2020): 16D10, 16D60, 16D99.

1. Introduction

Throughout the paper, R will be an associative ring with identity and all modules are unital right R-modules.

Let M be a right R-module. A submodule N of M is said to be essential in M if, for every nonzero submodule K of M, we have $K \cap N \neq 0$. The socle of M, denoted by $\mathrm{Soc}(M)$, is the sum of all minimal (i.e., simple) submodules of M and, consequently, is the largest semisimple submodule of M. It is also equal to the intersection of all essential submodules of M [1, Proposition 9.7]. It is also worth recalling that

$$\mathrm{Soc}(M) = \mathrm{Tr_M}(\mathcal{S}) = \mathrm{Tr_M}\left(\bigoplus_{\mathrm{S}\in\mathcal{S}} \mathrm{S}\right) = \sum_{\mathrm{S}\in\mathcal{S}} \mathrm{Tr_M}(\mathrm{S}),$$

where $\mathcal{S}$ is a set of representatives of the simple right R-modules, and $\mathrm{Tr_M}(\mathcal{S})$ denotes the trace of $\mathcal{S}$ in M [1, Proposition 9.11].

A submodule N of M is called a small submodule if, whenever $N + L = M$ for some submodule L of M, it follows that $M = L$. An R-module M is said to be small if it is a small submodule of some R-module. It was shown that M is small if and only if it is small in its injective hull [4].

In 2022, Baannon and Khalid introduced and studied the notion of e^*-essential submodules; that is, submodules N of a module M such that $N \cap K \neq 0$ for every nonzero cosingular submodule K [2]. Recall that a right R-module M is called cosingular if mR is a small R-module for every element $m \in M$ [5, Definition 2.5].

In this article, we investigate, as an e^*-version of semisimple modules and the socle, the concepts of e^*-semisimple modules and e^*-socle. The motivation behind this study is that the study of the socle of a modules has led to some interesting properties and has shed more light on their structure (see, for instance, [1, Theorem 10.4]).

The paper is organized as follows: In Section 2, we provide a definition of e^*-semisimple submodules of a module M, and establish some fundamental properties. In Section 3, we define the e^*-socle of a module M, denoted by $\mathrm{e^*\text{-}Soc(M)}$, as the largest e^*-semisimple submodule within M. We show that $\mathrm{e^*\text{-}Soc(M)}$ is exactly the intersection of all e^*-essential submodules of M (see Proposition 3.2). Once again, it turns out that

$$\mathrm{e^*\text{-}Soc(M)} = \mathrm{Tr_M}(\mathcal{S}_\mathrm{M}) = \mathrm{Tr_M}(\bigoplus_{\mathrm{S}\in\mathcal{S}_\mathrm{M}} \mathrm{S}) = \sum_{\mathrm{S}\in\mathcal{S}_\mathrm{M}} \mathrm{Tr_M}(\mathrm{S}),$$

where $\mathcal{S}_M$ is a set of representatives of the e^*-simple submodule of M, and $\mathrm{Tr_M}(\mathcal{S}_\mathrm{M})$ denote the trace of $\mathcal{S}_M$ in M.

From now on, for a module M, $\mathrm{Soc(M)}$ will denote the socle of M. We use $N \leqslant M$, $N \leqslant_e M$ and $N \leqslant_{e^*} M$ to mean that N is a nonzero submodule, an essential submodule and an e^*-essential submodule of M, respectively. General background material can be found in [1, 3].

2. e^*-Semisimple modules

In this section we define e^*-semisimple (sub)modules and we provide some useful results concerning this new class of modules. The main result of this section is Theorem 2.10, which provides a characterization of e^*-semisimple submodules in a module M. Recall from [6] that a module M is said to be simple if $M \neq 0$ and has no proper nonzero submodules. Moreover, M is said to be semisimple if it is a direct sum of (possibly infinitely many) simple modules.

Definition 2.1. Let N be a submodule of a module M.

(1) We say that N is e^*-simple in M if N is simple and contained in every e^*-essential submodule of M.

(2) We say that N is e^*-semisimple in M if it is a direct sum of e^*-simple submodules. In particular, if $M = N$, we say that M is e^*-semisimple.

Let $(S_i)_{i\in I}$ be an indexed set of e^*-simple submodules of M. If $N = \bigoplus_{i\in I} S_i$, then $\bigoplus_{i\in I} S_i$ is called an e^*-semisimple decomposition of N. Clearly every e^*-simple submodule in M (resp., e^*-semisimple submodule in M) is simple (resp., semisimple). However, the converse is not necessarily the case, as shown in Example 2.2 (1).

Example 2.2. (1) Let R be a ring and M be an artinian R-module such that M has a non-essential e^*-essential submodule N [2, Examples 1]. Then M has a simple submodule which is not e^*-simple in M. Indeed, by [1, Corollary 10.11], Soc(M), the intersection of all essential submodules of M, is an essential submodule of M. If $\mathrm{Soc(M)} \subseteq \mathrm{N}$, then N must be essential. Hence, $\mathrm{Soc(M)} \nsubseteq \mathrm{N}$. Since, by [1, Proposition 9.7], Soc(M) is the sum of all simple submodules of M, M has a simple submodule S such that $S \nsubseteq N$. Hence, S is not e^*-simple in M and Soc(M) is a semisimple module which is not e^*-semisimple in M.

(2) Since every essential submodule is e^*-essential,

$$H := \bigcap_{L \leq_{e^*} M} L \subseteq \mathrm{Soc(M)} = \bigcap_{\mathrm{L} \leq_{e} \mathrm{M}} \mathrm{L}.$$

Thus, if $H \neq 0$, every nonzero submodule of H is an e^*-semisimple in M.

Remark 2.3. If M is a simple R-module, then M is e^*-simple in itself in the sense of Definition 2.1. Indeed, this follows from the fact that M has no proper e^*-essential submodule. This is why we do not define the notion of an e^*-simple module. Moreover, one should be careful to avoid confusion between the notions of "e^*-semisimple" and "e^*-semisimple in".

Lemma 2.4. *Let $N \leq M$ be a submodule. It is easy to verify the following assertions.*

(1) *N is e^*-simple in M if and and only if N is a simple submodule of* $\bigcap_{L \leqslant_{e^*} M} L$.

(2) *N is e^*-semisimple in M if and and only if N a semisimple submodule of* $\bigcap_{L \leq_{e^*} M} L$.

We will use the following lemma to support the conclusions which we will reach in the rest of this paper

Lemma 2.5. (1) *Let K and L be nonzero submodules of a module M. Assume that K is isomorphic to L and denote this isomorphism by $f : K \to L$. Then, K is e^*-simple in M if and only if $f(K) = L$ is e^*-simple in M.*

(2) *Let $g : M \to N$ be an isomorphism of modules and let K be a submodule of M. Then, K is e^*-simple in M if and only if $g(K)$ is e^*-simple in N.*

Consequently, the class of e^*-semisimple modules is closed under isomorphisms.

Proof. 1. Let K and L be submodules of a module M as in (1). It suffices to show that if K is e^*-simple in M, then so is L. Two cases are then possible.

Case 1. M has no nonzero cosingular submodule (see for instance [2, Example 1]). Then, every submodule of M is e^*-essential. Let P be a nonzero submodule of M. Since K is e^*-simple, $K \subseteq P$ and $K \subseteq L$. Hence, $P \cap L \neq 0$; so $P \cap L = L$ because L is simple. Then $L \subset P$. Hence, L is e^*-simple.

Case 2. M has a nonzero cosingular submodule C. Assume that K is e^*-simple. If P be an e^*-essential submodule of M, then $K \subseteq P$. We need to show that $L \subseteq P$, but, since L is simple, it suffice to prove that $L \cap P \neq 0$. Consider the following diagram:

$$\begin{array}{ccccc} K & \xrightarrow{\ i\ } & M & & \\ \downarrow{\scriptstyle f} & & \| & \searrow{\scriptstyle h} & \\ L & \xrightarrow{\ j\ } & M & \xrightarrow{\ \sigma\ } & \mathrm{E}(M), \end{array}$$

where i and j are the inclusions, σ is the injective envelope of M and h is an induced map such that $hi = \sigma jf$. Since, by [3, Proposition 1.11], $\sigma(M) \leq_e \mathrm{E}(M)$, $\sigma(P) \leq_{e^*} \mathrm{E}(M)$ by [2, Proposition 6]. If $h^{-1}(\sigma(P)) = 0$, h must be injective; so, by [5, Lemma 2.6], $h(C)$ is a nonzero cosingular submodule of $\mathrm{E}(M)$, so $h(C) \cap \sigma(P) \neq 0$, a contradiction. Thus, $h^{-1}(\sigma(P)) \neq 0$ and $h^{-1}(\sigma(P)) \leq_{e^*} M$ by [2, Proposition 2]. Then $K \subseteq h^{-1}(\sigma(P))$; so, $h(K) \subseteq \sigma(P)$.

Now let x be a nonzero element in K. Then, $h(x) = \sigma jf(x) = \sigma f(x) \neq 0$ because f and σ are injective. On the other hand, $h(x) \in \sigma(P)$, so, $h(x) = \sigma f(x) = \sigma(p)$ for some nonzero $p \in P$. But σ is injective; this implies that $f(x) = p$. Finally, $L \cap P \neq 0$ as desired.

2. It suffices to prove that if $K \leqslant \bigcap\limits_{L \leq_{e^*} M} L$, then $g(K) \leqslant \bigcap\limits_{L \leq_{e^*} N} L$. Indeed, suppose that $K \leqslant \bigcap\limits_{L \leq_{e^*} M} L$. Let $x \in K$ and L be an e^*-essential submodule of N; we must show that $g(x) \in L$. By [2, Proposition 2], $g^{-1}(L) \leqslant_{e^*} M$ ($g^{-1}(L) \neq 0$ because g is an isomorphism). Then, $x \in g^{-1}(L)$. Hence, $g(x) \in L$. □

Proposition 2.6. *Let M be a module, and let N and K be nonzero submodules of M with $K \subseteq N$. Then the following assertions hold true:*

(1) *If N is e^*-semisimple in M, then so is K.*

(2) *If $N \leq_{e^*} M$ and K is e^*-semismple in M, then K is e^*-semisimple in N.*

Proof. By [1, Proposition 9.4], K is semisimple. Hence, K is e^*-semisimple by Lemma 2.4.

2. By Lemma 2.4, it suffices to show that $K \subseteq \bigcap\limits_{L \leqslant_{e^*} N} L$. If L is an e^*-essential submodule of N, then, by [2, Proposition 1], L is also an e^*-essential submodule of M. Since K is e^*-semisimple in M, $K \subseteq L$ by Lemma 2.4. □

Proposition 2.7. *Let $(S_i)_{i \in I}$ be an indexed set of e^*-simple submodule of M. If S is a simple submodule of M such that*

$$S \cap \sum_{i \in I} S_i \neq 0$$

then there is an $i \in I$ such that $S \cong S_i$.

Proof. This follows from [1, Corollary 9.5] □

Before giving the main result of this section which characterizes e^*-semisimple submodules in a given module M, we need to recall some notions. Let $(M_i)_{i \in I}$ be an indexed set of submodules of a module M. Recall from [1, page 93] that $(M_i)_{i \in I}$ is said to be independent in case for each $j \in I$

$$M_j \cap (\sum_{i \neq j} M_i) = 0.$$

One of the most important results in the study of semisimple structures is that: if $(S_i)_{i\in I}$ is an indexed set of simple submodules of a module M such that $M = \sum_{i\in I} S_i$, then for each submodule N of M there is a subset $J \subseteq I$ such that $(S_j)_{j\in J}$ is independent and

$$M = N \bigoplus (\bigoplus_{j\in J} S_j)$$

[1, Lemma 9.2]. Since every e^*-simple submodule is simple, the next result follows immediately.

Proposition 2.8. *Let $(S_i)_{i\in I}$ be an indexed set of e*-simple submodules of a module M. If $N = \sum_{i\in I} S_i$, then for each submodule K of N there is a subset $J \subseteq I$ such that $(S_j)_{j\in J}$ is independent and*

$$N = K \bigoplus (\bigoplus_{j\in J} S_j).$$

Proof. This is a particular case of [1, Lemma 9.2]. □

An interesting consequence of the Proposition 2.8 arises if the submodule $K = 0$.

Corollary 2.9. *Let M a module. Let $N = \sum_{i\in I} S_i$ for an indexed set $(S_i)_{i\in I}$ of e*-simple submodules of M, then for some $J \subset I$*

$$N = \bigoplus_{j\in J} S_j;$$

that is, N is e*-semisimple in M.

Let $\mathcal{X}$ be a class of modules. A module M is generated by $\mathcal{X}$ in case there is an indexed set $(X_i)_{i\in I}$ in $\mathcal{X}$ and an epimorphism $\oplus_{i\in I} X_i \to M$ [1, page 105].

Let $\mathcal{S}$ be a set of representatives of the simple modules. For a given module M, we define the subset $\mathcal{S}_M$ of S as follows:

$$\mathcal{S}_M := \{S \in \mathcal{S} \mid \ S \text{ is isomorphic to an } e^*\text{-simple submodule of M}\}.$$

Having finished all the preparatory work, we can now deduce the main result of this section.

Theorem 2.10. *Let M be a module, and let N be a submodule of M. The following statements are equivalent:*

(1) *N is e*-semisimple in M.*

(2) *N is generated by $\mathcal{S}_M$.*

(3) *N is the sum of some set of e*-simple submodules of M (e*-simple means e*-simple in M).*

(4) *N is the sum of all e*-simple submodules included in N (again, e*-simple means e*-simple in M).*

Proof. (1) ⇒ (2) Follows immediately by the definitions.

(2) ⇒ (3) There exists an epimorphism $h : \bigoplus_{i\in I} S_i \to N$ where, for each $i \in I$, $S_i \in \mathcal{S}_M$. Then, we have $N = \sum_{i\in I} h(S_i)$. Then, for each $i \in I$, $h(S_i)$ is either 0 or e^*-simple submodule of M by Lemma 2.5 (1).

(3) ⇒ (4) Obvious.

(4) ⇒ (1) Follow by Corollary 2.9. □

Corollary 2.11. *The following statements are equivalent for a module M:*

(1) *M is e*-semisimple;*

(2) *M is generated by $\mathcal{S}_M$;*

(3) *M is the sum of some set of e*-simple submodules;*

(4) *M is the sum of its e*-simple submodules;*

Proof. In Theorem 2.10, let $N = M$. □

3. The e^*-Socle

Let R be a ring, and let M be a right R-mdoule. The socle of M, denoted by Soc(M), is the sum of all minimal (i.e., simple) submodules of M and, therefore, is the largest semisimple submodule of M. It is equal to the intersection of all essential submodules of M [1, Proposition 9.7]. In this section, we define the e^*-Socle of a module M, denoted $\mathrm{e^*\text{-}Soc(M)}$, as the largest e^*-semisimple submodule of M and demonstrate that it shares many properties as the socle .

Definition 3.1. We define the e^*-socle of a module M as the largest e^*-semisimple submodule of M, and we denote it by $\mathrm{e^*\text{-}Soc(M)}$.

Proposition 3.2. *Let M be an R-module. Then,*

$$\mathrm{e^*\text{-}Soc(M)} = \sum_{\substack{S\leq M \\ S\ e^*\text{-simple}}} \mathrm{S} = \bigcap_{\mathrm{L}\leq_{\mathrm{e}^*}\mathrm{M}} \mathrm{L}.$$

Proof. The first equality is clear from Theorem 2.10. Then, it suffices to prove the second equality. The inclusion

$$\mathrm{e^*\text{-}Soc(M)} \subseteq \bigcap_{\mathrm{L}\leq_{\mathrm{e}^*}\mathrm{M}} \mathrm{L},$$

follows by definitions. For the reverse inclusion, by Lemma 2.4, we only need to prove that $\bigcap_{L\leq_{e^*}M} L$ is semisimple. But, Soc(M) is semisimple; then $\bigcap_{L\leq_{e^*}M} L$, being a submodule of Soc(M), is also semisimple. □

Unlike the socle [1, Proposition 9.8], the e^*-socle doesn't behave well under homomorphisms (see Example 3.7). But we have,

Proposition 3.3. *Let $f : M \to N$ be a homomorphism of R-modules. Assume that $f^{-1}(L) \neq 0$ for every* e*-essential submodule L of N. Then*

$$f(\mathrm{e^*\text{-}Soc(M)}) \leq \mathrm{e^*\text{-}Soc(N)}.$$

Proof. Let $x \in \mathrm{e^*\text{-}Soc(M)}$. If L is an e^*-essential submodule of N, then, by hypothesis, $f^{-1}(L) \neq 0$. Hence, by [2, Proposition 2], $f^{-1}(L)$ is an e^*-essential submodule of M. Then, by Proposition 3.2, $x \in f^{-1}(L)$; so $f(x) \in L$. Hence, again by Proposition 3.2, $f(x) \in \mathrm{e^*\text{-}Soc(N)}$. □

Corollary 3.4. *Let $f : M \to N$ be a homomorphism of R-modules. Assume that M has a nonzero cosingualr submodule C. Then*

$$f(\mathrm{e^*\text{-}Soc(M)}) \leq \mathrm{e^*\text{-}Soc(N)}.$$

Proof. Let L be an e^*-essential submodule of N. If $f^{-1}(L) = 0$, then f is injective since $\mathrm{ker(f)} \subset \mathrm{f^{-1}(L)} = 0$. Hence, by [5, Lemma 2.6], $f(C)$ is a nonzero cosingular submodule of N. Since $L \leq_{e^*} N$, $L \cap f(C) \neq 0$, which is clearly impossible. Thus, $f^{-1}(L) \neq 0$ for every e^*-essential submodule L of N. Now, the rsult follows from Proposition 3.3. □

Corollary 3.5. *Let M be a module. Suppose that $K \cap L \neq 0$ for every e^*-essential submodule L of M. If $K \leq_{e^*} M$, then*

$$\mathrm{e^*\text{-}Soc(K)} = \mathrm{K} \cap \mathrm{e^*\text{-}Soc(M)}.$$

In particular, if $\mathrm{e^*\text{-}Soc(M)} \leq_{\mathrm{e}^*} \mathrm{M}$, *then* $\mathrm{e^*\text{-}Soc(e^*\text{-}Soc(M))} = \mathrm{e^*\text{-}Soc(M)}$.

Proof. The inclusion $\mathrm{e^*\text{-}Soc(K)} \subseteq \mathrm{K} \cap \mathrm{e^*\text{-}Soc(M)}$ follows from Proposition 3.3. By Proposition 2.6 (1), $K \cap \mathrm{e^*\text{-}Soc(M)}$ is e^*-semisimple in M because $\mathrm{e^*\text{-}Soc(M)}$ is e^*-semisimple in M. Then, by Proposition 2.6 (2), $K \cap \mathrm{e^*\text{-}Soc(M)}$ is e^*-semisimple in K . Hence, by Proposition 3.2, $K \cap \mathrm{e^*\text{-}Soc(M)} \subseteq \mathrm{e^*\text{-}Soc(K)}$. □

Remark 3.6. Let $K \leq M$ be a submodule. Suppose that K contains a nonzero cosingular submodule, and $K \leq_{e^*} M$. Then, $\mathrm{e^*\text{-}Soc(K) = K \cap e^*\text{-}Soc(M)}$. Indeed, since K contains a nonzero consingular submodule, the zero submodule can't be e^*-essential. Therefore, for every e^*-essential submodule L, we have $K \cap L \neq 0$. Then Corollary 3.5 applies.

The promised example is as follows:

Example 3.7. Let $M = M_1 \oplus M_2$, where, for each i, M_i is an artinian modules wich has no nonzero cosingular submodule (see for instance [2, Example 1]). By [1, Corollary 10.11] M_1 has a simple module S_1. Notice that $\mathrm{e^*\text{-}Soc(S_1) = S_1}$ and $\mathrm{e^*\text{-}Soc(M)} = \bigcap_{\mathrm{K \leq M}} \mathrm{K}$ because every nonzero submodule of M is e^*-essential (M has no nonzero cosingular submodule because, for each i, M_i has no nonzero cosingular submodule, see [5, Lemma 2.6]). Now, suppose that $\mathrm{e^*\text{-}Soc(S_1) = S_1} \subseteq \bigcap_{\mathrm{K \leq M}} \mathrm{K}$; that is, $S \subseteq K$ for every submodule of M, which says that, $S_1 \subseteq M_2$, a contradiction. Therefore, $\mathrm{e^*\text{-}Soc(S_1)} \nsubseteq \mathrm{Soc(M)}$.

Now, the e^*-socle of a module M is the largest submodule of M that is contained in every e^*-essential submodule of M. In general, though, $\mathrm{e^*\text{-}Soc(M)}$ need not be e^*-essential in M; in fact, nonzero modules can have zero e^*-socle (because $\mathrm{e^*\text{-}Soc(M)}$ is submodule of $\mathrm{Soc(M)}$ which can be zero [1, Exercise 9.2]). However, we do have:

Proposition 3.8. *The following assertions are equivalent for a module M.*

(1) $\mathrm{e^*\text{-}Soc(M)} \leq_{\mathrm{e^*}} \mathrm{M}$

(2) *Every nonzero cosingular submodule of M contains an e^*-simple submodule.*

Proof. (1) $\Rightarrow$ (2) If C is a cosingular submodule of M, then $C \cap \mathrm{e^*\text{-}Soc(M)} \neq 0$. Hence, by Proposition 2.6, $C \cap \mathrm{e^*\text{-}Soc(M)} \neq 0$ is a nonzero e^*-semisimple submodule of M, which include in C.
(2) $\Rightarrow$ (1) Follows by Proposition 3.2. □

Let $\mathcal{X}$ be a class of R-modules. The trace of $\mathcal{X}$ in M is defined by

$$\mathrm{Tr_M}(\mathcal{X}) = \sum_{\substack{h \in \mathrm{Hom(X,M)} \\ X \in \mathcal{X}}} \mathrm{h(X)}.$$

It was shown that $\mathrm{Tr_M}(\mathcal{X})$ is the unique largest submodule of M generated by $\mathcal{X}$ [1, Proposition 8.12].
For a given module M, let $\mathcal{S}_M$ be a representatives of e^*-simple submodules of M as it is defined in Section 2. So, as an e^*-version of [1, Proposition 9.11], we have

Proposition 3.9. *Let M be a right R-modules. Then:*

$$\mathrm{e^*\text{-}Soc(M)} = \mathrm{Tr_M}(\mathcal{S}_\mathrm{M}) = \mathrm{Tr_M}(\bigoplus_{\mathrm{S} \in \mathcal{S}_\mathrm{M}} \mathrm{S}) = \sum_{\mathrm{S} \in \mathcal{S}_\mathrm{M}} \mathrm{Tr_M(S)}$$

Proof. The first equality follows from definitions and Theorem 2.10. The rest follows form well-know properties of the trace [1, Proposition 8.20]. □

Corollary 3.10. *For any ring R, the* $\mathrm{e^*\text{-}Soc(R_R)}$*, the e^*-socle of R as right R-module, is a two sided ideal of R.*

Proof. Follows by Proposition 3.9 and [1, Proposition 8.21]. □

Recall that a ring R is said to be *right cosingular* if it is cosingular as a (right) R-module [5, Definition 2.5]. It follows from [5, Corollary 2.7] that if R is right cosingular, then every right R-module is cosingular. Consequently, over a right cosingular ring, the notion of "e^*-semisimple submodules/modules" coincides with the classical notion of semisimple modules.
In the rest of this section, we provide some results concerning this situation, i.e., when these new relative concepts coincide with their classical counterparts.

Proposition 3.11. *Let M be an R-module. The following assertions are equivalent:*

(1) *Every simple submodule N of M is e^*-simple in M.*

(2) *Every semisimple submodule N of M is e^*-semisimple in M.*

(3) e^*-Soc(M) = Soc(M).

Proof. (1) $\Leftrightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) This follows from Proposition 3.2 and [1, Proposition 9.7].

(3) $\Rightarrow$ (1) If N is a simple submodule of M, then $N \subseteq$ Soc(M). By (3), we have $N \subseteq$ e^*-Soc(M). Then, by Proposition 3.2, N is contained in every e^*-essential submodule of M. Thus, N is e^*-simple in M. □

Using a similar argument as in the proof of Proposition 3.11, one can prove the following:

Proposition 3.12. *The following assertions are equivalent:*

(1) *Every semisimple R-module is e^*-semisimple.*

(2) *For every R-module M, every semisimple submodule of M is e^*-semisimple in M.*

(3) *For every R-module M, every simple submodule of M is e^*-simple in M.*

(4) *For any R-module M,* e^*-Soc(M) = Soc(M).

(5) *Every semisimple right R-module has no proper e^*-essential submodule.*

Acknowledgment. I am grateful to the referee for the careful and critical reading of the manuscript.

References

[1] Anderson F.W., Fuller K.R.; Rings and categories of modules. Graduate Texts in Mathematics. Springer, 1974.

[2] Baanoon H., Khalid W.; e^*-essential submodule. Eur. J. Pure Appl. Math.,–2022.–15.–P.224–228.

[3] Goodearl K.; Ring theory: Nonsingular rings and modules. CRC Press.–1976.

[4] Leonard W.W.; Small modules. Proc. Amer. Math. Soc.,–1966.–17.–P.527–531.

[5] zcan A.C.; Modules with small cyclic submodules in their injective hulls. Comm. Algebra, –2002.–30.–No.4.–P.1575–1589.

[6] Rotman J.; An introduction to homological algebra. Academic Press, New York.–1979.

Bouziri Ayoub,
Faculty of Sciences,
Mohammed V University in Rabat,
Rabat, Morocco
email: ayoubbouziri25@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.53-69
DOI: 10.29229/uzmj.2025-4-7

Complete systems of invariants of m-tuples for fundamental groups of a two-dimensional bilinear-metric space over the field of rational numbers

Beshimov G. R.

Abstract. Let Q be the field of rational numbers and Q^2 be the 2-dimensional linear space over Q. A classification of all non-degenerate symmetric bilinear-metric forms over Q^2 have obtained. Let φ be a non-degenerate symmetric bilinear form on Q^2. Denote by $O(2,\varphi,Q)$ the group of all φ-orthogonal (that is the form φ preserving) transformations of Q^2. Put $MO(2,\varphi,Q) = \{F : Q^2 \to Q^2 \mid Fx = gx + b, g \in O(2,\varphi,Q), b \in Q^2\}$, $SO(2,\varphi,Q) = \{g \in O(2,\varphi,Q) | detg = 1\}$ and $MSO(2,\varphi,Q) = \{F \in M(2,\varphi,Q) | detg = 1\}$.The present paper is devoted to solutions of problems of G-equivalence of m-tuples in Q^2 for groups $G = O(2,\varphi,Q), SO(2,\varphi,Q)$, $MO(2,\varphi,Q)$, $MSO(2,\varphi,Q)$. Complete systems of G-invariants of m-tuples in Q^2 for these groups are obtained.

Keywords: Invariant of m-tuple; m-point invariant.
MSC (2020): 14L24, 15A63, 15A72.

1. Introduction

Let N be the set of all natural numbers and $m \in N, m \geq 1$. Denote by $(Q^2)^m$ the set of all m-tuples $(u_1, u_2, \ldots, u_m)$ in Q^2, where $u_i \in Q^2, \forall i = 1, 2, \ldots, m$.

Let V be a finite dimensional vector space over a field B and ϕ be a bilinear form on V. Denote by $O(\phi, V)$ the group of all ϕ-orthogonal (that is the form ϕ preserving) transformations of V. Let $MO(\phi, V)$ be the group generated by the group $O(\phi, V)$ and all translations of V. In the paper [5], for the orthogonal group $O(\phi, V)$ in the Euclidean, spherical, hyperbolic and de-Sitter geometries, the orbit of m vectors is characterized by their Gram matrix and an additional subspace. In the book [1, Proposition 9.7.1], for the group $MO(\phi, V)$ in the Euclidean geometry, the orbit of m vectors is characterized by distances between m-vectors. A complete system of relations between elements of this complete system is also given in [1, Theorem 9.7.3.4]. In the paper [7], a complete system of invariants of m-tuples in the two-dimensional pseudo-Euclidean geometry of index 1 and a complete system relations between the obtained complete system of invariants are given. In the paper [8], a complete system of invariants of m-tuples in the one-dimensional projective space and a complete system relations between the obtained complete system of invariants are given. Invariants of m-points appear also in the theory of invariants of Bezier curves ([3], [19]. Complete systems of invariants for various geometric and topological settings have been developed in a series of works. In [9], the authors construct complete systems of invariants for m-tuples associated with the fundamental groups of the two-dimensional Euclidean space. The study in [10] presents complete systems of Galilean invariants describing the motion of parametric figures in three-dimensional Euclidean space. In [11], the authors investigate global invariants of topological figures in the two-dimensional Euclidean space, focusing on properties preserved under continuous deformations. Similarly, in [12], global invariants of objects are analyzed in the context of the two-dimensional Minkowski space, taking into account the Lorentzian structure. The papers [13] and [14] extend the study of invariants to immersions into n-dimensional affine manifolds and to mappings from arbitrary sets into the two-dimensional Euclidean space, respectively. Invariants of m-vectors in Lorentzian geometry are considered in [20], where algebraic invariants under Lorentz transformations are analyzed. Moreover, the concept of m-vector invariants appears prominently in applied disciplines such as computer vision ([16], [21]), where they are used for recognizing and comparing geometric configurations under affine or projective transformations, and in computational geometry ([18]), where such invariants aid in the analysis of shape and spatial relationships. General theory of m-point invariants considered in the invariant theory (see [2], [5], [6], [17], [23], [24]).This paper is a continuation of the paper [15]. The present paper is devoted to solutions of problems of G-equivalence of m-tuples in Q^2 for groups $G = O(2,\varphi,Q), SO(2,\varphi,Q)$, $MO(2,\varphi,Q)$, $MSO(2,\varphi,Q)$. Complete systems of G-invariants of m-tuples in Q^2 for these groups are obtained.

1.1. **A classification bilinear-metric spaces over the field of rational numbers.**

Let Q be the field of rational numbers, Q^2 be the 2-dimensional linear space over Q and $\varphi(x,y)$ be a symmetric bilinear form on Q^2.

If we replace the argument $y \in Q^2$ in the symmetric bilinear form $\varphi(x,y)$ by x, where $x=(x_1,x_2) \in Q^2$, we obtain the quadratic form $\varphi(x,x)$.

Theorem 1.1. *(see [4], p.196) For every quadratic form $\varphi(x,x)$ on Q^2, there exists a basis in Q^2 such that it has following form*

$$\varphi(x,x)=\lambda_1 x_1^2+\lambda_2 x_2^2$$

for some $\lambda_1,\lambda_2 \in Q$, where x_1,x_2 are the coordinates of the vector x in this basis.

In this case, there exist only following two cases: 1) $rank(\varphi(x,x))=1$ and 2) $rank(\varphi(x,x))=2$. In the case 1) $rank(\varphi(x,x))=1$, there exists a basis in Q^2 such that $\varphi(x,x)$ has following form: $\varphi(x,x)=\lambda_1 x_1$, where $\lambda_1 \in Q$ and $\lambda_1 \neq 0$.

Consider the case $rank(\varphi(x,x))=2$. In this case, there exists a basis e_1,e_2 in Q^2 such that $\varphi(x,x)$ has following form $\varphi(x,x)=\lambda_1 x_1^2+\lambda_2 x_2^2$, where $\lambda_1 \in Q$, $\lambda_1 \neq 0$ and $\lambda_2 \in Q$, $\lambda_2 \neq 0$. The equality $\varphi(x,x)=\lambda_1 x_1^2+\lambda_2 x_2^2$ implies following equality: $\varphi(x,x)=\lambda_1(x_1^2+\frac{\lambda_2}{\lambda_1}x_2^2)$. Since $\frac{\lambda_2}{\lambda_1}$ is a rational number, there are a,b integer numbers such that $\frac{\lambda_2}{\lambda_1}=\frac{a}{b}$. Then we have: $\varphi(x,x)=\lambda_1(x_1^2+\frac{a}{b}x_2^2)$.

We may then introduce a new basis e_1',e_2' by setting $e_1'=e_1, e_2'=be_2$, where b is the above integer number. This implies that the quadratic form $\varphi(x,x)$ can be written in this basis in the form $\varphi(x,x)=\lambda_1(x_1^2+abx_2^2)$. We now consider the case of a positive rational number $a\cdot b$. If the prime factors of the product ab have a square of an integer, then we create $\varphi(x,x)=\lambda_1(x_1^2+px_2^2)$ by introducing a new basis, where $p=1$ or $p=p_1\cdot p_2\cdot \ldots \cdot p_n$ such that $p_j, j=1,\ldots,n,-$ prime numbers and $p_k \neq p_l$ for all $l \neq k, k=1,\ldots,n, l=1,\ldots,n$. As a result, there are infinitely non-congruent symmetric bilinear forms over the field of rational numbers and bilinear-metric spaces relatively.

1.2. **A linear representation of the field $Q(\sqrt{-p})$ in two-dimensional linear space Q^2.**

Let Q be the field of rational numbers and $p=1$ or $p=p_1\cdot p_2\cdot\ldots\cdot p_n$, where p_j- prime numbers and $p_k \neq p_l$ for all $k \neq l$. Denote by $Q(\sqrt{-p})$ the set $\{a+b\sqrt{-p} \mid a,b \in Q\}$. Let $a=a_1+\sqrt{-p}a_2 \in Q(\sqrt{-p})$ and $b=b_1+\sqrt{-p}b_2 \in Q(\sqrt{-p})$. We define addition and multiplication operations on $Q(\sqrt{-p})$ as follows: put $a+b=(a_1+\sqrt{-p}a_2)+(b_1+\sqrt{-p}b_2)=(a_1+b_1)+\sqrt{-p}(a_2+b_2)$. A multiplication in $Q(\sqrt{-p})$ define as follows: $a\circ b=(a_1+\sqrt{-p}a_2)\circ(b_1+\sqrt{-p}b_2)=(a_1b_1-pa_2b_2)+\sqrt{-p}(a_1b_2+a_2b_1)$.

We will present the Propositions 1.2 - 1.9 mentioned in paper [15], as these propositions will be necessary for us

Proposition 1.2. *The set $Q(\sqrt{-p})$ is a field with respect to the defined above addition $a+b$ and multiplication $a\circ b$ operations.*

Let $a=a_1+\sqrt{-p}a_2$. We denote by M_a the matrix of the form $\begin{pmatrix} a_1 & -pa_2 \\ a_2 & a_1 \end{pmatrix}$. Let $M(Q,p)$ denote the set of all matrices M_a, where $a \in Q(\sqrt{-p})$. We consider on the set $M(Q,p)$ standard matrix operations: the component-wise addition and the multiplication operations of matrices. Then $M(Q,p)$ is a field with the unit element, where the unit element is the unit matrix. The following proposition is obvious.

Proposition 1.3. *The mapping $M: Q(\sqrt{-p}) \to M(Q,p)$, where $M: a \to M_a, \forall a \in Q(\sqrt{-p})$, is an isomorphism of fields $Q(\sqrt{-p})$ and $M(Q,p)$.*

For $a=a_1+\sqrt{-p}a_2, b=b_1+\sqrt{-p}b_2 \in Q(\sqrt{-p})$, we put $\langle a,b\rangle_p=a_1b_1+pa_2b_2$. Then $\langle a,b\rangle_p$ is a bilinear form on $Q(\sqrt{-p})$ and $\langle a,a\rangle_p=a_1^2+pa_2^2$ is a quadratic form on $Q(\sqrt{-p})$. For convenience, we denote by $\Psi(a)$ the quadratic form $\langle a,a\rangle_p$.

Proposition 1.4. *Let $M: Q(\sqrt{-p}) \to M(Q,p)$ be the isomorphism $M: x \to M_x$ of fields $Q(\sqrt{-p})$ and $M(Q,p)$. Then $\Psi(x)=det(M_x)$ and $\Psi(x\circ y)=\Psi(x)\Psi(y)$for all $x,y \in Q(\sqrt{-p})$.*

For an arbitrary element $a=a_1+\sqrt{-p}a_2 \in Q(\sqrt{-p})$, we set $W(a)=\overline{a}=a_1-\sqrt{-p}a_2$.

Proposition 1.5. *For an arbitrary element $a=a_1+\sqrt{-p}a_2 \in Q(\sqrt{-p})$ following equalities hold: $a+\overline{a}=2a_1, \langle a,a\rangle_p=a\circ\overline{a}=a_1^2+pa_2^2 \in Q$.*

Proposition 1.6. *The function $\Psi(x)$ has the following properties:*

(1) $\Psi(\lambda x) = \lambda^2 \Psi(x)$, $\forall \lambda \in Q, \forall x \in Q(\sqrt{-p})$;

(2) $\Psi(e) = 1$ *for the unit element* $e \in Q(\sqrt{-p})$;

(3) $\Psi(x) = x \circ \overline{x} = \overline{x} \circ x$ *hold for all* $x \in Q(\sqrt{-p})$;

(4) $\Psi(x) = \Psi(Wx) = \Psi(\overline{x})$ *hold for all* $x \in Q(\sqrt{-p})$.

Proposition 1.7. *Let $x \in Q(\sqrt{-p})$. Then the element x^{-1} exists if and only if $\Psi(x) \neq 0$. In the case $\Psi(x) \neq 0$, the equalities $x^{-1} = \frac{\overline{x}}{\Psi(x)}$ and $\Psi(x^{-1}) = \frac{1}{\Psi(x)}$ hold.*

Put $Q^*(\sqrt{-p}) = \{x \in Q(\sqrt{-p}) \mid \Psi(x) \neq 0\}$. $Q^*(\sqrt{-p})$ is a group with respect to the multiplication operation $\circ$ in the field $Q(\sqrt{-p})$. Denote by $M(Q^*, p)$ the set of all matrices M_a, where $a \in Q^*(\sqrt{-p})$. Consider elements $a = a_1 + \sqrt{-p}a_2 \in Q^*(\sqrt{-p})$ and $x = x_1 + \sqrt{-p}x_2 \in Q(\sqrt{-p})$ as column vectors $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Let M_a be the matrix $\begin{pmatrix} a_1 & -pa_2 \\ a_2 & a_1 \end{pmatrix}$. Since $a \in Q^*\sqrt{-p}$, we have $\Psi(a) = a_1^2 + pa_2^2 \neq 0$ and $\Psi(a) = det(M_a) \neq 0$.

Then the equality $a \circ x = (a_1 + \sqrt{-p}a_2) \circ (x_1 + \sqrt{-p}x_2) = (a_1x_1 - pa_2x_2) + \sqrt{-p}(a_1x_2 + a_2x_1)$ has the following form

$$a \circ x = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1x_1 - pa_2x_2 \\ a_1x_2 + a_2x_1 \end{pmatrix} = \begin{pmatrix} a_1 & -pa_2 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = M_a x, \tag{1.1}$$

where $M_a x$ is the multiplication of matrices M_a and x. Hence $M_a \in M(Q^*, p)$ and the mapping $M : Q^*(\sqrt{-p}) \to M(Q^*, p)$, where $M(a) = M_a$, is a linear representation of the group $Q^*(\sqrt{-p})$ in Q^2.

Proposition 1.8. *$M(Q^*, p)$ is a group with respect to the multiplication operation in the field $M(Q)$.*

Put $S(Q^*, \sqrt{-p}) = \{x \in Q(\sqrt{-p}) \mid \Psi(x) = 1\}$. It is a subgroup of the group $Q^*(\sqrt{-p})$.

Proposition 1.9. *Let $M : Q(\sqrt{-p}) \to M(Q, p)$ be the isomorphism $M : x \to M_x$ of fields $Q(\sqrt{-p})$ and $M(Q, p)$. Then $M(S(Q^*, \sqrt{-p}))$ is a subgroup of the group $M(Q^*, p)$ and the mapping M: $S(Q^*, \sqrt{-p}) \to M(Q^*, p)$, where $M(a) = M_a$ is a linear representation of the group $S(Q^*, \sqrt{-p})$ in Q^2.*

Let $p = 1$ or $p = p_1 \cdot p_2 \cdot ... \cdot p_n$, where p_j- prime numbers and $p_k \neq p_l$ for all $k \neq l$. The symmetric bilinear form $x_1y_1 + px_2y_2$ denote by $\langle x, y \rangle_p$. Denote by Q_p^2 the 2-dimensional linear space Q^2 over Q with the bilinear form $\langle x, y \rangle_p = x_1y_1 + px_2y_2$, where $x = (x_1, x_2), y = (y_1, y_2) \in Q^2$.

2. Fundamental groups of transformations of the 2-dimensional bilinear-metric space Q^2

Definition 2.1. A mapping $F : Q(\sqrt{-p}) \to Q(\sqrt{-p})$ is called p-orthogonal if $\langle Fx, Fy \rangle_p = \langle x, y \rangle_p$ for all $x, y \in Q(\sqrt{-p})$.

We denote the set of all p-orthogonal transformations of Q^2 by $O(2, p, Q)$. Let $I : Q^2 \to Q^2$ be the unit transformation $I(x) = x, \forall x \in Q^2$. Then $I \in O(2, p, Q)$. Let $T_1, T_2 \in O(2, p, Q)$ and $T_1 \cdot T_2 : Q^2 \to Q^2$ be such that $(T_1 \cdot T_2)(x) = T_1(T_2(x)), \forall x \in Q^2$. Then it is easy to see that $T_1 \cdot T_2 \in O(2, p, Q)$.

The following propositions are well known.

Proposition 2.2. *$O(2, p, Q)$ is a group with respect to the composition operation $T_1 \cdot T_2$, where $T_1, T_2 \in O(2, p, Q)$.*

Proposition 2.3. *([25], p.221) Every p-orthogonal transformation of Q_p^2 is linear.*

Let $x = (x_1, x_2) \in Q^2, y = (y_1, y_2) \in Q^2$. Denote the matrix of the bilinear form $\langle x, y\rangle_p = x_1y_1 + px_2y_2$ by $\Delta_p = \|\delta_{ij}\|_{i,j=1,2}$, where $\delta_{11} = 1, \delta_{12} = \delta_{21} = 0, \delta_{22} = p$. By Proposition 2.3, we can consider an element of $O(2, p, Q)$ as a 2×2-matrix. Let $H \in O(2, p, Q)$, where $H = \|h_{ij}\|_{i,j=1,2}$. Let H^T be the transpose matrix of H. It is known that the equality $\langle Hx, Hy\rangle_p = \langle x, y\rangle_p$ for all $x, y \in Q^2$ is equivalent to the equality

$$H^T \Delta_p H = \Delta_p. \tag{2.1}$$

The following proposition follows from the equation 2.1.

Proposition 2.4. *Let $H \in O(2, p, Q)$. Then $det(H) = 1$ or $det(H) = -1$.*

We denote by $SO(2, p, Q)$ the set $\{H \in O(2, p, Q) : det(H) = 1\}$. $SO(2, p, Q)$ is a subgroup of $O(2, p, Q)$. $O(2, p, Q) = SO(2, p, Q) \cup \{HW \mid H \in SO(2, p, Q)\}$, where HW is the multiplication of matrices H and W, where $W = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Theorem 2.5. *(see* [15]*). The equality $SO(2, p, Q) = M(S(Q^*, \sqrt{-p}))$ holds.*

Hence, we conclude from the above theorem that every special orthogonal transformations will be matrices $\begin{pmatrix} a & -pb \\ b & a \end{pmatrix}$ such that $a^2 + pb^2 = 1, a, b \in Q$. In that case, is the solution of the equation $a^2 + pb^2 = 1$ in the rational numbers field? We can answer this question by the following theorem.

Theorem 2.6. *The description of the elements of the group $SO(2, p, Q)$ is as follows.*
(i) There is no element $x = (x_1, x_2) \in Q^2$, such that $x_1 = 0$ and $M_x \in SO(2, p, Q)$, where $p \neq 1$. There are only two elements $(x_1, x_2) \in Q^2$, such that $x_2 = 0$ and $M_x \in SO(2, p, Q)$. These are $(1, 0)$ and $(-1, 0)$.
(ii) Assume that $x = (x_1, x_2) \in Q^2$ such that $x_2 \neq 0$ and $M_x \in SO(2, p, Q)$. Then there is the number $r \in Q$, where $r \neq 0$, such that the equalities are satisfied:

$$x_1 = \frac{p - r^2}{p + r^2}, \qquad x_2 = \frac{2r}{p + r^2} \quad (I).$$

(iii) Conversely, assume that r is an arbitrary nonzero element in Q and for $x = (x_1, x_2) \in Q^2$ the equalities are satisfied (I). Then $M_x \in SO(2, p, Q)$.

Proof. (i) This is obvious.

(ii) Assume that $x = (x_1, x_2) \in Q$ such that $x_2 \neq 0$ and $x_1^2 + px_2^2 = 1$.

First, we prove that in this case $x_1^2 \neq 1$. Suppose $x_1^2 = 1$. Then from the equation $x_1^2 + px_2^2 = 1$, we obtain that $x_2^2 = 0$. It follows that $x_2 = 0$. This contradicts to $x_2 \neq 0$. So we proved $x_1^2 \neq 1$, $x_1 \neq 1$ and $x_1 \neq -1$.

From the equation $x_1^2 + px_2^2 = 1$ and from the inequalities $x_1 \neq 1$, $x_1 \neq -1$ we obtain the following equalities: $1 - x_1^2 = px_2^2 \Rightarrow px_2^2 = (1 - x_1)(1 + x_1) \Rightarrow \frac{px_2}{1+x_1} = \frac{1-x_1}{x_2}$.
Put $r = \frac{px_2}{1+x_1}$. Then we have $r = \frac{1-x_1}{x_2}$. From these two equalities we obtain the following equalities $\frac{1}{x_2} + \frac{x_1}{x_2} = \frac{p}{r}$, $\frac{1}{x_2} - \frac{x_1}{x_2} = r$. From last equalities we obtain $\frac{2}{x_2} = \frac{p}{r} + r$, $\frac{2x_1}{x_2} = \frac{p}{r} - r$. We find x_1, x_2 from these two equalities and we obtain the following equalities $x_1 = \frac{p-r^2}{p+r^2}$, $x_2 = \frac{2r}{p+r^2}$. The (ii) is proved.

(iii) Conversely, let $r \in Q$ be an arbitrary nonzero rational number. Put $x_1 = \frac{p-r^2}{p+r^2}$, $x_2 = \frac{2r}{p+r^2}$. We have $x_1^2 + px_2^2 = (\frac{p-r^2}{p+r^2})^2 + p(\frac{2r}{p+r^2})^2 = \frac{p^2-2pr^2+r^4+4pr^2}{(p+r^2)^2} = \frac{p^2+2pr^2+r^4}{(p+r^2)^2} = 1$. Therefore, $M_x \in SO(2, p, Q)$.

Hence, all special orthogonal matrices given as follows:

$$SO(2, p, Q) = \left\{ \begin{pmatrix} \frac{p-r^2}{p+r^2} & \frac{-2pr}{p+r^2} \\ \frac{2r}{p+r^2} & \frac{p-r^2}{p+r^2} \end{pmatrix} \mid \ \forall r \in Q, r \neq 0 \right\}$$

and all orthogonal matrices are given as follows:

$$O(2,p,Q)=\left\{\begin{pmatrix} \frac{p-r^2}{p+r^2} & \frac{-2pr}{p+r^2} \\ \frac{2r}{p+r^2} & \frac{p-r^2}{p+r^2} \end{pmatrix} \mid \forall r\in Q, r\neq 0\right\}\cup\left\{\begin{pmatrix} \frac{p-r^2}{p+r^2} & \frac{2pr}{p+r^2} \\ -\frac{2pr}{p+r^2} & -\frac{p-r^2}{p+r^2} \end{pmatrix} \mid \forall r\in Q, r\neq 0\right\}.$$

□

2.1. **Complete systems of invariants of an m-tuple in Q_p^2 for groups $SO(2,p,Q)$ and $MSO(2,p,Q)$.**

Let N be the set of all natural numbers and $m\in N, m\geq 1$. Put $N_m=\{j\in N|1\leq j\leq m\}$.

Definition 2.7. A mapping $u:N_m\to Q^2$ will be called an m-tuple in Q^2. Denote it in the following form: $u=(u_1,u_2,\dots u_m)$.

Denote by $(Q^2)^m$ the set of all m-tuples in Q^2. Let G be a subgroup of the group $MO(2,p,R)$.

Definition 2.8. Two m-tuples $u=(u_1,u_2,\dots u_m)$ and $v=(v_1,v_2,\dots v_m)$ in Q^2 is called G-equivalent if there exists $g\in G$ such that $v_j=gu_j,\forall j\in N_m$. In this case, we write $v=g(u)$ or $u\overset{G}{\sim}v$.

Definition 2.9. A subset $C\subseteq(Q^2)^m$ is called G-invariant if $g(u)\in C,\forall u\in C,\forall g\in G$.

Definition 2.10. Let Ω be a set and it has at least two elements and C be a G-invariant subset of $(Q^2)^m$. A mapping $f:C\to\Omega$ is called G-invariant on C if $u\in C, v\in C$ and $u\overset{G}{\sim}v$, implies $f(u)=f(v)$.

Let C be a G-invariant subset of $(Q^2)^m$ and Ω be a set such that it has at least two elements. Denote the set of all G-invariant functions $f:C\to\Omega$ on C by $Map(C,\Omega)^G$.

Example 2.11. Definitions of the groups $H=O(2,p,Q), SO(2,p,Q)$ imply that the quadratic form $\Psi(x)=\langle x,x\rangle_p$ and the bilinear form $\langle x,y\rangle_p$ are H-invariant functions on the set Q^2.

Example 2.12. Let $[x\,y]$ be the determinant $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2\end{vmatrix}$ of $x=\begin{pmatrix} x_1\\ x_2\end{pmatrix}, y=\begin{pmatrix} y_1\\ y_2\end{pmatrix}\in Q^2$. Since $det(g)=1$ for all $g\in SO(2,p,Q)$, we have $[(gx)\,(gy)]=det(g)[x\,y]=[x\,y]$ for all $g\in SO(2,p,Q)$. Hence $[x\,y]$ is an $SO(2,p,Q)$-invariant function on the set $(Q^2)^2$.

Example 2.13. Definitions of the groups $H=MO(2,p,Q), MSO(2,p,Q)$ imply that the function $f(x,y)=\langle x-y,x-y\rangle_p$ is an H-invariant function on the set $(Q^2)^2$.

Definition 2.14. (see [22, 1.1]). Let C be a G-invariant subset of $(Q^2)^m$. A system $\{f_j|j\in J\}$, where $f_j\in Map(C,Q)^G,\forall j\in J$, will be called *a complete system* of G-invariant functions on C if $u\in C, v\in C$ and equalities $f_j(u)=f_j(v),\forall j\in J$, imply $u\overset{G}{\sim}v$.

Definition 2.15. (see [22, 1.1]) Let C be a G-invariant subset of $(Q^2)^m$ and $L=\{f_j|j\in J\}$ be a complete system of G-invariant functions on C. L is called *a minimal complete system* of G-invariant functions on C if $L\setminus\{f_j\}$ is not a complete system of G-invariant functions on C for any $j\in J$.

Put $\theta=(0,0)$, where $(0,0)\in Q^2$. Denote by θ_m the m-tuple $u=(u_1,u_2,\dots,u_m)\in(Q^2)^m$ such that $u_j=\theta,\forall j\in N_m$. Define the function $B:(Q^2)^m\to N_m\cup\{0\}$ as follows: put $B(0_m)=0$. Let $u=(u_1,u_2,\dots,u_m)\in(Q^2)^m$ be such that $u\neq\theta_m$. In this case, we put $B(u)=k$, where $k\in N_m$ such that $u_j=\theta,\forall j=1,\dots k-1$ and $u_k\neq\theta$.

Proposition 2.16. *Let G be a subgroup of $O(2,p,Q)$. The function $B(u)$ is a G-invariant function on $(Q^2)^m$.*

Proof. It is obvious. □

Denote by $U(m;0)$ the set $\{\theta_m\}$. Let $k\in N_m$. Denote by $U(m;k)$ the set $\{u\in(Q^2)^m|B(u)=k\}$.

Proposition 2.17. *Let G be a subgroup of $O(2,p,Q)$. Then:*

(1) *The set $U(m;k)$ is a G-invariant subset of $(Q^2)^m$ for $k=0$ and all $k\in N_m$.*

(2) $U(m;0) \cap U(m;l) = \emptyset, \forall l \in N_m$ *and* $U(m;k) \cap U(m;l) = \emptyset, \forall k,l \in N_m, k \neq l$.

(3) $U(m;0) \cup (\cup_{k \in N_m} U(m;k)) = (Q^2)^m$.

Proof. It is obvious. □

Proposition 2.18. *Let* $x, y \in Q(\sqrt{-p})$ *such that* $x \neq 0$. *Then*

(1) *The element* yx^{-1} *exists, the equality* $yx^{-1} = \frac{\langle x,y \rangle}{\Psi(x)} + \sqrt{-p}\frac{[x\, y]}{\Psi(x)}$ *and the following equality hold*

$$M_{yx^{-1}} = \begin{pmatrix} \frac{\langle x,y \rangle_p}{\Psi(x)} & -\frac{p[x\, y]}{\Psi(x)} \\ \frac{[x\, y]}{\Psi(x)} & \frac{\langle x,y \rangle_p}{\Psi(x)} \end{pmatrix}. \tag{2.2}$$

(2) $\det(M_{yx^{-1}}) = (\frac{\langle x,y \rangle_p}{\Psi(x)})^2 + p(\frac{[x\, y]}{\Psi(x)})^2 \neq 0$ *if and only if* $\Psi(y) \neq 0$.

Proof. (1) Let $x = x_1 + \sqrt{-p}x_2, y = y_1 + \sqrt{-p}y_2 \in Q(\sqrt{-p})$ such that $x \neq 0$. Then x^{-1} exists. Hence yx^{-1} exists. By Proposition 1.7, $x^{-1} = \frac{W(x)}{\Psi(x)}$. Using $W(x) = x_1 - \sqrt{-p}x_2$ and the multiplication in the field $Q(\sqrt{-p})$, we obtain the equalities $yx^{-1} = \frac{\langle x,y \rangle_p}{\Psi(x)} + \sqrt{-p}\frac{[x\, y]}{\Psi(x)}$ and $Eq.(2.2)$.

(2) Let $\Psi(x) \neq 0$. Using Proposition 1.4 and $Eq.(2.2)$, we obtain $(\frac{\langle x,y \rangle_p}{\Psi(x)})^2 + p(\frac{[x\, y]}{\Psi(x)})^2 = \det(M_{yx^{-1}}) = \Psi(yx^{-1}) = \Psi(y)\Psi(x^{-1}) = \frac{\Psi(y)}{\Psi(x)}$. Hence $(\frac{\langle x,y \rangle_p}{\Psi(x)})^2 + p(\frac{[x\, y]}{\Psi(x)})^2 = \frac{\Psi(y)}{\Psi(x)}$. This equality implies that $\det(M_{yx^{-1}}) = \frac{\Psi(y)}{\Psi(x)} \neq 0$ if and only if $\Psi(y) \neq 0$. □

Now we consider the G-equivalence problem of m-tuples for the group $SO(2,p,Q)$.

Proposition 2.19. *Let* G *be a subgroup of* $O(2,p,Q)$. *Assume that* $u = (u_1, \dots, u_m), v = (v_1, \dots, v_m) \in (Q^2)^m$ *be* m*-tuples such that* $u \overset{G}{\sim} v$. *Then* $B(u) = B(v)$.

Proof. Assume that $u \overset{G}{\sim} v$. By Proposition 2.16, the function $B(u)$ is G-invariant. The G-equivalence of u, v and the G-invariance of $B(u)$ imply the equality $B(u) = B(v)$. □

Let $u = (u_1, \dots, u_m), v = (v_1, \dots, v_m) \in (Q^2)^m$ be m-tuples such that $B(u) = B(v) = 0$. Then $u = v = \theta_m$. Hence $u \overset{G}{\sim} v$. Now we consider the case $B(u) = B(v) \neq 0$.

Theorem 2.20. *Let* $u = (u_1, u_2, \dots, u_m), v = (v_1, v_2, \dots, v_m) \in (Q^2)^m$ *be two* m*-tuples in* Q^2 *such that* $B(u) = B(v) = k$, *where* $k \in N_m$.

(*i*) *Assume that* $u \overset{SO(2,p,Q)}{\sim} v$. *Then*

(*i.1*) *In the case* $k = m$, *the equality* $\Psi(u_m) = \Psi(v_m)$ *holds.*

(*i.2*) *In the case* $k < m$, *the following equalities hold*

$$\begin{cases} \Psi(u_k) = \Psi(v_k), \\ \langle u_k, u_j \rangle_p = \langle v_k, v_j \rangle_p, \forall j \in N_m, k < j, \\ [u_k u_j] = [v_k\, v_j], \forall j \in N_m, k < j. \end{cases} \tag{2.3}$$

(*ii*) *Conversely, assume that the equality* $\Psi(u_m) = \Psi(v_m)$ *holds in the case* $k = m$ *and equalities* $Eq.(2.3)$ *hold in the case* $k < m$. *Then, in the every of these cases, there exists the unique matrix* $F \in SO(2,p,Q)$ *such that* $v_j = Fu_j, \forall j \in N_m$. *In these cases,* F *has the following form*

$$F = \begin{pmatrix} \frac{\langle u_k, v_k \rangle_p}{\Psi(u_k)} & -\frac{p[u_k\, v_k]}{\Psi(u_k)} \\ \frac{[u_k\, v_k]}{\Psi(u_k)} & \frac{\langle u_k, v_k \rangle_p}{\Psi(u_k)} \end{pmatrix}, \tag{2.4}$$

where $\det(F) = (\frac{\langle u_k, v_k \rangle_p}{\Psi(u_k)})^2 + p(\frac{[u_k\, v_k]}{\Psi(u_k)})^2 = 1$.

Proof. (i) Assume that $u \overset{SO(2,p,Q)}{\sim} v$. In the case $(i.1)$, the function $\Psi(u_m)$ is $SO(2,p,Q)$-invariant. Hence the equality $\Psi(u_m) = \Psi(v_m)$ holds.

In the case $(i.2)$, functions $\Psi(u_k)$, $\langle u_k, u_j\rangle_p$ and $[u_k\, u_j]$ are also $SO(2,p,Q)$-invariant for all $j \in N_m, k < j$. Hence equalities $Eq.(2.3)$ hold.

(ii) Conversely, assume that the equality $\Psi(u_m) = \Psi(v_m)$ holds in the case $k = m$ and equalities $Eq.(2.3)$ hold in the case $k < m$.

Let $k = m$. Consider the element $g = v_k u_k^{-1} \in Q^*(\sqrt{-p})$. Since $v_k = v_k(u_k^{-1}u_k) = (v_k u_k^{-1})u_k$, we have $v_k = gu_k$. Then by $Eq.(2.2)$, we obtain that $v_k = M_g u_k$, where $M_g \in M(Q^*(\sqrt{-p}))$. Using the equality $\Psi(u_k) = \Psi(v_k)$ and Proposition 1.4, we obtain $\det(M_g) = \Psi(g) = \Psi(v_k u_k^{-1}) = \Psi(v_k)\Psi(u_k^{-1}) = \Psi(v_k)\Psi(u_k)^{-1} = 1$. Hence $g \in S(Q^*(\sqrt{-p}))$. By Theorem 2.5, $M_g \in SO(2,p,Q)$. This implies that $v_k = M_g u_k$. Since $B(u) = B(v) = k$, we have $u_j = v_j = \theta, \forall j \in N_m, j < k$. These equalities, the equality $v_k = M_g u_k$ and the equality $k = m$ imply equalities $v_j = M_g u_j, \forall j \in N_m$. Hence $u \overset{SO(2,p,Q)}{\sim} v$ in the case $k = m$. By $g = v_k u_k^{-1}$ and Proposition 2.18, M_g has the form (2.4). By $det(M_g) = 1$ and Proposition 2.18, we obtain the equality $det(M_g) = (\frac{\langle u_k, v_k\rangle_p}{\Psi(u_k)})^2 + p(\frac{[u_k\, v_k]}{\Psi(u_k)})^2 = 1$.

Let $k < m$. Using equalities $Eq.(2.3)$ and equalities $u_k^{-1}u_j = \frac{\langle u_k, u_j\rangle_p}{\Psi(u_k)} + \sqrt{-p}\frac{[u_k\, u_j]}{\Psi(u_k)}, \forall j \in N_m, k < j$, equalities $v_k^{-1}v_j = \frac{\langle v_k, v_j\rangle_p}{\Psi(v_k)} + \sqrt{-p}\frac{[v_k\, v_j]}{\Psi(v_k)}, \forall j \in N_m, k < j$, in Proposition 2.18, we obtain following equalities

$$u_k^{-1}u_j = v_k^{-1}v_j, \forall j \in N_m, k < j. \tag{2.5}$$

Consider the element $g = v_k u_k^{-1} \in Q^*(\sqrt{-p})$. Since $v_k = v_k(u_k^{-1}u_k) = (v_k u_k^{-1})u_k$, we have $v_k = gu_k$. Using equalities $Eq.(2.5)$, we obtain $v_k(u_k^{-1}u_j) = v_k(v_k^{-1}v_j), \forall j \in N_m, k < j$. These equalities and the above equality $g = v_k u_k^{-1}$ imply $v_j = (v_k v_k^{-1})v_j = v_k(v_k^{-1}v_j) = v_k(u_k^{-1}u_j) = (v_k u_k^{-1})u_j = gu_j$ for all $j \in N_m, k < j$. Thus we have $v_j = gu_j, \forall j \in N_m, k < j$, where $g = v_k u_k^{-1} \in Q^*(\sqrt{-p})$. The equality $g = v_k u_k^{-1}$ implies $v_k = gu_k$. This equality and the equalities $v_j = gu_j, \forall j \in N_m, k < j$ imply equalities $v_j = gu_j, \forall j \in N_m, k \le j$. Then by $Eq.(1.1)$, we obtain that $v_j = M_g u_j, \forall j \in N_m, k \le j$, where $M_g \in M(Q^*(\sqrt{-p}))$. These equalities and the equality $B(u) = B(v) = k$ imply that $v_j = M_g u_j$ for all $j \in N_m$. So we obtain that $\det(M_g) = 1$. Since $\det(M_g) = 1$, by Theorem 2.5, $M_g \in SO(2, p.Q)$. Hence we obtain $u \overset{SO(2,p,Q)}{\sim} v$.

Prove the uniqueness of $U \in SO(2,p,Q)$ satisfying the conditions $v_j = Uu_j, \forall j \in N_m$. Assume that $U \in SO(2,p,Q)$ such that $v_j = Uu_j, \forall j \in N_m$. Then by $Eq.(1.1)$ and Theorem 2.5, there exists the unique $b \in S(Q^*(\sqrt{-p}))$ such that $U = M_b$. Hence we have $v_j = M_b u_j, \forall j \in N_m$. By $Eq.(1.1)$, we obtain $v_j = bu_j, \forall j \in N_m$. Since $\Psi(u_k) \neq 0$, the equality $v_k = bu_k$ implies that $b = v_k u_k^{-1} = g \in S(Q^*(\sqrt{-p}))$. The uniqueness of U is proved.

Let us obtain the evident form of M_g. By Proposition 2.18, the element $g = v_k u_k^{-1}$ is equal to $\frac{\langle u_k, v_k\rangle_p}{\Psi(u_k)} + \sqrt{-p}\frac{[u_k\, v_k]}{\Psi(u_k)}$. Hence the matrix M_g has the form $Eq.(2.4)$. Since $g \in S(Q^*(\sqrt{-p}))$, by Theorem 2.5, $\det(M_g) = 1$. □

Remark 2.21. Let $k, m \in N, m > 1, 1 \le k \le m$. By Theorem 2.20, the function $\Psi(u_k)$ is a complete system of $SO(2,p,Q)$-invariant functions on the set $U(k;k)$ in the case $k = m$. By Theorem 2.20, the system

$$\left\{\Psi(u_k), \langle u_k, u_j\rangle_p, [u_k\, u_j], j \in N_m, k < j\right\}. \tag{2.6}$$

is a complete system of $SO(2,p,Q)$-invariant functions on the set $U(m;k)$ in the case $k < m$.

Let $G = O(2,p,Q)$ or $G = SO(2,p,Q)$. Denote by $G \vee Tr(2,p,Q)$ the group of all transformations of Q^2 generated by elements of G and all translations of Q^2. In particularly, $MO(2,p,Q) = O(2,p,Q) \vee Tr(2,p,Q)$ and $MSO(2,p,Q) = SO(2,p,Q) \vee Tr(2,p,Q)$. Now we consider H-equivalence problem of m-tuples for the group $H = G \vee Tr(2,p,Q)$. Let u and v be m-tuples, where $m = 1$. Then it is obvious that they are $G \vee Tr(2,p,Q)$-equivalent.

Let $z \in Q^2$. Denote by $z \cdot 1_m$ the m-tuple $(y_1, y_2, \ldots, y_m)$ such that $y_j = z, \forall j \in N_m$. Let $u = (u_1, u_2, \ldots, u_m)$ be an m-tuple. Denote by $u - u_m \cdot 1_m$ the m-tuple $(u_1 - u_m, u_2 - u_m, \ldots, u_{m-1} - u_m, 0)$.

Proposition 2.22. *Let $G = O(2, p, Q)$ or $G = SO(2, p, Q)$. Assume that $m > 1$ and $u = (u_1, u_2, \dots, u_m), v = (v_1, v_2, \dots, v_m) \in (Q^2)^m$. Then $u \overset{G\vee Tr(2,p,Q)}{\sim} v$ if and only if m-tuples $u - u_m \cdot 1_m$ and $v - v_m \cdot 1_m$ are G-equivalent.*

Proof. $\Rightarrow$ Assume that $u \overset{G\vee Tr(2,p,Q)}{\sim} v$. Then there exists $F \in G$ and $a \in Q^2$ such that $v_j = Fu_j + a, \forall j \in N_m$. In particularly, for $j = m$, we have $v_m = Fu_m + a$. This equality implies $a = v_m - Fu_m$. This equality and equalities $v_j = Fu_j + a, \forall j \in N_m$, imply equalities $v_j = Fu_j + v_m - Fu_m, \forall j \in N_m$. These equalities imply equalities $v_j - v_m = F(u_j - u_m), \forall j \in N_m$. That is $(u_1 - u_m, u_2 - u_m, \dots, u_{m-1} - u_m, 0) \overset{G}{\sim} (v_1 - v_m, v_2 - v_m, \dots, v_{m-1} - v_m, 0)$.

$\Leftarrow$ Assume that $(u_1 - u_m, u_2 - u_m, \dots, u_{m-1} - u_m, 0) \overset{G}{\sim} (v_1 - v_m, v_2 - v_m, \dots, v_{m-1} - v_m, 0)$. Then there exists $F \in G$ such that $v_j - v_m = F(u_j - u_m), \forall j \in N_m$. Put $a = v_m - Fu_m$. This equality implies $v_m = Fu_m + a$. The equality $a = v_m - Fu_m$ and equalities $v_j - v_m = F(u_j - u_m), \forall j \in N_m$, $v_m = Fu_m + a$ imply equalities $v_j = Fu_j + a, \forall j \in N_m$. Hence $u \overset{G\vee Tr(2,p,Q)}{\sim} v$. □

Corollary 2.23. *Let $G = O(2, p, Q)$ or $G = SO(2, p, Q)$. Assume that $m > 1$ and $u = (u_1, u_2, \dots, u_m)$, $v = (v_1, v_2, \dots, v_m) \in (Q^2)^m$. Then $u \overset{G\vee Tr(2,p,Q)}{\sim} v$ if and only if $(m-1)$-tuples $(u_1 - u_m, u_2 - u_m, \dots, u_{m-1} - u_m)$ and $(v_1 - v_m, v_2 - v_m, \dots, v_{m-1} - v_m)$ are G-equivalent.*

Proof. It follows from Proposition 2.22. □

Proposition 2.24. *Let $G = SO(2, p, Q)$ or $G = O(2, p, Q)$. Assume that $u \overset{G\vee Tr(2,p,Q)}{\sim} v$. Then $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m)$.*

Proof. This statement follows from Propositions 2.19 and 2.22. □

Let $u = (u_1, u_2, \dots, u_m) \in (Q^2)^m$ and G denote either the special orthogonal group $SO(2, p, Q)$ or the orthogonal group $O(2, p, Q)$. By Proposition 2.24, the function $B(u - u_m \cdot 1_m)$ is a $G \vee Tr(2, p, Q)$-invariant function of $u \in (Q^2)^m$.

It is obvious that $B(u - u_m \cdot 1_m) \le m - 1, \forall u \in (Q^2)^m$. We note that $B(u - u_m \cdot 1_m) = 0$ if and only if $u - u_m \cdot 1_m = 0_m$ that is $u = u_m \cdot 1_m = (u_1, u_2, \dots, u_m)$, where $u_j = u_m, \forall j \in N_m$.

Proposition 2.25. *Let $G = SO(2, p, Q)$ or $G = O(2, p, Q)$. Assume that $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m) = 0$. Then $u \overset{G\vee Tr(2,p,Q)}{\sim} v$.*

Proof. In the case $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m) = 0$, the m-tuple u has the form $u = (u_m, u_m, \dots, u_m)$ and the m-tuple v has the form $v = (v_m, v_m, \dots, v_m)$. Then we have $v_j = F(u_j), \forall j \in N_m$, where $F \in Tr(2, p, Q)$ has the following form: $v_j = v_m = u_m + a = u_j + a, \forall j \in N_m$, where $a = v_m - u_m$. Hence u and v are $G \vee Tr(2, p, Q)$-equivalent. □

Denote by $\Omega(m; 0)$ the set of all $u \in (Q^2)^m$ such that $B(u - u_m \cdot 1_m) = 0$. Let $k = 0$ or $k \in N_m$ such that $k \le m - 1$. Put $\Omega(m; k) = \{u \in (Q^2)^m | B(u - u_m \cdot 1_m) = k\}$.

Proposition 2.26. (1) *Let $G = SO(2, p, Q)$ or $G = O(2, p, Q)$. Then every set $\Omega(m; k)$ is an $G \vee Tr(2, p, Q)$-invariant subset of $(Q^2)^m$ for $k = 0$ and all $k \in N_m, k \le m - 1$.*

(2) $\Omega(m; 0) \cap \Omega(m; l) = \emptyset, \forall l \in N_m, l \le m - 1$.

(3) $\Omega(m; k) \cap \Omega(m; l) = \emptyset, \forall k, l \in N_m$, *where* $k \ne l$, $k \le m - 1, l \le m - 1$.

(4) $\cup_{k=0}^{m-1} \Omega(m; k) = (Q^2)^m$.

Proof. It follows from Proposition 2.17 □

Let $u, v \in (Q^2)^m$ such that $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m) = 0$. Then, by Proposition 2.24, $u \overset{SO(2,p,Q)\vee Tr(2,p,Q)}{\sim} v$. Now we consider the case $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m) = k$, where $k \in N_m, k \le m - 1$.

Theorem 2.27. *Let $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in (Q^2)^m$ be two m-tuples such that $B(u - u_m \cdot 1_m) = B(v - v_m \cdot 1_m) = k$, where $k \in N_m, k \leq m-1$.*

(*i*) *Assume that $u \overset{MSO(2,p,Q)}{\sim} v$. Then*

(*i*.1) *In the case $m = k+1$, the equality $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ holds.*

(*i*.2) *In the case $k+1 < m$, the following equalities hold*

$$\begin{cases} \Psi(u_k - u_m) = \Psi(v_k - v_m); \\ \langle u_k - u_m, u_j - u_m\rangle_p = \langle v_k - v_m, v_j - v_m\rangle_p, \forall j \in N_m, k < j \leq m-1; \\ \left[(u_k - u_m)\,(u_j - u_m)\right] = \left[(v_k - v_m)\,(v_j - v_m)\right], \forall j \in N_m, k < j \leq m-1. \end{cases} \tag{2.7}$$

(*ii*) *Conversely, assume that the equality $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ holds in the case $k+1 = m$ and equalities Eq.(2.7) hold in the case $k+1 < m$. Then there exists the unique matrix $F \in SO(2,p,Q)$ and the unique element $b \in Q^2$ such that $v_j = Fu_j + b, \forall j \in N_m$. In this case, F has the following form*

$$F = \begin{pmatrix} \frac{\langle u_k - u_m, v_k - v_m\rangle_p}{\Psi(u_k - u_m)} & -p\frac{[(u_k - u_m)\,(v_k - v_m)]}{\Psi(u_k - u_m)} \\ \frac{[(u_k - u_m)\,(v_k - v_m)]}{\Psi(u_k - u_m)} & \frac{\langle u_k - u_m, v_k - v_m\rangle_p}{\Psi(u_k - u_m)} \end{pmatrix}, \tag{2.8}$$

where $\det(F) = (\frac{\langle u_k - u_m, v_k - v_m\rangle_p}{\Psi(u_k - u_m)})^2 + p(\frac{[(u_k - u_m)\,(v_k - v_m)]}{\Psi(u_k - u_m)})^2 = 1$ and element $b \in Q^2$ is equal to $v_m - Fu_m$.

Proof. (*i*) Assume that $u \overset{MSO(2,p,Q)}{\sim} v$. Then, by Proposition 2.22, the m-tuples $(u_1 - u_m, u_2 - u_m, \ldots, u_{m-1} - u_m, 0)$ and $(v_1 - v_m, v_2 - v_m, \ldots, v_{m-1} - v_m, 0)$ are $SO(2,p,Q)$-equivalent. This equivalence and Theorem 2.20 imply the equality $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ in the case $m = k+1$ and the equalities *Eq*.(2.7) in the case $k+1 < m$.

(*ii*) Conversely, assume that the equality $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ holds in the case $m = k+1$ and the equalities *Eq*.(2.7) hold in the case $k+1 < m$. Then, by Theorem 2.20, in every these cases there exists the unique matrix $F \in SO(2,p,Q)$ such that $v_j - v_m = F(u_j - u_m), \forall j \in N_m$. By Theorem 2.20, F has the form (2.8), where $\det(F) = (\frac{\langle u_k - u_m, v_k - v_m\rangle_p}{\Psi(u_k - u_m)})^2 + p(\frac{[(u_k - u_m)\,(v_k - v_m)]}{\Psi(u_k - u_m)})^2 = 1$. Put $b = v_m - Fu_m$. Then this equality and equalities $v_j - v_m = F(u_j - u_m), \forall j \in N_m$, imply equalities $v_j = F(u_j) + b, \forall j \in N_m$. The uniqueness of F such that $v_j - v_m = F(u_j - u_m), \forall j \in N_m$ implies the uniqueness of b such that $v_j = F(u_j) + b, \forall j \in N_m$. ☐

Remark 2.28. Let $k, m \in N, m > 1, 1 \leq k \leq m-1$. By Theorem 2.27, the function $\Psi(u_k - u_m)$ is a complete system of $MSO(2,p,Q)$-invariant functions on the set $\Omega(m;k)$ in the case $m = k+1$. By Theorem 2.27, the system

$$\{\Psi(u_k - u_m), \langle u_k - u_m, u_j - u_m\rangle_p, \left[(u_k - u_m)(u_j - u_m)\right], k+1 \leq j \leq m-1\} \tag{2.9}$$

is a complete system of $MSO(2,p,Q)$-invariant functions on the set $\Omega(m;k)$ in the case $k+1 < m$.

3. Complete systems of invariants of an m-tuple in Q^2 for groups $O(2,p,Q)$ and $MO(2,p,Q)$

First we consider the case $m = 1$.

Theorem 3.1. *Let $u, v \in Q^2$.*

(*i*) *Assume that $u \overset{O(2,p,Q)}{\sim} v$. Then the equality $\Psi(u) = \Psi(v)$ holds.*

(*ii*) *Conversely, assume that the equality $\Psi(u) = \Psi(v)$ holds. In this case, $\Psi(u) = 0$ or $\Psi(u) \neq 0$*

(*ii*.1) *Let $\Psi(u) = 0$. Then $u = \theta, v = \theta$, where $\theta = (0,0)$, and $u \overset{O(2,p,Q)}{\sim} v$.*

(*ii*.2) *Let $\Psi(u) \neq 0$. Then $u \neq \theta, v \neq \theta$ and $u \overset{O(2,p,Q)}{\sim} v$. In this case, only two matrices $F_1 \in$*

$O(2,p,Q)$ and $F_2 \in O(2,p,Q)$ exist such that $v = F_1 u$, and $v = F_2 u$. Here $F_1 \in SO(2,p,Q)$ and it has the following form

$$F_1 = \begin{pmatrix} \frac{\langle u,v\rangle_p}{\Psi(u)} & -\frac{p[u\, v]}{\Psi(u)} \\ \frac{[u\, v]}{\Psi(u)} & \frac{\langle u,v\rangle_p}{\Psi(u)} \end{pmatrix}, \tag{3.1}$$

where $det(F_1) = (\frac{\langle u,v\rangle_p}{\Psi(u)})^2 + p(\frac{[u\, v]}{\Psi(u)})^2 = 1$.

Here $F_2 \in O(2,p,Q)$ and it has the following form $F_2 = HW$, where $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, $H \in SO(2,p,Q)$ and H has the following form

$$H = \begin{pmatrix} \frac{\langle (Wu),v\rangle_p}{\Psi(Wu)} & -\frac{p[(Wu)\, v]}{\Psi(Wu)} \\ \frac{[(Wu)\, v]}{\Psi(Wu)} & \frac{\langle (Wu),v\rangle_p}{\Psi(Wu)} \end{pmatrix}, \tag{3.2}$$

where $det(H) = (\frac{\langle (Wu),v\rangle_p}{\Psi(Wu)})^2 + p(\frac{[(Wu)\, v]}{\Psi(Wu)})^2 = 1$ and $det(F_2) = -1$.

Proof. (i) Assume that $u \overset{O(2,p,Q)}{\sim} v$. By Example 2.11, the function $\Psi(x)$ is $O(2,p,Q)$-invariant. Hence the equality $\Psi(u) = \Psi(v)$ holds.

(ii) Assume that the equality $\Psi(u) = \Psi(v)$ holds.

$(ii.1)$ Let $\Psi(u) = 0$. Then $u = v = \theta$. Then it is obvious that $u \overset{O(2,p,Q)}{\sim} v$.

$(ii.2)$ Let $\Psi(u) \neq 0$. This inequality and the equality $\Psi(u) = \Psi(v)$ imply inequalities $u \neq \theta$ and $v \neq \theta$. By Theorem 2.20, there exists the unique $F_1 \in SO(2,p,Q)$ such that $v = F_1 u$. Since $F_1 \in SO(2,p,Q) \subset O(2,p,Q)$, we obtain that $u \overset{O(2,p,Q)}{\sim} v$. Put $g = vu^{-1}$. By this equality and Proposition 2.18, $F_1 = M_g$. Hence we have $v = M_g u$. By Theorem 2.20, we obtain that M_g has the form (3.1) and the properties $det(M_g) = (\frac{\langle u,v\rangle_p}{\Psi(u)})^2 + p(\frac{[u\, v]}{\Psi(u)})^2 = 1$, $M_g \in SO(2,p,Q)$, $v = M_g u$ hold.

Now we investigate an existence of $F_2 \in O(2,p,Q)$ of the form $F_2 = HW$ such that $v = F_2 u$, where $H \in SO(2,p,Q)$. For given above u, v, the equality $\Psi(u) = \Psi(v)$ holds. Using Proposition 1.6(4), we obtain the equality $\Psi(v) = \Psi(u) = \Psi(Wu)$. By the equality $\Psi(v) = \Psi(Wu)$ and Theorem 2.20, there exists the unique $H \in SO(2,p,Q)$ such that $v = H(Wu)$. Put $F_2 = HW$. Then $F_2 \in \{HW | H \in SO(2,p,Q)\}$. Hence there exists $F_2 \in O(2,p,Q)$ of the form $F_2 = HW$, where $H \in SO(2,p,Q)$, such that $v = F_2 u$.

Prove the uniqueness of $F_2 \in \{HW | H \in SO(2,p,Q)\}$ such that $v = F_2 u$. Assume that $F_2 = H_2 W \in \{HW | H \in SO(2,p,Q)\}$ and $F_3 = H_3 W \in \{HW | H \in SO(2,p,Q)\}$ such that $v = H_2 W u$ and $v = H_3 W u$, where $H_2, H_3 \in SO(2,p,Q)$. Then we have $v = H_2(Wu) = H_3(Wu)$. Using the uniqueness in Theorem 2.20, we obtain $H_2 = H_3$. This means that the unique $F_2 = H_2 W \in \{HW | H \in SO(2,p,Q)\}$ exists such that $v = F_2(u)$. By Theorem 2.20, we obtain that H_2 has the form (3.2) and the properties $det(H_2) = (\frac{\langle Wu,v\rangle_p}{\Psi(Wu)})^2 + p(\frac{[(Wu)\, v]}{\Psi(Wu)})^2 = 1$, $H_2 W \in O(2,p,Q)$, $det(F_2) = -1$, $v = H_2 W(u)$ hold. □

Remark 3.2. Theorem 3.1 means that the function $\Psi(u)$ is a complete system of $O(2,p,Q)$-invariant functions on the set $U(1;1)$.

Let $u = (u_1, u_2, \ldots, u_m) \in (Q^2)^m$. Denote by $r(u)$ the rank of the system $\{u_1, u_2, \ldots, u_m\}$ in the space Q^2. For $u = \theta_m$, we put $r(\theta_m) = 0$. Assume that $u \neq \theta_m$. Then $r(u) = 1$ or $r(u) = 2$. It is obvious that the rank $r(u)$ is $O(2,p,Q)$-invariant of u. Put $U_0(m) = U(m,0)$. For $k \in N_m$, $l = 1,2$, denote by $U_l(m,k)$ the set of elements $u \in U(m,k)$ such that $r(u) = l$. It is obvious that $U_0(m) \cap U_l(m,k) = \emptyset$ for $l = 1,2$ and $U_l(m,k) \cap U_q(m,k) = \emptyset$ for $l,q = 1,2$ such that $l \neq q$. The following equalities $U_1(m,k) \cup U_2(m,k) = U(m,k), \forall k \in N_m$ and $U_0(m) \cup (\cup_{k=1}^m U_1(m,k)) \cup (\cup_{k=1}^m U_2(m,k)) = (Q^2)^m$ hold.

Let $u \in U_0(m), v \in U_0(m)$. Then $u = v = \theta_m$. Hence $u \overset{O(2,p,Q)}{\sim} v$.

Theorem 3.3. *Let $m > 1$ and $u = (u_1, u_2, \ldots, u_m) \in U_1(m,k), v = (v_1, v_2, \ldots, v_m) \in U_1(m,k)$, where $k \in N_m$.*

(*i*) *Assume that* $k = m$ *and* $u \overset{O(2,p,Q)}{\sim} v$. *Then the equality* $\Psi(u_m) = \Psi(v_m)$ *holds. Conversely, assume that* $k = m$ *and the equality* $\Psi(u_m) = \Psi(v_m)$ *holds. In this case, only two matrices* $F_1 \in O(2,p,Q)$ *and* $F_2 \in O(2,p,Q)$ *exist such that* $v_j = F_1(u_j), \forall j \in N_m$, *and* $v_j = F_2(u_j), \forall j \in N_m$. *Here* $F_1 \in SO(2,p,Q)$ *and it has the following form*

$$F_1 = \begin{pmatrix} \frac{\langle u_k, v_k\rangle_p}{\Psi(u_k)} & -p\frac{[u_k\, v_k]}{\Psi(u_k)} \\ \frac{[u_k\, v_k]}{\Psi(u_k)} & \frac{\langle u_k, v_k\rangle_p}{\Psi(u_k)} \end{pmatrix}, \tag{3.3}$$

where $det(F_1) = (\frac{\langle u_k, v_k\rangle_p}{\Psi(u_k)})^2 + p(\frac{[u_k\, v_k]}{\Psi(u_k)})^2 = 1$.

Here $F_2 \in O(2,p,Q)$ *and it has the following form* $F_2 = HW$, *where* $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, $H \in SO(2,p,Q)$ *and* H *has the following form*

$$H = \begin{pmatrix} \frac{\langle (Wu_k), v_k\rangle_p}{\Psi(Wu_k)} & -\frac{p[(Wu_k)\, v_k]}{\Psi(Wu_k)} \\ \frac{[(Wu_k)\, v_k]}{\Psi(Wu_k)} & \frac{\langle (Wu_k), v\rangle_p}{\Psi(Wu_k)} \end{pmatrix}, \tag{3.4}$$

where $det(H) = (\frac{\langle (Wu_k), v_k\rangle_p}{\Psi(Wu_k)})^2 + p(\frac{[(Wu_k)\, v_k]}{\Psi(Wu_k)})^2 = 1$ *and* $det(F_2) = -1$.

(*ii*) *Assume that* $k < m$ *and* $u \overset{O(2,p,Q)}{\sim} v$. *Then the following equalities hold*

$$\begin{cases} \Psi(u_k) = \Psi(v_k), \\ \langle u_k, u_j\rangle_p = \langle v_k, v_j\rangle_p, \forall j \in N_m, k < j. \end{cases} \tag{3.5}$$

Conversely, assume that the equalities Eq.(3.5) *hold. In this case, only two matrices* $F_1 \in O(2,p,Q)$ *and* $F_2 \in O(2,p,Q)$ *exist such that* $v = F_1 u$ *and* $v = F_2 u$. *Here* $F_1 \in SO(2,Q)$ *and it is has the form Eq.*(3.3). *Here* $F_2 \in O(2,p,Q)$ *and it has the following form* $F_2 = HW$, *where* $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, $H \in SO(2,p,Q)$ *and* H *has the form Eq.*(3.4).

Proof. (*i*) In this case, m-tuples u and v have following forms: $u = (u_1, u_2, \ldots, u_m)$, where $u_j = \theta, \forall j \in N_m, j < m$, $u_m \neq \theta$, $v = (v_1, v_2, \ldots, v_m)$, where $v_j = \theta, \forall j \in N_m, j < m$, $v_m \neq \theta$. These forms imply that the statement (*i*) follows from Theorem 3.1.

(*ii*) Assume that the equalities *Eq.*(3.5) hold. Since $r(u) = r(v) = 1$, m-tuples u and v have following forms: $u = (u_1, u_2, \ldots, u_m)$, where $u_j = \theta, \forall j \in N_m, j < k$, $u_k \neq \theta$, $u_j = a_j u_k, \forall j \in N_m, k < j$, for some $a_j \in Q, \forall j \in N_m, k < j$, and $v = (v_1, v_2, \ldots, v_m)$, where $v_j = \theta, \forall j \in N_m, j < k$, $v_k \neq \theta$, $v_j = b_j v_k, \forall j \in N_m, k < j$ for some $b_j \in R, \forall j \in N_m, k < j$. It is easy to see that equalities *Eq.*(3.5) and the inequality $\Psi(u_k) \neq 0$ imply equalities $a_j = b_j, \forall j \in N_m, k < j$. Hence m-tuples u, v have following forms $u = (u_1, u_2, \ldots, u_m)$, where $u_j = \theta, \forall j \in N_m, j < k$, $u_k \neq \theta$, $u_j = a_j u_k, \forall j \in N_m, k < j$, and $v = (v_1, v_2, \ldots, v_m)$, where $v_j = \theta, \forall j \in N_m, j < k$, $v_k \neq \theta$, $v_j = a_j v_k, \forall j \in N_m, k < j$.

By using Eq.(3.5), we obtain equality $\Psi(u_k) = \Psi(v_k)$. Since $u_k \neq \theta$, we obtain that $\Psi(u_k) = \Psi(v_k) \neq 0$. Then, by Theorem 3.1(*ii*.2), only two matrices $F_1 \in O(2,p,Q)$ and $F_2 \in O(2,p,Q)$ exist such that $v_k = F_1(u_k)$ and $v_k = F_2(u_k)$. Here $F_1 \in SO(2,p,Q)$ and it has the form Eq.(3.3) and $F_2 \in O(2,p,Q)$ has the form $F_2 = HW$, where $H \in SO(2,p,Q)$ and H has the form Eq.(3.4). Equalities $v_k = F_1(u_k)$, $v_k = F_2(u_k)$ and equalities $u_j = \theta, \forall j \in N_m, j < k$, $u_k \neq \theta$, $u_j = a_j u_k, \forall j \in N_m, k < j$, $v_j = \theta, \forall j \in N_m, j < k$, $v_k \neq \theta$, $v_j = a_j v_k, \forall j \in N_m, k < j$ imply equalities $v_j = F_1(u_j), \forall j \in N_m$ and $v_j = F_2(u_j), \forall j \in N_m$.

Now we prove that if a matrix $A \in O(2,p,Q)$ such that $v_j = A(u_j), \forall j \in N_m$, then $A = F_1$ or $A = F_2$. Equalities $v_j = A(u_j), \forall j \in N_m$, implies the equality $v_k = A(u_k)$, where $k \in N_m$. Then, by Theorem 3.1(*ii*.2), $A = F_1$ or $A = F_2$. □

Theorem 3.4. *Let* $m > 1, k \in N_m$ *and* $u = (u_1, u_2, \ldots, u_m) \in (Q^2)^m$. *Let* $\left\{\Psi(u_k), \langle u_k, u_j\rangle_p \,|j \in N_m, k < j\right\}$ *be the complete system of* $O(2,p,Q)$*-invariants on the set* $U_1(m;k)$ *given in Theorem 3.3. Then* $\Psi(u_k) > 0$ *for all* $u \in U_1(m;k)$.

Proof. A proof is similar to the proof of Theorem 2.20 and it is omitted. □

Let $m > 1$ and $u = (u_1, u_2, \dots, u_m) \in (Q^2)^m$. Assume that $r(u) = 2$ and $B(u) = k$, where $k \in N_m$. Put $B_1(u) = B(u)$. Denote by $\{\lambda u_k | \lambda \in Q\}$ the linear subspace of Q^2 generated by u_k. Denote by $B_2(u)$ the smallest of $s, s \in N_m$, such that $u_s \notin \{\lambda u_k | \lambda \in Q\}$. Then $B_1(u) < B_2(u) \leq m$ for all $u \in U_2(m,k)$. The number $B_2(u)$ is an $O(2,p,Q)$-invariant of an m-tuple u. The pair $(B_1(u), B_2(u))$ will be called the type of an m-tuple $u \in U_2(m,k)$ of $r(u) = 2$. The type $(B_1(u), B_2(u))$ is an $O(2,p,Q)$-invariant of an m-tuple $u \in U_2(m,k)$. Let $k, l \in N_m$ such that $k < l$. Then there exists an m-tuple $u = (u_1, u_2, \dots, u_m) \in (Q^2)^m$ such that $B_1(u) = k$ and $B_2(u) = l$. In this case, vectors $u_k = (u_{k1}, u_{k2})$ and $u_l = (u_{l1}, u_{l2})$ are linearly independent.

Denote by $E(u; k, l)$ the following 2×2-matrix

$$\begin{pmatrix} u_{k1} & u_{k2} \\ u_{l1} & u_{l2} \end{pmatrix}.$$

Since vectors $u_k = (u_{k1}, u_{k2})$ and $u_l = (u_{l1}, u_{l2})$ are linearly independent, $det(E(u; k, l)) \neq 0$.

Denote by $\Phi(u; k, l)$ the following 2×2-matrix

$$\begin{pmatrix} u_{k1} & pu_{k2} \\ u_{l1} & pu_{l2} \end{pmatrix}. \tag{3.6}$$

We have $det(\Phi(u; k, l)) = pu_{k1}u_{i2} - pu_{k2}u_{l1} = pdet(E(u; k, l))$. Since $det(E(u; k, l)) \neq 0$, we obtain that $det(\Phi(u; k, l)) \neq 0$. Hence the inverse matrix $\Phi^{-1}(u; k, l)$ exits.

Denote by $U_2(m, k, l)$ the set of all m-tuples u such that $B_1(u) = k$ and $B_2(u) = l$.

Theorem 3.5. *Let $m > 1$ and $u = (u_1, u_2, \dots, u_m) \in U_2(m,k,l), v = (v_1, v_2, \dots, v_m) \in U_2(m,k,l)$, where $k, l \in N_m, k < l$, $u_k = (u_{k1}, u_{k2})$, $u_l = (u_{l1}, u_{l2})$, $v_k = (v_{k1}, v_{k2})$, $v_l = (v_{l1}, v_{l2})$.*

(*i*) *Assume that $u \overset{O(2,p,Q)}{\sim} v$. Then the following equalities hold:*

$$\begin{cases} \langle u_k, u_j \rangle_p = \langle v_k, v_j \rangle_p, \forall j \in N_m, k \leq j; \\ \langle u_l, u_j \rangle_p = \langle v_l, v_j \rangle_p, \forall j \in N_m, k < j. \end{cases} \tag{3.7}$$

(*ii*) *Conversely, assume that the equalities Eq.(3.7) hold. Then $u \overset{O(2,p,Q)}{\sim} v$. In this case, the unique $F \in O(2,p,Q)$ exists such that $v_j = F(u_j), \forall j \in N_m$, and F has the following form $F = \Phi^{-1}(v; k, l)\Phi(u; k, l)$.*

Proof. (*i*) Assume that $u \overset{O(2,p,Q)}{\sim} v$. Since the functions $\langle u_k, u_j \rangle_p$ and $\langle u_l, u_j \rangle_p$ are $O(2,p,Q)$-invariant, equalities Eq.(3.7) hold.

(*ii*) Conversely, assume that the equalities Eq.(3.7) hold. For $j \in N_m$, consider the vectors $u_j = (u_{j1}, u_{j2})$ and $v_j = (v_{j1}, v_{j2})$. Transposes of vectors u_j and v_j denote by $u_j^\top$ and $v_j^\top$.

$$u_j^\top = \begin{pmatrix} u_{j1} \\ u_{j2} \end{pmatrix}, v_j^\top = \begin{pmatrix} v_{j1} \\ v_{j2} \end{pmatrix}.$$

Consider the following vectors: $u_k = (u_{k1}, pu_{k2})$ and $u_l = (u_{l1}, pu_{l2})$. The multiplication of matrices u_k and $u_j^\top$ is equal to $\langle u_k, u_j \rangle_p$:

$$u_k \cdot u_j^\top = (u_{k1}, pu_{k2}) \begin{pmatrix} u_{j1} \\ u_{j2} \end{pmatrix} = \langle u_k, u_j \rangle_p, \forall j \in N_m. \tag{3.8}$$

Similarly we obtain

$$u_l \cdot u_j^\top = (u_{l1}, pu_{l2}) \begin{pmatrix} u_{j1} \\ u_{j2} \end{pmatrix} = \langle u_l, u_j \rangle_p, \forall j \in N_m. \tag{3.9}$$

Using equalities Eq.(3.8) and Eq.(3.9) to matrices $\Phi(u;k,l)$, $u_k^{\top}$ and vectors u_j, we obtain the following equalities

$$\Phi(u;k,l)u_j = \begin{pmatrix} u_{k1} & pu_{k2} \\ u_{l1} & pu_{l2} \end{pmatrix} \begin{pmatrix} u_{j1} \\ u_{j2} \end{pmatrix} = \begin{pmatrix} \langle u_k, u_j\rangle_p \\ \langle u_l, u_j\rangle_p \end{pmatrix}, \forall j \in N_m. \tag{3.10}$$

Similarly we obtain the following equalities

$$\Phi(v;k,l)v_j = \begin{pmatrix} v_{k1} & pv_{k2} \\ v_{l1} & pv_{l2} \end{pmatrix} \begin{pmatrix} v_{j1} \\ v_{j2} \end{pmatrix} = \begin{pmatrix} \langle v_k, v_j\rangle_p \\ \langle v_l, v_j\rangle_p \end{pmatrix}, \forall j \in N_m. \tag{3.11}$$

Since $B_1(u) = B_1(v) = k$, we have $u_j = v_j = 0, \forall j \in N_m, j < k$. These equalities imply the following equalities

$$\begin{cases} \langle u_k, u_j\rangle_p = \langle v_k, v_j\rangle_p = 0, \forall j \in N_m, j < k, \\ \langle u_l, u_j\rangle_p = \langle v_l, v_j\rangle_p = 0, \forall j \in N_m, j < l. \end{cases} \tag{3.12}$$

These equalities and the equalities Eq.(3.7) imply following equalities

$$\begin{cases} \langle u_k, u_j\rangle_p = \langle v_k, v_j\rangle_p, \forall j \in N_m, \\ \langle u_l, u_j\rangle_p = \langle v_l, v_j\rangle_p, \forall j \in N_m. \end{cases} \tag{3.13}$$

These equalities and equalities Eq.(3.10), Eq.(3.11) imply following equalities

$$\Phi(u;k,l)u_j = \begin{pmatrix} \langle u_k, u_j\rangle_p \\ \langle u_l, u_j\rangle_p \end{pmatrix} = \begin{pmatrix} \langle v_k, v_j\rangle_p \\ \langle v_l, v_j\rangle_p \end{pmatrix} = \Phi(v;k,l)v_j, \forall j \in N_m. \tag{3.14}$$

Since vectors $u_k = (u_{k1}, pu_{k2})$ and $u_l = (u_{l1}, pu_{l2})$ are linearly independent, the inverse of the matrix $\Phi^{-1}(u;k,l)$ exists. The equalities $\Phi(u;k,l)u_j = \Phi(v;k,l)v_j, \forall j \in N_m$ in Eq.(3.14) implies the following equalities

$$v_j = \Phi^{-1}(v;k,l))\Phi(u;k,l)u_j, \forall j \in N_m. \tag{3.15}$$

We prove that the matrix $\Phi^{-1}(v;k,l)\Phi(u;k,l)$ is orthogonal. Using the equalities Eq. (3.15) and Eq. (3.7), we obtain the following equality

$$\langle \Phi^{-1}(v;k,l)\Phi(u;k,l)u_j, \Phi^{-1}(v;k,l)\Phi(u;k,l)u_s\rangle = \langle v_j, v_s\rangle = \langle u_j, u_s\rangle \tag{3.16}$$

for $j = k,l$ and $s = k,l$. Since the system of two vectors u_k and u_l are a basis in Q^2, equalities Eq.(3.16) imply the following equalities

$$\langle \Phi^{-1}(v;k,l))\Phi(u;k,l)x, \Phi^{-1}(v;k,l))\Phi(u;k,l)y\rangle_p = \langle x, y\rangle_p, \forall x, y \in Q^2. \tag{3.17}$$

This means that the matrix $\Phi^{-1}(v;k,l))\Phi(u;k,l)$ is orthogonal.

Now we prove the uniqueness of a 2×2-orthogonal matrix F such that $v_j = Fu_j, \forall j \in N_m$. Assume that a 2×2-orthogonal matrix F such that $v_j = Fu_j, \forall j \in N_m$. In particularly, we have $v_j = Fu_j$ for $j = k,l$. These equalities and equalities Eq.(3.15) imply equalities

$$Fu_j = \Phi^{-1}(v;k,l)\Phi(u;k,l)u_j, \forall j = k,l. \tag{3.18}$$

Since the system of two vectors u_k and u_l are a basis in Q^2, equalities Eq.(3.18) imply following equalities:

$$Fx = \Phi^{-1}(v;k,l)\Phi(u;k,l)x, \forall x \in Q^2. \tag{3.19}$$

This means that $F = \Phi^{-1}(v;k,l)\Phi(u;k,l)$. The uniqueness of a 2×2-orthogonal matrix F such that $v_j = Fu_j, \forall j \in N_m$, is proved. □

Remark 3.6. By theorem 3.5, the system of $O(2,p,Q)$-invariants obtained in Theorem 3.5 is a complete system of $O(2,p,Q)$-invariants of m-tuples.

Now we investigate complete systems of invariants of the group $MO(2,p,Q) = O(2,p,Q) \vee Tr(2,p,Q)$ on the set $(Q^2)^m$.

Let $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in (Q^2)^m$. By Proposition 2.22, $u \overset{MO(2,p,Q)}{\sim} v$ if and only if $(u - u_m \cdot 1_m) \overset{O(2,p,Q)}{\sim} (v - v_m \cdot 1_m)$. By Proposition 2.24, the function $B(u - u_m \cdot 1_m)$ is an $MO(2,p,Q)$-invariant.

Let $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in U_0(m,k)$. Then $u = v = \theta_m$. This implies that $u \overset{MO(2,p,Q)}{\sim} v$.

Let $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in U_1(m,k)$. Assume that u and v be m-tuples such that $m = 1$. Then it is obvious that they are $MO(2,p,Q)$-equivalent. Assume that $u = (u_1, u_2, \ldots, u_m) \in U_1(m,m)$ and $v = (v_1, v_2, \ldots, v_m) \in U_1(m,m)$. Then it is obvious that they are $MO(2,p,Q)$-equivalent.

Let $u = (u_1, u_2, \ldots, u_m), v = (v_1, v_2, \ldots, v_m) \in (Q^2)^m$. By Corollary 2.23, $u \overset{MO(2,p,Q)}{\sim} v$ if and only if $(m-1)$-tuples $(u_1 - u_m, u_2 - u_m, \ldots, u_{m-1} - u_m)$ and $(v_1 - v_m, v_2 - v_m, \ldots, v_{m-1} - v_m)$ are $O(2,p,Q)$-equivalent. This Corollary 2.23 and Theorems 3.1, 3.3, 3.5 imply Theorems 3.7, 3.9, 3.10 given below.

Consider the case $m = 2$.

Theorem 3.7. *Let $u = (u_1, u_2) \in (Q^2)^2$ and $v = (v_1, v_2) \in (Q^2)^2$.*

- (i) *Assume that $u \overset{MO(2,p,Q)}{\sim} v$. Then the equality $\Psi(u_1 - u_2) = \Psi(v_1 - v_2)$ holds.*
- (ii) *Conversely, assume that the equality $\Psi(u_1 - u_2) = \Psi(v_1 - v_2)$ holds. In this case, $\Psi(u_1 - u_2) = 0$ or $\Psi(u_1 - u_2) \neq 0$*

  $(ii.1)$ *Let $\Psi(u_1 - u_2) = 0$. Then $u_1 - u_2 = v_1 - v_2 = 0$ and $u \overset{MO(2,p,Q)}{\sim} v$. In this case the unique $a \in Q^2$ exists such that $v_j = u_j + a, \forall j = 1,2$. It is equal to $v_2 - u_2$.*

  $(ii.2)$ *Let $\Psi(u_1 - u_2) \neq 0$. Then $u \overset{MO(2,p,Q)}{\sim} v$. In this case, only two elements $F_1 \in MO(2,p,Q)$ and $F_2 \in MO(2,p,Q)$ exist such that $v_j = F_1 u_j, \forall j = 1,2$, and $v_j = F_2 u_j, \forall j = 1,2$. Here $F_1(u_j) = H_1(u_j) + a_1, j = 1,2$, where $H_1 \in SO(2,p,Q)$, $a_1 \in Q^2$, and H_1 has the following form*

$$H_1 = \begin{pmatrix} \frac{\langle u_1-u_2, v_1-v_2\rangle_p}{\Psi(u_1-u_2)} & -p\frac{[(u_1-u_2)\ (v_1-v_2)]}{\Psi(u_1-u_2)} \\ \frac{[(u_1-u_2)\ (v_1-v_2)]}{\Psi(u_1-u_2)} & \frac{\langle u_1-u_2, v_1-v_2\rangle_p}{\Psi(u_1-u_2)} \end{pmatrix},$$

$$det(H_1) = \left(\frac{\langle u_1-u_2, v_1-v_2\rangle_p}{Q(u_1-u_2)}\right)^2 + p\left(\frac{[(u_1-u_2)\ (v_1-v_2)]}{Q(u_1-u_2)}\right)^2 = 1,\ a_1 = v_1 - H_1 u_1.$$

Here $F_2(u_j) = H_2 W(u_j) + a_2, j = 1,2$, where $H_2 \in SO(2,p,Q)$, $a_2 \in Q^2$, $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, and H_2 has the following form

$$H_2 = \begin{pmatrix} \frac{\langle W(u_1-u_2), v_1-v_2\rangle_p}{\Psi(W(u_1-u_2))} & -p\frac{[(W(u_1-u_2))\ (v_1-v_2)]}{\Psi(W(u_1-u_2))} \\ \frac{[(W(u_1-u_2))\ (v_1-v_2)]}{\Psi(W(u_1-u_2))} & \frac{\langle W(u_1-u_2), v_1-v_2\rangle_p}{\Psi(W(u_1-u_2))} \end{pmatrix},$$

$$det(H_2) = \left(\frac{\langle W(u_1-u_2), v_1-v_2\rangle_p}{\Psi(W(u_1-u_2))}\right)^2 + p\left(\frac{[(W(u_1-u_2))\ (v_1-v_2)]}{\Psi(W(u_1-u_2))}\right)^2 = 1,\ a_2 = v_1 - H_2 W u_1.$$

Proof. By Corollary 2.23, two 2-tuples $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are $MO(2,p,Q)$-equivalent if and only if vectors $u_1 - u_2$ and $v_1 - v_2$ are $O(2,p,Q)$-equivalent. Hence Theorem 3.1 implies this theorem. □

Let $m > 2$ and $u = (u_1, u_2, \ldots, u_m), v \in (Q^2)^m$ be two m-tuples such that $u = (u_1 - u_m, u_2 - u_m, \ldots, u_{m-1} - u_m) \in U_0(m-1,k), v = (v_1 - v_m, v_2 - v_m, \ldots, v_{m-1} - v_m) \in U_0(m-1,k)$, where $1 \leq k \leq m-1$. Then m-tuples u and v have forms $u = (u_m, u_m, \ldots, u_m)$ and $v = (v_m, v_m, \ldots, v_m)$. It is obvious that they are $MO(2,p,Q)$-equivalent.

Remark 3.8. Theorem 3.7 means that the function $\Psi(u_1 - u_2)$ is a complete system of $MO(2,p,Q)$-invariant functions on the set $(Q^2)^2$.

Theorem 3.9. *Let $m > 2$ and $u = (u_1, u_2, \dots, u_m), v = (v_1, v_2, \dots, v_m) \in (Q^2)^m$ be two m-tuples such that $u = (u_1 - u_m, u_2 - u_m, \dots, u_{m-1} - u_m), v = (v_1 - v_m, v_2 - v_m, \dots, v_{m-1} - v_m) \in U_1(m-1,k)$, where $1 \leq k \leq m-1$.*

(i) Assume that $k = m-1$ and $u \overset{MO(2,p,Q)}{\sim} v$. Then the equality $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ holds. Conversely, assume that $k = m-1$ and the equality $\Psi(u_k - u_m) = \Psi(v_k - v_m)$ holds. In this case, $\Psi(u_k - u_m) = 0$ or $\Psi(u_k - u_m) \neq 0$

(i.1) Let $\Psi(u_k - u_m) = 0$. Then $u_k - u_m = v_k - v_m = 0$ and $u \overset{MO(2,p,Q)}{\sim} v$. In this case the unique $a \in Q^2$ exists such that $v_j = u_j + a, \forall j \in N_m$. It is equal to $v_m - u_m$.

(i.2) Let $\Psi(u_k - u_m) \neq 0$. Then $u \overset{MO(2,p,Q)}{\sim} v$. In this case, only two elements $F_1 \in MO(2,p,Q)$ and $F_2 \in MO(2,p,Q)$ exist such that $v_j = F_1 u_j$ and $v_j = F_2 u_j, \forall j \in N_m$. Here $F_1(u_j) = H_1(u_j) + a_1, \forall j \in N_m$, where $H_1 \in SO(2,p,Q)$, $a_1 \in Q^2$, and H_1 has the following form

$$H_1 = \begin{pmatrix} \frac{\langle u_k - u_m, v_1 - v_m \rangle_p}{\Psi(u_k - u_m)} & -p\frac{[(u_k - u_m)\ (v_k - v_m)]}{\Psi(u_k - u_m)} \\ \frac{[(u_k - u_m)\ (v_k - v_m)]}{\Psi(u_k - u_m)} & \frac{\langle u_k - u_m, v_k - v_m \rangle_p}{\Psi(u_k - u_m)} \end{pmatrix}, \tag{3.20}$$

$det(H_1) = (\frac{\langle u_k - u_m, v_k - v_m \rangle_p}{\Psi(u_k - u_m)})^2 + p(\frac{[(u_k - u_m)\ (v_k - v_m)]}{\Psi(u_k - u_m)})^2 = 1$, $a_1 = v_k - H_1 u_k$.

Here $F_2(u_j) = H_2 W(u_j) + a_2, \forall j \in N_m$, where $H_2 \in SO(2,p,Q)$, $a_2 \in Q^2$, $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, and H_2 has the following form

$$H_2 = \begin{pmatrix} \frac{\langle W(u_k - u_m), v_k - v_m \rangle_p}{\Psi(W(u_k - u_m))} & -p\frac{[(W(u_k - u_m))\ (v_k - v_m)]}{\Psi(W(u_k - u_m))} \\ \frac{[(W(u_k - u_m))\ (v_k - v_m)]}{\Psi(W(u_k - u_m))} & \frac{\langle W(u_k - u_m), v_k - v_m \rangle_p}{\Psi(W(u_k - u_m))} \end{pmatrix}, \tag{3.21}$$

$det(H_2) = (\frac{\langle W(u_k - u_m), v_k - v_m \rangle_p}{\Psi(W(u_k - u_m))})^2 + p(\frac{[(W(u_k - u_m))\ (v_k - v_m)]}{\Psi(W(u_k - u_m))})^2 = 1$, $a_2 = v_k - H_2 W u_k$.

(ii) Assume that $k + 1 < m$ and $u \overset{MO(2,p,Q)}{\sim} v$. Then the following equalities hold

$$\begin{cases} \Psi(u_k - u_m) = \Psi(v_k - v_m), \\ \langle u_k - u_m, u_j - u_m \rangle_p = \langle v_k - v_m, v_j - v_m \rangle_p, \forall j \in N_m, k < j < m. \end{cases} \tag{3.22}$$

Conversely, assume that the equalities Eq.(3.22) hold. In this case, only two elements $F_1 \in MO(2,p,Q)$ and $F_2 \in MO(2,p,Q)$ exist such that $v_j = F_1 u_j, \forall j \in N_m$, and $v_j = F_2 u_j, \forall j \in N_m$. Here $F_1(u_j) = H_1(u_j) + a_1, \forall j \in N_m$, where $H_1 \in SO(2,p,Q)$, $a_1 \in Q^2$, and H_1 has the following form Eq.(3.20). Here $F_2(u_j) = H_2 W(u_j) + a_1, \forall j \in N_m$, where $H_2 \in SO(2,p,Q)$, $a_2 \in Q^2$, $W = \|w_{kl}\|_{k,l=1,2}$, $w_{11} = 1$, $w_{12} = w_{21} = 0$, $w_{22} = -1$, H_2 has the form Eq.(3.21) and $a_2 = v_k - H_2 W u_k$.

Proof. A proof follows from Corollary 2.23, Theorem 3.1 and Theorem 3.3. □

Let $m > 2$ and $u = (u_1, u_2, \dots, u_m) \in (Q^2)^m$ be an m-tuple in Q^2 such that $(u_1 - u_m, u_2 - u_m, \dots, u_{m-1} - u_m) \in U_2(m-1,k,l)$. In this case, vectors $u_k - u_m = (u_{k1} - u_{m1}, u_{k2} - u_{m2})$ and $u_l - u_m = (u_{l1} - u_{m1}, u_{l2} - u_{m2})$ are linearly independent. Denote by $E(u - u_m 1_m; k, l)$ the following 2×2-matrix

$$\begin{pmatrix} u_{k1} - u_{m1} & u_{k2} - u_{m2} \\ u_{l1} - u_{m1} & u_{l2} - u_{m2} \end{pmatrix}.$$

Since the vectors $u_k - u_m$ and $u_l - u_m$ are linearly independent, $det(E(u - u_m 1_m; k, l)) \neq 0$. Denote by $\Phi(u - u_m 1_m; k, l)$ the following 2×2-matrix

$$\begin{pmatrix} u_{k1} - u_{m1} & p(u_{k2} - u_{m2}) \\ u_{l1} - u_{m1} & p(u_{l2} - u_{m2}) \end{pmatrix}.$$

Since $det(\Phi(u - u_m 1_m; k, l) = p \cdot det(E(u - u_m 1_m; k, l))$ and $det(E(u - u_m 1_m; k, l)) \neq 0$, we obtain that $det(\Phi(u - u_m 1_m; k, l) \neq 0$. This implies that the inverse matrix $\Phi^{-1}(u - u_m 1_m; k, l)$ exists.

Theorem 3.10. *Let $m > 2$. Assume that $u = (u_1, u_2, \dots, u_m)$ and $v = (v_1, v_2, \dots, v_m)$ are $(m-1)$-tuples such that $(u_1 - u_m, u_2 - u_m, \dots, u_{m-1} - u_m) \in U_2(m-1, k, l), (v_1 - v_m, v_2 - v_m, \dots, v_{m-1} - v_m) \in U_2(m-1, k, l)$, where $1 \leq k < l \leq m-1$.*

(*i*) *Assume that $u \overset{MO(2,p,Q)}{\sim} v$. Then the following equalities hold:*

$$\begin{cases} \langle u_k - u_m, u_j - u_m \rangle_p = \langle v_k - v_m, v_j - v_m \rangle_p, \forall j \in N_m, k \leq j \leq m-1; \\ \langle u_l - u_m, u_j - u_m \rangle_p = \langle v_l - v_m, v_j - v_m \rangle_p, \forall j \in N_m, l \leq j \leq m-1. \end{cases} \tag{3.23}$$

(*ii*) *Conversely, assume that the equalities Eq.(3.23) hold. Then $u \overset{MO(2,p,Q)}{\sim} v$. In this case, the unique $F \in MO(2,p,Q))$ exists such that $v_j = F(u_j), \forall j \in N_m$, and F has the following form $F(u_j) = (\Phi^{-1}(v - v_m 1_m; k, l)\Phi(u - u_m 1_m; k, l))(u_j) + a, \forall j \in N_m$, where $a = v_k - (\Phi^{-1}(v - v_m 1_m; k, l)\Phi(u - u_m 1_m; k, l))(u_k)$.*

Proof. A proof follows from Corollary 2.23 and Theorem 3.5. □

References

[1] M. Berger, Geometry I, Springer-Verlag, Berlin, Heidelberg, 1987.

[2] J. A. Dieudonné, J.B. Carrell, Invariant Theory, Academic Press, New-York, London, 1971.

[3] O. Gűrsoy, M. İncesu, LS(2)-equivalence conditions of control points and application to planar Bezier curves, New Trends in Mathematical Sciences, 3, no. 5 (2017) 70 -84. https: //doi.org/10.20852/ntmsci.2017.186.

[4] R.Igor Shafarevich, O. Alexey Remizov, Linear Algebra and Geometry, Springer-Verlag, Berlin Heidelberg, 2013

[5] R. Hőfer, m-point invariants of real geometries, Beitrage Algebra Geom., 40 (1999) 261-266.

[6] D. Khadjiev, Application of the Invariant Theory to the Differential Geometry of Curves, Fan Publisher, Tashkent, 1988, [in Russian].

[7] D. Khadjiev, Y. Gőksal, Applications of hyperbolic numbers to the invariant theory in two-dimensional pseudo-Euclidean space, Adv. Appl. Clifford Algebras, 26 (2016) 645-668.https: //doi 10.1007/s00006-015-0627-9.

[8] D. Khadjiev, Projective invariants of m-tuples in the one-dimensional projective space, Uzbek Mathematical Journal, 1 (2019) 61-73. https: //doi:10.29229/uzmj.2019-1-6.

[9] D. Khadjiev, Sh. Ayupov, G. Beshimov, Complete systems of invariant of m-tuples for fundamental groups of the two-dimensional Euclidian space, Uzbek Mathematical Juornal, 1 (2020) 57-84. https://DOI: 10.29229/uzmj.2020-1-6

[10] D. Khadjiev, İ. Őren, G. Beshimov; Complete systems of Galileo invariants of a motion of parametric figure in the three dimensional Euclidean space, International Electronic Journal of Geometry, –2022. –15.–No2.–P.334-342.

[11] D. Khadjiev, Sh. Ayupov, G. Beshimov, Global invariants of topological figures in the two-dimensional Euclidean space, Uzbek Mathematical Juornal, 1 (2023) 52-62. https:// DOI: 10.29229/uzmj.2023-1-7 3.

[12] İ. Őren, D. Khadjiev, Ő. Peken, G. Beshimov; Global invariants of objects in two-dimensional Minkowski space, Journal of Dynamical Systems and Geometric Theories, 2 (2022) 249-273, DOI: 10.1080/1726037X.2022.2120271

[13] D. Khadjiev, İ. Őren, G. Beshimov; Invariants of immersions on n-dimensional affine manifold, G.U. J. Sci. 37(2) (2024) DOI: 10.35378/gujs.1037048

[14] D. Khadjiev, İ. Őren, G. Beshimov; Invariants of a mapping of a set to the two-dimensional Euclidean Space, Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat., 1 (2023) 137158, DOI:10.31801/cfsuasmas.1003511

[15] D. Khadjiev, G. R. Beshimov, Invariants of sequences for the group $SO(2,p,Q)$ of a two-dimensional bilinear-metric space over the field of rational numbers, Itogi Nauki i Tekhniki. Ser. Sovrem. Mat. i ee Pril. Temat. Obz., 2021, Vol. 197, pp. 4655 (in Russian).

[16] J. L. Mundy, A. Zisserman, D. Forsyth (Eds.), Applications of Invariance in Computer vision, Springer-Verlag, Berlin, Heidelberg, New York, 1994.

[17] D. Mumford, J. Fogarty, Geometric Invariant Theory, Springer-Verlag, Berlin, Heidelberg, 1994.

[18] J. O'Rourke, Computational Geometry in **C** , Cambridge University Press, 1997.

[19] İ. Őren, Equivalence conditions of two Bézier curves in the Euclidean geometry, Iranian Journal of Science and Technology, Transactions A: Science (ISTT), 42, no. 3 (2018) 1563-1577. https://doi.org/10.1007/s40995-016-0129-1.

[20] İ. Őren, Invariants of m-vectors in Lorentzian geometry, International Electronic Journal of Geometry, 9, no.1 (2016) 38-44.

[21] T. H. Reiss, Recognizing Planar Objects Using Invariant Image Features, Springer-Verlag, Berlin, Heidelberg, New York, 1993.

[22] K. S. Sibirskii, Introduction to the Algebraic Invariants of Differential Equations, Manchester University Press, New York, 1988.

[23] T. A. Springer, Invariant Theory, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[24] H. Weyl, The Classical Groups: Their Invariants and Representations, Princeton Univ. Press, Princeton, New Jersey, 1946.

[25] W. H. Greub, Linear Algebra, Springer-Verlag, New York Inc., 1967.

Beshimov Gayrat Ruzinazarovich,
Renaissance University of Education,
Tashkent, Uzbekistan,
Cyber University,
Tashkent, Uzbekistan
email: gayratbeshimov@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.70-82
DOI: 10.29229/uzmj.2025-4-8

The decreasing rearrangements of functions for vector-valued measures

Chilin V., Zakirova G.

Abstract. Let B be a complete Boolean algebra, let $Q(B)$ be the Stone compact of B, let $C_\infty(Q(B))$ be the commutative unital algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, assuming possibly the values $\pm\infty$ on nowhere-dense subsets of $Q(B)$. We consider Maharam measure m defined on B, which takes on value in the algebra $L^0(\Omega)$ of all real measurable functions on the measurable space (Ω, Σ, μ) with a σ-finite numerical measure μ. The decreasing rearrangements of functions from $C_\infty(Q(B))$, associated with such a measure m and taking values in the algebra $L^0(\Omega)$ are determined. The basic properties of such rearrangements are established, which are similar to the properties of classical decreasing rearrangements of measurable functions. As an application, with the help of the property of equimeasurablity of elements from $C_\infty(Q(B))$, associated with such a measure m, the notion of a symmetric Banach-Kantorovich space $(E, \|\cdot\|_E)$ over $L^0(\Omega)$ is introduced and studied in detail. Here $E \subset C_\infty(Q(B))$, and $\|\cdot\|_E - L^0(\Omega)$-valued norm in E, endowing it with the structure of the Banach-Kantorovich space. Examples of symmetric Banach-Kantorovich spaces are given, which are vector-valued analogues of classical L^p-spaces, $1 \le p \le \infty$, associated with a numerical σ-finite measure.

Keywords: vector integration, vector-valued measure, decreasing rearrangements, equimeasurablity, the Banach-Kantorovich lattice, symmetric space.

MSC (2020): 46B42, 46E30, 46G10.

1. Introduction

There are well-known examples of Banach-Kantorovich spaces constructed using integration theory for vector measures with values in order complete vector lattice (K-spaces), in particular, in the algebra $L^0(\Omega)$ of all classes of almost everywhere equal real measurable functions on the measurable space (Ω, Σ, μ) with a σ-finite numerical measure μ. Important examples of the Banach-Kantorovich spaces include the "vector-valued" analogues of the L_p-spaces, $1 \le p < \infty$ [1], [2], and the Orlicz spaces [3], [4], [5], [6].

If Ω is a singleton, then the class of Banach-Kantorovich spaces coincides with the class of real Banach spaces, important examples of which are functional symmetric spaces. The theory of symmetric spaces contains many profound results and has important applications in a wide variety of fields of function theory and functional analysis, in particular, in the theory interpolation of linear operators, ergodic theory and harmonic analysis (see for example, [7], [8], [9].

In the general theory of functional symmetric spaces, the notion non-increasing rearrangements plays an important role as shown in [7], [8]. For an measurable function $f \in L^0(\Omega)$ for which $\mu(\{|f| > \lambda\}) < \infty$ for some $\lambda > 0$, its non-increasing rearrangement $f^*(t)$ is defined by

$$f^*(t) = \inf\{\lambda > 0 : \mu(\{|f| > \lambda\}) \le t\} = \inf\{ \|f\chi_A\|_\infty : A \in \Sigma,\ f\chi_A \in L^\infty(\Omega),\ \mu\big(\Omega \setminus A\big) \le t\},\ t > 0,$$

where χ_A is a characteristic function of the set A, and $\|\cdot\|_\infty$ is a uniform norm in algebra $L^\infty(\Omega)$ of all essentially bounded functions from $L^0(\Omega)$.

In this paper, we consider a measure m defined on a complete Boolean algebra B, which takes on value in the algebra $L^0(\Omega)$. It is assumed that the measure m has the property Maharam, that is, for any $e \in B$, $f \in L^0(\Omega)$, $0 \le f \le m(e)$, there exists $q \in B$, $q \le e$, such that $m(q) = f$ (such measures are called Maharam measures). For the Maharam measure, there is always a unique injective completely additive Boolean homomorphism φ from the Boolean algebra $B(\Omega) = \{q \in L^0(\Omega) : q = q^2\}$ in Boolean algebra B such that $\nabla(m) = \varphi(B(\Omega))$ is a regular Boolean subalgebra of B, and $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$ [10]. Moreover, the algebra $L^0(\Omega)$ is identified with the algebra $L^0(\nabla(m)) := C_\infty(Q(\nabla(m)))$ of all continuous functions $x : Q(\nabla(m)) \to [-\infty, +\infty]$, defined on the Stone compact $Q(\nabla(m))$ of a Boolean algebra $\nabla(m)$, such that $x^{-1}(\{\pm\infty\})$ is a nowhere dense subsets of $Q(\nabla(m))$. It is clear that, the algebra $C_\infty(Q(\nabla(m)))$ can be considered as a subalgebra and as a regular vector sublattice of $L^0(B) = C_\infty(Q(B))$ (this means that the exact upper and lower bounds for bounded subsets of $L^0(\nabla(m))$ are the same in $L^0(B)$ and in $L^0(\nabla(m))$).

Let $L^0_{++}(\Omega)$ be the set of all positive elements $f \in L^0(\Omega)_+$ for which the support is $s(f) := \sup\limits_{n\geq 1}\{|f| > n^{-1}\} = \mathbf{1}$. With the help of the $L^0(\Omega)$-valued measure $m : B \to L^0(\Omega)$, non-increasing rearrangements

$$m(x,t) = \inf\{h \in L^0_{++}(\Omega) : m\{|x| > h\} \leq t \cdot \mathbf{1}\},\ t > 0,$$

for elements x from the algebra $L^0(B)$, is determined. Here $\mathbf{1}$ is a unit of Boolean algebra B.

In this paper, we study the properties of $L^0(\Omega)$-valued non-increasing rearrangements of $m(x,t)$, in particular, it is shown that for every fixed $x \in L^0(B)$ and $t > 0$, the equality is true

$$m(x,t) = \inf\{\|xe\|_{\infty,L^0(\Omega)} : e \in B,\ xe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1} - e) \leq t \cdot \mathbf{1}\},\ t > 0,$$

where $L^\infty(B, L^0(\Omega)) = \{x \in L^0(B) : |x| \leq f \text{ for some } f \in L^0(\Omega)_+\}$ is a Banach-Kantorovich space with an $L^0(\Omega)$-valued norm

$$\|x\|_{\infty,L^0(\Omega)} = \inf\{f \in L^0(\Omega)_+ : |x| \leq f\},\quad x \in L^\infty(B, L^0(\Omega)).$$

As an application, the notion of a symmetric Banach-Kantorovich space $(E, \|\cdot\|_E)$ over $L^0(\Omega)$ is introduced, where $E \subset L^0(B)$, and $\|\cdot\|_E$ is the $L^0(\Omega)$-valued norm in E, endowing it with the structure of the Banach-Kantorovich space. Examples of symmetric Banach-Kantorovich spaces are given, which are vector-valued analogues of classical L^p-spaces, $1 \leq p \leq \infty$, associated with a numerical σ-finite measure.

Throughout the paper, we use the terminology and notation of the theory of Boolean algebras [11], an order complete vector lattice [12], the theory of vector integration and the theory of Banach-Kantorovich spaces [1], as well as the terminology of the general theory of symmetric spaces [7].

2. Preliminaries

Let (Ω, Σ, μ) be a measurable space with σ-finite measure μ, and let $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ be the algebra of all real measurable functions on (Ω, Σ, μ) (functions coinciding almost everywhere are identified). $L^0(\Omega)$ is an order complete vector lattice with respect to the natural partial order $(f \leq g \Leftrightarrow g - f \geq 0$ almost everywhere). The weak unit is $\mathbf{1}(\omega) \equiv 1$, and the set $B(\Omega)$ of all idempotents in $L^0(\Omega)$ is a complete Boolean algebra. Denote $L^0(\Omega)_+ = \{f \in L^0(\Omega) : f \geq 0\}$.

Let X be a vector space over the field $\mathbb{R}$ of real numbers. A mapping $\|\cdot\| : X \to L^0(\Omega)$ is called an $L^0(\Omega)$-valued norm on X if the following relations hold for any $x, y \in X$ and $\lambda \in \mathbb{R}$:

(1) $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$;
(2) $\|\lambda x\| = |\lambda|\ \|x\|$;
(3) $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(X, \|\cdot\|)$ is called a lattice-normed space over $L^0(\Omega)$. A lattice-normed space X is said to be d-decomposable if for any $x \in X$ and any decomposition $\|x\| = f_1 + f_2$ into a sum of nonnegative disjoint elements $f_1, f_2 \in L^0(\Omega)$, there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$, $\|x_1\| = f_1$ and $\|x_2\| = f_2$.

A net $\{x_\alpha\}_{\alpha\in A}$ of elements of $(X, \|\cdot\|)$ is said to (bo)-converge to $x \in X$ if the net $\{\|x - x_\alpha\|\}_{\alpha\in A}$ (o)-converges to zero in $L^0(\Omega)$, that is, there exists a decreasing net $\{f_\gamma\}_{\gamma\in\Gamma}$ in $L^0(\Omega)$ such that $f_\gamma \downarrow 0$ and for each $\gamma \in \Gamma$ there is $\alpha(\gamma) \in A$ with $\|x - x_\alpha\| \leq f_\gamma$ $(\alpha \geq \alpha(\gamma))$ [1, 1.3.4] (note, that the o-convergence of a net in $L^0(\Omega)$ is equivalent to its convergence almost everywhere). A net $\{x_\alpha\}_{\alpha\in A} \subset X$ is called (bo)-fundamental if the net $\{x_\alpha - x_\beta\}_{(\alpha,\beta)\in A\times A}$ (bo)-converges to zero. A lattice normed space is called (bo)-complete if every (bo)-fundamental network in it (bo)-converges to an element of this space.

The Banach-Kantorovich space over $L^0(\Omega)$ is defined as a (bo)-complete d-decomposable lattice-normed space over $L^0(\Omega)$. If a Banach Kantorovich space $(X, \|\cdot\|)$ is in addition a vector lattice and the norm $\|\cdot\|$ is monotone (i.e. the condition $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for $x, y \in X$) then it is called a Banach-Kantorovich lattice over $L^0(\Omega)$ (see [1],[2]). Useful examples of Banach-Kantorovich lattices are constructed using vector integration theory. Let us recall some basic notions of the theory of vector integration (see [1],[2]).

Let B be a arbitrary complete Boolean algebra with zero $\mathbf{0}$ and unit $\mathbf{1}$. A mapping $m : B \to L^0(\Omega)$ is called a $L^0(\Omega)$-*valued measure* if it satisfies the following conditions: $m(e) \geq 0$ for all $e \in B$; $m(e \vee g) = m(e) + m(g)$ for any $e, g \in B$ with $e \wedge g = \mathbf{0}$; $m(e_\alpha) \downarrow 0$ for any net $e_\alpha \downarrow \mathbf{0}$, $\{e_\alpha\} \subset B$.

A measure m is said to be *strictly positive*, if $m(e) = 0$ implies $e = \mathbf{0}$. A strictly positive $L^0(\Omega)$-valued measure m is said to be *decomposable*, if for any $e \in B$ and a decomposition $m(e) = f_1 + f_2$, $f_1, f_2 \in L^0(\Omega)_+$ there exist $e_1, e_2 \in B$, such that $e = e_1 \vee e_2$, $m(e_1) = f_1$ and $m(e_2) = f_2$. A measure m is decomposable if and only if it is a Maharam measure, that is, the measure m is strictly positive and for any $e \in B$, $0 \leq f \leq m(e)$, $f \in L^0(\Omega)$, there exist $q \in B$, $q \leq e$, such that $m(q) = f$ [13].

The following statement shows that, in the case of the Maharam measure m, there is a natural embedding of the Boolean algebra $B(\Omega)$ into the Boolean algebra B.

Proposition 2.1. [10, Proposition 2.3]. *For each $L^0(\Omega)$-valued Maharam measure $m : B \to L^0(\Omega)$, there exists a unique injective completely additive homomorphism $\varphi : B(\Omega) \to B$ such that $\varphi(B(\Omega))$ is a regular Boolean subalgebra of B, and $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$.*

Let $Q(B)$ be the Stone compact of a complete Boolean algebra B, and let $L^0(B) := C_\infty(Q(B))$ be the algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, such that $x^{-1}(\{\pm\infty\})$ is a nowhere dense subsets of $Q(B)$. Denotes by $C(Q(B))$ the Banach algebra of all continuous real functions on $Q(B)$ with the uniform norm $\|x\|_\infty = \sup\limits_{t \in Q(B)} |x(t)|$.

We denote by $s(x) := \sup\limits_{n \geq 1}\{|x| > n^{-1}\}$, the support of an element $x \in L^0(B)$, where $\{|x| > \lambda\} \in B$ is the characteristic function χ_{E_λ} of the set E_λ which is the closure in $Q(B)$ of the set $\{t \in Q(B) : |x(t)| > \lambda\}$, $\lambda \in \mathbb{R}$.

Let $m : B \to L^0(\Omega)$ be a Maharam measure. We identify B with the complete Boolean algebra of all idempotents in $L^0(B)$, i.e., we assume $B \subset L^0(B)$. By Proposition 1, there exists a regular Boolean subalgebra $\nabla(m)$ in B and a Boolean isomorphism φ from $B(\Omega)$ onto $\nabla(m)$ such that $m(\varphi(q)e) = qm(e)$ for all $q \in B(\Omega)$, $e \in B$. In this case, the algebra $L^0(\Omega)$ is identified with the algebra $L^0(\nabla(m)) = C_\infty(Q(\nabla(m)))$ (the corresponding isomorphism will also be denoted by φ), and the algebra $C_\infty(Q(\nabla(m)))$ itself can be considered as a subalgebra and as a regular vector sublattice in $L^0(B) = C_\infty(Q(B))$ (this means that the exact upper and lower bounds for bounded subsets of $L^0(\nabla(m))$ are the same in $L^0(B)$ and in $L^0(\nabla(m))$. In addition, $L^0(B)$ is an $L^0(\nabla(m))$-module.

We now specify the vector integral of the [1] for elements of some abstract σ-Dedekind complete vector lattice. Take as a extended σ-Dedekind complete vector lattice the algebra $L^0(B)$. Consider in $L^0(B)$ the vector sublattice $\mathcal{S}(B)$ of all B-simple elements of $x = \sum\limits_{i=1}^{n} \alpha_i e_i$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $e_1, \ldots, e_n \in B$ are pairwise disjoint. Let $m : B \to L^0(\Omega)$ be a $L^0(\Omega)$-valued measure on B. If $x \in \mathcal{S}(B)$ then we put by definition

$$I_m(x) := \int x\, dm := \sum_{k=1}^{n} \alpha_k m(e_k).$$

As was described in [1], the integral I_m can be extended to the spaces of m-integrable elements $\mathcal{L}^1(B, m)$. On identifying equivalent elements, we obtain the K_σ-space $L^1(B, m)$. For each $x \in L^1(B, m)$ (the entry $x \in L^1(B, m)$ means that an equivalence class with a representative of x is considered) the formula

$$\|x\|_{1,m} := \int |x| dm$$

defines an $L^0(\Omega)$-valued norm, that is $(L^1(B, m), \|x\|_{1,m})$ is a lattice-normed space over $L^0(\Omega)$ (see [1, 6.1.3]). Moreover, in the case when $m : B \to L^0(\Omega)$ is a Maharam measure, the pair $(L^1(B, m), \|x\|_{1,m})$ is a Banach-Kantorovich space. In addition, $L^0(\nabla(m)) \cdot L^1(B, m) \subset L^1(B, m)$, $\int(\varphi(\alpha)x)dm = \alpha \int x dm$ for all $x \in L^1(B, m)$, $\alpha \in L^0(\Omega)$ [1, theorem 6.1.10].

Let $p \in [1, \infty)$, and let

$$L^p(B, m) = \{x \in L^0(B) : |x|^p \in L^1(B, m)\},$$

$$\|x\|_{p,m} := \left[\int |x|^p dm\right]^{\frac{1}{p}}, \quad x \in L^p(B,m).$$

It is known that for the Maharam measure m the pair $(L^p(B,m), \|x\|_{p,m})$ is the Banach-Kantorovich space [2, 4.2.2]. In addition,

$$\varphi(\alpha)x \in L^p(B,m) \ \forall \ x \in L^p(B,m), \ \alpha \in L^0(\Omega), \ 1 \le p < \infty,$$

and $\|\varphi(\alpha)x\|_{p,m} = |\alpha|\|x\|_{p,m}$.

In what follows we identify $\varphi(L^0(\Omega))$ and $L^0(\nabla(m))$, and instead of $\varphi(f)$ we will write $f \in L^0(\Omega)$.

The element $x \in L^0(B)$ is called $L^0(\Omega)$-*bounded,* if there exists an element $f \in L^0(\Omega)_+$ such that $|x| \le f$. Denote by $L^\infty(B, L^0(\Omega))$ the set of all $L^0(\Omega)$-bounded elements from $L^0(B)$. It is clear that $L^\infty(B, L^0(\Omega))$ is a subalgebra in $L^0(B)$, as well as order complete vector sublattice in $L^0(B)$, moreover, $L^0(\Omega) \subset L^\infty(B, L^0(\Omega))$, $C(Q(B)) \subset L^\infty(B, L^0(\Omega))$.

For each $x \in L^\infty(B, L^0(\Omega)$ put

$$\|x\|_{\infty, L^0(\Omega)} = \inf\{f \in L^0(\Omega)_+ : |x| \le f\}.$$

It follows directly from the definition of element $\|x\|_{\infty,L^0(\Omega)} \in L^0(\Omega)_+$ that $|x| \le \|x\|_{\infty,L^0(\Omega)}$. The following results follow from the work of [14].

Theorem 2.2. $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty,L^0(\Omega)})$ *is the Banach-Kantorovich lattice over* $L^0(\Omega)$.

3. Distribution functions and decreasing rearrangements, associated with a Maharam measure

Let m be $L^0(\Omega)$-valued Maharam measure on a complete Boolean algebra B. In the rest of this section we assume that $m(\mathbf{1}) = \mathbf{1}$.

Denote by $L^0_{++}(\Omega)$ the set of all positive elements $\lambda \in L^0_+(\Omega)$ such that $s(\lambda) = \mathbf{1}$. It is clear that for any $\lambda \in L^0_{++}(\Omega)$ there exists $\lambda^{-1} \in L^0_{++}(\Omega)$ such that $\lambda \cdot \lambda^{-1} = \mathbf{1}$.

The following notation is used below for the elements $x, y \in L^0(B)$
$\{x > y\} := s((x-y)_+)$, $\{x < y\} := s((x-y)_-)$, $\{x \ge y\} := \mathbf{1} - s((x-y)_-)$,
$\{x \le y\} := \mathbf{1} - s((x-y)_+)$.

Definition 3.1. Let $0 \le x \in L^0(B)$ and $h \in L^0_{++}(\Omega)$. The $L^0(\Omega)$-valued distribution function $d(.;x) : L^0_{++}(\Omega) \to L^0(\Omega)_+$ is defined by

$$d(h;x) := m(\{x > h\}),$$

where $\{x > h\} \in B$ is the idempotent in the algebra $L^0(B)$, which is the characteristic function $\chi_{E_h(x)}$ of the closure $E_h(x)$ of the set $\{s \in Q(B) : x(s) > h(s)\}$.

Note that $L^0(\Omega)$-valued distribution function $d(x)$ is also given by

$$d(h;x) = \int \chi_{E_h(x)} dm, \ h \in L^0_{++}(\Omega).$$

A mapping $d(.,x)$ is decreasing, and right-continuous, that is, if $h_n \in L^0_{++}(\Omega), n = 0,1,...,$ and $h_n \downarrow h_0$, then $d(h_0;x) = \sup\limits_{n\ge1} d(h_n;x)$.

Proposition 3.2. *Suppose* $x, y, x_n \, (n = 1,2,\ldots)$ *belong to* $L^0(B)$, *and let* $h, g \in L^0_{++}(\Omega)$. *Then*

(i). If $|x| \le |y|$, *then* $d(h;|x|) \le d(h;|y|)$;

(ii). $d(h;g|x|) = d(\frac{h}{g};|x|)$;

(iii.) If $x \ge 0, y \ge 0$, $h_1, h_2 \in L^0_{++}(\Omega)$, *then* $d(h_1 + h_2; x+y) \le d(h_1;x) + d(h_2;y)$;

(iv). If $|x_n| \uparrow |x|$, *then* $d(h;|x_n|) \uparrow d(h;|x|)$ *for every* $h \in L^0_{++}(\Omega)$.

Proof. (i). If $|x| \leq |y|$, then $\{|x| > h\} \leq \{|y| > h\}$. Consequently, $d(h;|x|) = m(\{|x| > h\}) \leq m(\{|y| > h\}) = d(h;|y|)$.

(ii). $d(h; g|x|) = m(\{g|x| > h\}) = m(\{|x| > \frac{h}{g}\}) = d(\frac{h}{g}; |x|)$.

(iii). If $s \in Q(B)$ and $x(s) + y(s) > h_1(s) + h_2(s)$, then either $x(s) > h_1(s)$ or $y(s) > h_2(s)$. Therefore $\{x + y > h_1 + h_2\} \leq \{x > h_1\} \vee \{y > h_2\}$. Consequently,

$$d(h_1 + h_2; x + y) = m(\{x + y > h_1 + h_2\}) \leq m(\{x > h_1\}) + m(\{y > h_2\}) = d(h_1; x) + d(h_2; y).$$

(iv). We fix $h \in L^0_{++}(\Omega)$ and put $G_h(x) = \{s \in Q(B) : |x(s)| > h(s)\}$, $G_h(x_n) = \{s \in Q(B) : |x_n(s)| > h(s)\}$, $(n = 1, 2, ...)$. Since $|x_n| \leq |x_{n+1}|$, then $G_h(x_n) \subset G_h(x_{n+1})$. Furthermore, the condition $|x_n| \uparrow |x|$ imply that $G_h(x) = \bigcup\limits_{n=1}^{\infty} G_h(x_n)$. Hence, by the monotone convergence property of measure m, we have

$$d(h; |x_n|) = m(\{|x_n| > h\}) \uparrow m(\{|x| > h\}) = d(h; |x|).$$

□

Example 3.3. Let $x = e \in B$ and $h \in L^0_{++}(\Omega)$. Then $d(h; e) = m(\{e > h\}) = m(e)$, if $h < \mathbf{1}$, and $d(h; e) = \mathbf{0}$, if $h \geq \mathbf{1}$.

It will be worthwhile to formally compute the $L^0(\Omega)$-valued distribution function $d(x)$ of a positive B-simple element $x \in \mathcal{S}(B)$:

Example 3.4. Let $x \in \mathcal{S}(B)_+$, i.e.

$$x = \sum_{k=1}^{n} \alpha_k e_k, \tag{1}$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+ = (0, \infty)$, and $e_1, \ldots, e_n$ are pairwise disjoint elements of B. Without loss of generality we may assume that $\alpha_1 > \alpha_2 > \ldots > \alpha_n > 0$. Then

$$d(h; x) = \sum_{i=1}^{k} m(e_i) \quad \text{if} \quad \alpha_{k+1} \cdot \mathbf{1} \leq h < \alpha_k \cdot \mathbf{1} \quad (h \in L^0_{++}(\Omega)), \tag{2}$$

where $k = 1, 2, \ldots, n$, and $\alpha_{n+1} = 0$.

Definition 3.5. Two positive elements $x, y \in L^0(B)$ are said to be m-equimeasurable, if they have the same distribution function, that is, if $d(h; x) = d(h; y)$ for all $h \in L^0_{++}(\Omega)$.

Example 3.6. Two idempotents $e_1, e_2 \in B$ are m-equimeasurable if and only if $m(e_1) = m(e_2)$ (see Example 3.3).

Example 3.7. Let $x, y \in \mathcal{S}(B)_+$, $x = \sum\limits_{k=1}^{n} \alpha_k e_k$ and $y = \sum\limits_{k=1}^{n} \beta_k g_k$, where $\alpha_k, \beta_k \in \mathbb{R}_+$, $\alpha_1 > \alpha_2 > \ldots > \alpha_n > 0$, $\beta_1 > \beta_2 > \ldots > \beta_n > 0$, and $\{e_k\}$, respectively, $\{g_k\}$ are pairwise disjoint elements of B. By Example 3.4, we have

$$d(h; x) = \sum_{i=1}^{k} m(e_i), \quad \text{if} \quad \alpha_{k+1} \cdot \mathbf{1} \leq h < \alpha_k \cdot \mathbf{1},$$

$$d(h; y) = \sum_{i=1}^{k} m(g_i), \quad \text{if} \quad \beta_{k+1} \cdot \mathbf{1} \leq h < \beta_k \cdot \mathbf{1} \ (h \in L^0_{++}(\Omega)),$$

where $k = 1, 2, \ldots, n$, and $\alpha_{n+1} = \beta_{n+1} = 0$.

From equality $d(h; x) = d(h; y)$ we get $\alpha_k = \beta_k$ and $\sum\limits_{i=1}^{k} m(e_i) = \sum\limits_{i=1}^{k} m(g_i)$ for all $k = 1, \ldots, n$. Of the last equalities by $k = 1$ we have $m(e_1) = m(g_1)$. Further, if $k = 2$ the equality $m(e_1) + m(e_2) = m(g_1) + m(g_2)$ is true, thus $m(e_2) = m(g_2)$, etc., when $k = n$ we get $m(e_n) = m(g_n)$.

Thus the elements x and y m-equimeasurable if and only if $\alpha_k = \beta_k$ and $m(e_k) = m(g_k)$ for all $k = 1, \ldots, n$.

Note that the statement from Example 4 holds for simple elements $a, b \in \mathcal{S}(B)_+$, $a = \sum_{k=1}^{n} t_k p_k$ and $b = \sum_{k=1}^{n} s_k q_k$, where $t_k, s_k \in \mathbb{R}_+$, $0 < t_1 < t_2 < \ldots < t_n$, $0 < s_1 < s_2 < \ldots < s_n$, and $\{p_k\}$, respectively, $\{q_k\}$ are pairwise disjoint elements of B. It is clear that $a = x = \sum_{k=0}^{n-1} t_{n-k} p_{n-k}$ and $b = y = \sum_{k=0}^{n-1} s_{n-k} g_{n-k}$, where $t_k, s_k \in \mathbb{R}_+$ and $t_n > t_{n-1} > \ldots > t_1 > 0$, $s_n > s_{n-1} > \ldots > s_1 > 0$. According to the above, we have that the elements x and y m-equimeasurable if and only if $t_k = s_k$ and $m(p_k) = m(q_k)$ for all $k = 1, \ldots, n$.

Definition 3.8. The $L^0(\Omega)$-valued decreasing rearrangement of $x \in L^0(B)$ is the mapping $m(.;x) : (0, \infty) \to L^0(\Omega)_+$, defined by

$$m(t;x) = \inf\{h \in L^0_{++}(\Omega) : d(h; |x|) \le t \cdot \mathbf{1}\},\ t > 0. \tag{3}$$

It is clear that $m(t;x) \le m(s;x)$ for $s < t$. In addition, the map $m(t;x)$ has the following useful continuity property.

Proposition 3.9. *If* $t_n, t > 0, n = 1, 2, ...,$ *and* $t_n \downarrow t$, *then* $m(t;x) = \sup_{n \ge 1} m(t_n; x)$.

Proof. Let $t_n, t > 0, n = 1, 2, ...,$ and $t_n \downarrow t$. Then

$$m(t;x) = \inf\{h \in L^0_{++}(\Omega) : d(h; |x|) \le \inf_{n \ge 1} t_n\} = \inf\left\{h \in L^0_{++}(\Omega) : d(h; |x|) \le t_n \text{ for all } n \ge 1\right\}$$

$$= \sup_{n \ge 1}\left\{\inf\{h \in L^0_{++}(\Omega) : d(h; |x|) \le t_n\}\right\} = \sup_{n \ge 1} m(t_n; x),$$

i.e. $m(t;x) = \sup_{n \ge 1} m(t_n; x)$. □

Example 3.10. Now we compute the $L^0(\Omega)$-valued decreasing rearrangement of the B-simple element x given by (1). Let $h \in L^0_{++}(\Omega)$ and let $\lambda_k = \sum_{i=1}^{k} m(e_i), (k = 1, 2, \ldots, n)$. Then according to (2) we have

$$d(h;x) = \begin{cases} \mathbf{0}, & h \ge \alpha_1 \cdot \mathbf{1}; \\ \lambda_k, & \alpha_{k+1} \cdot \mathbf{1} \le h < \alpha_k \cdot \mathbf{1},\ \ 1 \le k \le n-1; \\ \lambda_n, & \mathbf{0} < h \le \alpha_n \mathbf{1}. \end{cases}$$

Referring to (3), we see that $m(t;x) = \mathbf{0}$ if $t \cdot \mathbf{1} \ge \lambda_n$. Also, if $\lambda_n > t \cdot \mathbf{1} \ge \lambda_{n-1}$, then $m(t;x) = \alpha_n \cdot \mathbf{1}$, and if $\lambda_{n-1} > t \cdot \mathbf{1} \ge \lambda_{n-2}$, then $m(t;x) = \alpha_{n-1} \cdot \mathbf{1}$, and so on. Hence,

$$m(t;x) = \begin{cases} \alpha_1 \cdot \mathbf{1}, & \mathbf{0} < t \cdot \mathbf{1} \le \lambda_1; \\ \alpha_k \cdot \mathbf{1}, & \lambda_k > t \cdot \mathbf{1} \ge \lambda_{k-1},\ \ 2 \le k \le n-1; \\ \mathbf{0}, & t \cdot \mathbf{1} > \lambda_n. \end{cases} \tag{4}$$

Example 3.11. Let $x, y \in \mathcal{S}(B)_+$, $x = \sum_{k=1}^{n} \alpha_k e_k$ and $y = \sum_{k=1}^{n} \beta_k g_k$ (see Example 3.7). Let $m(t;x) = m(t;y)$ for all $t > 0$. Then the elements x and y are m-equimeasurable.

Indeed, if $\mathbf{0} < t \cdot \mathbf{1} \le \lambda_1$, then by (4), $m(t;x) = \alpha_1 \cdot \mathbf{1}$. Hence $m(t;y) = \alpha_1 \cdot \mathbf{1}$ if and only if $\beta_1 = \alpha_1$ and $\mu_1 = m(g_1) = m(e_1) = \lambda_1$. Also, if $\lambda_2 > t \cdot \mathbf{1} \ge \lambda_1$, then $m(t;x) = \alpha_2 \cdot \mathbf{1} = m(t;y)$. Because, $\beta_2 = \alpha_2$ and $\mu_2 = m(g_1) + m(g_2) = m(e_1) + m(e_2) = \lambda_2$, i.e. $m(g_2) = m(e_2)$, and so on. For $t \cdot \mathbf{1} > \lambda_n$ we have $\beta_n = \alpha_n$ and $m(g_n) = m(e_n)$. Therefore, the elements x and y are m-equimeasurable (see Example 3.7).

For fixed $x \in L^0(B)$, $t > 0$, we put

$$\xi(t;x) = \inf\{\|xe\|_{\infty, L^0(\Omega)} : e \in B,\ xe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1} - e) \le t \cdot \mathbf{1}\}. \tag{5}$$

Proposition 3.12. *For any* $x \in L^0(B)$, $t > 0$ *the equality*

$$\xi(t;x) = m(t;x),$$

is true, in addition, $m\{|x| > m(t;x)\} \leq t \cdot \mathbf{1}$.

Proof. Fix $t > 0$, and put

$$H(x) = \{h \in L^0_{++}(\Omega) : d(h;|x|) \leq t \cdot \mathbf{1}\}.$$

If $h_1, h_2 \in H(x)$, $e = \{h_1 < h_2\}$, then $h = h_1 \wedge h_2 = h_1 \cdot e + h_2 \cdot (\mathbf{1} - e) \in L^0_{++}(\Omega)$, in this case, by virtue of Proposition 2.1, we have

$$d(h;|x|) = m(\{|x| > h\}) = m(\{|x| > h_1\}) \cdot e + m(\{|x| > h_2\}) \cdot (\mathbf{1} - e) \leq t \cdot e + t \cdot (\mathbf{1} - e) = t \cdot \mathbf{1},$$

i.e. $h_1 \wedge h_2 \in H(x)$. Using mathematical induction, we obtain that for any finite set $\{h_i\}_{i=1}^n \subset H(x)$, the inclusion $\bigwedge\limits_{i=1}^n h_i \in H(x)$ is true. Since (Ω, Σ, μ) is a measurable space with σ-finite measure μ, the Boolean algebra $B(\Omega)$ has a countable type, i.e. any family of pairwise disjunct elements from $B(\Omega)$ is at most countable. Hence there exists a sequence $\{h_k\}_{k=1}^\infty \subset H(x)$ for which $h_k \downarrow g$(see, for example, [12, Chapter VI, §3]), where

$$g = m(t;x) = \inf\{h \in L^0_{++}(\Omega) : d(h;|x|) \leq t \cdot \mathbf{1}\} \in L^0(\Omega)_+.$$

Since $h_k \downarrow g$ and for all $k \in \mathbb{N}$ the inequality $d(h_k;|x|) \leq t \cdot \mathbf{1}$ is true, then

$$t \cdot \mathbf{1} \geq d(h_k;|x|) \uparrow d(g;|x|).$$

In particular, the inequality

$$m\{|x| > g\} = d(g;|x|) \leq t \cdot \mathbf{1} \text{ i.e. } m(\mathbf{1} - e) \leq t \cdot \mathbf{1},$$

is true, where $e = \{|x| \leq g\} \in B$. Since $|x|e \leq g \in L^0(\Omega)_+$, then $xe \in L^\infty(B, L^0(\Omega))$, in addition, the inequality $\|xe\|_{\infty, L^0(\Omega)} \leq g$ is true. Hence, $\xi(t;x) \leq g = m(t;x)$.

To prove the reverse inequality, we set

$$E(x) = \{e \in B : xe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1} - e) \leq t \cdot \mathbf{1}\}.$$

By (5), we have $\xi(t;x) = \inf\{\|xe\|_{\infty, L^0(\Omega)} : e \in E(x)\}$. Suppose that the inequality $m(t;x) = g \leq \xi(t;x)$ is not satisfied. Therefore, there exists an $e \in E(x)$ for which the inequality $g \leq \|xe\|_{\infty, L^0(\Omega)}$ does not hold. In particular, this means that there exist $0 \neq q \in B(\Omega)$, $\varepsilon > 0$, for which the inequalities

$$|xeq| \leq \|xeq\|_{\infty, L^0(\Omega)} = \|xe\|_{\infty, L^0(\Omega)} \cdot q \leq qg + \varepsilon q$$

are true.

Let $r = \{|xe| > qg + 2\varepsilon \cdot q\} \in B$. From the relations

$$|xe|rq \geq (qg + 2\varepsilon \cdot q)q = qg + 2\varepsilon \cdot q > qg + \varepsilon \cdot q \geq |xeq| = |xe|q$$

it follows that $|xe|rq > |xe|q$, which is impossible.

Thus, the inequality $m(t;x) \leq \xi(t;x)$ is satisfied. Consequently, the equality $\xi(t;x) = m(t;x)$ is true. □

In the next proposition, some basic properties of $L^0(\Omega)$-valued decreasing rearrangements are collected.

Proposition 3.13. *Let* $x, y \in L^0(B),\ t > 0$. *Then*

(i). $\xi(t;x) = \inf\{\|x-z\|_{\infty,L^0(\Omega)} : z \in R_t,\ (x-z) \in L^\infty(B, L^0(\Omega))\}$, *where* $R_t = \{z \in L^0(B) : m(s(|z|)) \le t \cdot \mathbf{1}\}$, $s(|z|)$ *the support of* $|z|$.

(ii). $m(t;x) = m(t;|x|)$ *and* $m(t;\lambda x) = |\lambda| m(t;x)$ *for all* $\lambda \in \mathbb{R}$;

(iii). *If* $m(s(x)) \le t \cdot \mathbf{1}$, *then* $m(t;x) = \mathbf{0}$;

(iv). *If* $|x| \le |y|$, *then* $m(t;x) \le m(t;y)$;

(v). $m(t_1 + t_2; x + y) \le m(t_1;x) + m(t_2;y)$ *for all* $t_1, t_2 > 0$.

(vi). *If* $x \in L^\infty(B, L^0(\Omega))$, $t_n > 0$, $n = 1, 2, ...,$ *and* $t_n \downarrow 0$, *then*

$$\|x\|_{\infty,L^0(\Omega)} = \sup_{n\ge1} m(t_n;x).$$

(vii). *If* $x_n, x \in L^0(B)$, $n \in \mathbb{N}$ *and* $0 \le x_n \uparrow x$, *then* $m(t,x_n) \uparrow m(t,x)$ *for all* $t > 0$.

Proof. (i). We put

$$h = \inf\{\|x-z\|_{\infty,L^0(\Omega)} : z \in R_t,\ (x-z) \in L^\infty(B, L^0(\Omega))\}.$$

If $z \in R_t$, $x - z \in L^\infty(B, L^0(\Omega))$ and $e = \mathbf{1} - s(|z|)$, then $e \in B$, $(x-z)e = xe$ and $|xe| \le |x-z|$. Hence $\|xe\|_{\infty,L^0(\Omega)} \le \|x-z\|_{\infty,L^0(\Omega)}$. Since $z \in R_t$, then $m(\mathbf{1}-e) = m(s(|z|)) \le t \cdot \mathbf{1}$. Therefore,

$$\xi(t;x) = \inf\{\|xe\|_{\infty,L^0(\Omega)} : e \in B,\ xe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1}-e) \le t \cdot \mathbf{1}\} \le h.$$

To obtain the reverse inequality, we fix $t > 0$ and for elements $u = s(x_+) - s(x_-) \in L^0(B)$, $p = \{|x| > \xi(t;x)\} \in B$ put $z = up|x|$, $e = \mathbf{1} - p$. Then $x - z = u|x| - up|x| = u(\mathbf{1}-p)|x| = uex \in L^\infty(B, L^0(\Omega))$, moreover,

$$\|x-z\|_{\infty,L^0(\Omega)} = \|ue|x|\,\|_{\infty,L^0(\Omega)} = \|ex\|_{\infty,L^0(\Omega)} = \|e|x|\|_{\infty,L^0(\Omega)} \le \xi(t;x).$$

By Proposition 3.12, we have

$$m(s(|z|)) = m\{|x| > \xi(t;x)\} \le t \cdot \mathbf{1},$$

i.e. $z \in R_t$. Therefore,

$$\inf\{\|x-z\|_{\infty,L^0(\Omega)} : z \in R_t,\ (x-z) \in L^\infty(B, L^0(\Omega))\} \le \xi(t;x).$$

(ii). The equality $m(t;x) = m(t;|x|)$ follows directly from the definition of the mapping $m(t;x)$. If $0 \ne \lambda \in \mathbb{R}$, then

$$m(t;\lambda x) = \inf\{h \in L^0_{++}(\Omega) : m\{|\lambda x| > h\} \le t \cdot \mathbf{1}\} =$$
$$= \inf\{h \in L^0_{++}(\Omega) : m\{|x| > |\lambda|^{-1} h\} \le t \cdot \mathbf{1}\} =$$
$$= \inf\{|\lambda| g \in L^0_{++}(\Omega) : m\{|x| > g\} \le t \cdot \mathbf{1}\} =$$
$$= |\lambda| \inf\{g \in L^0_{++}(\Omega) : m\{|x| > g\} \le t \cdot \mathbf{1}\} = |\lambda| m(t;x).$$

(iii). Since $m(s(x)) \le t \cdot \mathbf{1}$ and $\{|x| > \frac{1}{n} \cdot \mathbf{1}\} \le s(x)$, then $m\{|x| > \frac{1}{n} \cdot \mathbf{1}\} \le t \cdot \mathbf{1}$ for all $n \in \mathbb{N}$. Therefore, $m(t;x) = \mathbf{0}$.

(iv). If $|x| \le |y|$, then $d(h;|x|) \le d(h;|y|)$ for all $h \in L^0_{++}(\Omega)$, and therefore $m(t;x) \le m(t;y)$.

(v). For all $h_1, h_2 \in L^0(\Omega)_+$ the following inequality holds

$$\{|x+y| > h_1 + h_2\} \le \{|x| > h_1\} \vee \{|y| > h_2\}.$$

Hence $m\{|x+y| > h_1 + h_2\} \le m\{|x| > h_1\} + m\{|y| > h_2\}$.

We fix $\varepsilon > 0$ and set $h_1 = m(t_1; x)$, $h_2 = m(t_2; y) + \varepsilon \cdot \mathbf{1}$. Using the inequality $m\{|y| > h_2\} \leq m\{|y| > m(t_2; y)\}$ and Proposition 3.12, we have

$$m\{|x + y| > m(t_1; x) + m(t_2; y) + \varepsilon \cdot \mathbf{1}\} \leq m\{|x| > m(t_1; x)\} + m\{|y| > m(t_2; y) + \varepsilon \cdot \mathbf{1}\} \leq$$
$$\leq t_1 \cdot \mathbf{1} + m\{|y| > m(t_2; y)\} \leq (t_1 + t_2) \cdot \mathbf{1},$$

i.e. $m\{|x + y| > m(t_1; x) + m(t_2; y) + \varepsilon \cdot \mathbf{1}\} \leq (t_1 + t_2) \cdot \mathbf{1}$.

Since $m(t_1; x) + m(t_2; y) + \varepsilon \cdot \mathbf{1} \in L^0_{++}(\Omega)$, then from the definition of the mapping $m(t; x)$ the following inequality follows

$$m((t_1 + t_2); x + y) \leq m(t_1; x) + m(t_2; y) + \varepsilon \cdot \mathbf{1}.$$

From here, at $\varepsilon \downarrow 0$, we obtain the required inequality

$$m((t_1 + t_2); x + y) \leq m(t_1; x) + m(t_2; y).$$

(vi). First we show that for all $q \in B(\Omega)$, $x \in L^\infty(B, L^0(\Omega))$, $t > 0$ the equality $\xi(t; qx) = q\xi(t; x)$ is true. Since

$$\xi(t; qx) = \inf\{\|qxe\|_{\infty, L^0(\Omega)} : e \in B,\ qxe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1} - e) \leq t \cdot \mathbf{1}\} =$$
$$= \inf\{q\|qxe\|_{\infty, L^0(\Omega)} : e \in B,\ qxe \in L^\infty(B, L^0(\Omega)),\ m(\mathbf{1} - e) \leq t \cdot \mathbf{1}\} = q\xi(t; qx),$$

then from the inequality $|qx| \leq |x|$ and Proposition 3.12 follows the inequality $q\xi(t; qx) \leq q\xi(t; x)$ (see property (iv)).

On the other hand, if $e \in B$, $qxe \in L^\infty(B, L^0(\Omega))$ and $m(\mathbf{1} - e) \leq t \cdot \mathbf{1}$, then

$$\xi(t; x) = q\xi(t; x) + (\mathbf{1} - q)\xi(t; x) \leq q\|qxe\|_{\infty, L^0(\Omega)} + (\mathbf{1} - q)\xi(t; x).$$

Therefore,

$$q\xi(t; x) \leq q\|qxe\|_{\infty, L^0(\Omega)} \leq \|qxe\|_{\infty, L^0(\Omega)},$$

and hence $q\xi(t; x) \leq \xi(t; qx)$. Thus, the equality $\xi(t; qx) = q\xi(t; x)$ is true.

If $x \in L^\infty(B, L^0(\Omega))$, then $x \cdot \mathbf{1} \in L^\infty(B, L^0(\Omega))$, and it follows directly from the definition of the mapping $\xi(t; x)$ that $\xi(t; x) \leq \|x\|_{\infty, L^0(\Omega)}$ for all $t > 0$. Moreover, the inequality $\xi(t_1; x) \leq \xi(t_2; x)$ is true at $0 < t_2 < t_1$.

Thus, $\xi(t_n; x) \uparrow z \leq \|x\|_{\infty, L^0(\Omega)}$ at $t_n \downarrow 0$ for some $z \in L^0(\Omega)_+$.

If $z \neq \|x\|_{\infty, L^0(\Omega)}$, then for any $\varepsilon > 0$ exist $q_\varepsilon \in B(\Omega)$, that

$$\xi(t_n; xq_\varepsilon) = \xi(t_n; x)q_\varepsilon \leq zq_\varepsilon < q_\varepsilon \cdot \|x\|_{\infty, L^0(\Omega)}$$

for all $t_n \in (0, \varepsilon)$. Hence, by virtue of proposition 3.12, we get that

$$m(t_n; xq_\varepsilon) = \xi(t_n; xq_\varepsilon) \leq zq_\varepsilon \text{ for all } t_n \in (0, \varepsilon).$$

Again using proposition 3.12, we have,

$$m(\{|xq_\varepsilon| > zq_\varepsilon\}) \leq m(\{|xq_\varepsilon| > m(t_n, xq_\varepsilon)\}) \leq t_n \cdot \mathbf{1}$$

for all $t_n \in (0, \varepsilon)$. Therefore, $m(\{|xq_\varepsilon| > zq_\varepsilon\}) = 0$. This means that $|xq_\varepsilon| \leq zq_\varepsilon$, in particular, $\|xq_\varepsilon\|_{\infty, L^0(\Omega)} \leq zq_\varepsilon$, which is not the case. Thus,

$$\|x\|_{\infty, L^0(\Omega)} = \sup_{n \geq 1} \xi(t_n; x).$$

(vii). Since $m(t, x_n) \leq m(t, x_{n+1}) \leq m(t, x)$ for all $n = 1, 2, \ldots$ and $t > 0$, it is clear that $m(t, x_n) \uparrow_n$ and that

$$\sup_{n \geq 1} m(t, x_n) \leq m(t, x)$$

for all $t > 0$. For the proof of the reverse inequality, it may be assumed that $t > 0$ is such that $m(t, x_n) < h$, for some $h \in L^0_{++}(\Omega)$. Hence $d(h, x_n) \leq t \cdot \mathbf{1}$ for all n. Since $d(h, x_n) \uparrow d(h, x)$ (Proposition 3.2(iv)), it follows that $d(h, x) \leq t \cdot \mathbf{1}$. Thus $m(t, x) < h$. This suffices to show that $m(t, x) \leq \sup_{n \geq 1} m(t, x_n)$. Consequently $m(t, x) = \sup_{n \geq 1} m(t, x_n)$. $\square$

Corollary 3.14. *For any* $x, y \in L^\infty(B, L^0(\Omega))$, $t > 0$ *the inequality holds*

$$|\xi(t;x) - \xi(t;y)| \le \|x - y\|_{\infty, L^0(\Omega)}.$$

Proof. By proposition 3.13 (v) we have that

$$\xi(t_1 + t_2; x) = \xi(t_1 + t_2; y + (x - y)) \le \xi(t_1; y) + \xi(t_2; x - y) \le \xi(t_1; y) + \|x - y\|_{\infty, L^0(\Omega)}.$$

Similarly,

$$\xi(t_1 + t_2; y) = \xi(t_1 + t_2; x + (y - x)) \le \xi(t_2; x) + \xi(t_1; y - x) \le \xi(t_2; x) + \|x - y\|_{\infty, L^0(\Omega)}.$$

Assuming $t_1 = t$, $t_2 = 0$, in these inequalities, we obtain

$$\xi(t;x) \le \xi(t;y) + \|x - y\|_{\infty, L^0(\Omega)} \text{ and } \xi(t;y) \le \xi(t;x) + \|x - y\|_{\infty, L^0(\Omega)},$$

from which it follows that $|\xi(t;x) - \xi(t;y)| \le \|x - y\|_{\infty, L^0(\Omega)}$. □

4. Symmetric Banach-Kantorovich spaces

In this section, a class of symmetric Banach-Kantorovich spaces is introduced and examples of such spaces are given. We will need the following useful property about the equality of integrals for integrable m-equimeasurable elements.

Theorem 4.1. *Let* $0 \le x \in L^0(B)$ *and* $0 \le y \in L^1(B, m)$. *If* x *and* y m*-equimeasurable, then* $x \in L^1(B, m)$ *and* $\int x dm = \int y dm$.

Proof. Let $x, y \in \mathcal{S}(B)_+$, $x = \sum\limits_{k=1}^{n} \alpha_k e_k$ and $y = \sum\limits_{k=1}^{n} \beta_k g_k$. Then by m-equimeasurable x and y (see Example 4),

$$\int x dm = \sum_{k=1}^{n} \alpha_k m(e_k) = \sum_{k=1}^{n} \beta_k m(g_k) = \int y dm.$$

Let now $x \in L^0(B)_+$, $0 \le y \in L^1(B, m)$ and $d(.;x) = d(.;y)$. Let us, first assume that $y \in C(Q(B))$. Recall that by assumption $m(\mathbf{1}) = \mathbf{1}$, and therefore $C(Q(B)) \subset L^1(B, m))$, in this case, $\|y\|_{1,m} \le \|y\|_\infty \mathbf{1}$. Without loss of generality we may assume that $\|y\|_\infty \le 1$ (see Proposition 3.7(ii)). Since $d(.;x) = d(.;y)$, then $m\{x > \mathbf{1}\} = m\{y > \mathbf{1}\} = \mathbf{0}$, that is $\|x\|_\infty \le 1$, and therefore $x \in L^1(B, m)$.

Note that from the identities

$$m\{x \le t \cdot \mathbf{1}\} = m(\mathbf{1}) - m\{x > t \cdot \mathbf{1}\} = m(\mathbf{1}) - m\{y > t \cdot \mathbf{1}\} = m\{y \le t \cdot \mathbf{1}\}$$

using the equalities

$$\{s \cdot \mathbf{1} < x \le t \cdot \mathbf{1}\} = \{x > s \cdot \mathbf{1}\} - \{x > t \cdot \mathbf{1}\}, \{s \cdot \mathbf{1} < y \le t \cdot \mathbf{1}\} = \{y > s \cdot \mathbf{1}\} - \{y > t \cdot \mathbf{1}\}$$

for any $0 < s < t$, we obtain

$$m\{s \cdot \mathbf{1} < x \le t \cdot \mathbf{1}\} = m\{x > s \cdot \mathbf{1}\} - m\{x > t \cdot \mathbf{1}\} = m\{y > s \cdot \mathbf{1}\} - m\{y > t \cdot \mathbf{1}\} = m\{s \cdot \mathbf{1} < y \le t \cdot \mathbf{1}\}.$$

We now show that there are increasing sequences of positive simple elements $x_n \in C(Q(B))$ and $y_n \in C(Q(B))$ $(n = 1, 2, \ldots)$, such that $x_n \uparrow x$ and $y_n \uparrow y$ and the equalities $d(t; x_n) = d(t; y_n)$ are true for all $n = 1, 2, \ldots$.

Consider the following two sequences

$$x_n = (\sum_{k=1}^{2^n} \frac{k-1}{2^n} e_k) \uparrow x, \quad y_n = (\sum_{k=1}^{2^n} \frac{k-1}{2^n} g_k) \uparrow y,$$

where $e_k = \{\frac{k-1}{2^n} \cdot \mathbf{1} < x \leq \frac{k}{2^n} \cdot \mathbf{1}\}$, $g_k = \{\frac{k-1}{2^n} \cdot \mathbf{1} < y \leq \frac{k}{2^n} \cdot \mathbf{1}\}$. Since $d(.;x) = d(.;y)$, then $m(e_k) = m(g_k)$, and therefore $d(t;x_n) = d(t;y_n)$ (see Example 4). Hence,

$$\int x_n dm = \sum_{k=1}^{2^n} \frac{k-1}{2^n} m(e_k) = \sum_{k=1}^{2^n} \frac{k-1}{2^n} m(g_k) = \int y_n dm \uparrow \int ydm.$$

Thus,

$$\int xdm = (o)\text{-} \lim_{n\to\infty} \int x_n dm = \int ydm.$$

Now let y be an arbitrary positive element $L^1(B,m)$. Consider two increasing sequences of positive elements of $C(Q(B))$

$$x_n = xp_n \uparrow x, \quad y_n = yq_n \uparrow y,$$

where $p_n = \{x \leq n \cdot \mathbf{1}\}$, $q_n = \{y \leq n \cdot \mathbf{1}\}$. It is clear that

$$m\{x_n > t \cdot \mathbf{1}\} = m\{xp_n > t \cdot \mathbf{1}\} = m\{t \cdot \mathbf{1} < x \leq n \cdot \mathbf{1}\} = m\{t \cdot \mathbf{1} < y \leq n \cdot \mathbf{1}\} =$$

$$= m\{yq_n > t \cdot \mathbf{1}\} = m\{y_n > t \cdot \mathbf{1}\}$$

for any $t \in \mathbb{R}^+$. Since y_n is an integrable element of $C(Q(B))$, it follows from the above, that $\int x_n dm = \int y_n dm$. At the same time, there is a limit

$$(o)\text{-} \lim_{n\to\infty} \int x_n dm = (o)\text{-} \lim_{n\to\infty} \int y_n dm = \int ydm.$$

Hence $x \in L^1(B,m)$ and $\int xdm = \int ydm$. □

Corollary 4.2. *Let* $0 \leq x \in L^0(B)$ *and* $0 \leq y \in L^p(B,m)$, $p > 1$. *If* x *and* y *m-equimeasurable, then* $x \in L^p(B,m)$ *and* $\|x\|_{p,m} = \|y\|_{p,m}$.

Proof. Since $y^p \in L^1(B,m)$ and

$$m\{x^p > t \cdot \mathbf{1}\} = m\{x > t^{\frac{1}{p}} \cdot \mathbf{1}\} = m\{y > t^{\frac{1}{p}} \cdot \mathbf{1}\} = m\{y^p > t \cdot \mathbf{1}\}$$

for any $t \in \mathbb{R}^+$, $p > 1$, then for the elements x^p and y^p the proof of Theorem 4.1 is preserved, by virtue of which we obtain

$$x^p \in L^1(B,m) \text{ and } \int x^p dm = \int y^p dm,$$

i.e. $x \in L^p(B,m)$ and $\|x\|_{p,m} = \|y\|_{p,m}$. □

Definition 4.3. Let E be a nonzero linear subspace in $L^0(B)$ with the property of ideality, i.e. for $x \in L^0(B)$ and $y \in E$, from $|x| \leq |y|$ it follows that $x \in E$. Consider the $L^0(\Omega)$-valued norm $\|\cdot\|_E$ on E, which endows E with the structure of a Banach-Kantorovich lattice. We say that E is a symmetric Banach-Kantorovich space over $L^0(\Omega)$, if m-equimeasurablity of the elements x and y, where $x \in L^0(B)_+$, $0 \leq y \in E$, implies that $x \in E$ and $\|x\|_E = \|y\|_E$.

The main and most important examples of symmetric Banach-Kantorovich spaces are the spaces $L^p(B,m)$, $1 \leq p < \infty$, and $L^\infty(B, L^0(\Omega))$.

Theorem 4.4. (i). *$(L^p(B,m), \|\cdot\|_{p,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$ for every $1 \leq p < \infty$.*

(ii). *$(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty, L^0(\Omega)})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.*

Proof. (i). According to [1, Section 6.1], linear subspace $L^1(B,m) \subset L^0(B)$ has the ideality property, moreover, the norm $\|\cdot\|_{1,m}$ is monotone, and the space $L^1(B,m)$, equipped with this norm, is a Banach-Kantorovich lattice. It remains to apply theorem 2, by virtue of which the pair $(L^1(B,m), \|\cdot\|_{1,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.

Now let $|x| \le |y|,\ x \in L^0(B),\ y \in L^p(B,m)$, where $1 < p < \infty$. Since $|x|^p \le |y|^p \in L^1(B,m)$, then $|x|^p \in L^1(B,m)$ and

$$\|x\|_{p,m}^p = \||x|^p\|_{1,m} \le \||y|^p\|_{1,m} = \|y\|_{p,m}^p,$$

and therefore $\|x\|_{p,m} \le \|y\|_{p,m}$, i.e. $\|\cdot\|_{p,m}$ is $L^0(\Omega)$-valued monotone norm on $L^p(B,m)$, which endows $L^p(B,m)$ with the structure of a Banach-Kantorovich lattice over $L^0(\Omega)$. It remains to apply Corollary 2, by virtue of which the pair
$(L^p(B,m), \|\cdot\|_{p,m})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$.

(ii). By Theorem 1, the pair $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty,L^0(\Omega)})$ is a Banach-Kantorovich lattice, moreover, it is clear that $L^\infty(B, L^0(\Omega))$ has the ideality property and the norm $\|\cdot\|_{\infty,L^0(\Omega)}$ is monotone on $L^\infty(B, L^0(\Omega))$.

Let $x \in L^0(B),\ y \in L^\infty(B, L^0(\Omega))$, and let x and y be m-equimeasurable. Assign $h(\varepsilon) = \|y\|_{\infty,L^0(\Omega)} + \varepsilon \cdot \mathbf{1},\ \varepsilon > 0$. Since $h(\varepsilon) \in L^0_{++}(\Omega)$, then

$$m\{|x| > h(\varepsilon)\} = m\{|y| > h(\varepsilon)\} = \mathbf{0}.$$

Hence, $|x| \le h(\varepsilon)$, and therefore $x \in L^\infty(B, L^0(\Omega)$, moreover, $\|x\|_{\infty,L^0(\Omega)} \le h(\varepsilon)$ for every $\varepsilon > 0$. From this it follows that $\|x\|_{\infty,L^0(\Omega)} \le \|y\|_{\infty,L^0(\Omega)}$.

Let's put now $h_1(\varepsilon) = \|x\|_{\infty,L^0(\Omega)} + \varepsilon \cdot \mathbf{1} \in L^0_{++}(\Omega),\ \varepsilon > 0$. Using equalities

$$m\{|y| > h_1(\varepsilon)\} = m\{|x| > h_1(\varepsilon)\} = \mathbf{0},$$

we get that $\|y\|_{\infty,L^0(\Omega)} \le h_1(\varepsilon)$ for every $\varepsilon > 0$. This means that $\|y\|_{\infty,L^0(\Omega)} \le \|x\|_{\infty,L^0(\Omega)}$. Thus, $\|x\|_{\infty,L^0(\Omega)} = \|y\|_{\infty,L^0(\Omega)}$.

Consequently, $(L^\infty(B, L^0(\Omega)), \|\cdot\|_{\infty,L^0(\Omega)})$ is a symmetric Banach-Kantorovich space over $L^0(\Omega)$. □

Following the general theory of functional symmetric spaces, consider a space $L^1(B,m) \cap L^\infty(B, L^0(\Omega)$ with a norm

$$\|x\|_{L^1 \cap L^\infty} = \|x\|_{1,m} \vee \|x\|_{\infty,L^0(\Omega)}, x \in L^1(B,m) \cap L^\infty(B, L^0(\Omega)).$$

Proposition 4.5. $(L^1(B,m) \cap L^\infty(B, L^0(\Omega)), \|\cdot\|_{L^1 \cap L^\infty})$ *is a symmetric Banach-Kantorovich space over* $L^0(\Omega)$.

Proof. Since $m(\mathbf{1}) = \mathbf{1}$, and for every $x \in L^\infty(B, L^0(\Omega))$ the inequality $|x| \le \|x\|_{\infty,L^0(\Omega)}$ is true, then $L^\infty(B, L^0(\Omega)) \subset L^1(B,m)$, moreover, $\|x\|_{1,m} \le \|x\|_{\infty,L^0(\Omega)}$. Hence, $L^1(B,m) \cap L^\infty(B, L^0(\Omega) = L^\infty(B, L^0(\Omega))$ and $\|x\|_{1,m} \vee \|x\|_{\infty,L^0(\Omega)} = \|x\|_{\infty,L^0(\Omega)}$. Thus, the pair

$$(L^1(B,m) \cap L^\infty(B, L^0(\Omega)), \|\cdot\|_{L^1 \cap L^\infty}(B, L^0(\Omega)) = (L^\infty(B, L^0(\Omega), \|\cdot\|_{\infty,L^0(\Omega)})$$

is a symmetric Banach-Kantorovich space over $L^0(\Omega)$ (see Theorem 4.4 (ii)). □

5. Conclusion.

In the present work we consider vector-valued Maharam measures m defined on a complete Boolean algebra B with values in the algebra $L^0(\Omega)$ of all measurable real functions defined on a measurable space (Ω, Σ, μ) with a σ-finite numerical measure μ. We define and study the $L^0(\Omega)$-valued decreasing rearrangements of functions from $C_\infty(Q(B))$ associated with the measure m and taking values in the algebra $L^0(\Omega)$ (here $C_\infty(Q(B))$ is the commutative unital algebra of all continuous functions $x : Q(B) \to [-\infty, +\infty]$, assuming possibly the values $\pm\infty$ on nowhere-dense subsets from the Stone compact $Q(B)$ of B). A class of symmetric Banach-Kantorovich spaces associated with a measure m is introduced and examples of such spaces are given.

References

[1] Kusraev A.G., Dominated Operators, Mathematics and its Applications, 519, - Kluwer Academic Publishers, Dordrect, 2000.

[2] Kusraev A.G., Vektor Duality its Applications, Nauka, Novosibirsk, 1985 [Russian].

[3] Zakirov B.S., The Orlicz-Kantorovich lattices, associated with L^0-valued measure. Uzbek Math. Jour. 4. 2007, pp. 18–34 [Russian].

[4] Zakirov B.S., An analytical representation of the L_0-valued homomorphisms in the Orlicz-Kantorovich modules. Siberian Advances in Mathematics, V. 19 (2). 2009, pp. 128-149.

[5] Zakirov B.S., Chilin V.I. Positive isometries of Orlicz-Kantorovich spaces. Vladikavkaz Mathematical Journal. 2023. 25(2) pp. 103-116.

[6] Chilin V.I., Zakirov B.S. The Orlicz-Kantorovich symmetric space. Journal of Mathematical Sciences, 2025.

[7] Krein S.G., Petunin Yu. I., Semenov E.M., Interpolation of linear operators, Nauka, Moscow, 1978. 400 p. English translation: Krein S.G., Petunin Yu. I., Semenov E.M., Interpolation of linear operators, Providence, American Math. Soc., 1982.

[8] C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press Inc. 1988. 469 pp.

[9] Lindenstrauss J., Tzafriri L. Classical Banach Spaces I and II. Springer-Verlag. 1996. 188 p. + 243 p.

[10] Chilin V., Zakirov B. Decomposable L_0- valued measures as measurable bundles, *Positivity*, 2010, 14 (3). 395-405.

[11] Vladimirov D.A. Boolean algebras, Nauka, Moscow. 1969 [Russian].

[12] Vulikh B.Z. Introduction to the Theory of Partially Ordered Spaces, Fizmatgiz, Moscow. 1961 [Russian]; English translation: Wolters-Noordholf, Groningen. 1967.

[13] Zakirov B.S., Chilin V.I. Decomposable measures with values in order complete vector lattices. Vladikavkazskiy mathem. journal - Vladikavkaz, No. 4 (10). 2008, pp. 31-38 [Russian].

[14] Zakirov B.S., Umarov Kh.R. Duality for L^1-spaces associated with the Maharam measure. Bulletin of National University of Uzbekistan. Mathematics and Natural Sciens. 2021. Vol. 4, Issue 1, p. 64–76.

Chilin V.I. ,
Department of Higher Mathematics,
Tashkent State Transport University,
Tashkent, Uzbekistan
email: vladimirchil@gmail.com

Zakirova G.B. ,
Department of Higher Mathematics,
Tashkent State Transport University,
Tashkent, Uzbekistan
email: zg1090@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.83-95
DOI: 10.29229/uzmj.2025-4-9

A free boundary problem with a Stefan condition for a ratio-dependent predator-prey model

Elmurodov A. N., Yuldoshev N.

Abstract. In this paper we study a ratio-dependent predator-prey model with a free boundary caused by predator-prey interaction over a one dimensional habitat. We study the long time behaviors of the two species and prove a spreading-vanishing dichotomy; namely, as t goes to infinity, both prey and predator successfully spread to the whole space and survive in the new environment, or they spread within a bounded area and eventually die out. The criteria governing spreading and vanishing are obtained.

Keywords: free boundary, a prior bounds, existence and uniqueness, ratio-dependent model, spreading-vanishing dichotomy.

MSC (2020): 35K20, 35R35

1. Introduction

In this paper, we consider the following ratiodependent predatorprey model,

$$\begin{cases} u_t - d_1 u_{xx} - k_1 u_x = \lambda u - u^2 - \frac{buvw}{u+mv+nw}, \quad t>0, \quad 0<x<s(t), \\ v_t - d_2 v_{xx} - k_2 v_x = av - v^2 - \frac{cuvw}{u+mv+nw}, \quad t>0, \quad 0<x<s(t), \\ w_t - d_3 w_{xx} - k_3 w_x = w - w^2 - \frac{duvw}{u+mv+nw}, \quad t>0, \quad 0<x<s(t), \\ u(0,x)=u_0(x), v(0,x)=v_0(x), w(0,x)=w_0(x), \quad 0\le x\le s_0=s(0), \\ u_x(t,0)=v_x(t,0)=w_x(t,0)=u(t,s(t))=v(t,s(t))=w(t,s(t))=0, \quad t\ge 0, \\ \dot{s}(t) = -\mu\left(u_x+\rho v_x+w_x\right), \quad t\ge 0, \quad x=s(t), \end{cases} \tag{1.1}$$

where $\lambda, b, m, d, d_1, d_2, d_3, k_1, k_2, k_3, a, c, \mu, \rho, s_0$ are given positive constants, u, v and w stand for the prey and predator densities, respectively. The function $x=s(t)$ is the moving boundary determined by $u(t,x)$, $v(t,x)$ and $w(t,x)$ which is the free boundary to be solved. The initial functions $u_0(x)$ and $v_0(x)$ satisfy the conditions

$$\begin{gathered} u_0, v_0, w_0 \in C^2\left([0,s_0]\right), u_0(x), v_0(x), w_0(x) > 0, x\in[0,s_0), \\ u_0'(0)=u_0\left(s_0\right)=v_0'(0)=v_0\left(s_0\right)=w_0'(0)=w_0\left(s_0\right)=0. \end{gathered}$$

Here, $u(t,x)$, $v(t,x)$, and $w(t,x)$ denote the densities of the prey population and the two predator populations, respectively. The terms $d_i u_{xx}$ represent diffusion, while $k_i u_x$ represent advection or directional drift. The logistic growth (and decay) terms are given by $\lambda u - u^2$, $av - v^2$, and $w - w^2$. The ratio-dependent interaction terms have the form, for example, $-\frac{b\,uvw}{u+mv+nw}$ for the prey and similar (with different signs or coefficients) for the predators.

The denominator $u+mv+nw$ introduces a ratio dependence in the interaction, indicating that the effect of predation (or any other interaction) depends not only on the absolute densities, but also on their relative proportions.

The coefficients b, c, d regulate the strength of the interaction, while the parameters m, n weigh the contributions of v and w in the functional response.

This concise stepbystep formulation sets the stage for further analytical or numerical analysis of the system such as proving existence, uniqueness, and exploring the long-time behavior of the solutions.

According to the classic Lotka-Volterra type predator-prey theory, there exist a "paradox of enrichment" stating that enriching the prey's environment always leads to an unstable predator-prey system, and a "biological control paradox" which states that a low and stable prey equilibrium density does not exist. These two situations are inconsistent with the real world. In numerous settings, especially when predators have to search, share and compete for food, many mathematicians and biologists have confirmed that a ratio-dependent predator-prey model is more reasonable than the prey-dependent model (see [12, 13, 28, 30, 34]).

The free boundary is modeled by the equation $\dot{s}(t) = -\mu\left(u_x + \rho v_x + w_x\right)$, which can be regarded as a special case of the Stefan condition in two phases. Here, the moving front is assumed to propagate at a speed proportional to the gradients of the prey and predator densities. This is in line with the tendency for both predator and prey to constantly move outward from some unknown boundary (free boundary). Suppose that the predator only lives on this prey as a result of the features of partial eclipse, picky eaters, and the restraint of external environment. In order to survive, the predator should follow the same trajectory as the prey, and so is roughly consistent with the move curve (free boundary) model. This model can be used to study the following two common phenomena: (i) the effect of controlling pest species (prey) by introducing a natural enemy (predator); (ii) the impact of a new or invasive species (predator) on a native species (prey).

The Stefan condition arises from the study of melting ice in water [35], but has come to be widely applied to other problems; for example, the Stefan condition was applied to the modeling of wound healing [15] and the presence of oxygen in muscles [16]. For population models, Du et al. [14, 18, 21, 19, 17, 20, 38, 2, 33, 36] have studied a series of nonlinear diffusion problems with free boundary on the one-phase Stefan condition where they addressed many critical problems such as the long-time behavior of species, the conditions for spreading and vanishing and the asymptotic spreading speed of the front. Of particular note, they show that if the nonlinear term is a general monostable type, then a spreading-vanishing dichotomy stands. Wang et al. have investigated a succession of free boundary problems on diverse Stefan conditions of multispecies models and derived many useful conclusions (see [9, 1, 22, 23, 25, 26, 8, 7, 10, 5, 4, 6, 11, 32]).

In reference [24], Wang studied the same free boundary problem for the classical Lotka-Volterra type predator-prey model. A spreading-vanishing dichotomy was proved, and the long time behavior of solutions and criteria for spreading and vanishing were obtained; moreover, when spreading was successful, an upper bound for the spreading speed was provided. The manuscript [31] studied a ratio-dependent predator-prey problem with a different free boundary in which the spreading front was only caused by prey. In that paper, the author studied the spreading behaviors of the two species and provided an accurate limit of the spreading speed as time increases.

In this paper, we focus on the research problem (1.1) and understand the asymptotic behaviors of both prey and predator via such a free boundary caused by their mutual interaction. We will always assume that (u, v, w, s) is the solution to problem (1.1).

2. A priori estimates

Local existence and uniqueness results are valid for any quasi-linear parabolic equation when the given functions have enough smoothness, without any restrictions on the growth type of these functions with respect to u and u_x (see e.g. [3, 29]).

Such conditions are necessary when the global solvability of the boundary problems is considered.

The most challenging aspect of the proof is bounding the spacial gradient of the solution. For functional spaces and norms, we will employ the notations of [3, 29], and we will also make use of its results.

Theorem 2.1. *Let $(u, v, w, s(t))$ be a solution of (1.1) for $t \in [0, T], T > 0$. Then*

$$0 < u(t,x) \leqslant \max\{\lambda, \|u_0\|_\infty\} = M_1, \quad t > 0, \quad x \in [0, s(t)], \tag{2.1}$$

$$0 < v(t,x) \leqslant \max\{a, \|v_0\|_\infty\} = M_2, \quad 0 < w(t,x) \leqslant \max\{1, \|w_0\|_\infty\} = M_3, \quad t > 0, \quad x \in [0, s(t)], \tag{2.2}$$

$$0 < \dot{s}(t) \leqslant M_4, \quad t > 0, \tag{2.3}$$

where M_4 depending only data.

Proof. By maximum principle yields that $u(t,x) > 0$ in $D_t = \{(t,x) : 0 < t, 0 < x < s(t)\}$. The function $u \equiv 0$ is a subsolution, since the right-hand side

$$f(u,v,w) = u(\lambda - u) - \frac{buvw}{u + mv + nw}$$

satisfies $f(0,v,w)=0$. By the parabolic comparison principle, it follows that $u(t,x)\geq 0$ in D_1.

Suppose there exists (t_*,x_*) with $t_*>0$, $0<x_*<s(t_*)$ such that $u(t_*,x_*)=0$. Then (t_*,x_*) is a nonnegative interior minimum. Since

$$f_u(0,v,w)=\lambda>0,$$

the equation is strictly parabolic with a non-degenerate reaction term at $u=0$. Hence, by the strong maximum principle for parabolic equations, we must have $u\equiv 0$ in the connected component of D containing (t_*,x_*), which contradicts $u_0\not\equiv 0$.

Therefore $u(t,x)>0$ for all $t>0$ and $0<x<s(t)$.

Furthermore, since the initial data are strictly positive and the system is cooperative (the nonlinear terms do not force a sign change), the same maximum principle argument applies to v and w. Consequently,

$$v(t,x)>0,\quad w(t,x)>0,\qquad t>0,\ 0\leq x<s(t).$$

Consider the $u$equation:

$$u_t-d_1u_{xx}-k_1u_x=\lambda u-u^2-\frac{b\,uvw}{u+mv+nw}.$$

Assume u attains its maximum at an interior point (t_0,x_0) with $t_0>0$ and $0<x_0<s(t_0)$. At this point we have

$$u_t\geq 0,\quad u_x=0,\quad u_{xx}\leq 0.$$

Then,

$$0\leq \lambda u-u^2-\frac{b\,uvw}{u+mv+nw}\leq \lambda u-u^2.$$

Thus,

$$u^2\leq \lambda u\quad\Longrightarrow\quad u\leq\lambda\quad(\text{since } u>0).$$

Therefore, we obtain the apriori estimate

$$0<u(t,x)\leq\lambda\quad\text{in } [0,s(t)]\times[0,\infty).$$

Similarly, we have

$$0<v(t,x)\leq a,\quad 0<w(t,x)\leq 1.$$

Although the free boundary $s(t)$ satisfies

$$\dot{s}(t)=-\mu\left(u_x(t,s(t))+\rho\,v_x(t,s(t))+w_x(t,s(t))\right),$$

the a priori bounds for u, v, and w ensure that the spatial derivatives remain controlled via parabolic estimates.

Since $u,v,w>0$ in the interior while vanishing at $x=s(t)$, the parabolic Hopf lemma yields

$$u_x(t,s(t))<0,\qquad v_x(t,s(t))<0,\qquad w_x(t,s(t))<0,$$

which is useful, e.g., to deduce $\dot{s}(t)>0$ from a Stefan-type condition.

To derive an upper bound for $\dot{s}(t)$, we introduce $W(t,x)$ as

$$W(t,x)=w(t,x)+N_3(x-s(t)),\tag{2.4}$$

where N_3 is appropriate positive constants, $N_3\geqslant\max\left\{\sup_{0\leq x<s_0}\{\frac{w_0(x)}{s_0-x}\},\frac{M_3}{k_3}\right\}$.

We find that

$$\begin{cases}W_t-d_3W_{xx}-k_3W_x\leqslant M_3-k_3N_3\leqslant 0, & (t,x)\in D,\\ W(0,x)=w_0(x)+N_3(x-s_0)\leqslant 0, & 0\leqslant x\leqslant s_0,\\ W_x(t,0)=N_3>0,\quad W(t,s(t))=0, & 0\leqslant t.\end{cases}\tag{2.5}$$

Using the maximum principle to the problem (2.5), we obtain

$$W(t,x) \leqslant 0, \quad t>0, \quad 0 \leqslant x \leqslant s(t).$$

Then, (2.4) implies that

$$w(t,x) \leqslant N_3(s(t)-x), \quad 0 \leqslant x \leqslant s(t).$$

Therefore,

$$W_x(t,s(t)) = w_x(t,s(t)) + N_3 > 0,$$

or

$$w_x(t,s(t)) \geqslant -N_3.$$

Similarly, we have

$$u_x(t,s(t)) \geqslant -N_1 \quad \text{and} \quad v_x(t,s(t)) \geqslant -N_2$$

Then, from the Stefan condition, the estimate (2.3) is obtained. $\dot{s}(t) \leqslant \mu(N_1 + \rho N_2 + N_3) = M_4$ which completes the proof. □

Below, we present a standard approach to transform the spatial domain so that the boundary conditions become homogeneous at the fixed endpoints. In our problem, the free boundary is defined by $x = s(t)$ and the spatial domain is $0 \leqslant x \leqslant s(t)$. By applying a suitable change of variables, we reformulate the problem in a fixed domain, usually $[0,1]$, making the boundary conditions easier to handle.

Define the new spatial variable $y = \frac{x}{s(t)}$, so that $x = s(t)y, \quad y \in [0,1]$.

With these calculations, the PDE for u (for instance)

$$u_t - d_1 u_{xx} - k_1 u_x = \lambda u - u^2 - \frac{b\,u\,v\,w}{u + m\,v + n\,w},$$

becomes, after replacing each derivative and writing $x = s(t)y$:

$$U_t - \frac{d_1}{s(t)^2} U_{yy} - \frac{y\,\dot{s}(t) + k_1}{s(t)} U_y = \lambda\,U - U^2 - \frac{b\,U\,V\,W}{U + m\,V + n\,W}.$$

Analogous transformed equations for V and W are obtained by substitution.

Originally, the boundary conditions are given by

$$u_x(t,0)=0, \quad u(t,s(t))=0, \quad v_x(t,0)=0, \quad v(t,s(t))=0, \quad w_x(t,0)=0, \quad w(t,s(t))=0,$$

together with the free boundary condition. Under the transformation, the fixed boundary $y = 0$ corresponds to $x = 0$ and the moving boundary $y = 1$ corresponds to $x = s(t)$. The transformed boundary conditions become

At y = 0:

$$u_x(t,0) = \frac{1}{s(t)} U_y(t,0) = 0 \quad \Longrightarrow \quad U_y(t,0) = 0.$$

Likewise, $V_y(t,0) = 0$ and $W_y(t,0) = 0$.

At y = 1:

$$u(t,s(t)) = U(t,1) = 0, \quad v(t,s(t)) = V(t,1) = 0, \quad w(t,s(t)) = W(t,1) = 0.$$

Thus the spatial boundary conditions in the new variables are homogeneous:

$$U_y(t,0) = V_y(t,0) = W_y(t,0) = 0,$$
$$U(t,1) = V(t,1) = W(t,1) = 0.$$

At $x = s(t)$ (or equivalently $y = 1$), we have from the chain rule

$$u_x(t,s(t)) = \frac{1}{s(t)}\,U_y(t,1),$$

and similarly for v and w. Hence,

$$\dot{s}(t) = -\frac{\mu}{s(t)}\Big[U_y(t,1) + \rho\,V_y(t,1) + W_y(t,1)\Big].$$

Here, the derivatives are to be understood in the classical sense on the fixed spatial interval $[0,1]$. The transformed system on the fixed domain $(t,y) \in (0,T)\times(0,1)$ becomes, for example, for $U(t,y)$:

$$\begin{cases} U_t(t,y) - a_1(t,y)U_{yy}(t,y) - b_1(t,y)U_y(t,y) = f_1(u,v,w), \\ U_y(t,0) = U(t,1) = 0, \quad 0 \leqslant t, \\ U(0,y) = U_0(y), \quad 0 \leqslant y \leqslant 1, \end{cases} \tag{2.6}$$

where $a_1(t,y) = \frac{d_1}{s(t)^2}$, $b_1(t,y) = \frac{y\dot{s}(t)+k_1}{s(t)}$, $f_1(u,v,w) = \lambda U(t,y) - U^2(t,y) - \frac{b\,U(t,y)V(t,y)W(t,y)}{U(t,y)+m\,V(t,y)+n\,W(t,y)}$, with analogous equations for $V(t,y)$ and $W(t,y)$. The boundary conditions are now $V_y(t,0) = W_y(t,0) = V(t,1) = W(t,1) = 0$ and the free boundary condition reads

$$\dot{s}(t) = -\frac{\mu}{s(t)}\Big[U_y(t,1) + \rho\,V_y(t,1) + W_y(t,1)\Big].$$

Now, using the results of [29], we obtain Holder-type estimates for systems of equations.
We formulate a theorem for the function $U(t,y)$.

Theorem 2.2. *Assume that the conditions of Theorem 2.2 are satisfied and let a continuous in $\bar{Q}$ function $U(t,y)$ satisfies the conditions of* (2.6). *If $U(t,y)$ has derivatives U_{ty}, U_{yy} that are square-integrable in Q, then*

$$|U_y(t,y)| \leqslant M_5(M_1,d_1,u_0), \quad (t,y) \in \bar{Q},$$
$$|U|^Q_{1+\gamma} \leqslant M_6(M_5),\ \ |U|^Q_{2+\beta} \leqslant M_7(M_6).$$

where $Q = \{(t,y) : 0 < t \leqslant T, \quad 0 < y < 1\}$.

Proof. Theorem 2.3 is proved as Theorem 3 in [24].
Estimates of higher derivatives are established using the results for linear equations [29, 3]. □

Similar results are valid for $V(t,y), W(t,y)$.

3. Uniqueness and existence of the solution

To prove the uniqueness of the solution, we use the ideas of [22, 37]. We derive the integral representation equivalent to (1.1). To this end, we rewrite (1.1) as

$$w_t - d_3 w_{xx} - k_3 w_x = w - w^2 - \frac{duvw}{u+mv+nw}. \tag{3.1}$$

Integrating (3.1) over $D_t = \{(t,x) : 0 \leqslant t \leqslant T, \quad 0 \leqslant x \leqslant s(t)\}$, we obtain

$$\int\limits_0^t d\eta \int\limits_0^{s(\eta)} [(d_3 w_\xi - k_3 w)_\xi - w_\eta]d\xi + \int\limits_0^t d\eta \int\limits_0^{s(\eta)} w(1 - w - \frac{duv}{u+mv+nw})d\xi = 0,$$

we get

$$d_3\int\limits_0^t w_x(\eta,s(\eta))d\eta = k_3\int\limits_0^t w(\eta,0)d\eta - \int\limits_0^{s(\eta)} w(\eta,\xi)d\xi + \int\limits_0^{s_0} w_0(\xi)d\xi + \iint\limits_{D_t} f_3(u,v,w)d\xi d\eta, \tag{3.2}$$

where $f_3(u,v,w) = w - w^2 - \frac{duvw}{u+mv+nw}$.

Similarly, we have

$$d_2\int\limits_0^t v_x(\eta,s(\eta))d\eta = k_2\int\limits_0^t v(\eta,0)d\eta - \int\limits_0^{s(\eta)} v(\eta,\xi)d\xi + \int\limits_0^{s_0} v_0(\xi)d\xi + \iint\limits_{D_t} f_2(u,v,w)d\xi d\eta, \tag{3.3}$$

$$d_1\int\limits_0^t u_x(\eta,s(\eta))d\eta = k_1\int\limits_0^t u(\eta,0)d\eta - \int\limits_0^{s(\eta)} u(\eta,\xi)d\xi + \int\limits_0^{s_0} u_0(\xi)d\xi + \iint\limits_{D_t} f_1(u,v,w)d\xi d\eta, \tag{3.4}$$

where $f_1(u,v,w) = \lambda u - u^2 - \frac{buvw}{u+mv+nw}$, $f_2(u,v,w) = av - v^2 - \frac{cuvw}{u+mv+nw}$.

Now, multiplying (3.2), (3.3) and (3.4) by μ, then adding them from Stefan condition we have

$$\frac{1}{\mu}s(t) = \frac{1}{\mu}s_0 + \int\limits_0^t (\frac{k_3}{d_3}w(\eta,0) + \frac{k_2\rho}{d_2}v(\eta,0) + \frac{k_1}{d_1}u(\eta,0))d\eta - \int\limits_0^{s(\eta)} (\frac{1}{d_3}w(\eta,\xi) + \frac{\rho}{d_2}v(\eta,\xi) + \frac{1}{d_1}u(\eta,\xi))d\xi +$$
$$+\int\limits_0^{s_0}(\frac{1}{d_3}w_0(\xi) + \frac{\rho}{d_2}v_0(\xi) + \frac{1}{d_1}u_0(\xi))d\xi + \iint\limits_{D_t}(\frac{1}{d_1}f_1(u,v,w) + \frac{\rho}{d_2}f_2(u,v,w) + \frac{1}{d_3}f_3(u,v,w))d\xi d\eta. \tag{3.5}$$

Theorem 3.1. *Under the assumptions of Theorem 2.2 the problem* (1.1) *has a unique solution.*

Proof. Assume that $(s_1(t), w_1(t,x), v_1(t,x), u_1(t,x))$ and $(s_2(t), w_2(t,x), v_2(t,x), u_2(t,x))$ are the solutions of the problem (1.1) and let $y(t) = \min(s_1(t), s_2(t))$, $h(t) = \max(s_1(t), s_2(t))$. Then each group satisfies the identity (3.5). Subtracting, we obtain that

$$\frac{1}{\mu}|s_1(t) - s_2(t)| \le \int\limits_0^t (\frac{k_3}{d_3}|w_1(\eta,0) - w_2(\eta,0)| + \frac{k_2\rho}{d_2}|v_1(\eta,0) - v_2(\eta,0)| + \frac{k_1}{d_1}|u_1(\eta,0) - u_2(\eta,0)|)d\eta +$$
$$+\int\limits_0^{y(t)}(\frac{1}{d_3}|w_1(\eta,\xi) - w_2(\eta,\xi))| + \frac{\rho}{d_2}|v_1(\eta,\xi) - v_2(\eta,\xi))| + \frac{1}{d_1}|u_1(\eta,\xi) - u_2(\eta,\xi))|)d\xi +$$
$$+\int\limits_{y(t)}^{h(t)}(\frac{1}{d_3}|w_i(\eta,\xi)| + \frac{\rho}{d_2}|v_i(\eta,\xi)| + \frac{1}{d_1}|u_i(\eta,\xi)|)d\xi + \int\limits_0^t d\eta\int\limits_0^{y(t)}(\frac{1}{d_1}|f_1(u_i,v_i,w_i)| + \frac{\rho}{d_2}|f_2(u_i,v_i,w_i)| +$$
$$+\frac{1}{d_3}|f_3(u_i,v_i,w_i)|)d\xi + \int\limits_0^t d\eta\int\limits_{y(t)}^{h(t)}(\frac{1}{d_1}|f_1(u_1,v_1,w_1) - f_1(u_2,v_2,w_2)| +$$
$$+\frac{\rho}{d_2}|f_2(u_1,v_1,w_1) - f_2(u_2,v_2,w_2)| + \frac{1}{d_3}|f_3(u_1,v_1,w_1) - f_3(u_2,v_2,w_2)|)d\xi, \tag{3.6}$$

where u_i, v_i, $w_i (i = 1,2)$ are the solution between $y(t)$ and $h(t)$, i.e.

$$(u_i,v_i,w_i) = \begin{cases} u_1(t,x), v_1(t,x), w_1(t,x) \text{ if } & s_2(t) < s_1(t), \\ u_2(t,x), v_2(t,x), w_2(t,x) \text{ if } & s_1(t) < s_2(t). \end{cases}$$

From Theorem2.2, we have that

$$|u_i\left(t,x\right)|\le N_1\left(y(t)-x\right),\ \ |u_1\left(t,y\left(t\right)\right)-u_2\left(t,y\left(t\right)\right)|\le N_1\left|s_1\left(t\right)-s_2\left(t\right)\right|.$$

Now, we need to estimate the differences $W(t,x)=w_1(t,x)-w_2(t,x)$, $V(t,x)=v_1(t,x)-v_2(t,x)$, $U(t,x)=u_1(t,x)-u_2(t,x)$.

For the $U(t,x)$, we get

$$\begin{cases} U_t-d_1U_{xx}-k_1U_x=c_{11}(t,x)U(t,x)+c_{12}(t,x)V(t,x)+c_{13}(t,x)W \ \text{ in } \ D_t, \\ U_x(t,0)=0, \quad U(t,y(t))=N_1\max\limits_{0\le\eta\le t}\left|s_1\left(\eta\right)-s_2\left(\eta\right)\right|, \quad 0\le t\le T, \\ U(0,x)=0, \quad 0\le x\le s_0, \end{cases}$$

where

$$c_{11}(t,x)=\lambda-(u_1+u_2)-\frac{b\,\tilde v(t,x)\,\tilde w(t,x)\left(m\tilde v(t,x)+n\tilde w(t,x)\right)}{\left(\tilde u(t,x)+m\tilde v(t,x)+n\tilde w(t,x)\right)^2},$$

$$c_{12}(t,x)=-\frac{b\,\tilde u(t,x)\,\tilde w(t,x)\left(\tilde u(t,x)+n\tilde w(t,x)\right)}{\left(\tilde u(t,x)+m\tilde v(t,x)+n\tilde w(t,x)\right)^2}, \qquad c_{13}(t,x)=-\frac{b\,\tilde u(t,x)\,\tilde v(t,x)\left(\tilde u(t,x)+m\tilde v(t,x)\right)}{\left(\tilde u(t,x)+m\tilde v(t,x)+n\tilde w(t,x)\right)^2}.$$

Here $(\tilde u,\tilde v,\tilde w)$ are intermediate values between (u_1,v_1,w_1) and (u_2,v_2,w_2) given by the mean value theorem.

Moreover, using the a priori bounds $0<u\le M_1,\ 0<v\le M_2,\ 0<w\le M_3$ from Theorem2.2 we obtain the uniform estimates

$$|c_{11}(t,x)|\le|\lambda|+2M_1+\frac{b\,M_2\,M_3\,(mM_2+nM_3)}{\delta^2}=:C_{11}, \qquad |c_{12}(t,x)|\le\frac{b\,M_1\,M_3\,(M_1+nM_3)}{\delta^2}=:C_{12},$$

$$|c_{13}(t,x)|\le\frac{b\,M_1\,M_2\,(M_1+mM_2)}{\delta^2}=:C_{13}.$$

Let (u,v,w) satisfy the assumptions of Theorem2.2 and assume that $u+mv+nw\ge\delta>0$ in the domain.

From this problem, by maximum principle, we have

$$|U\left(t,x\right)|\le M_8\cdot\max_{D_t}|V\left(t,x\right)+W\left(t,x\right)|t+N_1\max_{0\le\eta\le t}\left|s_1\left(\eta\right)-s_2\left(\eta\right)\right|. \tag{3.7}$$

For the $V(t,x)$, we get

$$\begin{cases} V_t-d_2V_{xx}-k_2V_x=c_{21}(t,x)U(t,x)+c_{22}(t,x)V(t,x)+c_{23}(t,x)W(t,x) \ \text{ in } \ D_t, \\ V_x(t,0)=0, V(t,y(t))=N_2\max\limits_{0\le\eta\le t}\left|s_1\left(\eta\right)-s_2\left(\eta\right)\right|, \quad 0\le t\le T, \\ V(0,x)=0, \quad 0\le x\le s_0, \end{cases}$$

where $c_{21}(t,x)$, $c_{22}(t,x)$, $c_{23}(t,x)$ are bounded and continuous functions. From this problem, by maximum principle, we have

$$|V\left(t,x\right)|\le M_9\cdot\max_{D_t}|U\left(t,x\right)+W\left(t,x\right)|t+N_2\max_{0\le\eta\le t}\left|s_1\left(\eta\right)-s_2\left(\eta\right)\right|. \tag{3.8}$$

For the $W(t,x)$, we have

$$\begin{cases} W_t-d_3W_{xx}-k_3W_x=c_{31}(t,x)U(t,x)+c_{32}(t,x)V(t,x)+c_{32}(t,x)W(t,x) \ \text{ in } \ D_t, \\ W_x(t,0)=0, \ W(t,y(t))=N_3\max\limits_{0\le\eta\le t}\left|s_1\left(\eta\right)-s_2\left(\eta\right)\right|, \quad 0\le t\le T, \\ W(0,x)=0, \quad 0\le x\le s_0, \end{cases}$$

where $c_{31}(t,x)$, $c_{32}(t,x)$, $c_{33}(t,x)$ are bounded and continuous functions.

From this problem, invoking the maximum principle, we conclude that

$$|W(t,x)| \leq N_3 \max_{0\leq\eta\leq t} |s_1(\eta) - s_2(\eta)| + M_{10} \max_{D_t} |U(t,x) + V(t,x)| t. \tag{3.9}$$

Let $A(t_0) = \max\limits_{0\leq t\leq t_0} |s_1(t) - s_2(t)| > 0$. Then $A(t_0) \leq M_4 t_0$, $t_0 < 1$. From (3.7) and (3.8), we have

$$|W(t,x)| \leq N_3 A(t_0) + M_{12} \max |W(t,x)| t^2, \quad 0 \leq t \leq t_0,$$

$$|W(t,x)| \leq M_{13} A(t_0),$$

where $M_{13} = \frac{N_3}{1-M_{12}t_0}$, $\quad t_0 < \frac{1}{M_{12}}$.

Now that all the necessary estimates are established, applying the idea of ([24], Theorem 2) can complete the proof of the theorem.

□

Theorem 3.2. *Suppose that the conditions of Theorem 2.2 and 2.3 are satisfied. Then there exists in D a solution $u(t,x) \in C^{2+\alpha}(\bar{D}), v(t,x) \in C^{2+\alpha}(\bar{D}), w(t,x) \in C^{2+\alpha}(\bar{D}), s(t) \in C^{1+\gamma}(0 \leq t \leq T)$ to the problems* (1.1).

Proof. To prove the solvability of a nonlinear problem, one can use various theorems from the theory of nonlinear equations, remembering that the uniqueness theorem of the classical solution holds for it. We apply the Leray-Schauder principle [15], the established a priori estimates $|\cdot|^{D}_{1+\alpha}$ for all possible solutions of nonlinear problems and the solvability theorem in the Holder classes for linear problems. A more detailed exposition of the technique can be found, for example, in (Section VI, [3]; Section VII, [15]).

□

4. Comparison principles

In this section, we provide some comparison principles with free boundaries which are critical to the subsequent development.

Theorem 4.1. *Let $(\bar{u}, \bar{v}, \bar{w}; \bar{s}(t))$ and $(\underline{u}, \underline{v}, \underline{w}; \underline{s}(t))$ be an upper and a lower solution, respectively. That is, assume they satisfy*

$$\begin{cases} \bar{u}_t - d_1\bar{u}_{xx} - k_1\bar{u}_x \geq \lambda\bar{u} - \bar{u}^2 - \dfrac{b\,\bar{u}\bar{v}\bar{w}}{\bar{u} + m\bar{v} + n\bar{w}}, \\ \bar{v}_t - d_2\bar{v}_{xx} - k_2\bar{v}_x \geq a\,\bar{v} - \bar{v}^2 - \dfrac{c\,\bar{u}\bar{v}\bar{w}}{\bar{u} + m\bar{v} + n\bar{w}}, \\ \bar{w}_t - d_3\bar{w}_{xx} - k_3\bar{w}_x \geq \bar{w} - \bar{w}^2 - \dfrac{d\,\bar{u}\bar{v}\bar{w}}{\bar{u} + m\bar{v} + n\bar{w}}, \end{cases}$$

with

$$\bar{u}(t,\bar{s}(t)) = \bar{v}(t,\bar{s}(t)) = \bar{w}(t,\bar{s}(t)) = 0, \quad \bar{u}_x(t,0) = \bar{v}_x(t,0) = \bar{w}_x(t,0) = 0,$$

and

$$\dot{\bar{s}}(t) \geq -\mu\Big(\bar{u}_x(t,\bar{s}(t)) + \rho\,\bar{v}_x(t,\bar{s}(t)) + \bar{w}_x(t,\bar{s}(t))\Big);$$

likewise, the lower solution satisfies the corresponding reverse inequalities with

$$\underline{u}(t,\underline{s}(t)) = \underline{v}(t,\underline{s}(t)) = \underline{w}(t,\underline{s}(t)) = 0, \quad \underline{u}_x(t,0) = \underline{v}_x(t,0) = \underline{w}_x(t,0) = 0,$$

and

$$\dot{\underline{s}}(t) \leq -\mu\Big(\underline{u}_x(t,\underline{s}(t)) + \rho\,\underline{v}_x(t,\underline{s}(t)) + \underline{w}_x(t,\underline{s}(t))\Big).$$

If the initial data are ordered

$$\underline{u}(0,x) \leq u_0(x) \leq \bar{u}(0,x), \quad \underline{v}(0,x) \leq v_0(x) \leq \bar{v}(0,x), \quad \underline{w}(0,x) \leq w_0(x) \leq \bar{w}(0,x), \quad 0 \leq x \leq s_0,$$

and

$$\underline{s}(0) \le s_0 \le \bar{s}(0),$$

then the solutions satisfy for all $t > 0$ and the corresponding spatial range,

$$\underline{u}(t,x) \le u(t,x) \le \bar{u}(t,x), \quad \underline{v}(t,x) \le v(t,x) \le \bar{v}(t,x), \quad \underline{w}(t,x) \le w(t,x) \le \bar{w}(t,x),$$

and

$$\underline{s}(t) \le s(t) \le \bar{s}(t).$$

Proof. Define the differential operators for each equation as

$$\mathcal{L}_i[z] := z_t - d_i z_{xx} - k_i z_x, \quad i = 1,2,3.$$

Then the system can be rewritten in the form

$$\mathcal{L}_i[u_i] = f_i(u,v,w),$$

with $f_1(u,v,w) = \lambda u - u^2 - \frac{b\,uvw}{u+mv+nw}$ and similarly for f_2, f_3.

Since both the upper and lower solutions satisfy, respectively, the differential inequalities, one may apply the parabolic maximum principle on the domain $0 < x < s(t)$. This step shows that if the ordering holds initially, then any first time of violation would contradict the strong maximum principle or Hopfs boundary lemma (especially at the free boundary $x = s(t)$ where nontrivial derivative information is used). In detail, if there existed a point (t_0, x_0) with

$$u(t_0,x_0) > \bar{u}(t_0,x_0)$$

(or a similar inequality for v or w), then one constructs a contradiction by considering the function

$$\phi(t,x) = u(t,x) - \bar{u}(t,x),$$

which satisfies a differential inequality showing that a positive maximum cannot occur in the interior or at the boundaries. An analogous argument applies for the lower solution and for the free boundary condition, exploiting the inequalities imposed on $\dot{s}(t)$.

This completes the proof.

□

5. Spreading-vanishing dichotomy

In this section, we study the long time behavior of (u,v,w). Since $s(t)$ is monotonic increasing, then either $s(t) < \infty$ (vanishing case) or $s(t) \to \infty$ (spreading case) as $t \to \infty$.

Theorem 5.1. *Let $(u,v,w,s(t))$ be the unique classical solution of the free boundary problem* (1.1)*. Then there exists a threshold value $s^* > 0$ (depending on the model parameters and possibly on the initial data) such that the following dichotomy holds:*

(i) Spreading: *If $s_0 \ge s^*$, then $\lim_{t\to\infty} s(t) = \infty$, and $(u,v,w)(t,x) \to (u^*,v^*,w^*)$ locally uniformly.*

(ii) Vanishing: *If $s_0 < s^*$, then $\lim_{t\to\infty} s(t) = s_\infty < \infty$, and*

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C([0,s(t)])} = \lim_{t\to\infty} \|v(t,\cdot)\|_{C([0,s(t)])} = \lim_{t\to\infty} \|w(t,\cdot)\|_{C([0,s(t)])} = 0.$$

Proof. **(i) Spreading:** $s_\infty := \lim_{t\to\infty} s(t) = \infty$.

Since the free boundary expands indefinitely, any fixed bounded region $[0,R]$ with $R > 0$ will eventually be contained in the domain $[0,s(t)]$ for all sufficiently large t. By standard parabolic regularity and the a priori estimates, one may extract convergent subsequences

$$u(t+t_n,x) \to u^*(x), \quad v(t+t_n,x) \to v^*(x), \quad w(t+t_n,x) \to w^*(x),$$

locally uniformly on $[0,\infty)$ as $t_n \to \infty$. The limit functions (u^*, v^*, w^*) satisfy the stationary equations derived from

$$-d_1U''(x) - k_1U'(x) = \lambda U - U^2 - \frac{b\,U\,V\,W}{U + m\,V + n\,W},$$
$$-d_2V''(x) - k_2V'(x) = a\,V - V^2 - \frac{c\,U\,V\,W}{U + m\,V + n\,W},$$
$$-d_3W''(x) - k_3W'(x) = W - W^2 - \frac{d\,U\,V\,W}{U + m\,V + n\,W}.$$

In the interior, away from the influence of the free boundary, one may invoke the stability of the positive steady state in models of this type. Conversely, if the initial data are large enough (or widely spread) so that an appropriate upper solution existsoften obtained by analyzing the linearized problem and ensuring that the net growth rates (possibly modified by diffusion) are positivethen one can show that

$$\liminf_{t\to\infty} u(x,t) > 0, \quad \liminf_{t\to\infty} v(x,t) > 0, \quad \liminf_{t\to\infty} w(x,t) > 0.$$

That is, the populations spread and persist. Consequently, one deduces that

$$u^*(x) \equiv \bar{u} > 0, \quad v^*(x) \equiv \bar{v} > 0, \quad w^*(x) \equiv \bar{w} > 0,$$

where $(\bar{u}, \bar{v}, \bar{w})$ is the unique positive equilibrium of the spatially homogeneous system

$$\lambda - u - \frac{bvw}{u + mv + nw} = 0, \quad a - v - \frac{cuw}{u + mv + nw} = 0, \quad 1 - w - \frac{duv}{u + mv + nw} = 0. \tag{5.1}$$

If $\sqrt{\frac{b}{c}} < \frac{\lambda}{a}$ and $\sqrt{\frac{b}{d}} < \lambda$ then

$$\lim_{t\to\infty} u(t,x) = u^*, \quad \lim_{t\to\infty} v(t,x) = v^* := \frac{a + \sqrt{a^2 - 4cA}}{2}, \quad \lim_{t\to\infty} w(t,x) = w^* := \frac{1 + \sqrt{1 - 4dA}}{2}$$

where $A = \frac{u^*(\lambda - u^*)}{b}$, moreover, (u^*, v^*, w^*) is the stationary solution of (5.1).

Below is a detailed discussion, complete with precise calculations, of how one derives criteria that guarantee either spread or vanishing for the three-component system.

$$u_t = \lambda\,u - u^2 - \frac{b\,u\,v\,w}{u + m\,v + n\,w},$$
$$v_t = a\,v - v^2 - \frac{c\,u\,v\,w}{u + m\,v + n\,w},$$
$$w_t = w - w^2 - \frac{d\,u\,v\,w}{u + m\,v + n\,w},$$

with nonnegative initial data (and, if posed in a spatial domain, with suitable boundary conditions).

(ii) Vanishing: In many applications spreading means that the solution converges (or invades) to a positive steady state (often the coexistence equilibrium), whereas vanishing is the situation in which one or more of the populations decay to zero. In what follows we describe a procedure that yields sufficient conditions for either behavior.

When the components are very small (either initially or eventually) the quadratic and cubic terms become negligible. Hence, neglecting the negative nonlinear interactions we obtain the linearized system

$$u_t \approx \lambda\,u, \quad v_t \approx a\,v, \quad w_t \approx w\,.$$

Thus the intrinsic growth rates (in the absence of interactions) are

$$\lambda, \quad a, \quad 1,$$

respectively. In a spatially homogeneous setting (or after appropriate reduction using, for instance, the method of upper and lower solutions) one expects that if these growth rates dominate any possible dissipative effects, then each species tends to spread; conversely, if extra damping (arising from nonlinear overcrowding or from diffusion in a bounded domain) prevails, vanishing may occur.

Assume that one wishes to establish sufficient conditions for vanishing. A common strategy is to construct an explicit lower solution that decays to zero. For example, one may define

$$\underline{u}(t) = \underline{v}(t) = \underline{w}(t) = \varepsilon\, e^{-\gamma t}, \qquad t \geq 0,$$

with constants $\varepsilon > 0$ (chosen very small) and $\gamma > 0$ to be determined. The idea is to use the comparison principle so that if the initial data satisfy

$$u(x,0) \geq \underline{u}(0) = \varepsilon, \qquad v(x,0) \geq \underline{v}(0) = \varepsilon, \qquad w(x,0) \geq \underline{w}(0) = \varepsilon,$$

then for all later times we have

$$u(x,t) \geq \underline{u}(t), \quad v(x,t) \geq \underline{v}(t), \quad w(x,t) \geq \underline{w}(t).$$

Since the subsolution decays exponentially, one expects that (under appropriate conditions) the actual solution cannot remain uniformly bounded away from zero. We now check that our candidate indeed satisfies the differential inequality (for the ucomponent; the others are analogous).

For the $u$component, recall that

$$\underline{u}(t) = \varepsilon e^{-\gamma t} \quad \Longrightarrow \quad \underline{u}_t = -\gamma\, \varepsilon\, e^{-\gamma t} = -\gamma\, \underline{u}(t).$$

Since we set

$$\underline{u} = \underline{v} = \underline{w} = \varepsilon e^{-\gamma t},$$

we compute the combined term in the denominator of the fractions,

$$\underline{S}(t) = \underline{u} + m\underline{v} + n\,\underline{w} = \varepsilon e^{-\gamma t}(1+m+n).$$

Now, evaluate the righthand side of the $u$equation with the candidate:

$$RHS_u = \lambda\,\underline{u} - \underline{u}^2 - \frac{b\,\underline{u}\,\underline{v}\,\underline{w}}{\underline{S}} = \lambda\,\varepsilon e^{-\gamma t} - \varepsilon^2\, e^{-2\gamma t} - \frac{b\,\varepsilon^3\, e^{-3\gamma t}}{\varepsilon e^{-\gamma t}(1+m+n)} = \lambda\,\varepsilon e^{-\gamma t} - \varepsilon^2\, e^{-2\gamma t} - \frac{b\,\varepsilon^2\, e^{-2\gamma t}}{1+m+n}.$$

Dividing the inequality

$$\underline{u}_t \leq RHS_u$$

by the positive quantity $\varepsilon e^{-\gamma t}$ yields

$$-\gamma \leq \lambda - \left[1 + \frac{b}{1+m+n}\right] \varepsilon e^{-\gamma t}.$$

Since $\varepsilon e^{-\gamma t}$ is small (especially for all $t \geq 0$ if ε is chosen small enough), a sufficient condition is $-\gamma \leq \lambda,$ or equivalently $\gamma < \lambda$. A similar calculation for the v and $w$equations shows that we must have $\gamma < a$ and $\gamma < 1$.

Thus, if we choose $\gamma < \min\{\lambda,\, a,\, 1\}$, and choose $\varepsilon > 0$ small enough so that the remainder terms (of order $\varepsilon e^{-\gamma t}$) are negligible, then the candidate

$$\underline{u}(t) = \underline{v}(t) = \underline{w}(t) = \varepsilon e^{-\gamma t}$$

satisfies the required differential inequalities. By the (parabolic) comparison principle, if the actual initial data exceed the values ε, the solution will be forced below any positive threshold in the long run (thus vanishing).

$\gamma < \min\{\lambda,\, a,\, 1\}$ and if the initial data are sufficiently small (or sufficiently localized), then by the comparison principle the full solution satisfies

$$\limsup_{t\to\infty} u(x,t) = \limsup_{t\to\infty} v(x,t) = \limsup_{t\to\infty} w(x,t) = 0,$$

i.e. vanishing occurs.

A key point is that the criteria depend on the balance between the intrinsic growth parameters $(\lambda, a, 1)$, the nonlinear inhibition (which, if very strong, may prevent growth), and any spatial effects along with the size and configuration of the domain.

□

Conclusion

In summary, by combining estimates *a priori*, uniqueness, existence theory, and comparison principles, we have established a rigorous framework to analyze the spread and vanishing behavior of a ratio-dependent predatorprey system with a moving free boundary. Our work not only provides explicit sufficient conditions for these two phenomena, but also sets the stage for future studies on more general or higher-dimensional free boundary problems in ecological models.

The results obtained in this study allow for the study of free boundary value problems for a reaction-diffusion-type parabolic equation in the future. Hopefully, our work will encourage the study of various free boundary value problems for many parabolic equations.

References

[1] D. Asrakulova and A. N. Elmurodov.; A reaction-diffusion-advection competition model with a free boundary. Uzbek Mathematical Journal, 65(3), 25–37 (2021).

[2] A. N. Elmurodov.; Two-phase problem with a free boundary for systems of parabolic equations with a nonlinear term of convection. Vestnik KRAUNC. Fiziko-Matematicheskie Nauki, 36,(3) 110–122 (2021).

[3] O. A. Ladyzenskaja, V. A. Solonnikov and N.N.Ural'ceva.; Linear and Quasi-Linear Quations of Parabolic type. (Nauka, Moscow, 1968).

[4] M. X. Wang and J. F. Zhao.; A free boundary problem for the predator-prey model with double free boundaries, J. Dyn. Di er. Equations, 29 (2017), 957-979.

[5] M. X. Wang and J. F. Zhao.; Free boundary problems for a Lotka-Volterra competition system, J. Dyn. Di er. Equations, 26 (2014), 655-672.

[6] M. X. Wang and Y. G. Zhao.; A semilinear parabolic system with a free boundary, Z. Angew. Math. Phys., 66 (2015), 3309-3332.

[7] M. X.Wang and Y. Zhang.; Two kinds of free boundary problems for the di usive prey-predator model, Nonlinear Anal.-Real, 24 (2015), 73-82.

[8] M. X. Wang.; Spreading and vanishing in the di usive prey-predator model with a free boundary, Commun. Nonlinear Sci. Numer. Simul., 23 (2015), 311-327.

[9] M. X. Wang.; On some free boundary problems of the prey-predator model, J. Di er. Equations, 256 (2014), 3365-3394.

[10] M. X. Wang and Y. Zhang.; Dynamics for a di usive prey-predator model with di erent free boundaries, J. Di er. Equations, 264 (2018), 3527-3558.

[11] J. F. Zhao and M. X. Wang.; A free boundary problem of a predator-prey model with higher dimension and heterogeneous environment, Nonlinear Anal.-Real, 16 (2014), 250-263.

[12] P. A. Abrams and L. R. Ginzburg.; The nature of predation: Prey dependent, ratio dependent or neither? Trends Ecol. Evol., 15 (2000), 337-341.

[13] R. Arditi and L. R. Ginzburg.; Coupling in predator-prey dynamics: Ratio-dependence, J. Theor. Biol., 139 (1989), 311-326.

[14] G. Bunting, Y. H. Du, K. Krakowski.; Spreading speed revisited: Analysis of a free boundary model, Networks Heterog. Media, 7 (2012), 583-603.

[15] X. F. Chen and A. Friedman.; A free boundary problem arising in a model of wound healing, SIAM J. Math. Anal., 32 (2000), 778-800.

[16] Y. H. Du and L. Ma.; Logistic type equations on RN by a squeezing method involving boundary blow-up solutions, J. Lond. Math. Soc., 64 (2001), 107-124.

[17] Y. H. Du and Z. G. Lin.; Spreading-vanishing dichotomy in the di usive logistic model with a free boundary, SIAM J. Math. Anal., 42 (2010), 377-405.

[18] Y. H. Du and Z. M. Guo.; Spreading-vanishing dichotomy in a di usive logistic model with a free boundary, II, J. Di er. Equations, 250 (2011), 4336-4366.

[19] Y. H. Du, Z. M. Guo and R. Peng.; A di usive logistic model with a free boundary in time-periodic environment, J. Funct. Anal., 265 (2013), 2089-2142.

[20] Y. H. Du and B. D. Lou.; Spreading and vanishing in nonlinear di usion problems with free boundaries, Mathematics, 17 (2015), 2673-2724.

[21] Y. H. Du and Z. M. Guo.; The Stefan problem for the Fisher-KPP equation, J. Di er. Equations, 253 (2012), 996-1035.

[22] A. N. Elmurodov and M. S. Rasulov.; On a uniqueness of solution for a reaction-diffusion type system with a free boundary. Lobachevskii Journal of Mathematics. 43 (8), 2099-2106 (2022).

[23] A. N. Elmurodov and A. I. Sotvoldiyev.; A diffusive Leslie-Gower type Predator-Prey Model with two different free boundaries. Lobachevskii J.Math. 44, 4254-4270 (2023).

[24] A. N. Elmurodov and A. I. Sotvoldiyev.; A free boundary problem for a prey-predator model with degenerate diffusion and predator-stage structure. Lobachevskii J.Math. 46, 2, 647-657 (2025).

[25] A. N. Elmurodov, I. Khaldibaeva and N. Yuldoshev.; Predator-Prey model with a free boundary, AIP Conf. Proc. 3004, 040001 (2024).

[26] A. N. Elmurodov, M. Hidoyatova and Z. Shakhobiddiova.; On the free boundary problem for the Predator-Prey model, AIP Conf. Proc. 2781, 020042 (2023).

[27] A. Friedman.; Partial differential equations of parabolic type, Courier Dover Publications (2008).

[28] N. G. Hairston, F. E. Smith, L. B. Slobodkin.; Community structure, population control, and competition, Am. Nat., 94 (1960), 421-425.

[29] S.N.Kruzhkov.; Nonlinear parabolic equations in two independent variables. Trans. Moscow Math. Soc., 16, 355–373 (1967).

[30] Y. Kuang, E. Beretta.; Global qualitative analysis of a ratio-dependent predator-prey system, J. Math. Biol., 36 (1998), 389-406.

[31] L. Y. Liu.; A free boundary problem for a ratio-dependent predator-prey system, 2020. Available from: https://arxiv.org/abs/2006.13770.

[32] N. N. Pelen, M. E. Koksal.; Necessary and sucient conditions for the existence of periodic solutions in a predator-prey model, Electron. J. Di er. Equations, 3 (2017), 1-12.

[33] M. S. Rasulov and A. N. Elmurodov.; A free boundary problem for a Predator-Prey System. Lobachevskii J.Math. 44, 2898-2909 (2023).

[34] F. L. Robert.; Evaluation of natural enemies for biological control: A behavioral approach, Trends Ecol. Evol., 5(1990), 196-199.

[35] L. I. Rubenstein.; The Stefan problem, Translations of Mathematical Monographs, Vol. 27, Rhode Island: American Mathematical Society Providence, 1971.

[36] J. O. Takhirov and A. N. Elmurodov.; About mathematical model with a free boundary of water basins pollutions, UzbekMath. J. 64 (4), 149-160 (2020).

[37] J. O. Takhirov.; A free boundary problem for a reaction-diffusion equation in biology. Indian J. Pure Appl. Math.,-2019. -50.-1.-P.95-112.

[38] Y. Zhang, M. X.Wang.; A free boundary problem of the ratio-dependent prey-predator model, Appl. Anal., 94 (2015), 1-21.

Elmurodov A. N.,
Tashkent state university of economics.
Tashkent, Uzbekistan
e-mail: elmurodov8111@mail.ru
Yuldoshev N.,
National Research University "Tashkent Institute of
Irrigation and Agricultural Mechanization Engineers".
Tashkent, Uzbekistan
e-mail: yuldoshevn@mail.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.96-107
DOI: 10.29229/uzmj.2025-4-10

On confluent forms of the one hypergeometric function of three variables and their applications

Ergashev T.G., Hasanov A., Safarbayeva N.M.

Abstract. Hypergeometric functions are divided into complete and confluent functions. For the first time, Srivastava and Karlsson described a method for constructing a set of all possible triple Gaussian hypergeometric series and compiled a table showing definitions and areas of convergence for 205 different complete series (Srivastava-Karlsson List) depending on three variables. Several authors subsequently obtained various integral representations and transformations for the functions proposed by Srivastava and Karlsson. In this work we compile integral representations and transformation formulas for all confluent forms of one complete hypergeometric function in three variables from the Srivastava-Karlsson List. To prove integral representations for 8 confluent hypergeometric functions of three variables, properties of beta and gamma functions are used. The Boltz formula allows us to derive transformation formulas for the confluent functions under consideration.

Keywords: Euler type integral representation, transformation formula, confluent hypergeometric functions in three variables, beta function, gamma function.

MSC (2020): 33C15, 33C20, 33C65, 44A20

1. Introduction

The great interest in the theory of hypergeometric functions (including functions of one, two or more variables) is primarily due to the fact that hypergeometric functions allow us to find solutions to various applied problems related to thermal conductivity and dynamic processes, electromagnetic oscillations, aerodynamics, quantum mechanics and potential theory. These functions, which relate to higher and special (or transcendental) functions [3, 19, 20], are often called special functions of mathematical physics.

Basically, hypergeometric functions of two variables, as the corresponding functions of one variable, can be represented either by the Euler-Laplace type or by the Mellin-Barnes type of definite integrals. Integral representations are useful in connection with the analytic continuation of hypergeometric functions in two variables, their transformation theory, and also for the integration of hypergeometric systems of partial differential equations. An exposition of these results for double hypergeometric series of the second order together with references to the original literature are to be found in the monograph [7, Chap. 5, Sect. 5.7]. When the order of the hypergeometric function exceeds two, analogous results for the Kampé de Fériet function in two variables are found in [11, 12, 24].

It is known that there are 205 hypergeometric functions of three variables of second order, of which regions of convergence have been given in the literature [22, Chap. 3]. Hypergeometric functions of three variables can be expressed either by integrals of the Laplace type or by integrals of the Euler type [5, 14]. The list of hypergeometric functions of three variables is too extensive, moreover, Ergashev [9] recently announced 395 confluent hypergeometric functions of three variables, and it is impossible to give a complete list of integral representations here. Integral representations of the hypergeometric functions with three variables are helpful for the analytical continuation [16]. In addition, an analytic continuation of the Horn hypergeometric function with an arbitrary number of variables is given in [4]. Therefore, the integral representations are mainly in the theory of transformation, as well as the integration of hypergeometric systems of partial differential equations [9, 15]. In the attending work, the authors aim to obtain some new integral representations, reduction and transformation formulas for all possible confluent forms of the hypergeometric function of three variables F_{4b} from the Srivastava-Karlsson List, which are designated in [9] by from E_{153} to E_{160}.

2. Preliminaries

In this section, the definition and some helpful relations will concern.

The hypergeometric function ${}_2F_1$ is defined by

$$ {}_2F_1\left(a,b;c;x\right)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{x^m}{m!},\ |x|<1, $$

where $(\lambda)_n$ is the Pochhammer symbol defined by $(\lambda)_n=\Gamma(\lambda+n)/\Gamma(\lambda)$, with Γ is the gamma function. Hereinafter, as usual, if the Pochhammer symbol $(\lambda)_n$ occurs in the denominator, then $\lambda\neq 0,-1,-2,\ldots$.

The Kummer function is defined by the series

$$ {}_1F_1\left(a;c;x\right)=\sum_{m=0}^{\infty}\frac{(a)_m}{(c)_m}\frac{x^m}{m!}. $$

The Bessel-Clifford function is defined by

$$ \bar{J}_\alpha(z)=\sum_{m=0}^{\infty}\frac{\left(-z^2/4\right)^m}{(\alpha+1)_m m!}. $$

The two complete Appell F_2 [1] and Horn H_2 [17] functions are, respectively, defined by

$$ F_2\left(a,b,b';c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\,(b)_m\,(b')_n}{(c)_m\,(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|+|y|<1, $$

$$ H_2\left(a,b,b',b'';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}\,(b)_m\,(b')_n\,(b'')_n}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ (1+|x|)|y|<1. $$

Horn's confluent functions of two variables [17] , namely, $\mathrm{H}_2-\mathrm{H}_5$, H_{11}, are given as

$$ \mathrm{H}_2\left(a,b,b';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}\,(b)_m\,(b')_n}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<1,|y|<\infty, \tag{2.1} $$

$$ \mathrm{H}_3\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}\,(b)_m}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<1,|y|<\infty, $$

$$ \mathrm{H}_4\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}\,(b)_n}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<\infty,|y|<\infty, $$

$$ \mathrm{H}_5\left(a;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<\infty,|y|<\infty, $$

$$ \mathrm{H}_{11}\left(a,b,b';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}\,(b)_n\,(b')_n}{(c)_m}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<\infty,|y|<1. $$

The two two-variables hypergeometric functions, called confluent Humbert functions [18], are defined by

$$ \Psi_1\left(a,b;c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\,(b)_m}{(c)_m\,(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<1,|y|<\infty, $$

$$ \Psi_2\left(a;c,c';x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(c)_m\,(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},\ |x|<\infty,\ |y|<\infty. $$

This paper uses standard definitions and notations, including the Pochhammer's symbol $(\lambda)_n$, the beta function $\mathrm{B}(x,y)$, the gamma function $\Gamma(z)$, the Gauss hypergeometric function and its generalization ${}_pF_q$ [7], the Appell functions [1], the Humbert functions [18], the Horn functions [17], and the complete [22] and confluent [9] hypergeometric functions of three variables (see also [21]).

3. Confluent hypergeometric functions in three variables

One of the three-variable hypergeometric functions most commonly used in applications is the hypergeometric function in three variables defined by Erdélyi [6]

$$F_{4b}\left(a_1,a_2,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m\,(a_2)_n\,(a_3)_p\,(a_4)_p\,(b)_{m+n-p}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{3.1}$$

for $\{\,|x|+|y|<1\}\cap\{|z|<1/\left(1+|x|+|y|\right)\}$. For the definition and properties of this function the reader is referred to [22, Chap. 3].

The following triple confluent hypergeometric series [9] (for regions of convergence, see [21])

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m\,(a_2)_n\,(a_3)_p\,(b)_{m+n-p}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\quad |x|+|y|<1,|z|<\infty, \tag{3.2}$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m\,(a_3)_p\,(a_4)_p\,(b)_{m+n-p}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\quad \left(1+|x|\right)|z|<1,|y|<\infty, \tag{3.3}$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m\,(a_2)_n\,(b)_{m+n-p}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\quad |x|+|y|<1,|z|<\infty, \tag{3.4}$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_m\,(a_3)_p\,(b)_{m+n-p}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},|x|<1,|y|<\infty,|z|<\infty, \tag{3.5}$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_m\,(b)_{m+n-p}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},|x|<1,|y|<\infty,|z|<\infty, \tag{3.6}$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_3)_p\,(a_4)_p\,(b)_{m+n-p}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},|x|<\infty,|y|<\infty,|z|<1, \tag{3.7}$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_p\,(b)_{m+n-p}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},|x|<\infty,|y|<\infty,|z|<\infty, \tag{3.8}$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(b)_{m+n-p}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},|x|<\infty,|y|<\infty,|z|<\infty, \tag{3.9}$$

can be obtained from (3.1) by limit (confluence) formulas

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(a_1,a_2,a_3,\frac{1}{\varepsilon},b;c_1,c_2;x,y,\varepsilon z\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(a_1,\frac{1}{\varepsilon},a_3,a_4,b;c_1,c_2;x,\varepsilon y,z\right),$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(a_1,a_2,\frac{1}{\varepsilon},\frac{1}{\varepsilon},b;c_1,c_2;x,y,\varepsilon^2 z\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(a_1,\frac{1}{\varepsilon},a_3,\frac{1}{\varepsilon},b;c_1,c_2;x,\varepsilon y,\varepsilon z\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(a,\frac{1}{\varepsilon},\frac{1}{\varepsilon},\frac{1}{\varepsilon},b;c_1,c_2;x,\varepsilon y,\varepsilon^2 z\right),$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(\frac{1}{\varepsilon},\frac{1}{\varepsilon},a_3,a_4,b;c_1,c_2;\varepsilon x,\varepsilon y,z\right),$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(\frac{1}{\varepsilon},\frac{1}{\varepsilon},\frac{1}{\varepsilon},a,b;c_1,c_2;\varepsilon x,\varepsilon y,\varepsilon z\right),$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\lim_{\varepsilon\to0}F_{4b}\left(\frac{1}{\varepsilon},\frac{1}{\varepsilon},\frac{1}{\varepsilon},\frac{1}{\varepsilon},b;c_1,c_2;\varepsilon x,\varepsilon y,\varepsilon^2 z\right).$$

Note, in [13] the functions E_{153} and E_{155} are first defined and designated as A_1 and A_2, respectively. Analogues of the function E_{155} with four and more variables are found in [2, 8].

Using the series manipulation technique, we can obtain the following equivalent forms of (3.1) – (3.9):

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{(a_2)_n\,(a_3)_p\,(b)_{n-p}}{(c_2)_n}\frac{y^n}{n!}\frac{z^p}{p!}\;{}_2F_1\left(a_1,b+n-p;c_1;x\right), \tag{3.10}$$

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{(a_1)_m\,(a_2)_n\,(b)_{m+n}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\;{}_1F_1\left(a_3;1-b-m-n;-z\right),$$

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{(-1)^p\,(a_3)_p}{(1-b)_p}\frac{z^p}{p!}\;F_2\left(b-p,a_1,a_2;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{(a_2)_n\,(b)_n}{(c_2)_n}\frac{y^n}{n!}\;\mathrm{H}_2\left(b+n,a_1,a_3;c_1;x,z\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{(a_3)_p\,(a_4)_p\,(b)_{n-p}}{(c_2)_n}\frac{y^n}{n!}\frac{z^p}{p!}\;{}_2F_1\left(a_1,b+n-p;c_1;x\right), \tag{3.11}$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,p=0}^{\infty}\frac{(a_1)_m\,(a_3)_p\,(a_4)_p\,(b)_{m-p}}{(c_1)_m}\frac{x^m}{m!}\frac{z^p}{p!}\;{}_1F_1\left(b+m-p;c_2;y\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{(a_1)_m\,(b)_{m+n}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\;{}_2F_1\left(a_3,a_4;1-b-m-n;-z\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(-1)^p\,(a_3)_p\,(a_4)_p}{(1-b)_p}\frac{z^p}{p!}\;\Psi_1\left(b-p,a_1;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{(b)_n}{(c_2)_n}\frac{y^n}{n!}\;H_2\left(b+n,a_1,a_3,a_4;c_1;x,z\right),$$

$$\mathrm{E}_{154}\left(a_1,a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m=0}^{\infty}\frac{(a_1)_m\,(b)_m}{(c_1)_m}\frac{x^m}{m!}\;\mathrm{H}_{11}\left(b+m,a_3,a_4;c_2;y,z\right),$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{(b)_{n-p}\,(a_2)_n}{(c_2)_n}\frac{y^n}{n!}\frac{z^p}{p!}\;{}_2F_1\left(a_1+n-p,b;c_1;x\right), \tag{3.12}$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{(b)_{m+n}\,(a_1)_m\,(a_2)_n}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\;\bar{J}_{-b-m-n}\left(2\sqrt{z}\right),$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{(-1)^p}{(1-b)_p}\frac{z^p}{p!}\;F_2\left(b-p,a_1,a_2;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{155}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{(b)_n\,(a_2)_n}{(c_2)_n}\frac{y^n}{n!}\;\mathrm{H}_3\left(b+n,a_1;c_1;x,z\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{\left(a_3\right)_p\left(b\right)_{n-p}}{\left(c_2\right)_n}\frac{y^n}{n!}\frac{z^p}{p!}\ {}_2F_1\left(a_1,b+n-p;c_1;x\right), \tag{3.13}$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m,p=0}^{\infty}\frac{\left(a_1\right)_m\left(a_3\right)_p\left(b\right)_{m-p}}{\left(c_1\right)_m}\frac{x^m}{m!}\frac{z^p}{p!}\ {}_1F_1\left(b+m-p;c_2;y\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{\left(a_1\right)_m\left(b\right)_{m+n}}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}\ {}_1F_1\left(a_3;1-b-m-n;-z\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{\left(-1\right)^p\left(a_3\right)_p\left(b\right)_{-p}}{\left(1-b\right)_p}\frac{z^p}{p!}\ \Psi_1\left(b+n-p,a_1;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{\left(b\right)_n}{\left(c_2\right)_n}\frac{y^n}{n!}\ \mathrm{H}_2\left(b+n,a_1,a_3;c_1;x,z\right),$$

$$\mathrm{E}_{156}\left(a_1,a_3,b;c_1,c_2;x,y,z\right)=\sum_{m=0}^{\infty}\frac{\left(a_1\right)_m\left(b\right)_m}{\left(c_1\right)_m}\frac{x^m}{m!}\ \mathrm{H}_4\left(b+m,a_3;c_2;y,z\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{\left(b\right)_{n-p}}{\left(c_2\right)_n}\frac{y^n}{n!}\frac{z^p}{p!}\ {}_2F_1\left(a,b+n-p;c_1;x\right), \tag{3.14}$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,p=0}^{\infty}\frac{\left(a\right)_m\left(b\right)_{m-p}}{\left(c_1\right)_m}\frac{x^m}{m!}\frac{z^p}{p!}\ {}_1F_1\left(b+m-p;c_1;y\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{\left(a\right)_m\left(b\right)_{m+n}}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}\ \bar{J}_{-b-m-n}\left(2\sqrt{z}\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{\left(-1\right)^p}{\left(c_2\right)_n\left(1-b\right)_p}\frac{z^p}{p!}\ \Psi_1\left(b-p,a;c_1;x,y\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,p=0}^{\infty}\frac{\left(b\right)_n}{\left(c_2\right)_n}\frac{y^n}{n!}\ \mathrm{H}_3\left(b+n,a;c_1;x,z\right),$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m=0}^{\infty}\frac{\left(a\right)_m\left(b\right)_m\left(b+m\right)_{n-p}}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\ \mathrm{H}_5\left(b+m;c_2;y,z\right),$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{\left(a_3\right)_p\left(a_4\right)_p\left(b\right)_{n-p}}{\left(c_2\right)_n}\frac{y^n}{n!}\frac{z^p}{p!}\ {}_1F_1\left(b+n-p;c_1;x\right),$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{\left(b\right)_{m+n}}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}\ {}_2F_1\left(a_3,a_4;1-b-m-n;-z\right),$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{\left(-1\right)^p\left(a_3\right)_p\left(a_4\right)_p}{\left(1-b\right)_p}\frac{z^p}{p!}\ \Psi_2\left(b-p;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{158}\left(a_3,a_4,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{\left(b\right)_n}{\left(c_2\right)_n}\frac{y^n}{n!}\ \mathrm{H}_{11}\left(b+n,a_3,a_4;c_1;x,z\right),$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{\left(a\right)_p\left(b\right)_{n-p}}{\left(c_2\right)_n}\frac{y^n}{n!}\frac{z^p}{p!}\ {}_1F_1\left(b+n-p;c_1;x\right),$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{\left(b\right)_{m+n}}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}\ {}_1F_1\left(a;1-b-m-n;-z\right),$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{(-1)^p\,(a)_p}{(1-b)_p}\frac{z^p}{p!}\;\;\Psi_2\left(b-p;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{159}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{(b)_n}{(c_2)_n}\frac{y^n}{n!}\;\;\mathrm{H}_4\left(b+n,a;c_1;x,z\right),$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\sum_{n,p=0}^{\infty}\frac{(b)_{n-p}}{(c_2)_n}\frac{y^n}{n!}\frac{z^p}{p!}\;\;{}_1F_1\left(b+n-p;c_1;x\right),$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\sum_{m,n=0}^{\infty}\frac{(b)_{m+n}}{(c_1)_m\,(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\;\;\bar{J}_{-b-m-n}\left(2\sqrt{z}\right),$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\sum_{p=0}^{\infty}\frac{(-1)^p}{(1-b)_p}\frac{z^p}{p!}\;\;\Psi_2\left(b-p;c_1,c_2;x,y\right),$$

$$\mathrm{E}_{160}\left(b;c_1,c_2;x,y,z\right)=\sum_{n=0}^{\infty}\frac{(b)_n}{(c_2)_n}\frac{y^n}{n!}\;\;\mathrm{H}_5\left(b+n;c_1;x,z\right).$$

4. Integral representations

Theorem 4.1. *If $Re(\beta)>Re(\alpha)>0$, then the following integral representations hold true:*

$$\begin{aligned}&\mathrm{E}_{153}\left(\alpha,a_2,a_3,b;\beta,c;x,y,z\right)\\&\quad=\frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\left(1-x\xi\right)^{-b}\mathrm{H}_2\left(b,a_2,a_3;c;\frac{y}{1-x\xi},z\left(1-x\xi\right)\right)d\xi,\end{aligned}\tag{4.1}$$

$$\begin{aligned}&\mathrm{E}_{154}\left(\alpha,a_2,a_3,b;\beta,c;x,y,z\right)\\&\quad=\frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\left(1-x\xi\right)^{-b}\mathrm{H}_{11}\left(b,a_2,a_3;c;\frac{y}{1-x\xi},z\left(1-x\xi\right)\right)d\xi,\end{aligned}\tag{4.2}$$

$$\begin{aligned}&\mathrm{E}_{155}\left(\alpha,a,b;\beta,c;x,y,z\right)\\&\quad=\frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\left(1-x\xi\right)^{-b}\mathrm{H}_3\left(b,a;c;\frac{y}{1-x\xi},z\left(1-x\xi\right)\right)d\xi,\end{aligned}\tag{4.3}$$

$$\begin{aligned}&\mathrm{E}_{156}\left(\alpha,a,b;\beta,c;x,y,z\right)\\&\quad=\frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\left(1-x\xi\right)^{-b}\mathrm{H}_4\left(b,a;c;\frac{y}{1-x\xi},z\left(1-x\xi\right)\right)d\xi,\end{aligned}\tag{4.4}$$

$$\begin{aligned}&\mathrm{E}_{157}\left(\alpha,b;\beta,c;x,y,z\right)\\&\quad=\frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\left(1-x\xi\right)^{-b}\mathrm{H}_5\left(b;c;\frac{y}{1-x\xi},z\left(1-x\xi\right)\right)d\xi,\end{aligned}\tag{4.5}$$

$$\mathrm{E}_{158}\left(\alpha,a,b;c_1,c_2;x,y,z\right)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{158}\left(\beta,a,b;c_1,c_2;x,y,z\xi\right)d\xi,\tag{4.6}$$

$$\mathrm{E}_{159}\left(\alpha,b;c_1,c_2;x,y,z\right)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{159}\left(\beta,b;c_1,c_2;x,y,z\xi\right)d\xi,\tag{4.7}$$

$$\mathrm{E}_{160}\left(b;\beta,c;x,y,z\right)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-\alpha-1}\mathrm{E}_{160}\left(b;\alpha,c;x,y,z\xi\right)d\xi.\tag{4.8}$$

Proof. To prove the relation (4.1) conformed in Theorem 4.1, let I denotes its right-hand side. Then, from the definition of Horn's function H_2 in (2.1), we get

$$I=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\sum_{n,p=0}^{\infty}\frac{(b)_{n-p}(a_2)_n(a_3)_p}{(c)_n}\frac{y^n}{n!}\frac{z^p}{p!}\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-\alpha-1}(1-x\xi)^{-b-n+p}d\xi.$$

By applying the following integral representation of the Gaussian function [7, p. 59, Eq. (10)]

$${}_2F_1(a,b;c;x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1\xi^{a-1}(1-\xi)^{c-a-1}(1-x\xi)^{-b}d\xi,\ Re(c)>Re(a)>0,$$

in (4.9), we obtain

$$I=\sum_{n,p=0}^{\infty}\frac{(a_2)_n(a_3)_p(b)_{n-p}}{(c)_n}\frac{y^n}{n!}\frac{z^p}{p!}F(\alpha,b+n-p;\beta;x).\tag{4.9}$$

Now, by virtue of the expansion (3.10), we get the required result. Using the same manner, we find the identities (4.2) to (4.5).

Let J denotes right-hand side of (4.6). By applying the following integral representation of the beta function [7, p. 9, Eq. (1)]

$$\mathrm{B}(\alpha,\beta)=\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}d\xi,\ Re(\alpha)>0,\ Re(\beta)>0,$$

in

$$J=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\sum_{m,n,p=0}^{\infty}\frac{(\beta)_p(a)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\int_0^1\xi^{\alpha-1+p}(1-\xi)^{\beta-\alpha-1}d\xi,$$

we obtain

$$J=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\sum_{m,n,p=0}^{\infty}\frac{(\beta)_p(a)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\mathrm{B}(\alpha+p,\beta-\alpha)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.\tag{4.10}$$

Now, using the well-known expression for the beta function in terms of the gamma function [7, p. 9, Eq. (5)]

$$\mathrm{B}(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

in (4.10), we get the required integral representation (4.6). Using the same manner, we find the identities (4.7) and (4.8). □

Using a similar demonstration as the previous proof, we can give the following theorem.

Theorem 4.2. *If $Re(\alpha)>0$ and $Re(\beta)>0$, then the following integral representations hold true:*

$$\mathrm{E}_{153}(\alpha,a,\beta,b;c_1,c_2;x,y,z)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{176}(\alpha+\beta,a,b;c_2,c_1;y,x\xi,z(1-\xi))d\xi,$$

$$\mathrm{E}_{153}(\alpha,\beta,a,b;c_1,c_2;x,y,z)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{178}(\alpha+\beta,a,b;c_1,c_2;x\xi,y(1-\xi),z)d\xi,$$

$$\mathrm{E}_{154}(\alpha,\beta,a,b;c_1,c_2;x,y,z)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{177}(\alpha+\beta,a,b;c_2,c_1;y,x\xi,z(1-\xi))d\xi,$$

$$\mathrm{E}_{155}(\alpha,\beta,b;c_1,c_2;x,y,z)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\xi^{\alpha-1}(1-\xi)^{\beta-1}\mathrm{E}_{173}(\alpha+\beta,b;c_1,c_2;x\xi,y(1-\xi),z)d\xi,$$

$$\mathrm{E}_{156}\left(\alpha,\beta,b;c_1,c_2;x,y,z\right)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\xi^{\alpha-1}\left(1-\xi\right)^{\beta-1}\mathrm{E}_{174}\left(\alpha+\beta,b;c_2,c_1;y,x\xi,z\left(1-\xi\right)\right)d\xi,$$

where E_{173}, E_{174}, E_{176}, E_{177} *and* E_{178} *are the confluent functions defined in* [9]*:*

$$\mathrm{E}_{173}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \sqrt{|x|}+\sqrt{|y|}<1,\ |z|<\infty,$$

$$\mathrm{E}_{174}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{n+p}(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ |x|<\infty,\ \ |y|<1,\ |z|<\infty,$$

$$\mathrm{E}_{176}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_m(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ |x|+|y|<1,\ |z|<\infty,$$

$$\mathrm{E}_{177}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{n+p}(a_2)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ |x|<\infty,|z|+2\sqrt{|yz|}<1,$$

$$\mathrm{E}_{178}\left(a_1,a_2,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \sqrt{|x|}+\sqrt{|y|}<1,\ \ |z|<\infty.$$

Theorem 4.3. *The following double integral representations hold true:*

$$\mathrm{E}_{153}\left(\alpha,\gamma,a,b;\beta,\delta;x,y,z\right)=\frac{\Gamma\left(\beta\right)\Gamma\left(\delta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\gamma\right)\Gamma\left(\beta-\alpha\right)\Gamma\left(\delta-\gamma\right)}\int_0^1\int_0^1\xi^{\alpha-1}\eta^{\gamma-1}\left(1-\xi\right)^{\beta-\alpha-1}\times$$

$$\times\left(1-\eta\right)^{\delta-\gamma-1}\left(1-x\xi-y\eta\right)^{-b}{}_1F_1\left(a;1-b;-z\left(1-x\xi-y\eta\right)\right)d\xi d\eta,\tag{4.11}$$

$$Re(\beta)>Re(\alpha)>0,\ \ Re(\delta)>Re(\gamma)>0,\ \ b\neq0,\pm1,\pm2,\pm3,...;$$

$$\mathrm{E}_{153}\left(\alpha,\beta,\gamma,b;c_1,c_2;x,y,z\right)=\frac{\Gamma\left(\alpha+\beta+\gamma\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)\Gamma\left(\gamma\right)}\int_0^1\int_0^1\xi^{\beta+\gamma-1}\eta^{\beta-1}\left(1-\xi\right)^{\alpha-1}\times$$

$$\times\left(1-\eta\right)^{\gamma-1}\mathrm{E}_{184}\left(\alpha+\beta+\gamma,b;c_1,c_2;x\left(1-\xi\right),y\xi\eta,z\xi\left(1-\eta\right)\right)d\xi,\tag{4.12}$$

$$Re(\alpha)>0,\ \ Re(\beta)>0,\ \ Re(\gamma)>0,$$

where E_{184} *is the confluent hypergeometric function defined in* [9]*:*

$$\mathrm{E}_{184}\left(a,b;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b)_{m+n-p}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \sqrt{|x|}+\sqrt{|y|}<1,\ |z|<\infty.$$

Proof. To prove the relation (4.11) it is necessary to use the well-known integral representation for the Appell function F_2 (see [7, p.230, Eq.(2)]):

$$F_2\left(a,b,b';c,c';x,y\right)=\frac{\Gamma\left(c\right)\Gamma\left(c'\right)}{\Gamma\left(b\right)\Gamma\left(b'\right)\Gamma\left(c-b\right)\Gamma\left(c'-b'\right)}\times$$

$$\times\int_0^1\int_0^1\xi^{b-1}\eta^{b'-1}\left(1-\xi\right)^{c-b-1}\left(1-\eta\right)^{c'-b'-1}\left(1-x\xi-y\eta\right)^{-a}d\xi d\eta,$$

$$Re(c)>Re(b)>0,\ \ Re(c')>Re(b')>0.$$

□

5. Reduction formulas

Under exceptional circumstances, hypergeometric functions of three variables can be expressed in terms of simpler functions, notably in terms of hypergeometric functions of one or two variables or in terms of elementary functions. In such cases we speak of reducible hypergeometric functions and of reduction formulas. The exceptional circumstances arise either if the parameters in a hypergeometric series satisfy one or several relations, or if the two variables are connected by a relation. In the latter case the relation is usually the equation of a singular curve of the system of partial differential equations associated with the series in question.

Certain trivial reduction formulas are obvious: if $a_1 = 0$ in (3.2) – (3.8), if $z = 0$ in any of the series, the hypergeometric series of three variables can be expressed in terms of series of two variables: such trivial reductions are disregarded in the sequel.

The following reduction formulas can be proved either by expanding in infinite series and comparing coefficients, or by manipulating integral representations:

$$\mathrm{E}_{153}\left(c_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{H}_2\left(b, a_2, a_3; c_2; \frac{y}{1-x}, (1-x)\,z\right),$$

$$\mathrm{E}_{154}\left(c_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{H}_{11}\left(b, a_2, a_3; c_2; \frac{y}{1-x}, (1-x)\,z\right),$$

$$\mathrm{E}_{155}\left(c_1, a_2, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{H}_3\left(b, a_2; c_2; \frac{y}{1-x}, (1-x)\,z\right),$$

$$\mathrm{E}_{156}\left(c_1, a_2, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{H}_4\left(b, a_2; c_2; \frac{y}{1-x}, (1-x)\,z\right),$$

$$\mathrm{E}_{157}\left(c_1, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{H}_5\left(b, c_2; \frac{y}{1-x}, (1-x)\,z\right).$$

6. Transformations

Although there is essentially only one hypergeometric series of the second order in one variable (namely Gauss' series), its transformation theory is quite extensive (see, Sections 2.9 to 2.11 in [7]). With the considerable number of hypergeometric series of the second order in two and three variables, the complete set of transformations would run into the hundreds, and only a few examples can be given here. The best means for deriving these (and other) transformations is the integral representations of the functions concerned where a change of variables of integration, or a deformation of the contour of integration will often yield the desired results.

First we have transformations of a series into a series of the same type:

$$\mathrm{E}_{153}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{E}_{153}\left(c_1-a_1, a_2, a_3, b; c_1, c_2; \frac{x}{x-1}, \frac{y}{1-x}, (1-x)\,z\right), \tag{6.1}$$

$$\mathrm{E}_{153}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = (1-x-y)^{-b}\times$$
$$\times\,\mathrm{E}_{153}\left(c_1-a_1, c_2-a_2, a_3, b; c_1, c_2; \frac{x}{x+y-1}, \frac{y}{x+y-1}, \frac{(1-x-y)^2}{1-x}z\right), \tag{6.2}$$

$$\mathrm{E}_{154}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{E}_{154}\left(c_1-a_1, a_2, a_3, b; c_1, c_2; \frac{x}{x-1}, \frac{y}{1-x}, (1-x)\,z\right), \tag{6.3}$$

$$\mathrm{E}_{155}\left(a_1, a_2, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{E}_{155}\left(c_1-a_1, a_2, b; c_1, c_2; \frac{x}{x-1}, \frac{y}{1-x}, (1-x)\,z\right), \tag{6.4}$$

$$\mathrm{E}_{156}\left(a_1, a_2, b; c_1, c_2; x, y, z\right) = (1-x)^{-b}\,\mathrm{E}_{156}\left(c_1-a_1, a_2, b; c_1, c_2; \frac{x}{x-1}, \frac{y}{1-x}, (1-x)\,z\right), \tag{6.5}$$

$$\mathrm{E}_{157}\left(a,b;c_1,c_2;x,y,z\right)=(1-x)^{-b}\,\mathrm{E}_{157}\left(c_1-a,b;c_1,c_2;\frac{x}{x-1},\frac{y}{1-x},(1-x)\,z\right). \tag{6.6}$$

All these correspond to Euler's transformation of the ordinary hypergeometric series:

$${}_2F_1(a,b;c;x)=(1-x)^{-a}{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right)=(1-x)^{-b}{}_2F_1\left(c-a,b;c;\frac{x}{x-1}\right). \tag{6.7}$$

The transformations (6.1) – (6.6) can be proved by applying the Boltz formula (6.7) to the expansions (3.10) – (3.14). To give an example, by the expansion (3.10) for E_{153}, we have

$$\begin{aligned}&\mathrm{E}_{153}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)\\&=\sum_{n,p=0}^{\infty}\frac{(a_2)_n\,(a_3)_p\,(b)_{n-p}}{(c_2)_n}\frac{y^n}{n!}\frac{z^p}{p!}(1-x)^{-b-n+p}{}_2F_1\left(c_1-a_1,b+n-p;c_1;\frac{x}{x-1}\right)\\&=(1-x)^{-b}\,\mathrm{E}_{153}\left(c_1-a_1,a_2,a_3,b;c_1,c_2;\frac{x}{x-1},\frac{y}{1-x},(1-x)\,z\right).\end{aligned}$$

No simple transformations of this type seem to be known for any of the confluent hypergeometric functions E_{158}, E_{159}, E_{160}, defined in (3.7) – (3.9).

7. Applications

Confluent hypergeometric functions of three variables have important applications.

1) Fundamental solutions of the generalized bi-axially symmetric Helmholtz equation

$$\sum_{k=1}^{n}\frac{\partial^2 u}{\partial x_k^2}+\frac{2\alpha}{x_1}\frac{\partial u}{\partial x_1}+\frac{2\beta}{x_2}\frac{\partial u}{\partial x_2}-\lambda^2 u=0,\ 0<2\alpha,2\beta<1$$

in the domain $\{(x_1,...,x_n):x_1>0,x_2>0\}$ are expressed by a confluent function E_{155}, for instance, one of which has an explicit form (for details, see [10]):

$$q_1=k_1 r^{-2\alpha-2\beta}\mathrm{E}_{155}\left(\alpha,\beta,\alpha+\beta;2\alpha,2\beta;-\frac{4x_1\xi_1}{r^2},-\frac{4x_2\xi_2}{r^2},-\frac{\lambda^2}{4}r^2\right),\ r^2=\sum_{k=1}^{n}\left(x_k-\xi_k\right)^2.$$

Moreover, to construct fundamental solutions of the equation

$$\sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial x_k^2}+\frac{2\alpha_k}{x_k}\frac{\partial u}{\partial x_k}\right)-\lambda^2 u=0,\ 0<2\alpha_k<1$$

a multivariable analogue of the function E_{155} defined as

$$\mathrm{H}_A^{(n,1)}\left[\begin{array}{l}a,b_1,...,b_n;\\ c_1,...,c_n;\end{array}x_1,...,x_n;y\right]=\sum_{k_1,...,k_n,l=0}^{\infty}(a)_{k_1+...+k_n-l}\prod_{j=1}^{n}\frac{(b_j)_{k_j}}{(c_j)_{k_j}}\frac{x_j^{k_j}}{k_j!}\cdot\frac{y^l}{l!},\ \sum_{j=1}^{n}|x_j|<1,$$

is used [23].

2) Consider the three-dimensional singular Helmholtz equation

$$u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z-\lambda^2 u=0,\ 0<2\alpha,2\beta,2\gamma<1 \tag{7.1}$$

in the first octant $\Omega:=\{(x,y,z):x>0,y>0,z>0\}$.

Dirichlet problem. Find a regular solution $u\left(x,y,z\right)\in C\left(\overline{\Omega}\right)\cap C^2(\Omega)$ to the singular Helmholtz equation (7.1) , satisfying the following conditions

$$u(x,y,0)=\tau_1(x,y),\quad 0\le x,y<\infty,\ u(x,0,z)=\tau_2(x,z),\quad 0\le x,z<\infty, \tag{7.2}$$

$$u(0,y,z)=\tau_3(y,z),\quad 0\le y,z<\infty,\ \lim_{R\to\infty}u(x,y,z)=0,\ \ R=\sqrt{x^2+y^2+z^2}, \tag{7.3}$$

where $\tau_{1,2,3}\left(t,s\right)$ are given functions.

Theorem 7.1. *The following function*

$$u(x,y,z) = (1-2\gamma)\,k_3 x^{1-2\alpha} y^{1-2\beta} z^{1-2\gamma} \int_0^\infty \int_0^\infty \frac{\tau_1(t,s)\,ts}{r_1^{2a}} \mathrm{E}_{155}\left(1-\alpha, 1-\beta, a; 2-2\alpha,\ 2-2\beta; X\right) dtds$$

$$+ (1-2\beta)\,k_3 x^{1-2\alpha} y^{1-2\beta} z^{1-2\gamma} \int_0^\infty \int_0^\infty \frac{\tau_2(t,s)\,ts}{r_2^{2a}} \mathrm{E}_{155}\left(1-\alpha, 1-\gamma, a; 2-2\alpha,\ 2-\gamma; Y\right) dtds$$

$$+ (1-2\alpha)\,k_3 x^{1-2\alpha} y^{1-2\beta} z^{1-2\gamma} \int_0^\infty \int_0^\infty \frac{\tau_3(t,s)\,ts}{r_3^{2a}} \mathrm{E}_{155}(1-\gamma, 1-\beta, a; 2-2\gamma,\ 2-2\beta; Z)\, dtds,$$

where $a = 7/2 - \alpha - \beta - \gamma$,

$$r_1^2 = (x-t)^2 + (y-s)^2 + z^2,\ r_2^2 = (x-t)^2 + y^2 + (z-s)^2\ r_3^2 = x^2 + (y-t)^2 + (z-s)^2;$$

$$X = \left(-\frac{4xt}{r_1^2}, -\frac{4ys}{r_1^2}, -\frac{1}{4}\lambda^2 r_1^2\right),\ Y = \left(-\frac{4xt}{r_2^2}, -\frac{4zs}{r_2^2}, -\frac{1}{4}\lambda^2 r_2^2\right),\ Z = \left(-\frac{4yt}{r_3^2}, -\frac{4zs}{r_3^2}, -\frac{1}{4}\lambda^2 r_3^2\right),$$

is a regular solution of equation (7.1) *in* Ω, *satisfying the conditions* (7.2) *and* (7.3).

Proof. The validity of the statements of theorem 7.1 is verified by direct calculation. □

References

[1] Appell P.; Sur les séries hypergéométriques de deux variables, et sur des équations différentielles linéaires aux dérivées partielles. C.R. Acad. Sci., Paris. – 1880. –90.–P. 296 - 298.

[2] Arzikulov Z.O., Ergashev T.G.; Some systems of PDE associated with the multiple confluent hypergeometric functions and their applications. Lobachevskii Journal of Mathematics.–2024.–45(2).–P.591 - 603.

[3] Bers L.; Mathematical Aspects of Subsonic and Transonic Gas Dynamics. New York: Wiley.–1958.

[4] Bezrodnykh S.I.; Analytic continuation of the Horn hypergeometric series with an arbitrary number of variables. Integral Transforms and Special Functions.–2020.–31(10).–P.788 - 803.

[5] Bin-Saad M.G., Shahwan J.S., Younis J.A., Aidi H., Abd El Salam M.A.; On Gaussian hypergeometric functions of three variables: some new integral representations. Hindawi Journal of Mathematics. 2022. –Article ID 1914498, –P. 1 - 16.

[6] Erdélyi A.; Integraldarstellungen für Produkte Whittakerscher Funktionen. Nieuw Archief voor Wiskunde. – 1939. 2(20).–P.1 - 34.

[7] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G.; Higher Transcendental Functions, 1. New York, Toronto and London: McGraw-Hill.– 1953.

[8] Ergashev T.G.; On fundamental solutions for multidimensional Helmholtz equation with three singular coefficients. Computers and Mathematics with Applications. –2019.–77(1).–P.69 - 76.

[9] Ergashev T.G.; Confluent hypergeometric functions in three variables and associated systems of partial differential equations. Bulletin of the Institute of Mathematics.–2024.–7(4).–P.121 - 215.

[10] Ergashev T.G., Hasanov A.; Fundamental solutions of the bi-axially symmetric Helmholtz equation. Uzbek Mathematical Journal. –2018.–1.–P. 55 - 64.

[11] Ergashev T.G., Hasanov A., Yuldashev T.K.; Multiple Euler type integral representations for the Kampé de Fériet functions. Chelyabinsk Physical and Mathematical journal. –2023.– 8(4).–P.553-567.

[12] Ergashev T.G., Hasanov A., Yuldashev T.K.; Euler type integral representations for the Kampé de Fériet functions. Journal of Mathematical Sciences.–2024.– 279(1).–P.22-36.

[13] Hasanov A.; Fundamental solutions of generalized bi-axially symmetric Helmholtz equation. Complex Variables and Elliptic Equations. –2007.–52(8).–P.673 - 683.

[14] Hasanov A., Ruzhansky M.; Euler-type integral representations for the hypergeometric functions in three variables of second order. Bulletin of the Institute of Mathematics.–2019.–6(2). –P.73 - 223.

[15] Hasanov A., Ruzhansky M.; Systems of differential equations of Gaussian hypergeometric functions in three variables and their linearly-independent solutions. Bulletin of the Institute of Mathematics.–2022.–5(3).–P.50 - 142.

[16] Hasanov A., Yuldashev T.K.; Analytic continuation formulas for the hypergeometric functions in three variables of second order. Lobachevskii Journal of Mathematics.–2022. –43(2).–P.386 - 393.

[17] Horn J.; Hypergeometrische Funktionen zweier Veränderlichen. Math. Ann.–1931.–105.–P.381 - 407.

[18] Humbert P.; The confluent hypergeometric functions of two variables. Proc. Roy. Soc. Edinburgh.–1920.–41(21).–P.73 - 96.

[19] Lohofer G.; Theory of an electromagnetically deviated metal sphere. 1: Abcorbed power. SIAM J. Appl. Math.–1989.–49.–P.567 - 581.

[20] Niukkanen A.W.; Generalised hypergeometric series ${}^{N}F\left(x_1, ..., x_N\right)$ arising in physical and quantum chemical applications. J. Phys. A: Math. Gen. –1983.–16.–P.1813 - 1825.

[21] Ruzhansky M., Hasanov A., Ergashev T.G.; PDE-Systems associated with the hypergeometric functions in three variables and their particular solutions near the origin. *ArXiv:2410.00748v1*, 245 pages. http://arxiv.org/abs/2410.00748v1

[22] Srivastava H.M., Karlsson P.W.; Multiple Gaussian Hypergeometric Series. New York, Chichester, Brisbane and Toronto: Halsted Press (Ellis Horwood Limited, Chichester), Wiley,–1985.

[23] Urinov A.K., Ergashev T.G.; Confluent hypergeometric functions of many variables and their application to the finding of fundamental solutions of the generalized Helmholtz equation with singular coefficients. Vestnik Tomskogo Gosudarstvennogo Universiteta, Matematika i Mekhanika, –2018. – 55. P.45-56.

[24] Van der Jeugt J., Pitre S.N., Srinivasa Rao K.; Tramsformation and summation formulas for double hypergeometric series. Journal of Computational and Applied Mathematics. –1997. 83.–P.185-193.

Ergashev Tuhtasin
Tashkent Institute of Irrigation and Agricultural
Mechanization Engineers
National Research University,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
University of Ghent, Belgium
email: ergashev.tukhtasin@gmail.com

Hasanov Anvardjan
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
University of Ghent, Belgium
email: anvarhasanov@yahoo.com

Safarbayeva Nigora Mustafayevna
Tashkent Institute of Irrigation and Agricultural
Mechanization Engineers
National Research University,
email: kh23031995@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.108-112
DOI: 10.29229/uzmj.2025-4-11

On the Non-Smooth Optimal Control Problem for a Parametrized Dynamic System under Conditions of Uncertainty

Haydarov T. T.

Abstract. In the paper, we consider a class dynamic control system with a discrete parameter and under conditions of uncertainty in the initial data. The optimal control problem of the minimax type is formulated for a non-smooth terminal functional using the principle of the best-guaranteed result. This problem is studied by methods of multivalued and convex analysis. For this non-smooth control problem the necessary and sufficient conditions for optimality are obtained.

Keywords: control system, discrete parameter, terminal functional, minimax, non-smooth control problem, conditions for optimality.

MSC (2020): 34A60,34H05,49J21.

1. Introduction

The issues of decision making in the economic planning, in the design of technical devices and control processes lead to various optimization problems. Non-smooth optimization problems constitute a special class of mathematical models [1]-[3]. They are often represented with non-smooth objective functionals. As a result of studies of non-smooth optimization models, special methods for solving them have been developed, and sections of non-smooth and multi-valued analysis are being developed [1]-[6].

One of the approaches used in decision-making in conditions of incomplete information about the initial data of the system and external influences is the principle of optimization by the minimax criterion, which assumes obtaining a guaranteed value of the quality criterion [7],[8]. This usually leads to non-smooth optimal control problems in form minimax or maximin [9]-[14]. They are closely related to the problems of controlling an ensemble of trajectories dynamic systems[15]-[17].

In this paper, we consider a dynamic control system with a discrete parameter under conditions of incomplete information about the initial state. The goal of management is to achieve a guaranteed result under such conditions of inaccuracy of information. A terminal functional of the minimum function type is considered as a criterion for evaluating the quality of management. Necessary and sufficient optimality conditions are obtained. They develop research of work [12]-[13].

2. Statement of the problem

Let $\mathbb{R}^n$ be an n- dimensional Euclidean space; (x, y) is the scalar product of vectors $x, y \in \mathbb{R}^n$,$||x||$ is the norm of the vector $x \in \mathbb{R}^n$; $\sigma(X, \psi) = \sup\{(x, \psi) : x \in X\}$ is the support function of a limited set $X \subset \mathbb{R}^n$.

Consider a dynamic control system of the form

$$\dot{x} = A(t, y)x + B(t, u, y), t \in T, u \in U, \tag{2.1}$$

where x is state n-vector, u is control m -vector, y is k - dimensional parameter of external influences, $A(t, y)$ is $n \times n$ - matrix, $b(t, u, y) \in \mathbb{R}^n$, $T = [t_0, t_1]$ is given time interval.The initial state of the system is inaccurate, that is, only set possible values are known: $x(t_0) \in D$, where D is convex compact subset of space $\mathbb{R}^n$; U is compact set of space $\mathbb{R}^m$; the parameter y accepts discrete values, i.e. $y \in Y = \{y_1, y_2, ..., y_v\} \subset \mathbb{R}^k$.

We assume that the following conditions are met:

1) the elements of the matrix $A(t, y)$ are summable on variable $t \in T = [t_0, t_1]$ for each $y \in Y$;

2) the mapping $(t, u, y) \to b(t, u, y)$ is measurable on $t \in T$ and continuous on $u \in U$ for every $y \in Y$, moreover $||b(t, u, y)|| \leq \beta(t), \beta(\cdot) \in L_1(T)$.

The admissible controls for system (2.1) are measurable and bounded m -vector function $u = u(t)$, $t \in T$, which accepts values from U almost everywhere on $T = [t_0, t_1]$.

Let U_T be the set of all admissible controls $u(\cdot)$, and $H_T(u(\cdot), y)$ be the set of all absolutely continuous solutions $x = x(t, u(\cdot), x_0, y)$ of equation (2.1) with the initial condition $x(t_0) = x_0 \in D$ for given admissible control $u(\cdot) \in U_T(y)$ and discrete parameter $y \in Y$. Under given conditions, $H_T(u(\cdot), y)$ is compact set in the space of continuous n -vector functions $C^n(T)$ [17].

Let the quality of control of a dynamical system be evaluated by a non-smooth terminal functional

$$J(u(\cdot), x_0) = \min_{l\in L} \sum_{y\in Y} \big(P(y)x(t_1, u(\cdot), x_0, y), l\big), \tag{2.2}$$

where $P(y)$ is an $s \times n$ -matrix, L is a bounded and closed set of $\mathbb{R}^s$. Typically, with a given initial state of the control system, optimal control is determined from the conditions for minimizing terminal functionality. Since the initial state of the control system (2.1) is inaccurate, then the goal of management can be considered to achieve a guaranteed value of the quality criterion $J(u(\cdot), x_0)$ in form (2.2). The best guaranteed values of the functional (2.2) we will assume the minimum value of the following functional:

$$J(u(\cdot)) = \max_{x(\cdot)\in H_T(u(\cdot),y), y\in Y} \min_{l\in L} \sum_{y\in Y} \big(P(y)x(t_1), l\big). \tag{2.3}$$

So, for the system (2.1), we will consider the non-smooth optimal control problem of the minimax type:

$$\max_{x(\cdot)\in H_T(u(\cdot),y), y\in Y} \min_{l\in L} \sum_{y\in Y} \big(P(y)x(t_1), l\big) \to \min, u(\cdot) \in U_T. \tag{2.4}$$

From the view of the posed minimax is clear that it is a control problem of terminal state of the ensemble of trajectories of a dynamical system (2.1) under conditions of indeterminacy of initial data. We will study the necessary and sufficient optimality conditions for the minimax problem (2.4).

3. Conditions for optimality

Consider the set $X_T(t_1, u(\cdot), y) = \{\xi \in \mathbb{R}^n | \xi = x(t_1, u(\cdot), x(\cdot), y), x_0 \in D\}$. It is clear that

$$X_T(t_1, u(\cdot), y) = \{\xi \in \mathbb{R}^n \mid \xi = x(t_1),\ x(\cdot) \in H_T(u(\cdot), y)\}. \tag{3.1}$$

Due to the results of [13],[17], $X_T(t_1, u(\cdot), y)$ is a convex compact set of $\mathbb{R}^n$ and for the set $X_T(t_1, u(\cdot), y)$ following formula is valid

$$X_T(t_1, u(\cdot), y) = \left\{\xi : \xi = F(t_1, t_0, y)x_0 + \int_{t_0}^{t_1} F(t_1, t, y)b(t, u(t), y)\, dt, \quad x_0 \in D\right\}, \tag{3.2}$$

here $F(t, \tau, y)$ is the fundamental matrix of solutions to the differential equation

$$\frac{dx}{dt} = A(t, y)x,$$

i.e. $\frac{\partial F(t,\tau,y)}{\partial t} = A(t, y)F(t, \tau, y), t \in T, \tau \in T, F(\tau, \tau, y) = E, E$ is an identity $n \times n$- matrix.

Using the formulas (3.1) and (3.2) we have:

$$J(u(\cdot)) = \max_{x(\cdot)\in H_T(u(\cdot),y), y\in Y} \min_{l\in L} \sum_{y\in Y} (P(y)x(t_1), l) = \max_{\xi\in X_T(t_1,u(\cdot),y), y\in Y} \min_{l\in L} \sum_{y\in Y} (P(y)\xi, l) =$$

$$\max_{x_0\in D} \min_{l\in L} \left(\sum_{y\in Y} P(y)\Big(F(t_1, t_0, y)x_0 + \int_{t_0}^{t_1} F(t_1, t, y)\, b(t, u(t), y)\, dt\Big), l\right) = \max_{\xi\in \sum_{y\in Y} P(y)\, X_T(t_1,u(\cdot),y)} \min_{l\in L} (\xi, l). \tag{3.3}$$

where

$$\sum_{y\in Y} P(y)X_T(t_1,u(\cdot),y) = \sum_{y\in Y} P(y)F(t_1,t_0,y)D + \int\limits_{t_0}^{t_1} \sum_{y\in Y} P(y)F(t_1,t,y)b(t,u(t),y)\,dt. \tag{3.4}$$

Due to the properties of the set $X_T(t_1,u(\cdot),y)$, the set in form (3.4) is closed compact set of space $\mathbb{R}^s$. Now, using the minimax theorem known from convex analysis [18],we obtain that the equality is valid

$$\max_{\xi\in\sum\limits_{y\in Y} P(y)\,X_T(t_1,u(\cdot),y)} \min_{l\in L}(\xi,l) = \min_{l\in \mathrm{co}\,L} \max_{\xi\in\sum\limits_{y\in Y} P(y)\,X_T(t_1,u(\cdot),y)} (\xi,l) = \min_{l\in \mathrm{co}\,L} \sigma\left(\sum_{y\in Y} P(y)\,X_T(t_1,u(\cdot),y),\, l\right), \tag{3.5}$$

here $\mathrm{co}\,L$ is convex hull of the set L , $\sigma(\sum\limits_{y\in Y} P(y)X(t_1,u(\cdot),y),l)$ is the support function of the set $\sum\limits_{y\in Y} P(y)X_T(t_1,u(\cdot),y)$. So, from (3.3) (3.5) we obtain following statement:

Lemma 3.1. *The minimax problem (2.4) can be written as follows:*

$$\min_{l\in \mathrm{co}\,L} \sigma\left(\sum_{y\in Y} P(y)X(t_1,u(\cdot),y),\, l\right) \to \min,\quad u(\cdot)\in U_T. \tag{3.6}$$

In that way, the minimax problem (2.4) is reduced to the problem of repeated minimize (3.6). By virtue (3.3), the flowing formula is valid:

$$\sigma\left(\sum_{y\in Y} P(y)X_T(t_1,u,y),l\right) = \sigma(D,\psi^0(l)) + \int_{t_0}^{t_1} \sum_{y\in Y}\langle b(t,u(t),y),\psi(t,y,l)\rangle\,dt, \tag{3.7}$$

where $\psi(t,y,l) = F'(t_1,t,y)P'(y)l,\ \psi^0(l) = \sum\limits_{y\in Y}\psi(t_0,y,l)$.

Consider the function

$$\gamma(l) = \sigma(D,\psi^0(l)) + \int_{t_0}^{t_1} \min_{u\in U}\sum_{y\in Y}\langle b(t,u,y),\psi(t,y,l)\rangle dt.$$

Theorem. 3.1.The existence of the point $l^0\in \mathrm{co}\,L$ of the global minimum of the function $\gamma(l), l\in \mathrm{co}\,L$ and the fulfillment almost everywhere on T the conditions of a minimum

$$\min_{u\in U}\sum_{y\in Y}\left(b(t,u,y),\psi(t,y,l^0)\right) = \sum_{y\in Y}\left(b(t,u^0(t),y),\psi(t,y,l^0)\right), \tag{3.8}$$

is necessary and sufficient for optimality of the admissible control $u^0(t), t\in T$.

Proof. Necessity. According to Lemma 3.1 and formula (3.7), minimax problem (2.4) can be written as problem of minimizing of the following functional

$$\mu(u(\cdot)) = \min_{l\in \mathrm{co}\,L}\left[\sigma(D,\psi^0(l)) + \int_{t_0}^{t_1}\sum_{y\in Y}(b(t,u(t),y),\psi(t,y,l))\,dt\right], u(\cdot)\in U_T(y).$$

Therefore, if $u^0(\cdot)$ is optimal control in problem (2.4), then

$$\mu(u^0(\cdot)) = \min_{u(\cdot)\in U_T}\mu(u(\cdot)).$$

We have:

$$\min_{u(\cdot)\in U_T}\mu(u(\cdot)) = \min_{u(\cdot)\in U_T}\min_{l\in L}\left[\sigma(D,\psi^0(l)) + \int_{t_0}^{t_1}\sum_{y\in Y}(b(t,u(t),y),\psi(t,y,l))\,dt\right] =$$

$$= \min_{l \in \operatorname{co} L} \left[\sigma(D, \psi^0(l)) + \min_{u(\cdot) \in U_T} \int_{t_0}^{t_1} \sum_{y \in Y} (b(t, u(t), y), \psi(t, y, l))\, dt \right] = \min_{l \in \operatorname{co} L} \gamma(l).$$

Therefore, $\mu(u^0(\cdot)) = \min_{u(\cdot) \in U_T} \mu(u(\cdot)) = \min_{l \in \operatorname{co} L} \gamma(l)$.

Let $l^0 \in coL$ be a point of global minimum of continuous function

$$\eta^0(l) = \sigma(D, \psi^0(l)) + \int_{t_0}^{t_1} \sum_{y \in Y} (b(t, u^0(t), y), \psi(t, y, l))\, dt, \quad l \in \operatorname{co} L.$$

Then we have:

$$\gamma(l^0) = \sigma(D, \psi^0(l^0)) + \int_{t_0}^{t_1} \min_{u \in U} \left[\sum_{y \in Y} (b(t, u, y), \psi(t, y, l^0)) \right] dt \geq$$

$$\geq \min_{l \in \operatorname{co} L} \left[\sigma(D, \psi^0(l)) + \int_{t_0}^{t_1} \min_{u \in U} \sum_{y \in Y} (b(t, u, y), \psi(t, y, l))\, dt \right] = \min_{l \in coL} \gamma(l) = \min_{u(\cdot) \in U_T} \mu(u(\cdot)) = \mu(u^0(\cdot)) =$$

$$= \min_{l \in \operatorname{co} L} \left[\sigma(D, \psi^0(l)) + \int_{t_0}^{t_1} \sum_{y \in Y} (b(t, u^0(t), y), \psi(t, y, l))\, dt \right] = \min_{l \in \operatorname{co} L} \eta^0(l) = \eta(l^0) = \sigma(D, \psi^0(l^0)) +$$

$$+ \int_{t_0}^{t_1} \sum_{y \in Y} \big(b(t, u^0(t), y), \psi(t, y, l^0)\big)\, dt \geq \sigma(D, \psi^0(l^0)) + \int_{t_0}^{t_1} \min_{u \in U} \sum_{y \in Y} \big(b(t, u, y), \psi(t, y, l^0)\big)\, dt = \gamma(l^0).$$

From these ratios it follows that

$$\gamma(l^0) = \min_{l \in \operatorname{co} L} \gamma(l), \tag{3.9}$$

i.e. the point $l^0 \in \operatorname{co} L$ is point of the global minimum of the function $\gamma(l), l \in \operatorname{co} L$, and

$$\int_{t_0}^{t_1} \min_{u \in U} \sum_{y \in Y} (b(t, u, y), \psi(t, y, l^0))dt = \int_{t_0}^{t_1} \sum_{y \in Y} (b(t, u^0(t), y), \psi(t, y, l^0))dt.$$

By virtue of the properties of the Lebesgue integral from the latter we get that the condition (3.8) is true for almost everyone $t \in T$.

Sufficiently. Let $l^0 \in \operatorname{co} L$ be the point of the global minimum of the function $\gamma(l), l \in \operatorname{co} L$, and the conditions of a minimum (3.8) be true almost everywhere on T.

Then:

$$\mu(u^0(\cdot)) = \min_{l \in \operatorname{co} L} \left[\sigma(D, \psi^0(l)) + \int_{t_0}^{t_1} \sum_{y \in Y} (b(t, u^0(t), y), \psi(t, y, l))dt \right] \leq \sigma(D, \psi^0(l^0)) +$$

$$+ \int_{t_0}^{t_1} \sum_{y \in Y} (b(t, u^0(t), y), \psi(t, y, l^0))dt = \sigma(D, \psi^0(l^0)) + \int_{t_0}^{t_1} \min_{u \in U} \sum_{y \in Y} (b(t, u, y), \psi(t, y, l^0))dt = \gamma(l^0)$$

$$= \min_{l \in \operatorname{co} L} \gamma(l) \leq \mu(u^0(\cdot)) = \min_{l \in \operatorname{co} L} \left[\sigma(D, \psi^0(l)) + \int_{t_0}^{t_1} \sum_{y \in Y} (b(t, u(t), y), \psi(t, y, l))dt \right] = \mu(u(\cdot))\ \forall u(\cdot) \in U_T.$$

Therefore, $\mu(u^0(\cdot)) = \min_{u(\cdot) \in U_T} \mu(u(\cdot))$. So, $u^0(\cdot)$ is the optimal control in the minimax problem (2.4). The theorem is proven.

4. Conclusion

In this paper, we studied the problem of controlling an ensemble of trajectories of a system (2.1) with discrete parameter and under conditions of uncertainty in the initial data. The problem formulated in the form of a non-smooth control problem of the minimax type. Using methods from multivalued and convex analysis, we derived the necessary and sufficient optimality conditions. They make up the theoretical basis for the method of constructing a solution to problem (2.4) by solving finite-dimensional problems of the form (3.8) and (3.9).

References

[1] Clark F. Optimization and non-smooth analysis. John Wiley and Sons, New York, Toronto, Singapore,1983.

[2] Demyanov V.F., Vasilyev L.V. Nonsmooth optimization. Nauka, Moskow, 1981.

[3] Demyanov V.F., Rubinov A.M. Fundamentals of nonsmooth analysis and quasidifferential calculus. Nauka, Moskow, 1990.

[4] M. Gaudioso and G. Giallobaro, G. Miglionco, Essentials of numerical nonsmooth optimization, Annals of Operations Research, 314, 2022. pp.213-254. Doi: 10.1007/s10479-021-04498-y.

[5] Y. Cui, J.-S. Rang, Modern Nonconvex Nondifferentiable Optimization, Society for linear inverse problems with a sparsity constraint, Communications on Pure and Applied Mathematics 57(11), 2021.pp.1413-1457. Doi: 10.1002/cpa.20042.

[6] Tan H. Cao and B. S. Mordukhovich. Optimal Control of a Nonconvex Perturbed Sweeping Process. Journal of Differential Equations, 2017, No 11. pp. 137.

[7] Kurzhanskiy A.B. Control and observation under conditions of uncertainty. Nauka, Moskow, 1977.

[8] Keyn V. N. Optimization of control systems by minimax criterion. - Nauka, Moscow, 1985.

[9] R. Gabasov, F.M. Kirillova, Control of a Dynamic Object in Real Time Under Conditions of Contant Disturbances, Doklady of the National Academy of Sciences of Belarus, 2017, vol. 61, No. 6, pp. 7–13

[10] Natalia M. Dmitruk, Optimal control strategy in the problem of guaranteed optimization of a linear system with disturbances. Doklady of the National Academy of Sciences of Belarus, 2017, vol. 61, No. 6, pp. 28–34.

[11] Otakulov S, Kholiyarova F, Nonsmooth optimal control problem for model of system with delay under conditions of uncertainty external influence, International Conference on Information Science and Communications Technologies: Applications, Trends and Opportunities (ICISCT-2021). Tashkent, 3-5 November 2021, Publisher: IEEE. pp.1-3.

[12] Otakulov S, Haydarov T. The nonsmoth optimal control problem for model dynamic system under conditions of incomplete information, Science and Innovation, 2022, Vol.1, No 1, pp. 349-359.

[13] Otakulov S., Haydarov T. Non-smooth Optimization Problem for Model of Parameterized Dynamic Control System under Conditions of Uncertainty, 2024, AIP Conf. Proc. 3244, 020031, pp. 1-7.

[14] Otakulov S., Kholiyarova F. On the Method for Constructing Optimal Control in the Problem of Minimizing the Terminal Functional on Trajectories of Differential Inclusions with Delay. 2024, AIP Conf. Proc. 3244, 020053 (Scopus). pp. 1-8.

[15] Jr-Shin Li. Ensemble control of finite-dimensional time-varying linear systems, IEEE: Transactions on automatic control, 2011, Vol. 56, No 2, pp.345357.

[16] A. Scagliotti, Optimal control of ensembles of dynamical systems, ESAIM: Control, Optimisation and Calculus of Variations, 2023, Vol. 29, No 22, pp.139.

[17] Otakulov S. Problems of controlling an ensemble of trajectories of differential inclusions. Monograph. Lambert Academic Publishing, 2019.

[18] Pshenichny B.N, Convex analysis and extreme problems, Moskow, Nauka, 1980.

Haydarov T.T.
Jizzakh Polytechnic Institute,
Jizzakh,Uzbekistan.
email: omad2015@inbox.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.113-127
DOI: 10.29229/uzmj.2025-4-12

Differential game of one evader and multiple pursuers with exponential integral constraints

Ibragimov G.I., Tursunaliev T.G.

Abstract. We analyze an evasion differential game involving one evader and multiple pursuers in $\mathbb{R}^n$. The control functions of the players are subject to exponential integral constraints to ensure bounded energy consumption. Evasion is considered possible if, for any time t, the position of the evader differs from the positions of all the pursuers. In this work, we establish a sufficient condition for the possibility of evasion. We construct an admissible evasion strategy and demonstrate that, for any number of pursuers m, evasion is possible. Additionally, we show that the number of maneuvers required for evasion does not exceed m.

Keywords: Differential game, evasion, control function, exponential integral constraints, evasion strategy, evader, multiple pursuers.
MSC (2020): 91A23; 49N75.

1. Introduction

Pursuit-evasion games have been a significant topic in differential game theory, with various approaches and results developed over the years. An enormous amount of work has been devoted to studying problems (for example, Azamov [1], Azamov et al. [2, 3, 4, 5] Pontryagin [21], Petrosyan [19]).

Several studies considered pursuit-evasion differential games with many players such as Chen et al. [6], Garcia et al. [7], Ibragimov and Salimi [9], Ibragimov [11], Ibragimov and Tursunaliev [13], Kumkov et al. [16], Kuchkarov et al. [14], Petrov [20], Ruziboev et al. [22, 23], Salimi and Ferrara [31], and Von Moll et al. [34].

Further extensions of the pursuit-evasion problem have been considered in various works. Ibragimov et al. [8] studied an evasion differential game that involves one evader and many pursuers. The dynamics of the players are described by linear differential equations, with integral constraints applied to the control functions of the players. They demonstrated that evasion is possible for any positive integer m by showing that the total energy of the pursuers does not exceed the energy of the evader. Ibragimov et al. [12], Pansera et al. [18], Sharifi et al. [30] and Mamadaliev et al. [17] contributed to previous results in pursuit-evasion games and extended the analysis by considering integral constraints on the motion capabilities of the players.

Many studies have considered different variations of the above problem. Kuchkarov et al. [15] analyzed a differential game of the approach of many pursuers and one evader described by linear systems of the same type. They obtained estimates for the payoff function of the game that players can ensure and provide an explicit description of strategies. Ibragimov et al. [10] explored admissible and adaptive strategies in multi-agent interactions.

Samatov and Soyibboev [25] studies a pursuit differential game in which players move under inertial dynamics controlled by acceleration vectors. Using the parallel approach strategy, optimal interception is ensured against any evader action. The capture set is shown to be a linear combination of two Apollonius sets defined by the players' initial positions and velocities.

In addition, comparisons with existing work help illustrate the novelty of the approach and its potential applications in real-world scenarios. Rilwan et al. [24], Satimov [28, 29], Samatov and Uralova [26, 27], Scott and Leonard [33], Shchelchkov [32], Zhao et al. [35], Zhang et al. [36], and Zhou et al. [37] provided further insight into related topics.

In many practical scenarios, the accumulated heat in a system depends on the control effort applied over time, but past inputs contribute less to the current thermal state due to heat dissipation. To model this behavior, we impose an exponentially weighted integral constraint on the control input

$$\int_0^t e^{-k(t-s)}|u(s)|^2 ds \le \rho^2, \quad \forall t \ge 0, \tag{1.1}$$

where $u(s)$ is the control input (e.g., power in a heating system), $k > 0$ is the thermal dissipation rate, which governs how fast past control inputs lose their impact due to heat dissipation, ρ^2 is a bound on the effective thermal load, the exponential weight $e^{-k(t-s)}$ ensures that older control inputs contribute less to the current heat state. If we multiply the inequality (1.1) by e^{kt} and denote $e^{ks/2}u(s)$ by $\bar{u}(s)$, then the inequality (1.1) takes the form

$$\int_0^t |\bar{u}(s)|^2 ds \le \rho^2 e^{kt}, \quad \forall t \ge 0.$$

Clearly, the control $u(s)$ is uniquely defined by the control $\bar{u}(s)$. In the present paper, we consider thermal type (exponential) constraints on the control functions of players.

We show that evasion is possible from any initial position of the players. In addition, we construct an explicit strategy for the evader and then prove the admissibility of the strategy. To the best of our knowledge, no prior research has addressed the specific simple motion evasion differential game with exponential integral constraints. The main difficulties in solving the problem are constructing an evasion strategy and proving that the constructed strategy guarantees evasion.

In this work, the construction of strategy requires the identification of approach times θ_i. Furthermore, our approach requires θ_i to be bounded, as well as new techniques to estimate the distance between a pursuer $x_p(t)$ and the evader. Note that according to the strategy constructed, the evader moves with a positive speed in a vicinity of the y-axis, for any control functions of the pursuers on the time interval $[0, T]$. The fact that each maneuvering interval of the evader is contained within $[0, T]$ plays a crucial role in establishing key estimates required for the proof of the main result.

2. Statement of problem

We consider a simple motion evasion differential game of one evader y and m pursuers x_i, $i = 1, ..., m$, in $\mathbb{R}^n$, $n \ge 2$. Game is described by the following equations:

$$\begin{aligned} &\dot{x}_i = u_i, \quad x_i(0) = x_i^0, \quad i = 1, \dots, m, \\ &\dot{y} = v, \quad y(0) = y^0, \end{aligned} \tag{2.1}$$

where $x_i, x_i^0, y, y^0, u_i, v \in \mathbb{R}^n$, $n \ge 2$, $x_i^0 \ne y^0$, $i = 1, ..., m$ and $u_1, \dots, u_m$ are the control parameters of pursuers and v is that of evader.

Definition 2.1. A measurable function $u_i(t)$, $t \ge 0$, is called an admissible control of the pursuer x_i if

$$\int_0^t |u_i(s)|^2 ds \le \rho_i^2 e^{2kt}, \ i = 1, \dots, m, \tag{2.2}$$

where $\rho_1, \rho_2, ..., \rho_m$ and k are given positive numbers.

Definition 2.2. A measurable function $v(t)$, $t \ge 0$, is called an admissible control of the evader y if

$$\int_0^t |v(s)|^2 ds \le \sigma^2 e^{2kt}, \tag{2.3}$$

where σ is a given positive number.

Definition 2.3. A function $V : [0, \infty) \times \mathbb{R}^{(2m+1)n} \to \mathbb{R}^n$,

$$(t, y, x_1, ..., x_m, u_1, ..., u_m) \mapsto V(t, y, x_1, ..., x_m, u_1, ..., u_m),$$

is called a strategy of evader if the following initial value problem

$$\begin{aligned} &\dot{x}_i = u_i, \quad x_i(0) = x_i^0, \quad i = 1, 2, \dots, m, \\ &\dot{y} = V(t, y, x_1, ..., x_m, u_1, ..., u_m), \quad y(0) = y^0, \end{aligned} \tag{2.4}$$

has a unique solution $(x_1(t), \ldots, x_m(t), y(t))$, $t \geq 0$, for any admissible controls of the pursuers $u_i = u_i(t)$, $i = 1, ..., m$, and along this solution the following inequality

$$\int_0^t |V(s, y(s), x_1(s), \ldots x_m(s), u_1(s), \ldots, u_m(s))|^2 ds \leq \sigma^2 e^{2kt}.$$

holds.

Definition 2.4. If there exists a strategy V of the evader such that for any admissible controls of pursuers $x_i(t) \neq y(t)$ for all $t \geq 0$, and $i = 1, \ldots, m$, then we say that evasion is possible.

Problem 1. Construct a strategy V for the evader y and find a condition for the parameters ρ_i, $i = 1, ..., m$, σ, which guarantees the evasion in game (2.1)-(2.3).

Note that the evader knows the values $y(t), x_1(t), \ldots, x_m(t)$, $u_1(t), \ldots, u_m(t)$ at the current time t. During the game, the pursuers apply arbitrary controls $u_1(t), \ldots, u_m(t)$, $t \geq 0$, and attempt to realize the equation $x_i(t) = y(t)$ at least for one $i \in \{1, 2, ..., m\}$, whereas the evader strives to ensure the inequalities $x_i(t) \neq y(t)$ for all $i = 1, ..., m$ and $t \geq 0$.

3. The main result

We consider the evasion problem in the case where $n = 2$ and prove a theorem on evasion. The following presents the main result of this paper.

Theorem 3.1. *If*

$$\rho_1^2 + \cdots + \rho_m^2 < \sigma^2, \tag{3.1}$$

then evasion is possible in game (2.1)-(2.3).

Without loss of generality, we assume that $y^0 = (0, 0)$, that is, the evader is at the origin at the initial time. Then, we construct a strategy for the evader which guarantees evasion. There is no restriction in assuming that $J = \{1, 2, ..., m\}$ meaning that only the first m pursuers $x_1^0, x_2^0, ..., x_m^0$ are in the upper half plane. Let $J = \{i \mid x_{i2}^0 > 0,\ 1 \leq i \leq m\}$. In Fig. 3, this set of indices is $J = \{1, 2, 3, 4, 5\}$.

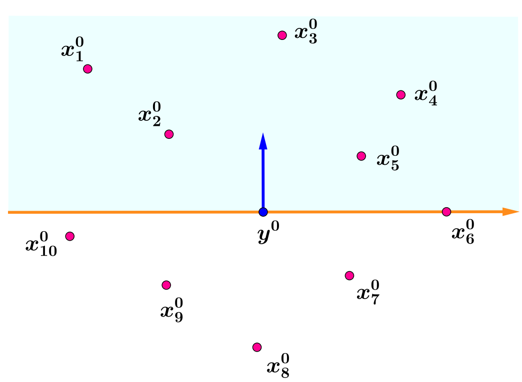


Figure 3. Example of initial states of players.

The solutions of the initial value problem (2.1) are given by

$$x_i(t) = x_i^0 + \int_0^t u_i(s) ds, \quad i = 1, ..., m, \quad y(t) = y^0 + \int_0^t v(s) ds. \tag{3.2}$$

We prove the theorem in several subsections.

3.1. **Notations.** Let α be any number satisfying the condition

$$0<\alpha<\frac{(\sigma-\rho)^2}{2(\max_{1\le i\le m}|y_2^0-x_{i2}^0|+1)},\quad \rho=(\rho_1^2+...+\rho_m^2)^{1/2}. \tag{3.3}$$

We choose a number a_1 from the condition

$$0<a_1<\min\left\{\frac{1}{2},\frac{(\sigma-\rho)^2}{4\alpha},\frac{\sigma^2}{8\alpha},\min_{1\le i\le m}|y^0-x_i^0|\right\}. \tag{3.4}$$

Let

$$T_0=\frac{1}{\alpha}\max_{1\le i\le m}|y_2^0-x_{i2}^0|,\quad T=T_0+\frac{2a_1}{\alpha},\quad \beta=\frac{\alpha^3}{4\cdot 6^4\sigma^6e^{6kT}}. \tag{3.5}$$

We observe $\beta<\frac{1}{2}$ since $\alpha<\sigma,\ k>0$.

Let a sequence $\{a_k\}_{k=1}^{\infty}$ be defined by the formula $a_{k+1}=\beta\cdot a_k^4,\ k=1,2,...$ It is not difficult to prove that this sequence has the following:

Property 3.2. $\sum_{k=p+1}^{\infty}a_k\le 2a_{p+1}$ for any $p\ge 1$.

Proof. Since $\beta<\frac{1}{2},\ a_1<\frac{1}{2}$, we have $a_{k+1}<a_k^4,\ k=1,2,...$, and hence, $a_k<1,\ k=1,2,...$ Then

$$\sum_{k=p+1}^{\infty}a_k=a_{p+1}+a_{p+2}+...<a_{p+1}+a_{p+1}^4+a_{p+1}^{16}+...<a_{p+1}+a_{p+1}^2+a_{p+1}^3+...=\frac{a_{p+1}}{1-a_{p+1}}<2a_{p+1}.$$

The proof of the property is complete. □

3.2. **Definitions of approach times.** Let $\theta_0=0$ and $\theta_1>0$ be the first time at which

(*i*) $|x_i(\theta_1)-y(\theta_1)|=a_1$,

(*ii*) $x_{i2}(\theta_1)>y_2(\theta_1)$,

for some $i\in J$. Note that such a time θ_1 may not exist. If there are several pursuers x_i that satisfy conditions (*i*) and (*ii*), (for example, θ_1 is an a_1-approach time for both pursuers x_1 and x_2, but θ_1 cannot be an a_1-approach time for both pursuers x_3 and x_4 because it does not satisfy condition (*ii*) (see Fig. 4)), then we can assume, by relabeling if necessary, that one of such pursuers x_i is x_1. We call θ_1 the a_1-approach time. The times θ_i are unspecied and depend on the evaders strategy and the controls of the pursuers. It is important to note the fact that all the numbers θ_i will be in the interval $[0,T]$, which will be established in Subsection 3.4.

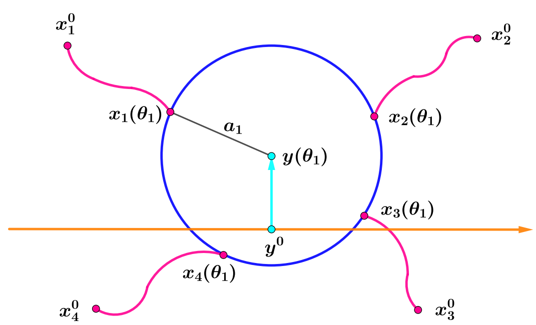


FIGURE 4. a_1-approach time.

If $\theta_1, \theta_2, ..., \theta_{k-1},\ \theta_1 < \theta_2 < ... < \theta_{k-1}$, are $a_1-, a_2-, ..., a_{k-1}-$ approach times, respectively, then we define $\theta_k > \theta_{k-1}$ to be the a_k-approach time if the following conditions are satisfied

(*i*) $|x_i(\theta_k) - y(\theta_k)| = a_k$,

(*ii*) $x_{i2}(\theta_k) > y_2(\theta_k)$,

for some $i \in J$. If there are more than one such pursuers x_i, then we assume without loss of generality that one of them is x_k. In this way, we define a_k-approach times, θ_k, $k \in J_0 = \{1, 2, ..., m_0\}$, i.e., $\theta_1, \theta_2, ..., \theta_{m_0}$, where m_0 is a positive integer. Note that a_k-approach times θ_k will not necessarily be defined for all pursuers $x_i, i \in J$. We will establish that at most one approach time will be defined for each pursuer $x_i, i \in J$, and therefore $m_0 \le m$.

Let

$$\theta'_k = \theta_k + \frac{2a_k}{\alpha}, \quad k = 1, 2, ..., m_0.$$

Note that we have defined θ_k and θ'_k only for $k = 1, 2, ..., m_0$.

3.3. **A function assigning a maneuver for the evader.** Denote $I_k = \cup_{j=k}^{m_0} [\theta_j, \theta'_j)$, $I_{m_0+1} = \emptyset$. We define a function $r : [0, T] \to \{0, 1, \ldots, m_0\}$, which plays a key role in assigning a maneuver for the evader. Set

$$r(t) = \begin{cases} 0, & t \in [0, T] \setminus I_1, \\ k, & t \in [\theta_k, \theta'_k] \setminus I_{k+1},\ k = 1, ..., m_0. \end{cases} \tag{3.6}$$

The function $r(t)$ has the following property.

Property 3.3. Let $m_0 > 1$. Then, for $k = 1, 2, \ldots, (m_0 - 1)$,

(*i*) If $\theta'_k \le \theta_{k+1}$, then $r(t) = k$ for $\theta_k \le t < \theta'_k$,

(*ii*) If $\theta_{k+1} \le \theta'_k$, then $r(t) = k$ for $\theta_k \le t < \theta_{k+1}$.

Proof. Assume that $\theta'_k \le \theta_{k+1}$. Then $[\theta_k, \theta'_k) \setminus I_{k+1} = [\theta_k, \theta'_k)$ since $\theta'_k \le \theta_{k+1} < \theta_{k+2} < ...$ and $[\theta_k, \theta'_k) \cap I_{k+1} = \emptyset$. Therefore, $r(t) = k$ for $t \in [\theta_k, \theta'_k)$. This proves item (*i*).

To prove item (*ii*), suppose that $\theta_{k+1} \le \theta'_k$. Since $\theta_k < \theta_{k+1} < \cdots < \theta_{m_0}$, we have $[\theta_k, \theta_{k+1}) \subset [\theta_k, \theta'_k) \setminus I_{k+1}$. Therefore, $r(t) = k$ for $t \in [\theta_k, \theta_{k+1})$ by the definition of $r(t)$ (3.6). □

Example 3.4. If

$$0 = \theta_0 < \theta_1 < \theta_2 < \theta'_2 < \theta'_1 < \theta_3 < \theta_4 < \theta'_3 < \theta'_4 < \theta_5 < \theta'_5,$$

then $r(t)$ has the graph shown in Fig. 5.

3.4. **Construction and admissibility of strategy for the evader.** We now construct a strategy for the evader. Let $u_i(t)$, $i = 1, ..., m$, be arbitrary controls of pursuers. Set

$$v(t) = V_0(t) = \left(0, \alpha + \left(\sum_{i=1}^{m} |u_i(t)|^2\right)^{1/2}\right), \quad t \in [0, T] \setminus I_1, \tag{3.7}$$

$$v(t) = V_r(t) = (V_{r1}(t), U(t)),\ t \in [0, T] \cap I_1, \tag{3.8}$$

where $r = r(t)$, $V_k(t) = (V_{k1}(t), U(t))$, $\theta_k \le t < \theta'_k$, $k = 1, \ldots, m_0$, is defined as follows

$$V_{k1}(t) = \begin{cases} \alpha + |u_{k1}(t)|, & y_1(\theta_k) \ge x_{k1}(\theta_k), \\ -(\alpha + |u_{k1}(t)|), & y_1(\theta_k) < x_{k1}(\theta_k), \end{cases} \tag{3.9}$$

$$U(t) = \alpha + \left(\sum_{i=1}^{m} u_{i2}^2(t)\right)^{1/2}.$$

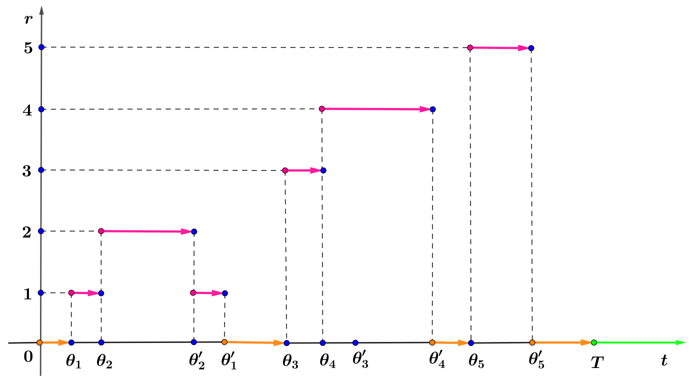


FIGURE 5. The graph of function $r(t)$.

Note that $U(t)$ doesn't depend on k. Finally, let

$$v(t) = \left(0, \left(\sum_{i=1}^{m} |u_i(t)|^2\right)^{1/2}\right), \quad t > T. \tag{3.10}$$

Equation (3.8) shows that the function $r = r(t)$ assigns the control $V_r(t)$ for $v(t)$.
We now show that the strategy defined by equations (3.7)-(3.10) is admissible. Indeed, let we denote

$$\varphi(t) = \begin{cases} (0, \alpha), & t \in [0, T) \setminus I_1 \\ (\alpha, \alpha), & t \in I_1 \\ (0, 0), & t > T \end{cases}, \ \psi(t) = \begin{cases} \left(0, \left(\sum_{i=1}^m |u_i(t)|^2\right)^{1/2}\right), & t \in [0, T] \setminus I_1, \\ \left(|u_{r1}(t)|, \left(\sum_{i=1}^m u_{i2}^2(t)\right)^{1/2}\right), & t \in I_1, \\ \left(0, \left(\sum_{i=1}^m |u_i(t)|^2\right)^{1/2}\right), & t > T. \end{cases}$$

Note that

$$\int_0^t |\varphi(s)|^2 ds \le 2\alpha^2 T, \quad |\psi(t)|^2 \le \sum_{i=1}^m |u_i(t)|^2. \tag{3.11}$$

Clearly, for $v(t)$ defined by (3.7)-(3.10) we have $v_1^2(t) + v_2^2(t) = |\varphi(t) + \psi(t)|^2$. Therefore, using the Minkowskii inequality and (3.11) we obtain, for $t \ge 0$,

$$\begin{aligned} \left(\int_0^t |v(s)|^2 ds\right)^{1/2} &= \left(\int_0^t |\varphi(s) + \psi(s)|^2 ds\right)^{1/2} \le \left(\int_0^t |\varphi(s)|^2 ds\right)^{1/2} + \left(\int_0^t |\psi(s)|^2 ds\right)^{1/2} \\ &\le (2\alpha^2 T)^{1/2} + \left(\int_0^t \sum_{i=1}^m |u_i(s)|^2 ds\right)^{1/2} \le \alpha\sqrt{2T} + \left(\sum_{i=1}^m \rho_i^2 e^{2kt}\right)^{1/2} = \alpha\sqrt{2T} + \rho e^{kt} \le \sigma e^{kt}, \end{aligned}$$

since by definition of T, T_0 and α

$$\begin{aligned} \alpha\sqrt{2T} &= \alpha\sqrt{2\left(T_0 + \frac{2a_1}{\alpha}\right)} = \sqrt{2\alpha\left(\max_{i=1,\dots,m} |y_2^0 - x_{i2}^0| + 2a_1\right)} \\ &\le \sqrt{2\alpha\left(\max_{i=1,\dots,m} |y_2^0 - x_{i2}^0| + 1\right)} \le \sigma - \rho. \end{aligned}$$

Here, in the last inequality we used (3.3). Thus, the evasion strategy (3.7)-(3.10) is admissible.
Next, we prove the following statement.

3.5. **One characteristics of the strategy.**

Lemma 3.5. *If the evader uses strategy* (3.7)-(3.10), *then*

(a) *For all* $k \in J_0 = \{1, ..., m_0\}$, *we have* (i) $\theta_k \le T_0$ *and* (ii) $\theta_k' \le T$.

(b) *If* $y_2^0 \ge x_{i2}^0$ *for some* $i \in \{1, ..., m\}$, *then* $y_2(t) > x_{i2}(t)$ *for all* $t > 0$.

Proof. We first show that $y_2(T_0) \ge x_{i2}(T_0)$ for all $i = 1, \ldots, m$. Indeed, by (3.7)-(3.9) we have

$$v_2(t) \ge \alpha + \left(\sum_{i=1}^{m} u_{i2}^2(t) \right)^{1/2} \ge \alpha + |u_{i2}(t)|, \; 0 \le t \le T, \tag{3.12}$$

and therefore,

$$\frac{d}{dt}(y_2(t) - x_{i2}(t)) = (v_2(t) - u_{i2}(t)) \ge (v_2(t) - |u_{i2}(t)|) \ge \alpha > 0. \tag{3.13}$$

Hence, $y_2(t) - x_{i2}(t)$, $0 \le t \le T$, increases strictly. Since $T_0 < T$, by (3.13) we have

$$y_2(T_0) - x_{i2}(T_0) = y_2^0 - x_{i2}^0 + \int_0^{T_0} (v_2(s) - u_{i2}(s))ds \ge y_2^0 - x_{i2}^0 + \alpha T_0 = y_2^0 - x_{i2}^0 + \max_{1 \le j \le m} |y_2^0 - x_{j2}^0| \ge 0.$$

Thus, $y_2(T_0) \ge x_{i2}(T_0)$ for all $i = 1, \ldots, m$.

Next, since $v_2(t) \ge \alpha + |u_{i2}(t)|$ for $T_0 \le t \le T$ (see Fig.4a), and $v_2(t) \ge |u_{i2}(t)|$ for $t > T$ (see Fig.4b), therefore for $t > T_0$ we have

$$\begin{aligned} y_2(t) - x_{i2}(t) &= y_2(T_0) - x_{i2}(T_0) + \int_{T_0}^{t} (v_2(s) - u_{i2}(s))ds \\ &\ge y_2(T_0) - x_{i2}(T_0) + \int\limits_{[T_0,t]\cap[T_0,T]} (\alpha + |u_{i2}(s)| - u_{i2}(s))ds \\ &> y_2(T_0) - x_{i2}(T_0) \ge 0. \end{aligned} \tag{3.14}$$

Thus, $y_2(t) > x_{i2}(t)$ for all $t > T_0$ and $i = 1, \ldots, m$.

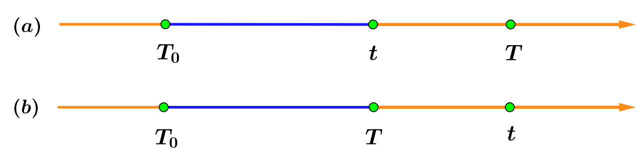


FIGURE 6. The location of t relative to T.

In particular, we obtain that there is no a_k-approach time θ_k in the time interval $[T_0, \infty)$, since by definition of an a_k-approach time θ_k, one has to have $y_2(\theta_k) < x_{k2}(\theta_k)$. This is impossible for $\theta_k \ge T_0$ since as proved above $y_2(t) \ge x_{k2}(t)$ for all $t \ge T_0$. Hence, $\theta_k \le T_0$ for all $k = 1, .., m_0$.

Next, by definition of θ_k' we have

$$\theta_k' = \theta_k + \frac{2a_k}{\alpha} \le T_0 + \frac{2a_1}{\alpha} = T, \tag{3.15}$$

and the proof of item (a) of Lemma 3.5 follows. In particular, (3.15) implies that $I_1 \subset [0, T]$.

Remark 3.6. Due to the inclusion $I_1 \subset [0, T]$ the set $[0, T] \cap I_1$ in (3.8) is equal to I_1.

To show item (b), using $y_2^0 \ge x_{i2}^0$ we observe that for $t > 0$

$$y_2(t) - x_{i2}(t) = y_2^0 - x_{i2}^0 + \int_0^t (v_2(s) - u_{i2}(s))ds \ge \int\limits_{[0,t]\cap[0,T]} (\alpha + |u_{i2}(s)| - u_{i2}(s))ds > 0.$$

The intersection $[0, t] \cap [0, T]$ in (3.16) is equal to either $[0, t]$ (see Fig.5a) or $[0, T]$ (see Fig.5b).

Thus, we have $y_2(t) > x_{i2}(t)$ for all $t > 0$ by (3.16). This completes the proof of Lemma 3.5. □

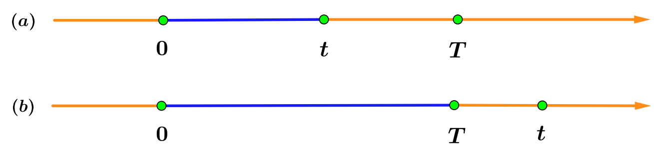


FIGURE 7. The location of t relative to T.

3.6. **Fictitious evader z_p.** Take any integer $p \in \{1, \dots, m_0\}$ and assume that θ_p is the a_p-approach time of the pursuer x_p to the evader y. We will estimate the distance between $x_p(t)$ and $y(t)$ on $[\theta_p, \theta'_p]$. To this end, we introduce a fictitious evader (FE) z_p whose motion is described by the following equation

$$\dot{z}_p = w_p, \quad z_p(\theta_p) = y(\theta_p),$$

where w_p is the control parameter of z_p. The fictitious evader $z_p(t)$ is defined only on the interval $[\theta_p, \theta'_p]$ and

$$w_p(t) = V_p(t) = (V_{p1}(t), U(t)), \quad \theta_p \le t \le \theta'_p. \tag{3.16}$$

The trajectory of FE

$$z_p(t) = y(\theta_p) + \int_{\theta_p}^t V_p(s)ds, \quad \theta_p \le t \le \theta'_p.$$

Since by (3.16) $v_2(t) = U(t)$, therefore

$$z_{p2}(t) = z_{p2}(\theta_p) + \int_{\theta_p}^t U(s)ds = y_2(\theta_p) + \int_{\theta_p}^t v_2(s)ds = y_2(t), \quad \theta_p \le t \le \theta'_p.$$

We now demonstrate that

$$\int_{\theta_p}^{\theta'_p} |V_p(s)|^2 ds \le \sigma^2 e^{2kT}. \tag{3.17}$$

By denoting

$$\varphi_1(t) = (\alpha, \alpha), \quad \psi_1(t) = \left(|u_{p1}(t)|, \left(\sum_{i=1}^m u_{i2}^2(t) \right)^{1/2} \right)$$

we have

$$|V_p(t)|^2 = V_{p1}^2(t) + U^2(t) = (\alpha + |u_{p1}(t)|)^2 + \left(\alpha + \left(\sum_{i=1}^m u_{i2}^2(t) \right)^{1/2} \right)^2 = |\varphi_1(t) + \psi_1(t)|^2, \quad \theta_p \le t \le \theta'_p.$$

Therefore, using the Minkowskii inequality we obtain

$$\begin{aligned} \left(\int_{\theta_p}^{\theta'_p} |V_p(s)|^2 ds \right)^{1/2} &= \left(\int_{\theta_p}^{\theta'_p} |\varphi_1(s) + \psi_1(s)|^2 ds \right)^{1/2} \le \left(\int_{\theta_p}^{\theta'_p} |\varphi_1(s)|^2 ds \right)^{1/2} + \left(\int_{\theta_p}^{\theta'_p} |\psi_1(s)|^2 ds \right)^{1/2} \\ &\le (2\alpha^2(\theta'_p - \theta_p))^{1/2} + \left(\int_{\theta_p}^{\theta'_p} \sum_{i=1}^m |u_i(s)|^2 ds \right)^{1/2}, \end{aligned} \tag{3.18}$$

since by (2.2) and (3.15) we have $\int_{\theta_p}^{\theta'_p} |u_i(s)|^2 ds \le \rho_i^2 e^{2k\theta'_p} \le \rho_i^2 e^{2kT}$, and then it follows from (3.18) that

$$\left(\int_{\theta_p}^{\theta'_p} |V_p(s)|^2 ds \right)^{1/2} \le 2\sqrt{\alpha a_p} + \left(\sum_{i=1}^m \rho_i^2 e^{2kT} \right)^{1/2} = 2\sqrt{\alpha a_p} + \left(\rho^2 e^{2kT}\right)^{1/2} = 2\sqrt{\alpha a_p} + \rho e^{kT} < \sigma e^{kT}.$$

since by (3.4) $a_p \le a_1 < \frac{(\sigma - \rho)^2}{4\alpha}$, and hence (3.17) is true.

3.7. **Distance between fictitious evader and pursuer.**

Lemma 3.7. *Let the pursuer x_p apply an arbitrary admissible control $u_p(t)$ on $\theta_p \le t \le \theta'_p$. Then*

$$|z_p(t) - x_p(t)| > \frac{\alpha a_p^2}{6\sigma^2 e^{2kT}}, \quad \theta_p \le t \le \theta'_p, \quad \textit{and} \quad y_2(t) - x_{p2}(t) \ge a_p, \quad t \ge \theta'_p. \tag{3.19}$$

Proof. Let $\theta_p \le t < \theta'_p$ and for definiteness assume that $x_{p1}(\theta_p) \le y_1(\theta_p)$. Then by (3.9) we have $V_{p1}(t) = \alpha + |u_{p1}(t)|$. Therefore,

$$|z_p(t) - x_p(t)| \ge z_{p1}(t) - x_{p1}(t) = y_1(\theta_p) - x_{p1}(\theta_p) + \int_{\theta_p}^{t} (V_{p1}(s) - u_{p1}(s))ds$$
$$\ge \int_{\theta_p}^{t} (\alpha + |u_{p1}(s)| - u_{p1}(s))ds \ge \alpha(t - \theta_p). \tag{3.20}$$

On the other hand,

$$|z_p(t) - x_p(t)| \ge |z_p(\theta_p) - x_p(\theta_p)| - \int_{\theta_p}^{t} |V_p(s) - u_p(s)|ds. \tag{3.21}$$

The integral in (3.21) can be estimated by using the Cauchy-Schwartz inequality as follows

$$\int_{\theta_p}^{t} |V_p(s) - u_p(s)|ds \le \left(\int_{\theta_p}^{t} 1^2 ds \int_{\theta_p}^{t} |V_p(s) - u_p(s)|^2 ds\right)^{1/2} \le \left((t - \theta_p) \int_{\theta_p}^{t} 2(|V_p(s)|^2 + |u_p(s)|^2)ds\right)^{1/2}. \tag{3.22}$$

Since $t - \theta_p < \theta'_p - \theta_p \le T$ by (3.15), we have

$$\int_{\theta_p}^{t} |V_p(s)|^2 ds \le \sigma^2 e^{2kT}, \quad \int_{\theta_p}^{t} |u_p(s)|^2 ds \le \rho_p^2 e^{2kT} \le \sigma^2 e^{2kT},$$

then it follows from (3.22) that

$$\int_{\theta_p}^{t} |V_p(s) - u_p(s)|ds \quad \le \quad (t - \theta_p)^{1/2} \left(4\sigma^2 e^{2kT}\right)^{1/2} = 2\sigma(t - \theta_p)^{1/2} e^{kT}.$$

By using (3.23) and the equation $|z_p(\theta_p) - x_p(\theta_p)| = a_p$, (3.21) yields that

$$|z_p(t) - x_p(t)| \ge a_p - 2\sigma(t - \theta_p)^{1/2} e^{kT}. \tag{3.23}$$

It is easily seen from (3.20) and (3.23) that

$$|z_p(t) - x_p(t)| \ge h(t) = \max\{h_1(t), h_2(t)\}, \quad t \ge \theta_p, \tag{3.24}$$

where

$$h_1(t) = \alpha(t - \theta_p), \quad h_2(t) = a_p - 2\sigma(t - \theta_p)^{1/2} e^{kT}.$$

Note that the funcion $h_1(t)$, $t \ge \theta_p$, is increasing, and the function $h_2(t)$, $t \ge \theta_p$, is decreasing, therefore, it is not difficult to see that the function $h(t)$, $t \ge \theta_p$, attains its minimum at the point $t = t_*$ where

$$h_1(t) = h_2(t), \quad t \ge \theta_p. \tag{3.25}$$

Let $(t - \theta_p)^{1/2} = d$. Then equation (3.25) takes the form

$$\alpha d^2 = a_p - 2\sigma d e^{kT},$$

or $\alpha d^2 + 2\sigma e^{kT} d - a_p = 0$. This equation has the following positive root

$$d_* = \frac{-\sigma e^{kT} + \sqrt{\sigma^2 e^{2kT} + \alpha a_p}}{\alpha} = \frac{a_p}{\sigma e^{kT} + \sqrt{\sigma^2 e^{2kT} + \alpha a_p}}.$$

Then

$$\min_{t \geq \theta_p} h(t) = h(t_*) = h_1(t_*) = \alpha d_*^2 = \frac{\alpha a_p^2}{\left(\sigma e^{kT} + \sqrt{\sigma^2 e^{2kT} + \alpha a_p}\right)^2}. \tag{3.26}$$

Since by (3.4) $a_1 < \frac{1}{2} < e^{2kT}$, and in view of $\alpha < \sigma^2$, we have $\alpha a_p \leq \alpha a_1 < \sigma^2 e^{2kT}$, therefore (3.26) implies that

$$|z_p(t) - x_p(t)| \geq \min_{t \geq \theta_p} h(t) > \frac{\alpha a_p^2}{6\sigma^2 e^{2kT}}, \quad \theta_p \leq t \leq \theta_p'. \tag{3.27}$$

Next, using the fact that $y_2(\theta_p) - x_{p2}(\theta_p) \geq -|y(\theta_p) - z_p(\theta_p)| = -a_p$, and the equality $z_{p2}(\theta_p) = y_2(\theta_p)$ by (3.17) we obtain

$$z_{p2}(\theta_p') - x_{p2}(\theta_p') = y_2(\theta_p) - x_{p2}(\theta_p) + \int_{\theta_p}^{\theta_p'} (U(s) - u_{p2}(s))ds \geq -a_p + \int_{\theta_p}^{\theta_p'} \left(\alpha + \left(\sum_{i=1}^m u_{i2}^2(s)\right)^{1/2} - u_{p2}(s)\right) ds$$

$$\geq -a_p + \alpha \int_{\theta_p}^{\theta_p'} ds = -a_p + \alpha\left(\theta_p' - \theta_p\right) = -a_p + \alpha\left(\theta_p + \frac{2a_p}{\alpha} - \theta_p\right) = a_p. \tag{3.28}$$

Finally, let $t \geq \theta_p'$. By (3.17) $y_2(\theta_p') = z_{p2}(\theta_p')$, and by (3.7), (3.8) and (3.10), $v_2(t) \geq |u_{p2}(t)|$. Then using (3.10), (3.28) we get

$$y_2(t) - x_{p2}(t) = z_{p2}(\theta_p') - x_{p2}(\theta_p') + \int_{\theta_p'}^{t} (v_2(s) - u_{p2}(s))ds \geq z_{p2}(\theta_p') - x_{p2}(\theta_p') \geq a_p, \quad t \geq \theta_p'.$$

Thus, we have the following inequalities:

$$|z_p(t) - x_p(t)| > \frac{\alpha a_p^2}{6\sigma^2 e^{2kT}}, \quad \theta_p \leq t \leq \theta_p', \tag{3.29}$$

$$y_2(t) - x_{p2}(t) \geq a_p, \quad t \geq \theta_p'. \tag{3.30}$$

This completes the proof of Lemma 3.7. □

3.8. **Distance between real and fictitious evader.**

Lemma 3.8. *The following estimate holds*

$$|y(t) - z_p(t)| \leq 6\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}, \quad \theta_p \leq t \leq \theta_p'. \tag{3.31}$$

Proof. Since $z_p(\theta_p) = y(\theta_p)$, we have

$$|y(t) - z_p(t)| = \left| \int_{\theta_p}^{t} (v(s) - V_p(s))ds \right|, \quad \theta_p \leq t \leq \theta_p'. \tag{3.32}$$

By (3.8) and (3.17)

$$v(t) = (V_{r1}(t), U(t)), \quad V_p(t) = (V_{p1}(t), U(t)), \quad \theta_p \leq t < \theta_p'. \tag{3.33}$$

Consider two cases: (i) $\theta_p' \leq \theta_{p+1}$ and (ii) $\theta_{p+1} \leq \theta_p'$.

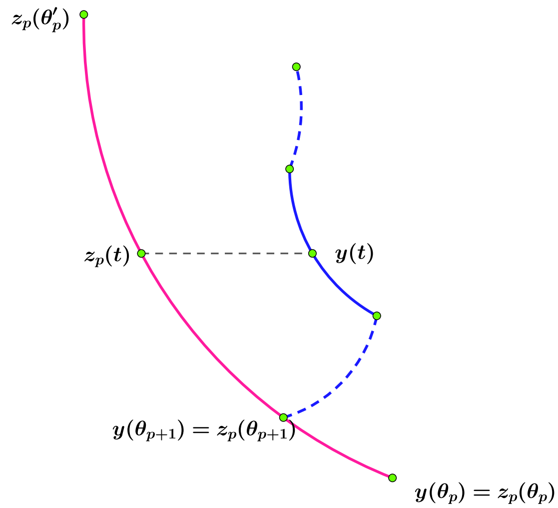


FIGURE 8. Points $y(t)$ and $z_p(t)$ are on one horizontal line.

Case (i). Let $\theta'_p \le \theta_{p+1}$. Then by item (i) of Property 3.3 $r = r(t) = p$ for $\theta_p \le t < \theta'_p$. Therefore by (3.11) we have $v(t) = V_p(t)$, $\theta_p \le t < \theta'_p$. Hence, by (3.10)

$$|y(t) - z_p(t)| = 0. \tag{3.34}$$

Case (ii). Assume now $\theta_{p+1} \le \theta'_p$. Then by item (ii) of Property 3.3 we have $v(t) = V_p(t)$, $\theta_p \le t < \theta_{p+1}$, therefore, $y(t) = z_p(t)$, for $t \in [\theta_p, \theta_{p+1}]$, and so (3.31) satisfied. This means that the trajectories of $y(t)$ and $z_p(t)$ coincide on $[\theta_p, \theta_{p+1}]$ (see Fig.8). Then, starting from the time θ_{p+1} the evader applies the maneuver $V_{p+1}(t)$ against the pursuer x_{p+1}.

Next, we estimate $|y(t) - z_p(t)|$ for $t \in [\theta_{p+1}, \theta'_p)$, we then obtain

$$\begin{aligned} |y(t) - z_p(t)| &= \left| \int_{\theta_{p+1}}^{t} (v(s) - V_p(s))ds \right| \le \int_{\theta_{p+1}}^{t} |v(s) - V_p(s)|ds \\ &\le \int\limits_{[\theta_{p+1},t)\setminus I_{p+1}} |v(s) - V_p(s)|ds + \int\limits_{[\theta_{p+1},t)\cap I_{p+1}} |v(s) - V_p(s)|ds. \end{aligned} \tag{3.35}$$

Since by definition $r(t) = p$ for $t \in [\theta_p, \theta'_p) \setminus I_{p+1}$, and $[\theta_{p+1}, t) \setminus I_{p+1} \subset [\theta_p, \theta'_p) \setminus I_{p+1}$ for $t \in [\theta_p, \theta'_p)$, therefore we have $r = r(t) = p$, and hence, $v(t) = V_p(t)$ for $t \in [\theta_{p+1}, t) \setminus I_{p+1}$. Consequently, the first integral in (3.35) is 0, and so (3.35) takes the form

$$|y(t) - z_p(t)| \le \int_{[\theta_{p+1},t]\cap I_{p+1}} |v(s) - V_p(s)|ds. \tag{3.36}$$

By (3.9) and (3.11)

$$|v(s) - V_p(s)| = |V_{r1}(s) - V_{p1}(s)| \le 2\alpha + |u_{r1}(s)| + |u_{p1}(s)|,$$

and therefore (3.36) implies that

$$|y(t) - z_p(t)| \le \int_{I_{p+1}} (2\alpha + |u_{r1}(s)| + |u_{p1}(s)|)ds. \tag{3.37}$$

To estimate the integral in (3.37), we need to estimate the integrals

$$\int_{I_{p+1}} 2\alpha ds, \ \int_{I_{p+1}} |u_{r1}(s)|ds, \ \text{and} \ \int_{I_{p+1}} |u_{p1}(s)|ds. \tag{3.38}$$

The first integral can be estimated using the definition of θ'_k and Property 3.2 as follows

$$\int_{I_{p+1}} 2\alpha ds \le \sum_{i=p+1}^{m} \int_{\theta_i}^{\theta'_i} 2\alpha ds = 2\alpha \sum_{i=p+1}^{m} (\theta'_i - \theta_i) = 2\alpha \sum_{i=p+1}^{m} \frac{2a_i}{\alpha} \le 8a_{p+1}. \tag{3.39}$$

Next, we estimate the second integral in (3.38). Using the Cauchy-Schwartz inequality we have

$$\int_{I_{p+1}} |u_{r1}(s)| ds \le \left(\int_{I_{p+1}} ds \right)^{1/2} \left(\int_{I_{p+1}} |u_{r1}(s)|^2 ds \right)^{1/2}. \tag{3.40}$$

Since $\theta_p \le t \le \theta'_p \le T$, we have

$$\int_{I_{p+1}} |u_{r1}(s)|^2 ds \le \sum_{i=1}^{m} \int_0^t |u_i(s)|^2 ds \le \sigma^2 e^{2kt} \le \sigma^2 e^{2kT}$$

and similar to (3.39) for the first integral in the right hand side of (3.40) we get

$$\int_{I_{p+1}} ds \le \sum_{i=p+1}^{m} \int_{\theta_i}^{\theta'_i} ds \le \frac{4a_{p+1}}{\alpha}.$$

Then it follows from (3.40) that

$$\int_{I_{p+1}} |u_{r1}(s)| ds \le 2\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}. \tag{3.41}$$

Similarly, for the third integral in (3.38), we have

$$\int_{I_{p+1}} |u_{p1}(s)| ds \le 2\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}. \tag{3.42}$$

Combining (3.39), (3.41), and (3.42) we obtain from (3.37) that

$$|y(t) - z_p(t)| \le 8a_{p+1} + 4\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}} \le 6\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}$$

using the inequality

$$16a_{p+1} < 2\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}$$

which follows from the inequalities $a_{p+1} \le a_1 < \frac{\sigma^2}{8\alpha}$ (see (3.4)).

Thus,

$$|y(t) - z_p(t)| \le 6\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}}. \tag{3.43}$$

The proof of the lemma is complete. □

3.9. **Distance between evader and pursuer.** Using (3.29) and (3.43) we obtain

$$|y(t) - x_p(t)| \quad \ge \quad |x_p(t) - z_p(t)| - |z_p(t) - y(t)| \ge \frac{\alpha a_p^2}{6\sigma^2 e^{2kT}} - 6\sigma e^{kT} \sqrt{\frac{a_{p+1}}{\alpha}} = \frac{\alpha a_p^2}{12\sigma^2 e^{2kT}}$$

for $t \in [\theta_p, \theta'_p]$ since by (3.5)

$$a_{p+1} = \frac{\alpha^3}{4 \cdot 6^4 \sigma^6 e^{6kT}} a_p^4.$$

Also, it follows from the definition of β and the inequality $a_p < 1$ that

$$a_{p+1} \le \frac{\alpha}{16\sigma^2 e^{2kT}} a_p^4 \le \frac{\alpha}{12\sigma^2 e^{2kT}} a_p^2.$$

Therefore, (3.44) implies that $|y(t) - x_p(t)| > a_{p+1},\ \theta_p \le t \le \theta'_p$. Also, by (3.30)

$$y_2(t) - x_{p2}(t) \ge a_p, \quad t \ge \theta'_p.$$

Thus, starting from the time θ'_p we can ignore the pursuer x_p since $x_p(t) \ne y(t)$ for all $t \ge \theta'_p$ for this pursuer. We now can conclude that

(1) if $y_2^0 \ge x_{i2}^0$ for the pursuer x_i, then by item (ii) of Lemma 3.5 $y_2(t) > x_{i2}(t)$ for all $t > 0$ and hence $x_i(t) \ne y(t)$ for all $t > 0$. This means the evader ensures evasion from such a pursuer.

(2) if $x_{i2}^0 > y_2^0$ for all $i \in \{1, 2, ..., m\}$, then the a_i-approach of the pursuer x_i may occur at some θ_i. Then, as proved, we have

$$|y(t) - x_p(t)| \ge a_p,\ \text{for } 0 \le t \le \theta_p, \quad (\text{by definition of } \theta_p) \tag{3.44}$$

$$|y(t) - x_p(t)| \ge \frac{\alpha a_p^2}{12\sigma^2 e^{2kT}} > a_{p+1},\ \text{for } \theta_p \le t \le \theta'_p, \quad (\text{by } (3.44)) \tag{3.45}$$

$$y_2(t) - x_{p2}(t) \ge a_p,\ \text{for } t \ge \theta'_p, \quad (\text{by } (3.30)) \tag{3.46}$$

Based on these relations, we summarize as follows:

If an a_p-approach time θ_p of pursuer x_p to the evader y occurs, then $x_p(t) \ne y(t)$, for all $t \ge 0$ (see (3.44)- (3.46)). Moreover, for any $i \ge p+1$, there is no a_i-approach time θ_i of the pursuer x_p to the evader y. This means that even all the pursuers are in the upper half-plane, and the evader ensures evasion by applying its own maneuver.

The proof of Theorem 3.1 is completed.

4. Conclusion

We have analyzed a differential game of evasion from many pursuers. The control functions of the players are subject to exponential integral constraints. We have constructed a strategy for the evader and demonstrated that evasion is possible. The evader uses the control $v(t) = \left(0, \alpha + \left(\sum_{i=1}^m |u_i(t)|^2\right)^{1/2}\right)$ on the set $[0, T] \setminus I_1$ and applies a maneuver on the set I_1. The measure of the set I_1 can be made by choosing the parameters a_1 and α as small as we wish. We have also shown that all the approach times θ_i for each pursuer can occur only before a specific time T_0, and the approach times θ'_i satisfy $\theta'_i \le T$. The total number of approach times θ_i associated with all pursuers does not exceed the total number of pursuers, m. The evader uses the control $v(t) = \left(0, \left(\sum_{i=1}^m |u_i(t)|^2\right)^{1/2}\right)$ for $t \ge T$, and the approach time no longer occurs.

References

[1] Azamov A.A.; About the quality problem for simple pursuit games with constraint. Publ.Sofia: Serdica Bulgariacae math.,–1986.–12.–1.–P.38–43.

[2] Azamov A.A, Ibragimov G.I., Mamayusupov K., Ruziboev M.; On the Stability and Null-Controllability of an Infinite System of Linear Differential Equations. Journal of Dynamical and Control Systems.,–2023.–29.–P.595–605.

[3] Azamov A.A., Samatov B.T., Soyibboev U.B.; The Π-strategy when Players Move under Repulsive Forces. Contributions to Game Theory and Management, XVII.,–2025.–P.7–17.

[4] Azamov A.A., Turgunboyeva M.A.; The differential game with inertial players under integral constraints on controls. Vestnik Sankt-Peterburgskogo universiteta. Prikladnaya matematika. Informatika. Prosess upravleniya.,–2025.–21.–1.P.122–138.

[5] Azamov A.A., Ibaydullaev T.T., Ibragimov G.I.; Differential game with slow pursuers on the edge graph of a simplex. International Game Theory Review.,–2021.–23.–04.P.2250006.

[6] Chen J., Zha W., Peng Z., Gu D.; Multi-player pursuit-evasion games with one superior evader. Automatica.,–2019.–71.–P.24–32.

[7] Garcia E., Casbeer D.W., Von Moll A., and Pachter M.; Multiple pursuer multiple evader differential games. IEEE Transactions on Automatic Control.,–2021.–66.–5.–P.2345–2350.

[8] Ibragimov G.I., Ferrara M., Ruziboev M., and Pansera B.A.; Linear evasion differential game of one evader and several pursuers with integral constraints. International Journal of Game Theory.,–2021.–50.–P.729–750.

[9] Ibragimov G.I., Salimi M., and Amini M.; Evasion from many pursuers in simple motion differential game with integral constraints. European Journal of Operational Research.,–2012.–218.–P.505–511.

[10] Ibragimov G.I., Tursunaliev T.G., and Luckraz S.; Evasion in a linear differential game with many pursuers. Discrete and Continuous Dynamical Systems - Series B, (DCDS-B).,–29.–12.–P.4929–4945.

[11] Ibragimov G.I.; Evasion Differential Game of One Evader and Many Slow Pursuers. Dynamic Games and Applications.,–2024.–14.–3.–P.665–685.

[12] Ibragimov G.I., Salleh Y., Alias I.A., Pansera B.A., Ferrara M.; Evasion from Several Pursuers in the Game with Coordinate-wise Integral Constraints. Dynamic Games and Applications.,–2023.–13.–3.–P.819–842.

[13] Ibragimov G.I., Tursunaliev T.G.; Evasion problem in a differential game with geometric constraints. Uzbek Mathematical Journal.,–2024.–68.–2.–P.81–91.

[14] Kuchkarov A., Ibragimov G.I., Ferrara M.; Simple motion pursuit and evasion differential games with many pursuers on manifolds with Euclidean metric. Discrete Dynamics in Nature and Society.,–2016.–2016.–1.–P.1386242.

[15] Kuchkarov A.Sh., Ibragimov G.I., and Khakestari M.; On a linear differential game of optimal approach of many pursuers with one evader. Journal of Dynamical and Control Systems.,–2013.–19.–P.1–15.

[16] Kumkov S.S., Le Mnec S., Patsko V.S.; Zero-Sum Pursuit-Evasion Differential Games with Many Objects: Survey of Publications. Dynamic Games and Applications.,–2017.–7.–P.609–633.

[17] Mamadaliev N.A., Ibaydullaev T.T., Abdualimova G.M.; On Game Problems of Controlling Pencil Trajectories under Integral Constraints on the Controls of Players. Russian Mathematics.,–2023.–67.–6.P.34-45.

[18] Pansera B.A., Ibragimov G.I and Luckraz S.; On the Existence of an Evasion Strategy in a Linear Differential Game with Integral Constraints. Dynamic Games Applications.,–2025.–P.1–16.

[19] Petrosyan L.A. (1993). Differential Games of Pursuit. World Scientific, Singapore, London.

[20] Petrov N.N. (2024). Double capture of coordinated evaders in recurrent differential games. News of the Institute of Mathematics and Informatics of Udmurt State University. 63: 49–60. https://doi.org/10.35634/2226-3594-2024-63-04.

[21] Pontryagin L.S.; Selected works. Nauka, Moscow.,–1988.

[22] Ruziboev M., Ibragimov G., Mamayusupov Kh., Khaitmetov A., Pansera B.A.; On a Linear Differential Game in the Hilbert Space l_2. Mathematics.,–2023.–11.–24.–P.49–87.

[23] Ruziboev M., Zaynabiddinov I.; Pursuit Evasion Differential Games in on a Finite Time Interval. Lobachevskii Journal of Mathematics.,–2025.–46.–2.–P.852–862.

[24] Rilwan J., Ferrara M., Badakaya A.J., Bruno A.P.; On pursuit and evasion game problems with Grounwall-type constraints. Quality and Quantity.,–2023.–57.–6.–P.5551–5562.

[25] Samatov B.T., Soyibboev U.B.; Applications of the Π-strategy when players move with acceleration. Proceedings of the IUTAM Symposium on Optimal Guidance and Control for Autonomous Systems.,–2024.–40.–P.165–181.

[26] Samatov B.T, Uralova S.I.; Pursuit-Evasion linear differential games with exponential integral constraints. Scientific Bullettin of NamSU.,–2022.–8.–P.9–14.

[27] Samatov B.T, Uralova S.I.; Pursuit-Evasion problems for one linear case under La-Constraints. Bulletin of the Institute of Mathematics.,–2023.–6.–1.–P.48–57.

[28] Satimov N.Yu.; One Pursuit Method in Linear Differential Games. Dokl Akad Nauk R Uz.,–1981.–6.–P.7–8.

[29] Satimov N.Yu.; Methods of Solving the Pursuit Problems in the Theory of Differential Games. Izd-vo NBRUz.,–2019. Tashkent.

[30] Sharifi S., Badakaya A.J., Salimi M.; On game value for a pursuit-evasion differential game with state and integral constraints. Japan Journal of Industrial and Applied Mathematics.,–2022.–39.–P.653–668.

[31] Salimi M., Ferrara M.; Differential game of optimal pursuit of one evader by many pursuers. International Journal of Game Theory.,–2019.–48.–P.481–490.

[32] Shchelchkov K.A.; Estimate of the capture time and construction of the pursuers strategy in a nonlinear two-person differential game. Differential Equations.,–2022.–58.–2.–P.264–274.

[33] Scott W.L., Leonard N.E.; Optimal evasive strategies for multiple interacting agents with motion constraints. Automatica J. IFAC.,–2018.–94.–P.26–34.

[34] Von Moll A., Casbeer D., Garcia E., Milutinovic D., Pachter M.; The Multi-pursuer Single-Evader Game. Journal of Intelligent & Robotic Systems.,–2019.–96.–P.193–207.

[35] Zhao S., Zhang H., Lv R.; Optimal Evasion Strategy of Three-Dimensional Differential Game Based on RHC. Proceedings of 2023 7th Chinese Conference on Swarm Intelligence and Cooperative Control. CCSICC 2023. Springer, Singapore.,–2024.–1203.

[36] Zhang H., Zhao W., Ge H., Xie X., Yue D.; Distributed Model-Free Optimal Control for Multiagent Pursuit-Evasion Differential Games. IEEE Transactions on Network Science and Engineering.,–2024.–11.–4.–P.3800–3811.

[37] Zhou P., Chen B.M.; Distributed Optimal Solutions for Multiagent Pursuit-Evasion Games. 2023 62nd IEEE Conference on Decision and Control (CDC), Singapore, Singapore.,–2023.–P.6424–6429.

Ibragimov G.I. ,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan,
University of Public Safety of
the Republic of Uzbekistan,
Tashkent, Uzbekistan.
email: ibragimov.math@gmail.com

Tursunaliev T.G.,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan,
Tashkent International University of
Financial Management and Technologies,
Department of Mathematics,
Tashkent, Uzbekistan.
email: toychivoytursunaliyev@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.128-140
DOI: 10.29229/uzmj.2025-4-13

On a Boundary Value Problem for a Loaded Parabolic-Hyperbolic Equation of the Second Kind Degenerating Inside the Domain

Islomov B.I., Nasirova D.A.

Abstract. This paper studies an analog of the Tricomi problem for a loaded parabolic-hyperbolic equation of the second kind, which degenerates inside the domain. In proving the existence and uniqueness theorem for a classical solution to the Tricomi-type problem, a general representation of the solution to the loaded parabolic-hyperbolic equation degenerating within the domain is derived. The uniqueness of the solution is established using the extremum principle and the energy integral method. The existence of the solution is equivalently reduced to integral equations of the second kind, specifically Volterra and Fredholm equations, which remain relatively unexplored. Furthermore, a class of prescribed functions is determined to ensure the solvability of the obtained integral equations.

Keywords: Second-kind equation, loaded equation, extremum principle, method of energy integrals, Fredholm integral equation of the second kind.

MSC (2020): 35M10, 35M12, 35L80, 35K65, 35K20.

1. Introduction

Many important problems in mathematical physics and biology, particularly long-term groundwater prediction and regulation problems [1], modeling of particle transport processes [2], heat and mass transfer problems with finite velocity, liquid filtration modeling in porous media [3], inverse problem studies [4], and various optimal control problems in agroecosystems [5] lead to boundary value problems for loaded partial differential equations.

The term "loaded equation" first appeared in [6]. The general definition of loaded equations, which is now widely accepted in scientific literature, was introduced by A.M. Nakhushev in 1976. In [7], a more comprehensive definition and detailed classification of various types of loaded equations were provided, including loaded differential, integral, integro-differential, and functional equations, along with their numerous applications.

Boundary value problems for non-degenerate loaded equations of mixed type of the second and third orders, where the loaded part contains a trace or derivative of the unknown function, have been studied in [8–20]. For degenerate-loaded equations of hyperbolic, parabolic, and mixed types, investigations have been carried out in [21–26]. A three-dimensional analog of the Tricomi problem for a loaded parabolic-hyperbolic equation was examined in [27]. The theory of boundary value problems for second-order loaded equations with integro-differential operators has been explored in [28–33].

In the study of degenerate loaded equations of mixed type of the second kind, difficulties arise associated with the lack of a general representation of the solution, as well as the impossibility of direct application of classical methods. This problem is solved in this paper. A new method for constructing a representation of the general solution of a loaded parabolic-hyperbolic equation of the second kind in a form convenient for further studies of various boundary value problems is developed, and a new type of extremum principle for a degenerate loaded parabolic-hyperbolic equation of the second kind is proved. The analysis of the state of affairs in this direction shows that boundary value problems for degenerate loaded equations leading to less studied integral Volterra and Fredholm equations with shifts.

In this paper, we study problems with the Tricomi condition for a loaded parabolic-hyperbolic equation of the second kind, degenerating inside the domain. We prove the existence and uniqueness theorems of the classical solution of the problems posed. The proofs of the theorem are based on energy identities and the extremum principle, as well as on the theory of Volterra and Fredholm integral equations.

2. Formulation of Problem A_T

Let Ω be a finite simply connected domain in the plane of variables x, y bounded by curves:

$$S_j: \quad (-1)^{j-1}x = 1,\ 0 < y < 1, \quad S_3: \quad 0 < x < 1,\ y = 1, \quad S_4: \quad -1 < x < 0,\ y = 1,$$

$$\Gamma_j: (-1)^{j-1}x - \frac{2}{2-m}(-y)^{2/(2-m)} = 0,\ \Gamma_{j+2}: (-1)^{j-1}x + \frac{2}{2-m}(-y)^2/(2-m) = 1,\ y < 0, (j = 1, 2).$$

We introduce denotations

$$\Omega_j^+ = \Omega \cap \left\{(x,y) : (-1)^{j-1}x > 0,\ y > 0\right\}, \Omega_j^- = \Omega \cap \left\{(x,y) :\ (-1)^{j-1}x > 0,\ y < 0\right\},$$

$$I_j = \left\{(x,y) :\ 0 < (-1)^{j-1}x < 1,\ y = 0\right\},\ \Omega_j = \Omega_j^+ \cup \Omega_j^- \cup I_j,\ I_3 = \left\{(x,y) :\ x = 0,\ 0 < y < 1\right\},$$

$$\Omega_3 = \Omega_1^+ \cup \Omega_2^+ \cup I_3, \quad A_j\left((-1)^{j-1}, 0\right) = \bar{I}_j \cap \bar{S}_j,\ O\,(0,0) = \bar{I}_1 \cap \bar{I}_2,\ B_1\,(1,1) = \bar{S}_1 \cap \bar{S}_3,$$

$$B_2\,(-1,1) = \bar{S}_2 \cap \bar{S}_4,\ B_0\,(0,1) = \bar{S}_3 \cap \bar{S}_4,\ C_j\left[\tfrac{(-1)^{j-1}}{2}; -\left((-1)^{j-1}\,\tfrac{2-m}{4}\right)^{2/(2-m)}\right] = \bar{\Gamma}_j \cap \bar{\Gamma}_{j+2}, (j = 1, 2).$$

Next, we assume that the domains Ω_1^+ and Ω_1^- should be symmetric domains of Ω_2^+ and Ω_2^- with respect to the axes Oy.

In the domain Ω we consider the following equation

$$0 = \begin{cases} u_{xx} - ((-1)^{j-1}x)^p u_y - \rho_j u\,(x,0), & (x,y) \in \Omega_j^+, \\ u_{xx} - (-y)^m u_{yy} + \mu_j u\,(x,0), & (x,y) \in \Omega_j^-, \end{cases} \tag{2.1}$$

where m, p, ρ_j, μ_j $(j = 1, 2)$ are any real numbers, and

$$0 < m < 1, \quad p > 0, \quad \rho_j > 0, \quad \mu_j > 0, \quad (j = 1, 2). \tag{2.2}$$

In the domain Ω for equation (2.1) we investigate the following problem.

Problem A_T. Find a function $u(x,y)$ that satisfies the following properties:

1) $u(x,y) \in (\overline{\Omega}) \cap C^1(\Omega)$; 2) $u(x,y) \in C^{2,1}_{x,y}(\Omega_1^+ \cup \Omega_2^+)$ and it is a regular solution of equation (2.1) in the domain Ω_j^+; 3) $u(x,y)$ is a generalized solution of equation (2.1) from the class R_2 [34] in the domain Ω_j^- $(j = 1, 2)$; 4) $u(x,y)$ satisfies the boundary conditions:

$$u|_{S_j} = \varphi_j(y), \quad 0 \le y \le 1, \tag{2.3}$$

$$u|_{\Gamma_j} = \psi_j(x), \qquad 0 \le (-1)^{j+1}x \le \frac{1}{2}, \tag{2.4}$$

5) $u(x,y)$ satisfies the matching conditions on the degeneration line I_i $\left(i = \overline{1,3}\right)$:

$$\lim_{y\to+0} u(x,y) = \lim_{y\to-0} u(x,y), \quad \lim_{y\to+0} u_y(x,y) = \lim_{y\to-0} u_y(x,y), \quad (x,0) \in I_j\ (j = 1, 2), \tag{2.5}$$

$$\lim_{x\to+0} u(x,y) = \lim_{x\to-0} u(x,y), \quad \lim_{x\to+0} u_x(x,y) = \lim_{x\to-0} u_x(x,y), \quad (0,y) \in I_3, \tag{2.6}$$

where $\varphi_j(y)$, $\psi_j(x) (j = 1, 2)$ are given function, and $\psi_1\,(0) = \psi_2\,(0)$,

$$\varphi_j(y) \in C(\bar{I}_3) \cap C^1(I_3), \tag{2.7}$$

$$\psi_1(x) \in C^2\left[0, \frac{1}{2}\right], \ \ \psi_2(x) \in C^2\left[-\frac{1}{2}, 0\right]. \tag{2.8}$$

3. Investigation of Problem A_T for equation (2.1)

If the conditions 1) and 2) of Problem A_T, any regular solution of equation (2.1) can be represented as in [16], ([35],p.3-6):

$$u(x,y) = \upsilon(x,y) + \omega(x) \tag{3.1}$$

where

$$\upsilon(x,y) = \begin{cases} \upsilon_j(x,y), & (x,y) \in \Omega_j^+, \\ w_j(x,y) & (x,y) \in \Omega_j^-, \end{cases} \tag{3.2}$$

$$\omega(x) = \begin{cases} \omega_j^+(x), & (x,+0) \in \bar{I}_j, \\ \omega_j^-(x), & (x,-0) \in \bar{I}_j, \end{cases} \tag{3.3}$$

here $\upsilon_j(x,y)$ and $w_j(x,y)(j=1,2)$ are regular solutions of the equation

$$L\upsilon_j \equiv \upsilon_{j\,xx} - ((-1)^{j-1}x)^p \upsilon_{jy} = 0, \qquad (x,y) \in \Omega_j^+, \tag{3.4}$$

$$Lw_j \equiv w_{j\,xx} - (-y)^m w_{j\,yy} = 0, \qquad (x,y) \in \Omega_j^- \quad (j=1,2) \tag{3.5}$$

and $\omega_j^+(x)$ and ω_j^- $(j=1,2)$ are arbitrary twice continuously differentiable solutions of the equations

$$\omega_j^{+''}(x) - \rho_j \omega_j^+(x) = \rho_j \upsilon_j(x,0), \qquad (x,0) \in I_j, \tag{3.6}$$

$$\omega_j^{-''}(x) + \mu_j \omega_j^-(x) = -\mu_j w_j(x,0), \qquad (x,0) \in I_j. \tag{3.7}$$

Since the function $ax+b$ is a solution to equations (3.4) and (3.6), arbitrary functions $\omega_j^+(x)$ and $\omega_j^-(x)$ can be chosen to satisfy the conditions

$$\omega_j^+\left((-1)^{j+1}\right) = \omega_j^{+'}\left((-1)^{j+1}\right) = 0, \tag{3.8}$$

$$\omega_j^-(0) = \omega_j^{-'}(0) = 0, \quad (j=1,2). \tag{3.9}$$

The solution to the Cauchy problem (3.7), (3.9) and (3.8), (3.10) is given by:

$$\omega_j^+(x) = \sqrt{\rho_j} \int_{(-1)^{j-1}}^{x} \tau_j(t)\, sh\sqrt{\rho_j}(x-t)dt, \qquad (x,0) \in \bar{I}_j, \tag{3.10}$$

and

$$\omega_j^-(x) = -\sqrt{\mu_j} \int_0^x \tau_j(t) sin\sqrt{\mu_j}\,(x-t)\,dt, \qquad (x,0) \in \bar{I}_j, \tag{3.11}$$

where

$$\tau_j(x) \equiv \upsilon_j(x,+0) = w_j(x,-0), \qquad (x,0) \in \bar{I}_j (j=1,2). \tag{3.12}$$

In view of (2.1),(2.3), (2.4), (3.2), (3.3), (3.9), (3.10) Problem A_T reduces to Problem A_T^* for the equation

$$0 = \begin{cases} L\,\upsilon_j, & (x,y) \in \Omega_j^+, \\ L\,w_j, & (x,y) \in \Omega_j^- \end{cases} \tag{3.13}$$

with boundary conditions

$$\upsilon_j|_{S_j} = \varphi_j(y), \;\; 0 \le y \le 1, \tag{3.14}$$

$$w_j|_{\Gamma_j} = \psi_j(x) - \omega_j^-(x), \qquad 0 \le (-1)^{j+1}x \le \frac{1}{2}, \tag{3.15}$$

where $\omega_j^-(x)$ is determined from (3.12), $(j=1,2)$.

4. Uniqueness of solution of the Problem A_T

To prove the uniqueness of the solution of Problem A_T, we first prove the uniqueness of the solution of Problem A_T^* for equation (3.14). The following lemmas play an important role in proving the uniqueness of the solution to Problem A_T^* for equation (3.14).

Lemma 1. *If the conditions (2.2), $-1 < 2\beta < 0$, $p + 2\beta > 0$, $\varphi_1(y) \equiv \varphi_2(y) \equiv 0$, $\forall y \in [0,1]$ and $\psi_1(x) \equiv 0$, $\forall x \in \left[0; \frac{1}{2}\right]$, $\psi_2(x) \equiv 0$, $\forall x \in \left[-\frac{1}{2}; 0\right]$ are satisfied, then*

$$\tau_j(x) \equiv 0, \quad \forall (x,0) \in \bar{I}_j, \tag{4.1}$$

where $\tau_j(x)$ is defined from (3.13) $(j = 1,2)$, $2\beta = m/(m-2)$.

Lemma 1 is proven in exactly the same way as in ([25], p. 39-41, Lemma 3.1).

From (4.1), using (3.11) and (3.12) , we obtain

$$\omega_j^+(x) = \omega_j^-(x) = 0, \quad \forall x \in \bar{I}_j.$$

Thus, from (3.3), we have

$$\omega(x) \equiv 0, \ \ \forall\, x \in \bar{I}_1 \ \cup \bar{I}_2\,. \tag{4.2}$$

Lemma 2. *The solution $\upsilon(x,y) \in C\left(\bar{\Omega}_3\right) \cap C^1(\Omega_3) \cap C_{x,y}^{2,1}(\Omega_1^+ \cup \Omega_2^+)$ of equation (3.4) in the closed domain $\bar{\Omega}_3$ attains its positive maximum and negative minimum only on $\overline{A_1B_1} \cup \overline{A_2B_2} \cup I_1 \cup I_2$.*

Proof. By the extremum principle for parabolic equations [36, 37, 38], the solution of equation (3.4) inside the domain Ω_1^+ and Ω_2^+ cannot attain its positive maximum and negative minimum. We will show that the solution $\upsilon(x,y)$ of equation (3.4) in the domain Ω_3 does not attain its positive maximum (or negative minimum) on $\bar{I}_3$.

Assume the contrary. Suppose that $\upsilon(x,y)$ attains its positive maximum (or negative minimum) at some point $(0, y_0)$ on the interval I_3. Then, based on the extremum principle [36, 39] from the domain Ω_1^+ **,** we have

$$\upsilon_x(+0, y_0) < 0 \ (> 0). \tag{4.3}$$

On the other hand, from the domain Ω_2^+ **,** we obtain

$$\upsilon_x(-0, y_0) > 0 \ (< 0).$$

This inequality, due to the matching condition $\upsilon_x(+0,y) = \upsilon_x(-0,y)$, $(0,y) \in I_3$, contradicts the relation (4.3). Therefore, $\upsilon(x,y)$ does not attain its positive maximum (or negative minimum) on the interval I_3.

By condition (3.10) from (3.16), and considering (3.2) and $\psi_1(0) = \psi_2(0) = 0$, it follows that $\upsilon(0,0) = w_j(0,0) = \upsilon_j(0,0) = 0$. Thus, $\upsilon(x,y)$ does not attain its extremum at the point $O(0,0)$**.**

Using Lemmas 1.1 and 1.2 ([36], ch. 2, § 2.3. p. 93-94), it can be proven that at the point $B_0\,(0,1)$, there is no positive maximum (or negative minimum).

Therefore, $\upsilon\,(x,y)$ does not attain its positive maximum (or negative minimum) on the interval $\bar{I}_3$. □

Theorem 1. *If the conditions of Lemmas 1-2 and (4.2) are satisfied, then the solution of Problem A_T^* for equation (3.14) is unique in the domain Ω.*

Proof. According to the maximum principle for parabolic equations [37, 38, 40], and considering Lemma 2, the Problem with conditions(3.13) and (3.15) for equation (3.4) in the domain $\bar{\Omega}_3$ with $\tau_j(x) \equiv \varphi_j(y) \equiv 0 (j = 1,2)$, has no non-zero solution, i.e.

$$\upsilon_j(x,y) \equiv 0 \quad in \quad \bar{\Omega}_j^+ \ (j = 1,2). \tag{4.4}$$

Due to the uniqueness of a solution of the Cauchy problem with homogeneous conditions $w_j(x,y)\big|_{y=0} = 0$, $(x,0) \in \bar{I}_j$, $w_{j\,y}(x,y)\big|_{y=0} = 0$, $(x,0) \in I_j$ for equation (3.6) in the domain Ω_j, it follows that

$$w_j(x,y) \equiv 0, (x,y) \in \bar{\Omega}_j. \tag{4.5}$$

Due to (4.4) and (4.5), from (3.2), we have

$$\upsilon(x,y) \equiv 0, \ \ (x,y) \in \bar{\Omega}\ . \tag{4.6}$$

From (4.6) the uniqueness of the solution of Problem A_T^* for equation (3.14) follows. □

Theorem 2. *If the conditions of Theorem 1 are satisfied, then the solution of Problem* A_T *for equation (2.1) is unique in the domain* Ω.

Proof. From (4.2), (4.6) and (3.1), it follows that $u(x, y) \equiv 0, \ (x, y) \in \bar{\Omega}$. Thus, the uniqueness of the solution of Problem A_T for equation (2.1) follows. □

5. Existence of a Solution to Problem A_T

The existence of a solution to Problem A_T is proven using the method of integral equations. To prove the existence of a solution to Problem A_T, we first prove the existence of a solution to Problem A_T^* for equation (3.14) with the conditions (3.15) and (3.16).

Theorem 3. *If the conditions (2.2), (2.8), (2.9) and*

$$-1 < 2\beta < 0, \ \ p + 2\beta > 0, \ \ \psi_1(0) = \psi_2(0) = 0 \tag{5.1}$$

are satisfied, then the solution to Problem A_T^* *exists in the domain* Ω .

Proof. The proof of Theorem 3 relies on the following problems, which have independent interest.

Problem $B_j (j = 1, 2)$. Find a solution $\upsilon(x, y) \in C(\bar{\Omega}_j) \cap C^1(\Omega_j) \cap C^{2,1}(\Omega_j^+) \cap C^2(\Omega_j^-)$ to equation (3.14), satisfying the conditions (3.15), (3.16) and

$$\upsilon(0, y) = \tilde{\tau}_3(y) - \omega_j^+(0), \quad (0, y) \in \bar{I}_3, (j = 1, 2), \tag{5.2}$$

where $\tilde{\tau}_3(y) = u(0, y), \ (0, y) \in \bar{I}_3$ is a given function, and $\tilde{\tau}_3(0) = \psi_1(0) = \psi_2(0) = 0$,

$$\tilde{\tau}_3(y) \in C(\bar{I}_3) \cap C^1(I_3), \tag{5.3}$$

and $\omega_j^+(0)$ is determined from (3.11) .

Problem B_3. Find a solution $\upsilon(x, y) \in C(\bar{\Omega}_3) \cap C^1(\Omega_3) \cap C^{2,1}_{x,y}(\Omega_1^+ \cup \Omega_2^+)$ to equation (3.4), satisfying the conditions (3.15) and

$$\upsilon(x, y)|_{y=0} = \tau_j(x) - \omega_j^+(x), \qquad (x, 0) \in \bar{I}_j, \tag{5.4}$$

where $\tau_j(x)$ and $\omega_j^+(x)(j = 1, 2)$ are determined respectively from (3.11), and

$$\tau_j(x) = w_j(x, -0) = \upsilon_j(x, +0), \quad (x, 0) \in \bar{I}_j.$$

5.1. Investigation of Problem $B_j, \ (j = 1, 2)$.

Theorem $4_j, (j = 1, 2)$. *If the conditions (2.2), (2.8), (2.9), (5.1), and (5.3) are satisfied, then a solution to Problem* B_j, *exists and is unique in the domain* Ω_j.

Proof. By Lemma 1 and from the extremum principle for degenerate parabolo-hyperbolic equations [36], it follows that the solution $\upsilon(x, y)$ to Problem B_j with $\psi_j(x) \equiv 0$ has a positive maximum (PM) and a negative minimum (NM) in the closed domain $\bar{\Omega}_j^+$, which is attained only at $\overline{S}_j \cup \bar{I}_3, \ (j = 1, 2)$.

According to the extremum principle, the homogeneous Problem B_j, taking into account conditions (3.3) and (4.2), i.e., the problem with zero boundary conditions, has no solution other than zero. This implies the uniqueness of the solution to Problem B_j.

Now we proceed to prove the existence of a solution to the problem with conditions (3.15), (3.16) and (5.2).

The generalized solution of the class R_2 [25, 34] of the Cauchy problem with initial conditions

$$\lim_{y \to 0-} w_j(x, y) = \tau_j(x), (x, 0) \in \bar{I}_j, \quad \lim_{y \to 0-} w_{jy}(x, y) = \nu_j(x), \quad (x, 0) \in I_j,$$

for equation (3.6) in the domain Ω_j^- $(j = 1, 2)$ is given by the formula

$$w_j(\xi, \eta) = (-1)^{j-1} \left[\int_0^\xi (\eta - t)^{-\beta} (\xi - t)^{-\beta} T_j(t) \, dt + \int_\xi^\eta (\eta - t)^{-\beta} (t - \xi)^{-\beta} N_j(t) \, dt \right], \tag{5.5}$$

where $\xi=(-1)^{j-1}x-\frac{2}{2-m}(-y)^{\frac{2-m}{2}}, \qquad \eta=(-1)^{j-1}x+\frac{2}{2-m}(-y)^{\frac{2-m}{2}},$

$$N_j\left(x\right)=T_j\left(x\right)/2\cos\pi\beta-\gamma_2\nu_j\left(x\right), \quad \gamma_2=\left[2\left(1-2\beta\right)\right]^{2\beta-1}\frac{\Gamma\left(2-2\beta\right)}{\Gamma^2\left(1-\beta\right)}. \tag{5.6}$$

$$\tau_j(x)=(-1)^{j-1}\int_0^x\frac{T_j(t)dt}{((-1)^{j-1}(x-t))^{2\beta}}=\Gamma(1-2\beta)(-1)^{j-1}D_{0x}^{2\beta-1}T_j(x),(x,0)\in\bar{I}_j, \tag{5.7}$$

where the function $T_j(x)$ is continuous in I_j and integrable on $\bar{I}_j$, and $\tau_j\left(x\right)$ and becomes zero of order no less than at $1-2\beta$ when $x\to 0$, and $D_{0x}^{2\beta-1}\left[\bullet\right]$ is the integral-differential operator of fractional order ([41], p. 42-43).

By setting $\xi=0$, $\eta=x$, in (5.5) considering (3.16), (5.6), and the properties of integral-differential operators of fractional order ([41], p. 42-43), ([42], p. 16-21), we obtain

$$T_j(x)=\gamma_3\nu_j(x)+\frac{2(-1)^{j-1}\cos\pi\beta}{\Gamma(1-\beta)}((-1)^{j-1}x)^{\beta}D_{0x}^{1-\beta}\left[\psi_j(x)-\omega_j^-(x)\right], \quad (x,0)\in I_j, \tag{5.8}$$

where $\gamma_3=2\gamma_2\cos\pi\beta$, and $\omega_j^-(x)$ is determined from (3.12) $(j=1,2)$,

Substituting (5.8) into (5.7), we find the first functional relationship between $\tau_j(x)$ and $\nu_j(x)$, transferred from the domain Ω_j^- to domain I_j $(j=1,2)$:

$$\tau_j(x)=\Gamma(1-2\beta)\gamma_3(-1)^{j-1}D_{0x}^{2\beta-1}\nu_j(x)+\Psi_j(x), \quad (x,0)\in\bar{I}_j, \tag{5.9}$$

where

$$\Psi_j(x)=\frac{2\Gamma(1-2\beta)\cos\pi\beta}{\Gamma(1-\beta)}D_{0x}^{-(1-2\beta)}(\pm x)^{\beta}D_{0x}^{1-\beta}\left[\psi_j(x)-\omega_j^-(x)\right].$$

Therefore, as in the work ([43], p.39-48), according to conditions 1) - 2) of Problem A_T, by taking the limit as $y\to+0$ in equation (3.4) and condition (3.15), (5.4) considering (5.1), we obtain

$$\upsilon_j(x,+0)=\tau_j(x), \ \ (x,0)\in\bar{I}_j, \quad \upsilon_{j\,y}(x,+0)=\nu_j(x), \quad (x,0)\in I_j$$

it follows that

$$\tau_j''(x)=\left((-1)^{j-1}x\right)^p\nu_j(x), \tag{5.10}$$

$$\tau_j(0)=-\omega_j^+(0),\tau_j\left((-1)^{j-1}\right)=\varphi_j(0), \tag{5.11}$$

where $\omega_j^+(x)\,(j=1,2)$ is determined from (3.11).

Solving the problem (5.10) and (5.11), we obtain the functional relationship between $\tau_j(x)$ and $\nu_j(x)$, transferred from the domain Ω_j^+ to domain $I_j(j=1,2)$:

$$\tau_j(x)=(-1)^{j-1}\int_0^{(-1)^{j-1}}G_j\left(x,t\right)\left((-1)^{j-1}t\right)^p\nu_j(t)dt+f_j(x), \quad x\in\bar{I}_j, \tag{5.12}$$

where

$$G_1\left(x,t\right)=\begin{cases}(x-1)\,t, & 0\le t\le x,\\ (t-1)\,x, & x\le t\le 1,\end{cases} \qquad G_2\left(x,t\right)=\begin{cases}(t+1)\,x, & -1\le t\le x,\\ (x+1)\,t, & x\le t\le 0,\end{cases}$$

$$f_j(x)=-\omega_j^+(0)+(-1)^{j-1}x\left[\varphi_j\left(0\right)+\omega_j^+(0)\right]. \tag{5.13}$$

By virtue of (3.11), from (5.12) and (5.13), we obtain the Fredholm integral equation of the second kind with respect to: $\tau_j(x)$ $(j=1,2)$:

$$\tau_j(x)+\int_0^{(-1)^{j-1}}K_j\left(x,t\right)\tau_j(t)dt=\Phi_j(x), \quad (x,0)\in\bar{I}_j, \tag{5.14}$$

where

$$K_j\left(x,t\right)=\sqrt{\rho_j}\left(1-(-1)^{j-1}x\right)sh\sqrt{\rho_j}\;t\,,$$

$$\Phi_j(x) = (-1)^{j-1}\int_0^{(-1)^{j-1}} G_j(x,t)\left((-1)^{j-1}t\right)^p \nu_j(t)dt + (-1)^{j-1}x\varphi_j(0). \tag{5.15}$$

According to the theory of Fredholm integral equations ([44], ch.1, § 15) and from the uniqueness of the solution to Problem B_j, we conclude that the integral equation (5.19) is uniquely solvable in the class $C^1(\bar{I}_j)\cap C^2(I_j)$, $(j=1,2)$ and its solution is given by the formula:

$$\tau_j(x) = \Phi_j(x) - \int_0^{(-1)^{j-1}} K_j^*(x,t)\Phi_j(t)dt, \quad (x,0)\in\bar{I}_j, \tag{5.16}$$

where $K_j^*(x,t)$ is the resolvent of the kernel $K_j(x,t)$ $(j=1,2)$ (see [44], p. 81-88).

Eliminating $\tau_j(x)$ from (5.9) and (5.16), considering the matching condition and (3.12), we obtain the integral equation with respect to: $\nu_j(x)\,(j=1,2)$:

$$\nu_j(x) - \int_0^{(-1)^{j-1}} M_j(x,t)\,\nu_j(t)dt = F_j(x), \quad (x,0)\in I_j, \tag{5.17}$$

where

$$M_j(x,t) = \frac{\left((-1)^{j-1}t\right)^p}{\gamma_3\Gamma(1-2\beta)}D_{0x}^{1-2\beta}G_j(x,t) - \frac{\left((-1)^{j-1}t\right)^p}{\gamma_3\Gamma(1-2\beta)}\int_0^{(-1)^{j-1}} G_j(z,t)\,D_{0x}^{1-2\beta}K_j^*(x,z)dz-$$

$$-\frac{2(-1)^{j+1}\sqrt{\mu_j}\cos\pi\beta\cdot\left((-1)^{j-1}x\right)^\beta}{\gamma_3\Gamma(1-\beta)}\left((-1)^{j+1}t\right)^p D_{0x}^{1-\beta}\int_0^x sin\sqrt{\mu_j}\,(x-z)\,G_j(z,t)\,dz+$$

$$+\frac{2(-1)^{j-1}\sqrt{\mu_j}\cos\pi\beta\cdot\left((-1)^{j-1}x\right)^\beta}{\gamma_3\Gamma(1-\beta)}\left((-1)^{j-1}t\right)^p\times$$

$$\times\; D_{0x}^{1-\beta}\int_0^x sin\sqrt{\mu_j}\,(x-z)\,dz\int_0^{(-1)^{j-1}} K_j^*(t,s)G_j(s,z)\,ds, \tag{5.18}$$

$$F_j(x) = \frac{\varphi_j(0)}{\gamma_3\Gamma(1-2\beta)}D_{0x}^{1-2\beta}x - \frac{\varphi_j(0)}{\gamma_3\Gamma(1-2\beta)}\int_0^{(-1)^{j-1}} t\,D_{0x}^{1-2\beta}K_j^*(x,t)dt-$$

$$-\frac{2\cos\pi\beta}{\gamma_3\Gamma(1-\beta)}\left((-1)^{j-1}x\right)^\beta D_{0x}^{1-\beta}\psi_j(x) - \frac{2(-1)^{j-1}\sqrt{\mu_j}\varphi_j(0)\cos\pi\beta}{\gamma_3\Gamma(1-\beta)}\left((-1)^{j-1}x\right)^\beta\times$$

$$\times D_{0x}^{1-\beta}\int_0^x t\,sin\sqrt{\mu_j}\,(x-t)\,dt + \frac{2(-1)^{j-1}\sqrt{\mu_j}\varphi_j(0)\cos\pi\beta}{\gamma_3\Gamma(1-\beta)}\left((-1)^{j-1}x\right)^\beta\times$$

$$\times\; D_{0x}^{1-\beta}\int_0^x sin\sqrt{\mu_j}(x-t)dt\int_0^{(-1)^{j-1}} zK_j^*(t,z)dz. \tag{5.19}$$

Based on (2.2), (2.8), (2.9) and (5.1), considering the properties of the integral-differentiation operator, Beta, the hypergeometric function ([42], ch. 1, § 1, 2 and 4, p.4-32) and function $G_j(x,t)$ $(j=1,2)$ from (5.18) and (5.19), it follows that the kernel and the right-hand side of equation (5.17) admit estimates

$$|M_j(x,t)| \le C_j((-1)^{j-1}x)^{2\beta-1}((-1)^{j-1}t)^p, \tag{5.20}$$

$$|F_j(x)| \le C_{j+2}((-1)^{j-1}x)^{2\beta}, \quad C_k = const > 0, \;\; (k=\overline{1,4}). \tag{5.21}$$

Based on (2.2), (2.8), (2.9) and considering (5.21), we conclude that $F_j(x)\in C^1(I_j)$, where the function $F_j(x)\,(j=1,2)$ may have a singularity of order less than -2β as $(-1)^{j-1}x\to 0$, and it is bounded as $x\to(-1)^{j-1}$.

By virtue of (2.2),(5.20), (5.21), equation (5.17) is a Fredholm integral equation of the second kind. According to the theory of Fredholm integral equations ([44], ch.1, § 15) and from the uniqueness of the solution to the problem$j$, we conclude that the integral equation (5.17) is uniquely solvable in the class $C^1(I_j)$, and the function $\nu_j(x)$ may have a singularity of order less than -2β as $(-1)^{j-1}x \to 0$, and it is bounded as $x \to (-1)^{j-1}$, and its solution is given by the formula:

$$\nu_j(x) = F_j(x) + \int_0^{(-1)^{j-1}} M_j^*(x,t)\ F_j(t)dt, \quad (x,0) \in I_j, \tag{5.22}$$

where $M_j^*(x,t)\,(j=1,2)$ is the resolvent of the kernel $M_j(x,t)$ (see [44], p. 81-88).

Substituting (5.22) into (5.15) and (5.16), we find the function $\tau_j(x)$:

$$\tau_j(x) = (-1)^{j-1}\int_0^{(-1)^{j-1}} G_j(x,t)\left((-1)^{j-1}t\right)^p F_j(t)dt + (-1)^{j-1}\int_0^{(-1)^{j-1}} G_j(x,t)\left((-1)^{j-1}t\right)^p dt\times$$

$$\times\int_0^{(-1)^{j-1}} M_j^*(t,z)\ F_j(z)dz - (-1)^{j-1}\int_0^{(-1)^{j-1}} K_j^*(x,t)dt\int_0^{(-1)^{j-1}} G_j(t,s)\left((-1)^{j-1}s\right)^p F_j(s)ds-$$

$$(-1)^{j-1}\int_0^{(-1)^{j-1}} K_j^*(x,t)dt\int_0^{(-1)^{j-1}} G_j(t,z)\left((-1)^{j-1}z\right)^p dz\int_0^{(-1)^{j-1}} M_j^*(z,s)\ F_j(s)ds-$$

$$(-1)^{j-1}\varphi_j(0)\int_0^{(-1)^{j-1}} t\,K_j^*(x,t)dt + (-1)^{j-1}x\varphi_j(0),$$

and it belongs to the class

$$\tau_j(x) \in C^1(\bar{I}_j) \cap C^2(I_j),\ \ (j=1,2)\,. \tag{5.23}$$

Therefore, Problem B_j is uniquely solvable due to the equivalence of its Fredholm integral equation of the second kind (5.17).

Thus, the solution to Problem B_j can be recovered in the domain Ω_j^+ as the solution of the first boundary value problem for equation (3.4) [45], and in the domain Ω_j^- as the generalized solution of the Cauchy problem for equation (3.6) (see (5.5)).

From this, it follows that in the domain Ω_j, the solution to Problem B_j for equation (3.14) exists and is unique. *Theorem* $4_j, (j=1,2)$ *is proven.* □

5.2. **Investigation of Problem B_3 for Equation (3.4).**

Theorem 4_3. *Let the conditions (2.2), (2.8), (5.23) and* $-1 < 2\beta < 0,\quad p + 2\beta > 0$ *be satisfied, then in the domain* Ω_3*, the solution to Problem* B_3 *exists and is unique.*

Proof of Theorem 4_3. The solution to the first boundary value problem with conditions (3.15), (5.4) for equation (3.4) in the domain Ω_j^+ has the form:

$$\upsilon_j(x,y) = (-1)^{j+1}\left\{\int_0^{(-1)^{j+1}} R_j(x,t,y;\delta)\ \left((-1)^{j+1}t\right)^p \tau_j(t)dt+\right.$$

$$\left.+\frac{\partial}{\partial y}\int_0^y R_j^{(1)}(x,y-t;\delta)\left(\tilde{\tau}_3(t)-\omega_j^+(0)\right)d\,t - \frac{\partial}{\partial y}\int_0^y R_j^{(2)}(x,y-t;\delta)\ \varphi_j(t)d\,t\right\},\ \ (j=1,2) \tag{5.24}$$

and belongs to the class $\upsilon_j(x,y) \in C(\bar{\Omega}_j^+) \cap C_{x,y}^{2,1}(\Omega_j^+)$, if the conditions (2.8), (5.3), (5.23) are satisfied, where $R_j(x,t,y;\alpha)$ is the Green's function for the first boundary value problem for equation (3.4) in the domain $\Omega_j^+$is:

$$R_j(x,\xi,y;\delta) = \sum_{k=0}^{\infty} exp\left\{-\frac{\lambda_k^2 y}{4}\right\}\frac{(1-\delta)\sqrt{x\xi}}{J_{2-\delta}^2(\lambda_k)}\times$$

$$\times J_{1-\delta}\left(\lambda_k(1-\delta)\left((-1)^{j+1}x\right)^{1/2(1-\delta)}\right)J_{1-\delta}\left(\lambda_k(1-\delta)\left((-1)^{j+1}\xi\right)^{1/2(1-\delta)}\right), \tag{5.25}$$

$$R_j^{(1)}(x,y;\delta) = 1 + (-1)^j(1-\delta)^{2(1-\delta)}x-$$

$$-\int_0^{(-1)^{j+1}} R_j(x,t,y;\delta)\left[1+(-1)^j(1-\delta)^{2(1-\delta)}\xi\right]\left((-1)^{j+1}\xi\right)^p d\xi, \tag{5.26}$$

$$R_j^{(2)}(x,y;\delta)=(-1)^{j+1}(1-\delta)^{2(1-\delta)}x-$$

$$-\int_0^{(-1)^{j+1}} R_j(x,t,y;\delta)\left[(-1)^{j+1}(1-\delta)^{2(1-\delta)}\xi\right]\left((-1)^{j+1}\xi\right)^p d\xi, \tag{5.27}$$

$J_\theta(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\theta+2k}}{k!\Gamma(k+\theta+1)}$ is the Bessel function of the first kind [46], λ_k are the positive roots of the equation $J_{1-\delta}(\lambda_k)=0,\ \ k\in N\cup\{0\},\quad \delta=\frac{p+1}{p+2}$, where

$$\frac{1}{2}<\delta<1. \tag{5.28}$$

Note that in [46] the convergence of Bessel series and (5.25) was shown, and in [36], [45] the existence of integrals (5.26) and (5.27) was proven.

Differentiating (5.24) with respect to x and taking the limit as $(-1)^{j-1}x\to\ 0$, we obtain

$$\nu_3(y)=\frac{\partial}{\partial y}\int_0^y \mathrm{N}_j(y-t;\delta)\ \tilde{\tau}_3(t)dt+\mathrm{H}_j(y),\qquad (0,y)\in I_3, \tag{5.29}$$

where $\nu_3(y)=\upsilon_{j\,x}(0,y),\ \ (0,y)\in I_3$,

$$\mathrm{H}_j(y)=\lim_{x\to 0}(-1)^{j+1}\frac{\partial}{\partial x}\left\{\int_0^1 R_j(x,t,y;\delta)\ ((-1)^{j+1}t)^p\tau_j(t)dt-\right.$$

$$\left.-\,\omega_j^+(0)\frac{\partial}{\partial y}\int_0^y R_j^{(1)}(x,y-t;\delta)\,dt-\frac{\partial}{\partial y}\int_0^y R_j^{(2)}(x,y-t;\delta)\ \varphi_j(t)\,dt\right\}, \tag{5.30}$$

$$\mathrm{N}_j(y-t;\delta)\ \equiv(1-\delta)^{2\delta-1}(-1)^{j+1}\lim_{x\to 0}\frac{\partial}{\partial x}\left[R_j^{(1)}(x,y-t;\delta)\right]=$$

$$=(-1)^j\left\{(1-\delta)+\sum_{k=0}^{\infty}exp\left\{-\frac{\lambda_k^2(y-t)}{4}\right\}\frac{2^{2\delta}\lambda_k^{-2\delta}}{\Gamma^2(1-\delta)J_{2-\delta}^2(\lambda_k)}\right\},$$

where $\omega_j^+(0)\,(j=1,2)$ are determined from (3.11).

Based on the properties of the function $J_\theta(z)$ and $\mathrm{N}_j(y-t;\alpha)$ can be represented as ([46], p. 12):

$$\mathrm{N}_j(y-t;\delta)\ =\frac{(-1)^j}{\Gamma(1-\alpha)}(y-t)^{\delta-1}+B_j(y-t), \tag{5.31}$$

where the function $B_j(y-t)$ is continuously differentiable at $y\geq t$.

Substituting (5.31) into (5.29), we obtain the functional relationship between $\tau_3(y)$ and $\nu_3(y)$, transferred from the domain Ω_j^+ to domain I_3:

$$\nu_3(y)=\frac{(-1)^j\Gamma(\delta)}{(1-\delta)}D_{0y}^{1-\delta}\tilde{\tau}_3(y)+B_j(0)\,\tilde{\tau}_3(y)+\int_0^y B_j'(y-t)\,\tilde{\tau}_3(t)dt+\mathrm{H}_j(y). \tag{5.32}$$

Eliminating $\nu_3(y)$ from the relations (5.32) as $j=1$ and $j=2$, and then applying the integral operator $D_{0y}^{\delta-1}[\bullet]$ considering $\tilde{\tau}_3(0)=0$ and $D_{0y}^{\delta-1}D_{0y}^{1-\delta}\tilde{\tau}_3(y)=\tilde{\tau}_3(y)$, we obtain

$$\tilde{\tau}_3(y)-\int_0^y M_3(y,t)\,\tilde{\tau}_3(t)dt=\mathrm{H}_3(y),\quad (0,y)\in\bar{I}_3, \tag{5.33}$$

where

$$M_3(y,t)=\frac{1}{2\Gamma(\alpha)}\left\{\frac{B_2(0)-B_1(0)}{(y-t)^\delta}-\int\limits_t^y[B'_2(z-t)-B'_1(z-t)](y-z)^{-\delta}dz\right\}, \tag{5.34}$$

$$\mathrm{H}_3(y) = \frac{\Gamma(1-\delta)}{2(\delta)} D_{0y}^{\delta-1}\left[\mathrm{H}_2(y) - \mathrm{H}_1(y)\right], \tag{5.35}$$

here $\mathrm{H}_j(y)\ \ (j=1,2)$ are determined from (5.30).

By virtue of (2.2), (2.8), (5.28), (5.32) and the properties of the function $B_j\,(y-t)$ from (5.34) and (5.35), it follows that:

1) The kernel $M_3\,(y,t)$ is continuous in $\{(y,t): \ 0 \le t < y \le 1\}$ and, for $y \to t$ admit the estimate

$$|M_3\,(y,t)| \le C_5\,(y-t)^{-\delta}; \tag{5.36}$$

2) The function $H_3(y)$ belongs to the class $C\left(\bar{I}_3\right)\bigcap C^1\left(I_3\right)$ and admits the estimate

$$|\mathrm{H}_3(y)| \le C_6 y^{1-\delta} \tag{5.37}$$

where C_5 and C_6 are arbitrary positive constants.

From (5.36) and (5.37), it follows that the integral equation (5.33) is a Volterra integral equation of the second kind with a weak singularity. According to the theory of Volterra integral equations of the second kind [44], we conclude that the integral equation (5.33) is uniquely solvable in the class $C\left(\overline{I_3}\right)\bigcap C^1\left(I_3\right)$, and its solution is given by the formula:

$$\tilde{\tau}_3(y) = \int_0^y M_3{}^*\,(y,t)\,\mathrm{H}_3(t)dt + \mathrm{H}_3(y), \quad (0,y)\in \bar{I}_3, \tag{5.38}$$

where $M_3^*\,(y,t)$ is the resolvent kernel of $M_3\,(y,t)$.

Substituting (5.38) into (5.32), considering (5.36) and (5.37), we determine the function $\nu_3(y)$ from the class

$$\nu_3(y) \in C^1\left(I_3\right),$$

where the function $\nu_3(y)$ may have a singularity of order less than $1-\delta$ at $y \to 0$ and is bounded at $y \to 1$. Therefore, Problem B_3 is uniquely solvable.

Thus, the solution to Problem B_3 can be recovered in the domain $\Omega_j^+\ \ (j=1,2)$ as the solution to the first boundary value problem for equation (3.4) [45].

This completes the investigation of the existence of the solution to Problem B_3 for equation (3.4) in the domain Ω_3. *Theorem* 4_3 *is proven.* □

From Theorems 4_j and 4_3, it follows the existence of the solution to Problem A_T^* for equation (3.14) in the domain Ω. *Theorem 3 is proven.* □

We proceed to the proof of the existence of the solution to Problem A_T **.**

Theorem 5. *If the conditions (2.2), (2.8), (2.9) and (5.1) are satisfied, then the solution to Problem* A_T *exists in the domain* Ω *.*

*Proof.*Let the solution $u\,(x,y)$ to Problem A_T in the domain Ω with conditions (2.3), (2.4), (2.5), (2.7) exist, then, using the results of Theorems 4_1 and 4_2 (see Section 5.1), we recover the solution to the problem A_T . By virtue of (5.12), (5.22) from (3.11), (3.12), considering (3.1), (3.2), (3.3), we determine the functions $\omega_j^+(x)$ and $\omega_j^-(x)$. Then, in the domain Ω_j^+, the solution to Problem A_T is expressed as

$$u\,(x,y) = \upsilon_j\,(x,y) + \omega_j^+\,(x)\,,$$

where $\upsilon_j\,(x,y)$ is the solution to the first boundary value problem with conditions (3.15) and (5.2) for equation (3.4) [36, 45], here $\tilde{\tau}_3(y)$ is determined from formula (5.38), and in the domain Ω_j^- , it is expressed as:

$$u(x,y) = w_j(x,y) + \omega_j^-(x), (j=1,2)\,,$$

where $w_j(x,y)$ is the generalized solution to the Cauchy problem for equation (3.6) in the domain Ω_j^- $(j=1,2)$ (see (5.5)).

Thus, in the domain Ω the solution to Problem A_T for equation (2.1) exists.

Theorem 5 is proven. □

This concludes the study of the problem A_T for equation (2.1).

6. Conclusion

In the study of degenerate loaded equations of mixed type of the second kind, difficulties arise associated with the absence of a general representation of the solution, as well as the impossibility of direct application of classical methods. This problem is solved in this paper. A new method for constructing a representation of the general solution of a loaded parabolic-hyperbolic equation of the second kind in a form convenient for further studies of various boundary value problems is developed, and a new type of extremum principle for a degenerate loaded parabolic-hyperbolic equation of the second kind is proved. The analysis of the state of affairs in this direction shows that boundary value problems for degenerate loaded equations leading to less studied integral equations of Volterra and Fredholm with shifts. Moreover, boundary value problems for loaded parabolic-hyperbolic equations of the second kind, degenerating inside the domain, have not yet been studied.

In this paper, we study problems with the Tricomi condition for a loaded parabolic-hyperbolic equation of the second kind, degenerating inside the domain. Theorems of existence and uniqueness of the classical solution of the problems are proved. The proofs of the theorem are based on energy identities and the extremum principle, as well as on the theory of Volterra and Fredholm integral equations. A class of given functions is determined that ensures the solvability of the obtained integral equations. The studied boundary value problems for such equations are effectively used in modeling processes that are associated with the dynamics of soil moisture, groundwater and biology.

References

[1] Nakhushev A.M. ; Equations of Mathematical Biology. Higher School, Moscow,–1995. - 301 p.

[2] Wiener J., Debnath L.; A survey of partial differential equations with piecewise continuous arguments // *Internet Journal of Mathematics and Mathematical Sciences.* – 1995. -**18**(2). -P. 209–228.

[3] Shkhanukov M.Kh.; On some boundary value problems for third-order equations arising in the modeling of liquid filtration in porous media. // *Differential Equations.* –1982. -**18**(4). -P. 689–699.

[4] Kozhanov A.I.; On a nonlinear loaded parabolic equation and its related inverse problem. // *Mathematical Notes.* –2004. -**76**(6). -P. 840–853.

[5] Jenaliev M.T., Ramazanov M.I.; *Loaded Equations as Perturbations of Differential Equations.* Almaty: "FYLIM", –2010. – 334 p.

[6] Gunter N.M., Gabib-zade A.Sh.; Investigation of a class of linear and nonlinear loaded integral equations with different parameters. // *Scientific Notes of AGU. Ser. Phys. Math.and Chem. Sciences.* Baku. –1958. –No. 1. –P. 41–59.

[7] Nakhushev A.M.; Loaded Equations and Their Applications. // *Differential Equations.* –1983. -**19**(1). -P. 86–94.

[8] Nakhushev A.M.; Boundary value problems for loaded integro-differential equations of hyperbolic type and some applications to soil moisture forecasting. // *Differential Equations.* –1979. -**15**(1). -P. 96–105.

[9] Jenaliev M.T.; On the theory of linear boundary value problems for loaded differential equations. Almaty, –1995. –270 p.

[10] Jenaliev M.T., Ramazanov M.I, Kosmakova M.T, Tuleutaeva Zh.M.; *On the solution to a two-dimensional heat conduction problem in a degenerate domain.* Eurasian Math. J. –**11**(2020). –No. 3. –P.334 .

[11] Sabitov K.B., Melisheva E.P.; Dirichlet's problem for a loaded mixed-type equation in a rectangular domain. // *Izvestiya VUZ. Mathematics.* –2013. –No. 7. –P. 62–76.

[12] Zikirov O.S., Kholikov D.K.; Solvability of some nonlocal problems for a third-order loaded equation. // *Siberian Electronic Mathematical Journal*, –2020. –**17**(2020). –P. 77–88.

[13] Pulkina L.S.; Nonlocal problem for a loaded hyperbolic equation. // *Transactions of the Mathematical Institute of the Academy of Sciences*, –2002. –Vol. 236. –P. 298–303.

[14] Khubiev K.U.; Analogue of the Tricomi problem for a loaded hyperbolic-parabolic type equation with variable coefficients. // *Bulletin of the Samara State Technical University. Series: Physical and Mathematical Sciences.* –2007. –**2**(15).– P. 155–157.

[15] Lanina I.N.; Boundary value problem for a loaded hyperbolic-parabolic third-order equation. // *Differential Equations*. –1981. –**17**(1). –P. 97–106.

[16] Islomov B., Kuryazov D.M.; On a boundary value problem for a loaded second-order equation. // *Proceedings of the Academy of Sciences of Uzbekistan*. –1996. –No. 1-2. –P. 3–6.

[17] Islomov B., Boltaeva U.I.; Boundary value problem for a loaded third-order equation with a parabolic-hyperbolic operator. // *Uzbek Mathematical Journal*. –2007. –No. 2. –P. 45–55.

[18] Islomov B.I., Kholbekov Zh.A.; On a nonlocal boundary value problem for a loaded parabolic-hyperbolic equation with three lines of type change. // *Bulletin of Samara State Technical University. Series: Physical and Mathematical Sciences*. –2021. -**25**(3). –P. 407–422.

[19] Abdullaev O.Kh.; Nonlocal problem with an integral gluing condition for a loaded parabolic-hyperbolic equation with a fractional Caputo derivative. // *Differential Equations*. –2023. –**59**(3). –P. 350–357.

[20] Islomov B., Kuryazov D.M.; Boundary value problems for a mixed loaded third-order equation of parabolic-hyperbolic type. // *Uzbek Mathematical Journal*. –2000. -No. 2. –P. 29–35.

[21] Nakhushev A.M.; On the Darboux problem for a degenerate loaded integrodifferential equation of the second order. // *Differential Equations*. –1976. -**12**(1). –P. 103–108.

[22] Attaev A.Kh.; Gurs' problem for a loaded degenerate hyperbolic equation of the second order with a Gellerstedt operator in the main part. // *Bulletin of Samara State Technical University. Series: Physical and Mathematical Sciences*. –2016. –**20**(1). –P. 7–21.

[23] Dzhamalov S.Z., Ashurov R.R.; On a nonlocal boundary-value problem for a second-kind second-order mixed-type loaded equation in a rectangle. // *Uzbek Mathematical Journal*.– 2018. –No. 3. –P. 63–72.

[24] Islomov B., Juraev F.; Local boundary value problems for a loaded parabolic-hyperbolic type equation, degenerate inside the domain. // *Ufa Mathematical Journal*. –2022. –**14**(1). –P. 41–56.

[25] Islomov B.I., Nasirova D.A.; A problem with Gellerstedt conditions on different characteristics for a mixed loaded equation of the second kind. // *Eurasian Mathematical Journal*. –2024. -**15**(1). –P. 34–48.

[26] Islomov B.I., Yuldashev T.K., Yunusov O.M.; Nonlocal boundary problem for a loaded equation of mixed type in a special domain. // *Lobachevskii Journal of Mathematics*. –2024. -**45**(7). –P. 3304–3313.

[27] Islomov B.I., Alikulov Y.K.; Boundary value problem for a loaded parabolic-hyperbolic equation of the third order in an infinite three-dimensional domain. // *International Journal of Applied Mathematics*. –2021. -**34**(2). –P. 158–170.

[28] Abdullaev O.Kh., Agarwal P.; A nonlocal problem with an integral gluing condition for a third-order loaded equation with parabolic-hyperbolic operator involving fractional derivatives. // *Mathematical Methods in the Applied Sciences*. –2020. -**43**(6). -P. 3716-3726.

[29] Islomov B., Baltaeva U.I.; Boundary value problems for the classical and mixed integro-differential equations with Riemann-Liouville operators. // *International Journal of Partial Differential Equations*. –2013. -Article ID 157947. –P. 11–17.

[30] Gekkieva S.Kh. Second boundary value problem for a loaded equation with a fractional derivative. // *Reports of the Academy of Mathematical Sciences*. –2008. -**10**(2). –P. 17–19.

[31] Ubaidullaev U.Sh.; Inverse problem for a mixed loaded equation with the Riemann-Liouville operator in a rectangular domain. // *Bulletin of the KRAUNC. Physical and Mathematical Sciences*. –2020. –**31**(2). –P. 18–31.

[32] Islomov B.I., Baltaeva U.I.; Boundary value problems for loaded differential equations of hyperbolic and mixed types of third order. // *Ufa Mathematical Journal*. –2011. -**3**(3). –P. 15–25.

[33] Islomov B.I., Baltaeva U.; Boundary-value problems for a third-order loaded parabolic-hyperbolic equation with variable coefficients. // *Electronic Journal of Differential Equations*. –2015. –No. 221. –P. 1–10.

[34] Karol I.L.; On a boundary value problem for a mixed elliptic-hyperbolic type equation. // *Reports of the USSR Academy of Sciences*, –1953. –**88**(2). –P. 197–200.

[35] Salahitdinov M.S.; Mixed-type equations. Fan, Tashkent: –1974. –156 p.

[36] Akbarova S.Kh.; Boundary value problems for equations of mixed parabolic-hyperbolic and elliptic-parabolic types with two lines and different orders of degeneration: Ph.D. Dissertation, Tashkent: Institute of Mathematics, Academy of Sciences of Uzbekistan, –1995. –120 p .

[37] Ilyin A.M., Kalashnikov A.S., Oleinik O.A.; Linear second-order equations of parabolic type. // *Journal of Mathematical Physics and Mechanics.* –1965. –**4**(6). –P. 1006–1024.

[38] Salahitdinov M.S., Akbarova S.Kh.; Boundary value problem for a mixed elliptic-parabolic equation with different orders of degeneration inside the domain. // *Uzbek Mathematical Journal.* –1994. –No. 4. –P. 51–60.

[39] Tersenov S.A.; Introduction to the theory of equations degenerating at the boundary. Novosibirsk: NSU, –1973. –144 p.

[40] Djuraev T.D., Sopuev A., Mamajonov M.; Boundary value problems for equations of parabolic-hyperbolic type. Tashkent: Fan, –1986. –220 p.

[41] Samko S.G., Kilbas A.A., Marichev O.I.; Fractional integrals and derivatives and some of their applications. Minsk: Science and Technology. –1987. -P.688.

[42] Smirnov M.M.; Equations of mixed type. Moscow: Higher School. 1985. –301 p .

[43] Djuraev T.D.; Boundary value problems for equations of mixed and mixed-composite type. Tashkent: Fan, –1979. –240 p.

[44] Mikhlin S.G.; Lectures on linear integral equations. Fizmatgiz, Moscow, 1959. –232 p.

[45] Tersenov S.A.; First boundary value problem for a parabolic type equation with a changing time direction. Novosibirsk: NSU, –1978.

[46] Bateman G., Erdelyi A.; Higher Transcendental Functions. Science, Moscow, –1966. -Vol. 2. –295 p.

Islomov Bozor Islomovich
Department of Differential Equations and
Mathematical Physics
National University of Uzbekistan,
Tashkent, Uzbekistan
email: islomovbozor@yandex.com

Nasirova Dildora Anvarovna
Tashkent State Technical University
Tashkent, Uzbekistan
e-mail: ndildora0909@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.141-151
DOI: 10.29229/uzmj.2025-4-14

The Samarskii-Ionkin type problem for the fourth-order ordinary differential equation

Kadirkulov B. J., Uzaqbaeva D. E.

Abstract. In this paper the spectral properties of the nonlocal SamarskiiIonkin type problems for a fourth-order ordinary differential equation are investigated. The eigenvalues and corresponding eigenfunctions are found and their completeness are studied. The spectral properties of the adjoint problem are also studied. Further, the Riesz basis property of the systems of root functions of these problems is proved.

Keywords: Ionkin-Samarskii type problems; non-self-adjoint problem; eigenvalues and eigenfunctions; completeness; Riesz basis.
MSC (2020): 34B05, 34B09, 34B10, 34L10

1. Introduction and statement of the problem

As is known, the application of the Fourier method in solving boundary value problems for partial differential equations leads to an eigenvalue problem, and the main issue here is the expansion of an arbitrary function into a series in terms of the system of eigenfunctions (and associated functions) of this problem. In the case of self-adjoint operators, in general, the system of root functions forms an orthonormal basis, and, in this case, the indicated problem is theoretically solved [1].

As for non-self-adjoint operators, the solution to the above problems is ambiguous [2]. In this case, the system of eigenfunctions may be incomplete, and the problem arises in supplementing them with so-called associated functions. It should be noted that the system of eigenfunctions and associated functions of non-self-adjoint operators is defined ambiguously; that is, there are different approaches to constructing systems of eigenfunctions and related functions of such operators.

Let us note the well-known works [3], [4], where the theory of associated functions was constructed and the completeness of the system of eigenfunctions and associated functions of a wide class of non-self-adjoint differential equations was proven. We also note the work [5], where another method for constructing associated functions of non-self-adjoint differential operators is proposed. In the works [6]-[7], a constructive method for constructing the so-called reduced system of eigen- and associated functions such operators is proposed, and necessary and sufficient conditions of its basis property are also proved.

We note the works [2], [8]-[10], where new formulas for constructing chains of associated functions of non-self-adjoint differential operators are proposed and substantiated. More detailed information on the spectral properties of non-self-adjoint differential operators can be found in the monograph [11].

As noted above, the spectral problems of self-adjoint problems for ordinary differential equations of the fourth order, in the case of a model equation of the form

$$y^{IV}(x) - \lambda y(x) = 0 \qquad (*)$$

have been theoretically solved [1], [12]. In this direction, we note the work [13], where the spectral properties of a differential operator defined by differential equations of even (in particular, fourth) order with piecewise smooth weight functions, as well as with separated boundary conditions, are studied.

As for non-classical problems, here we can only note the works of [14],[15], where for equation (*) investigates the spectral properties of a Samarskii-Ionkin type problem. Using the spectral method, eigenvalues and the corresponding root functions are found, and their completeness and basis properties are proven. The associated adjoint problem is also studied. We also note the works [16], where spectral issues of multidimensional spectral problem is studied. It is shown that the system of eigenfunctions is complete and forms a Riesz basis in Sobolev spaces.

As far as we know, the spectral properties of Bitsadze-Samarskii type problems, or interior boundary value problems for fourth-order equations, have not been studied.

In the proposed work, spectral issues of two nonlocal problems of the Samarskii-Ionkin type for a non-self-adjoint fourth-order differential operator are investigated. Problems of this type for the heat equation were first formulated and investigated by N.I. Ionkin [5].

Problem 1. It is required to find such values λ for which problem

$$X^{IV}(x) - \lambda X(x) = 0,\ 0 < x < 1 \tag{1.1}$$

$$X(1) = 0, X''(0) = 0, \tag{1.2}$$

$$X'(0) = X'(1), X'''(0) = X'''(1). \tag{1.3}$$

has a non-trivial solution.

Problem 2. It is required to find such values λ for which problem (1.1), (1.2) and

$$X'(0) + X'(1) = 0, X'''(0) + X'''(1) = 0, \tag{1.4}$$

has a non-trivial solution. Here λ is a spectral parameter.

The necessity of studying such problems arises when studying boundary value problems for partial differential equations by the spectral method, when conditions of the form (1.2), (1.3) are given with respect to one of the spatial variables.

2. Some auxiliary information about the Riesz Basis

Let $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$ are two complete system of functions in $L_2(a,b)$. Let $(\varphi,\psi)_0$ denote the scalar product of functions $\varphi(x)$ and $\psi(x)$ in $L_2(a,b)$, that is

$$(\varphi,\psi)_0 = (\varphi,\psi)_{L_2(a,b)} = \int\limits_a^b \varphi(x)\psi(x)dx.$$

Definition 2.1. (see [17]) Two system of functions $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$ form a biorthonormal system on some interval $[a,b]$, if

$$(\varphi_n,\psi_k)_0 = \int_a^b \varphi_n\psi_k dx = \delta_{nk} = \begin{cases} 0,\ n \neq k, \\ 1,\ n = k. \end{cases}$$

Thus, the system $\{\psi_n(x)\}$ is called biorthogonally adjoint to the system $\{\varphi_n(x)\}$.

Definition 2.2. (see [17]) System is called minimal if none of the functions of this system is included in the linear envelope of other functions of this system.

The minimality of the system ensures the existence of a biorthogonally adjoint system.

Definition 2.3. (see [17]) The biorthogonal expansion of the function $f \in L_2(a,b)$ in the system $\{\varphi_n(x)\}$ is the series

$$f(x) \sim \sum_{n=1}^{\infty} c_n\varphi_n(x),$$

where $c_n = (f,\psi_n)_0$.

Definition 2.4. (see [17]) We call complete and minimal system of functions $\{\varphi_n(x)\}$ the Bessel system, if for any $f \in L_2(a,b)$ the series of squared coefficients of its biorthogonal expansion in $\{\varphi_n(x)\}$ converges, i.e. if $f \in L_2(a,b)$ implies that

$$\sum_{n=1}^{\infty} \left|(f,\psi_n)_0\right|^2 < \infty,$$

where $\{\psi_n\}$ is biorthogonal conjugate system to $\{\varphi_n(x)\}$.

Definition 2.5. (see [17]) We call complete and minimal system of functions $\{\varphi_n(x)\}$ the Hilbert system, if for any sequences of numbers c_n, such that $\sum\limits_{k=1}^{\infty} c_n^2 < \infty$, there is one and only one $f \in L_2(a,b)$ for which these are the coefficients of its biorthogonal expansion in $\{\varphi_n(x)\}$, i.e.

$$c_n = (f, \psi_n)_0, \; n = 1, 2, \ldots .$$

Definition 2.6. We call complete and minimal system a Riesz basis, if it is both Bessel and Hilbert system.

Theorem 2.7. *(see [18]) Following statements are equivalent:*

1) Sequence $\{\psi_j\}_1^\infty$ forms a basis in the space R, which is equivalent to an orthonormal one.

2) Sequence $\{\psi_j\}_1^\infty$ will be an orthonormal basis in the space R at the corresponding replacement of the scalar product (f,g) with the new $(f,g)_1$, which topologically equivalent to the previous.

3) Sequence $\{\psi_j\}_1^\infty$ is complete in R and there exist constants $a_1, a_2\ (>0)$, such that, for any natural n and for any complex numbers $\gamma_1, \gamma_2, \ldots, \gamma_n$

$$a_2 \sum_{j=1}^{n} |\gamma_j|^2 \le \sum_{j=1}^{n} |\gamma_j \psi_j|^2 \le a_1 \sum_{j=1}^{n} |\gamma_j|^2.$$

4) Sequence $\{\psi_j\}_1^\infty$ complete in R and its matrices of Gramm $(\psi_j, \psi_k)_1^\infty$ generate a bounded invertible operator in the space l_2.

5) Sequence $\{\psi_j\}_1^\infty$ complete in R, it corresponds to a complete bi-orthogonal sequence $\{\chi_j\}_1^\infty$ and for any $f \in R$

$$\sum_{j=1}^{n} |(f, \psi_j)|^2 < \infty, \sum_{j=1}^{n} |(f, \chi_j)|^2 < \infty.$$

Lemma 2.8. *(see [19]) Let $f(x) \in L_2(0,1)$, $a_n = \int\limits_0^1 f(x) e^{-\lambda n x} dx$, $b_n = \int\limits_0^1 f(x) e^{\lambda n (x-1)} dx$, where λ is any complex number, $\operatorname{Re} \lambda > 0$. Then series $\sum\limits_{n=1}^{\infty} |a_n|^2$, $\sum\limits_{n=1}^{\infty} |b_n|^2$ converge.*

3. The solution of Problem 1

Let us find the eigenvalues and eigenfunctions of Problem 1. For this aim, we write the characteristic equation

$$k^4 - \lambda = 0 \Leftrightarrow k^4 = \lambda. \tag{3.1}$$

We have the consider three cases: $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, from which depends the general expression of the solution of the equation (1.1). Let $\lambda = 0$. Then equation (3.1) has multiple roots $k_{1,2,3,4} = 0$, therefore general solution has a form

$$X(x) = C_1 x^3 + C_2 x^2 + C_3 x + C_4,$$

where $C_i, i = \overline{1,4}$ are some numbers. Substituting this solution into conditions (1.2),(1.3), obtain $C_1 = C_2 = 0, C_3 = -C_4$. As consequence, all the solutions of Problem 1 with $\lambda = 0$ are given by the expression $X(x) = C_4(1-x)$, with any real number C_4. Thus, we can call $X_0(x) = 1 - x$ the eigenfunction associated to the single eigenvalue $\lambda_0 = 0$.

Let $\lambda < 0$. Assume $\lambda = -4\mu^4$, $(\mu > 0)$ and we write characteristic equation as follows $k^4 = -4\mu^4$, roots of which are

$$k_1 = (1+i)\mu,\ k_2 = (-1+i)\mu,\ k_3 = (1-i)\mu,\ k_4 = (-1-i)\mu.$$

Obviously, the general solution of equation (1.1) has the form

$$X(x) = C_1 \cosh \mu x \cos \mu x + C_2 \cosh \mu x \sin \mu x + C_3 \sinh \mu x \cos \mu x + C_4 \sinh \mu x \sin \mu x.$$

Substituting this expression into conditions (1.2) and (1.3), to find $C_i, i=\overline{1,4}$, we obtain a system of equations

$$\begin{cases} C_1\cosh\mu\cos\mu + C_2\cosh\mu\sin\mu + C_3\sinh\mu\cos\mu = 0, \\ C_1\sinh\mu\cos\mu + C_2\sinh\mu\sin\mu + C_3(\cosh\mu\cos\mu - 1) = 0, \\ C_1\cosh\mu\sin\mu + C_2(1-\cosh\mu\cos\mu) + C_3\sinh\mu\sin\mu = 0, \\ C_4 = 0. \end{cases}$$

which has only a trivial solution $C_i = 0, i=\overline{1,4}$, so that Problem 1 also has only a trivial solution $X(x)\equiv 0$.

Consider the case $\lambda > 0$. Assume, $\lambda = \mu^4$, $(\mu > 0)$ and we write the characteristic equation with its roots $k^4 = \mu^4$, $k_{1,2} = \pm\mu$, $k_{3,4} = \pm\mu i$. The following functions correspond to these roots

$$X_1(x) = e^{\mu x},\ X_2(x) = e^{-\mu x},\ X_3(x) = \cos\mu x,\ X_4(x) = \sin\mu x,$$

and the general solution is form

$$X(x) = C_1e^{\mu x} + C_2e^{-\mu x} + C_3\cos\mu x + C_4\sin\mu x.$$

Substituting this solution into conditions (1.2) and (1.3), obtain following system

$$\begin{cases} C_1 + C_2 - C_3 = 0, \\ C_1e^{\mu} + C_2e^{-\mu} + C_3\cos\mu + C_4\sin\mu = 0, \\ C_1(e^{\mu}-1) - C_2(e^{-\mu}-1) - C_3\sin\mu + C_4(\cos\mu - 1) = 0, \\ C_1(e^{\mu}-1) - C_2(e^{-\mu}-1) + C_3\sin\mu - C_4(\cos\mu - 1) = 0. \end{cases} \tag{3.2}$$

The resulting system of equations has a nontrivial solution only for those values of μ at which its determinant goes to zero. The determinant of this system is $\Delta(\mu) = 4(2 - e^{\mu} - e^{-\mu})(1-\cos\mu)$. Equating this determinant to zero, we find that the numbers $\lambda_n = \mu_n^4 = (2\pi n)^4, n = 1,2,...$ are the eigenvalues of Problem 1.

Let us study the multiplicity of the found eigenvalues. It is easy to see that for $\lambda_n = (2\pi n)^4$ the rank of the main matrix of system (3.2) is equal to 2. It follows that the geometric multiplicity of the eigenvalues is equal to 2, therefore, each eigenvalue corresponds to a pair of eigenfunctions. Since $\Delta'(\mu_k) = 0, \Delta''(\mu_k) \neq 0$, we find that the algebraic multiplicity of the eigenvalues is equal to 2. Consequently, all eigenvalues of the problem under consideration have multiplicity equal to two (geometrically and algebraically), and the eigenfunctions are the functions

$$X_{1n}(x) = -\sin 2\pi nx,\ \ X_{2n}(x) = \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n}-1} - \cos 2\pi nx.$$

Thus, the eigenvalues of Problem 1 are given by

$$\lambda_0 = 0,\ \lambda_n = (2\pi n)^4,\ n\in N, \tag{3.3}$$

and corresponding eigenfunctions has the form

$$X_0(x) = 2(1-x), X_{1n}(x) = -2\sin 2\pi nx, X_{2n}(x) = \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n}-1} - \cos 2\pi nx. \tag{3.4}$$

Problem 1 is non-self-adjoint and it is easy to see that the following problem will be adjoint to it

$$Y^{IV}(x) - \lambda Y(x) = 0,\ 0 < x < 1, \tag{3.5}$$

$$Y(0) = Y(1), Y'(1) = 0, Y''(0) = Y''(1), Y'''(0) = 0. \tag{3.6}$$

It is not difficult to show that problem (3.5), (3.6) has eigenvalues (3.3), and the corresponding eigenfunctions have the form

$$Y_0(x) = 1,\ Y_{1n}(x) = \frac{e^{2\pi nx} + e^{2\pi n(1-x)}}{e^{2\pi n}-1} - \sin 2\pi nx,\ Y_{2n}(x) = -2\cos 2\pi nx. \tag{3.7}$$

It should be noted that (3.4) and (3.7) are a non-orthogonal system of functions. Indeed, let us consider, for example, system (3.4) and calculate

$$\left(X_0(x)\,,\,X_n^{(1)}(x)\right)_0 = -4\int_0^1 (1-x)\sin 2\pi nx dx = -\frac{2}{\pi n} \neq 0.$$

We proceed to study the questions of the basis property of systems (3.4) and (3.7) in $L_2(0,1)$.

Lemma 3.1. *System of functions (3.4) and (3.7) are biorthogonal system in $L_2(0,1)$, that is*

$$(X_0, Y_0)_{L_2(a,b)} = 1, (X_{ik}, Y_{jn}) = \begin{cases} 1, & k = n, i = j \\ 0, & k \neq n, i \neq j \end{cases}, i, j = 1, 2; k, n = 1, 2, ...$$

Proof. We present the proof of Lemma 3.1 for the functions $X_{1n}(x)$ and $Y_{1n}(x)$. According to Definition 2.1, we calculate the integral

$$(X_{1k}, Y_{1n}) = -2\int_0^1 \sin 2\pi kx \left(\frac{e^{2\pi nx} + e^{2\pi n(1-x)}}{e^{2\pi n}-1} - \sin 2\pi nx\right) dx =$$

$$-\frac{2}{e^{2\pi n}-1}\int_0^1 (e^{2\pi nx} + e^{2\pi n(1-x)})\sin 2\pi kx dx + 2\int_0^1 \sin 2\pi nx \sin 2\pi kx dx = I_{kn} + J_{kn}.$$

Simple calculations show that

$$I_{kn} = -\frac{2}{e^{2\pi n}-1}\int_0^1 (e^{2\pi nx} + e^{2\pi n(1-x)})\sin 2\pi kx dx = 0, k, n \in N,$$

$$J_{kn} = 2\int_0^1 \sin 2\pi kx \cdot \sin 2\pi nx dx = \begin{cases} 1, k = n \\ 0, k \neq n \end{cases}, k, n \in N$$

and so $(X_{1n}, Y_{1n}) = 1$ at $k = n$ and $(X_{1n}, Y_{1n}) = 0$ at $k \neq n$, which is needed what to proven. □

Lemma 3.2. *The system of functions (3.4) and (3.7) are minimal in $L_2(0,1)$.*

The proof of Lemma 3.2 follows from the existence of a biorthonormal system, which was established in Lemma 3.1.

Theorem 3.3. *The system of functions (3.4) and (3.7) are complete in $L_2(0,1)$.*

Proof. First, we prove the completeness of (3.4). Assume, on the contrary, that the system of functions (3.4) is not complete in $L_2(0,1)$. Then there exists a nontrivial function $\varphi(x)$ in $L_2(0,1)$, that is orthogonal to all functions of system (3.4). Let us expand the function $\varphi(x)$ into a Fourier series

$$\varphi(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos 2\pi nx + b_n \sin 2\pi nx),$$

which converge in $L_2(0,1)$. Since $\varphi(x)$ is orthogonal to the system $\{-2\sin 2\pi nx\}_{n=1}^{\infty}$, then the last expansion can be written as

$$\varphi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nx. \tag{3.8}$$

By assumption, $\varphi(x)$ is orthogonal to all functions of the form $X_0(x)$, $X_{2k}(x)$. Then, multiplying series (3.8) sequentially by these functions and integrating along $[0,1]$, we have

$$0 = 2\int_0^1 \varphi(x)(1-x)dx = 2a_0\int_0^1 (1-x)dx + 2\sum_{n=1}^{\infty} a_n \int_0^1 (1-x)\cos 2\pi nx dx = a_0,$$

$$0=\int_0^1 \varphi(x)\cdot\left(\frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1}-\cos 2\pi kx\right)dx=$$
$$=\sum_{n=1}^{\infty} a_n\int_0^1\left(\frac{e^{2\pi kx}-e^{2\pi k(1-x)}}{e^{2\pi k}-1}-\cos 2\pi kx\right)\cos 2\pi nxdx=-\frac{1}{2}a_k, k=1,2,3,....$$

From here, it follows that $a_k=0$, $k=0,1,2,....$ Therefore, from (3.8) we conclude that $\varphi(x)=0$ in $[0,1]$, which opposing conditions $\varphi(x)\neq 0$. Thus, system (3.4) is complete in the space $L_2(0,1)$.

We prove the completeness of the system (3.7). Let there exists a nontrivial function $\varphi(x)$ in $L_2(0,1)$, that is orthogonal to all functions of system (3.7). Since the function $\varphi(x)$ is orthogonal to the system $\{-2cos2\pi nx\}_{n=0}^{\infty}$, it can be represented in $L_2(0,1)$ as a series of sinus, i.e.

$$\varphi(x)=\sum_{n=1}^{\infty} b_n sin2\pi nx. \tag{3.9}$$

Then multiplying last series to $Y_{1k}(x)$ and integrating along $[0,1]$, taking into account the orthogonality of the functions $\varphi(x)$ and $Y_{1k}(x)$, we obtain

$$0=\int_0^1 \varphi(x)\cdot\left(\frac{e^{2\pi kx}+e^{2\pi k(1-x)}}{e^{2\pi k}-1}-\sin 2\pi kx\right)dx=$$
$$=\sum_{n=1}^{\infty} b_n\int_0^1\left(\frac{e^{2\pi kx}+e^{2\pi k(1-x)}}{e^{2\pi k}-1}-\sin 2\pi kx\right)\sin 2\pi nxdx=-\frac{1}{2}b_k, k=1,2,...,$$

that is $b_k=0$, $n=1,2,....$ Then from (3.9) it follows that $\varphi(x)=0$ in $[0,1]$, i.e. system (3.7) is complete in $L_2(0,1)$. Theorem 3.3 is proven. □

Theorem 3.4. *The system of functions (3.4) and (3.7) are two bases of Riesz in* $L_2(0,1)$.

Proof. In order to prove this statement, it is sufficient to prove the completeness of systems (3.4) and (3.7), and the convergence of the following series for $\varphi(x)\in L_2(0,1)$ according to Theorem 2.7:

$$(\varphi(x),2(1-x))_0^2+\sum_{n=1}^{\infty}(\varphi(x),-2\sin 2\pi nx)_0^2+\sum_{n=1}^{\infty}\left(\varphi(x),\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\right)_0^2, \tag{3.10}$$

$$(\varphi(x),1)_0^2+\sum_{n=1}^{\infty}(\varphi(x),-2\cos 2\pi nx)_0^2+\sum_{n=1}^{\infty}\left(\varphi(x),\frac{e^{2\pi nx}+e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\sin 2\pi nx\right)_0^2. \tag{3.11}$$

Since the completeness of systems (3.4) and (3.7) has been proven in Lemma 3.1 we only must verify the convergence of the previous series. To this end, we consider (3.10) and introduce the following notations

$$I_1=4\left(\varphi(x),(1-x)\right)_0^2, I_2=4\sum_{n=1}^{\infty}(\varphi(x),\sin 2\pi nx)_0^2,$$
$$I_3=\sum_{n=1}^{\infty}\left(\varphi(x),\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\right)_0^2.$$

From I_1, applying the Cauchy-Bunyakovsky inequality we obtain

$$I_1=4\left(\int_0^1(1-x)\varphi(x)dx\right)^2\leq 4\int_0^1(1-x)^2dx\cdot\int_0^1\varphi^2(x)dx=\frac{4}{3}\|\varphi(x)\|^2{}_{L_2(0,1)},$$

i.e. I_1 is finite.

I_2 will be represented in the form

$$I_2=4\sum_{n=1}^{\infty}(\varphi(x),\sin 2\pi nx)_0^2=2\sum_{n=1}^{\infty}\left(\varphi(x),\sqrt{2}\sin 2\pi nx\right)^2=2\sum_{n=1}^{\infty}c_n^2,$$

where $c_n = \left(\varphi(x), \sqrt{2}\sin 2\pi nx\right)$ are the coefficients of Fourier of the function $\varphi(x)$ on the orthonormal system $\{\sqrt{2}\sin 2\pi nx\}$. From here, applying Bessels inequality, we obtain that $I_2 = 2\sum\limits_{n=1}^{\infty} c_n^2 \leq 2\left\|\varphi(x)\right\|^2_{L_2(0,1)}$, i.e. I_2 is finite.

Consider I_3. Let

$$A = \left(\varphi(x), \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \cos 2\pi nx\right)^2_0.$$

Since

$$A = \left(\left(\varphi(x), \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right) - (\varphi(x), \cos 2\pi nx)\right)^2_0,$$

from here, applying inequality

$$(a+b)^2 \leq 2(a^2+b^2)$$

obtain that

$$A \leq 2\left(\varphi(x), \frac{e^{2\pi nx} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)^2_0 + 2\left(\varphi(x), \cos 2\pi nx\right)^2_0$$

$$= 2\left(\left(\varphi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1}\right) - \left(\varphi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)\right)^2_0 + 2\left(\varphi(x), \cos 2\pi nx\right)^2_0.$$

Applying the previous inequality again, we get that

$$A \leq 4\left(\varphi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1}\right)^2_0 + 4\left(\varphi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)^2_0 + 2\left(\varphi(x), \cos 2\pi nx\right)^2_0.$$

Thus

$$I_3 \leq 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1}\right)^2_0 + 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{2\pi n(1-x)}}{e^{2\pi n} - 1}\right)^2_0 + 2\sum_{n=1}^{\infty}\left(\varphi(x), \cos 2\pi nx\right)^2_0 =$$

$$= J_1 + J_2 + J_3.$$

Consider J_3.

$$J_3 = 2\sum_{n=1}^{\infty}\left(\varphi(x), \cos 2\pi nx\right)^2 = \sum_{n=1}^{\infty} a_n^2,$$

where $a_n = \left(\varphi(x), \sqrt{2}\cos 2\pi nx\right)$ are the coefficients of Fourier of the function on the orthonormal system $\{\sqrt{2}\cos 2\pi nx\}$. Then applying Bessel's inequality, we get

$$J_3 = \sum_{n=1}^{\infty} a_n^2 \leq \left\|\varphi(x)\right\|^2_{L_2(0,1)}.$$

Consider J_1. Since

$$\left(\varphi(x), \frac{e^{2\pi nx}}{e^{2\pi n} - 1}\right)^2_0 = \left(\int_0^1 \varphi(x)\cdot\frac{e^{2\pi n}}{e^{2\pi n} - 1}e^{2\pi n(x-1)}dx\right)^2_0 =$$

$$= \left(\int_0^1 \varphi(x)\left[1 + \frac{1}{e^{2\pi n} - 1}\right]e^{2\pi n(x-1)}dx\right)^2 \leq 4\left(\int_0^1 \varphi(x)e^{2\pi n(x-1)}dx\right)^2,$$

hence

$$J_1 \leq 16\sum_{n=1}^{\infty}\left(\int_0^1 \varphi(x)e^{2\pi n(x-1)}dx\right)^2 = 16\sum_{n=1}^{\infty} b_n^2, b_n = \left(\int_0^1 \varphi(x)e^{2\pi n(x-1)}dx\right).$$

Here, from Lemma 3.2, it follows that J_1 is finite. Similarly, we can prove that J_2 is finite too. Thus, the series I_1 and I_2 are converge, and therefore series (3.10) also converges. The convergence of series (3.11) is proved similarly. Theorem 3.4 is proved. □

4. The solution of Problem 2

Let us find the eigenvalues and eigenfunctions of Problem 2. After similar calculations, as in the case of Problem 1, it is easy to see that the eigenvalues of Problem 2 are given by

$$\lambda_n = (\pi(2n-1))^4, \; n \in N \tag{4.1}$$

and corresponding eigenfunctions has the form

$$X_{1n}(x) = 2\sin(\pi(2n-1)x), \; X_{2n}(x) = \frac{e^{\pi(2n-1)x} + e^{\pi(2n-1)(1-x)}}{e^{\pi(2n-1)}+1} + \cos(\pi(2n-1)x). \tag{4.2}$$

Note that Problem 2 is a non self-adjoint problem. On the contrary, it is not difficult to verify that the following problem is self-adjoint.

$$Y^{IV}(x) - \lambda Y(x) = 0, \; 0 < x < 1, \tag{4.3}$$

$$Y(0) + Y(1) = 0, Y'(1) = 0, Y''(0) + Y''(1) = 0, Y'''(0) = 0. \tag{4.4}$$

Problem (4.3), (4.4) have eigenvalues (4.1), and the corresponding eigenfunctions have the form

$$Y_{1n}(x) = \frac{e^{\pi(2n-1)x} - e^{\pi(2n-1)(1-x)}}{e^{\pi(2n-1)}+1} + \sin(\pi(2n-1)x), Y_{2n}(x) = 2\cos(\pi(2n-1)x). \tag{4.5}$$

It is not difficult to show that systems (4.2) and (4.5) are bi-orthonormalized in $L_2(0,1)$. This also implies the minimality of these systems in $L_2(0,1)$.

Theorem 4.1. *The system of functions (4.2) and (4.5) are complete in the space* $L_2(0,1)$.

Proof. We prove the completeness of system (4.2). Assume, on the contrary, that the system of functions (4.2) is not complete in $L_2(0,1)$. Then there exists a nontrivial function $\varphi(x)$ in $L_2(0,1)$, that is orthogonal to all functions of system (4.2). Let us expand the function $\varphi(x)$ into a Fourier series

$$\varphi(x) = \sum_{n=1}^{\infty}(a_n\cos(2n-1)\pi x + b_n\sin(2n-1)\pi x),$$

according to the complete orthogonal system $\{\cos(2n-1)\pi x, \sin(2n-1)\pi x\}_{n=1}^{\infty}$, which converges in $L_2(0,1)$. Since $\varphi(x)$ is orthogonal to the system $\{\sin(2n-1)\pi x\}_{n=1}^{\infty}$, then the last expansion can be written as

$$\varphi(x) = \sum_{n=1}^{\infty} a_n\cos(2n-1)\pi x. \tag{4.6}$$

Further, multiplying last series by the function $X_{2k}(x)$, and integrating along $[0,1]$, by using the orthogonality of this last function and $\varphi(x)$, we obtain the following equality:

$$0 = \int_0^1 \varphi(x)\cdot\left(\frac{e^{(2k-1)\pi x} + e^{\pi(2k-1)(1-x)}}{e^{(2k-1)\pi}+1} + \cos(2k-1)\pi x\right)dx =$$

$$= \sum_{n=1}^{\infty} a_n\int_0^1\left(\frac{e^{(2k-1)\pi x} + e^{\pi(2k-1)(1-x)}}{e^{(2k-1)\pi}+1} + \cos(2k-1)\pi x\right)\cos(2n-1)\pi x dx = \frac{1}{2}a_k, k = 1,2,3,....$$

From here, it follows that $a_k = 0$, $k = 1,2,....$ Therefore, from (4.6) we conclude that $\varphi(x) = 0$ in $[0,1]$, which opposing conditions $\varphi(x) \neq 0$. Thus, the system (4.2) is complete in the space $L_2(0,1)$. The completeness of the system (4.5) is proved similarly. Theorem 4.1 is proven. □

Theorem 4.2. *The system of functions (4.2) and (4.5) are two bases of Riesz in* $L_2(0,1)$.

Proof. In order to prove this statement, it is sufficient to prove the completeness of systems (4.2) and (4.5), and the convergence of the following series for $\varphi(x) \in L_2(0,1)$ according to Theorem 2.7:

$$\sum_{n=1}^{\infty} (\varphi(x), 2\sin(2n-1)\pi x)_0^2 + \sum_{n=1}^{\infty} \left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1} + \cos(2n-1)\pi x\right)_0^2, \tag{4.7}$$

$$\sum_{n=1}^{\infty} (\varphi(x), 2\cos(2n-1)\pi x)_0^2 + \sum_{n=1}^{\infty} \left(\frac{e^{(2n-1)\pi x} - e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1} + \sin(2n-1)\pi x, \varphi(x)\right)_0^2.$$

Since the completeness of systems (4.2) and (4.5) has been proven in Theorem 4.1, we only must verify the convergence of the previous series. Let us prove the convergence of series (4.7). To this end, we consider (4.7) and introduce the following notations

$$I_1 = 4\sum_{n=1}^{\infty} (\varphi(x), \sin(2n-1)\pi x)_0^2,$$

$$I_2 = \sum_{n=1}^{\infty} \left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1} + \cos(2n-1)\pi x\right)_0^2.$$

I_1 represent in the form

$$I_1 = 4\sum_{n=1}^{\infty} (\varphi(x), \sin(2n-1)\pi x)_0^2 = 2\sum_{n=1}^{\infty} \left(\varphi(x), \sqrt{2}\sin(2n-1)\pi x\right)_0^2 = 2\sum_{n=1}^{\infty} c_n^2,$$

where $c_n = (\varphi(x), \sqrt{2}\sin(2n-1)\pi x)_0$ are the Fourier coefficients of the function $\varphi(x)$ on the orthonormal system $\{\sqrt{2}\sin(2n-1)\pi x\}$. From here, applying Bessels inequality, we obtain that $I_1 = 2\sum\limits_{n=1}^{\infty} c_n^2 \le 2\,\|\varphi(x)\|^2_{L_2(0,1)}$, i.e. I_1 is finite.

Consider I_2. Let

$$A = \left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1} + \cos(2n-1)\pi x\right)_0^2.$$

Since

$$A = \left(\left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0 + (\varphi(x), \cos(2n-1)\pi x)_0\right)^2,$$

from here, applying inequality $(a+b)^2 \le 2(a^2+b^2)$ we obtain that

$$A \le 2\left(\varphi(x), \frac{e^{(2n-1)\pi x} + e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 + 2\,(\varphi(x), \cos(2n-1)\pi x)_0^2 =$$

$$= 2\left(\left(\varphi(x), \frac{e^{(2n-1)\pi x}}{e^{(2n-1)\pi}+1}\right)_0 + \left(\varphi(x), \frac{e^{2\pi n(1-x)}}{e^{(2n-1)\pi}+1}\right)_0\right)^2 + 2\,(\varphi(x), \cos(2n-1)\pi x)_0^2.$$

Applying the previous inequality again, we get that

$$A \le 4\left(\varphi(x), \frac{e^{(2n-1)\pi x}}{e^{(2n-1)\pi}+1}\right)_0^2 + 4\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 + 2\,(\varphi(x), \cos(2n-1)\pi x)_0^2.$$

Thus

$$I_2 \le 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{(2n-1)\pi x}}{e^{(2n-1)\pi}+1}\right)_0^2 + 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 + 2\sum_{n=1}^{\infty}(\varphi(x), \cos(2n-1)\pi x)_0^2 = J_1+J_2+J_3.$$

Consider $J_3 = 2\sum\limits_{n=1}^{\infty}(\varphi(x),\cos(2n-1)\pi x)_0^2 = \sum\limits_{n=1}^{\infty} a_n^2$, where $a_n = (\varphi(x),\sqrt{2}\cos(2n-1)\pi x)_0$ are the Fourier coefficients of the function $\varphi(x)$ on the orthonormal system $\{\sqrt{2}\cos(2n-1)\pi x\}$. Then applying Bessel's inequality, we get $J_3 = \sum\limits_{n=1}^{\infty} a_n^2 \le \|\varphi(x)\|^2_{L_2(0,1)}$. Consider J_2. Since

$$\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{\pi(2n-1)}+1}\right)_0 = \frac{e^{\pi(2n-1)}}{e^{\pi(2n-1)}+1}\left(\varphi(x)e^{\pi x}, e^{-2\pi n x}\right)_0,$$

hence

$$J_2 = 4\sum_{n=1}^{\infty}\left(\varphi(x), \frac{e^{\pi(2n-1)(1-x)}}{e^{(2n-1)\pi}+1}\right)_0^2 \le 4\sum_{n=1}^{\infty}\left(\varphi(x)e^{\pi x}, e^{-2\pi n x}\right)_0^2 = 4\sum_{n=1}^{\infty} c_n^2,$$

where $c_n = (\varphi(x)e^{\pi x}, e^{-2n\pi x})_0$. Then from Lemma 3.2, it follows that J_2 is finite. Similarly, we can prove that J_1 is finite too. Thus, the series I_1 and I_2 are converge, and therefore series (4.7) also converges. The convergence of the second series is proved similarly. Theorem 4.2 is proven. □

References

[1] M.A.Neymark. Linear Differential Operators. - New York: Ungar, -1968.

[2] M.A.Sadybekov, A.M.Sarsenbi. New definition of associated functions of non-self-adjoint differential operators. Reports of the NAS RK. 2006. pp.13-17.

[3] M.V.Keldysh. On the completeness of eigenfunctions of some classes of non-self-adjoint linear operators. UMN. 1971. Volume 26, Issue 4(160), pp.1541.

[4] M.V.Keldysh. On eigenvalues and eigenfunctions of some classes of non-self-adjoint equations. DAN USSR. 1951. Volume 77, No. 1, pp. 1114.

[5] N.I. Ionkin, The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition. Differential equations. 1977. Volume 13, No. 2, pp. 294-304.

[6] V.A.Ilin. On the existence of a reduced system of eigen and associated functions for a non-self-adjoint ordinary differential operator. Tr. Steklov Mathematical Institute of the USSR. 1976. Volume 142, pp. 148-155.

[7] V.A.Ilin. On the connection between the type of boundary conditions and the properties of basisity and equiconvergence with the trigonometric series of expansions in root functions of a non-self-adjoint differential operator. Differential equations. 1994. Volume 30, No. 9, pp.1516-1529.

[8] M. Sadybekov, G, Dildabek, A. Tengaeva. Constructing a basis from systems of eigenfunctions of one not strengthened regular boundary value problem. Filomat. 2017. Vol.31, Issue 4, pp.981-987.

[9] M.A.Sadybekov, A. M. Sarsenbi. On a necessary condition for a system of normalized elements to be a basis in a Hilbert space. Vestnik Tomskogo Gosudarstvennogo Universiteta. Matematika i Mekhanika, 2011, Number 1(13), pp. 4446.

[10] M.Sadybekov, N. Imanbaev, On System of Root Vectors of Perturbed Regular Second-Order Differential Operator Not Possessing Basis Property, Mathematics, 11:20 (2023), 4364 , 10 p.

[11] M.Ruzhansky, M.Sadybekov, D.Suragan, Spectral Geometry of Partial Differential Operators, Taylor Francis, 2020 , 378 p.

[12] E. Kamke. Handbook on Ordinary Differential Equations, Moscow, Nauka, 1976 (in Russian).

[13] S.I. Mitrokhin. Spectral properties of an even-order differential operator with a discontinuous weight function. Vestnik rossiyskikh universitetov. Matematika Russian Universities Reports. Mathematics, 2022, vol. 27, no. 137, pp. 3757.

[14] A.S. Berdyshev, B.J. Kadirkulov. On a Nonlocal Problem for a Fourth-Order Parabolic Equation with the Fractional DzhrbashyanNersesyan Operator. Differential Equations, 2016, Vol. 52, No. 1, pp. 122127. 2016.

[15] A.S. Berdyshev, A. Cabada, B.J. Kadirkulov. The Samarskii-Ionkin type problem for the fourth order parabolic equation with fractional differential operator, Computers and Mathematics with Applications, 2011, 67, 3884-3893.

[16] Sh.G. Kasimov, U.S. Madrakhimov On the unique solvability for initial - boundary problems of vibrations of a beam, one end of which is fixed and the other is free, in the Sobolev classes in the multidimensional case. Uzbek Mathematical Journal, 2019 , 4, pp.89-103.

[17] N.K.Bary Biorthogonal systems and bases in Hilbert space. Uch. Zap. MSU, 1951. Volume 148, Issue 4, pp. 69-107.

[18] I.S. Gohberg, M.G. Krein, Introduction to theory of linear non-adjoint operators, Moscow, 1965.

[19] G.M. Keselman, On non-conditional convergence of expansions by eigenfunctions of some differential operators, Izv. vuzov. Matem. , 1964. Volume 39, No. 2, pp.82-93.

Kadirkulov B. J,
Alfraganus university, Tashkent, Uzbekistan.
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Science,
Tashkent, Uzbekistan.
e-mail: b.kadirkulov@afu.uz

Uzaqbaeva D. E,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Science,
Tashkent, Uzbekistan.
e-mail: uzaqbaevadilfuza1606@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.152-164
DOI: 10.29229/uzmj.2025-4-15

An initial-boundary value problem with local and nonlocal conditions for a degenerate partial differential equation of high even order

Karimov K., Oripov D.

Abstract. In this paper, an initial-boundary value problem with local and nonlocal conditions for a degenerate partial differential equation of high even order in a rectangle is formulated and investigated. The method of energy integrals is used to prove the uniqueness of the solution to the problem, and using spectral analysis methods and the Green function, the solution to the considered problem is constructed as the sum of a Fourier series.

Keywords: degenerate equation, initial-boundary value problem, method of separation of variables, spectral problem, Green's function method, integral equation, Fourier series.

MSC (2020): 35G16

1. Introduction. Formulation of the problem

In a rectangle $\Omega = \{(x,t):\ 0 < x < 1;\ 0 < t < T\}$, we consider the following degenerate equation of high even order

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^{2n}}{\partial x^{2n}}\left(x^{\alpha}\frac{\partial^{2n} u}{\partial x^{2n}}\right) = f(x,t),\ \ n \in \mathbb{N}, \tag{1.1}$$

where $u = u(x,t)$ is an unknown function, $f(x,t)$ is a given function, and $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$.

Many problems on vibrations of beams and plates, which are of great importance in structural mechanics, lead to differential equations of even order [1, 2, 3].

In 1956, F.I. Frankl in his work [4], considering the flow around a finite symmetrical profile by a subsonic flow with a supersonic zone ending in a normal shock wave, formulated a problem for the Chaplygin equation in a mixed domain with a non-local condition of type $u(0,y) = u(0,-y)$. In this case, local condition $u_x(0,y) = 0$ was additionally specified. In the work [5] N.I. Ionkin proved the existence of a solution to a non-local problem with conditions $u_x(0,y) = u_x(1,y),\ u(0,y) = 0,\ 0 \le y \le T$ and $u(x,0) = \tau(x),\ 0 \le x \le 1$ for the heat conduction equation using the spectral analysis method. In his work [6] he substantiated the uniqueness of the solution to this problem. Such conditions are encountered, for example, when solving problems describing the process of particle diffusion in turbulent plasma, as well as in the processes of heat propagation in a thin heated rod, if the law of change in the total amount of heat of the rod is given. Due to its wide range of applications, scientific research in these areas continues to advance.

This paper aims to study a nonlocal boundary value problem for an even-order higher-order differential equation. We first provide a brief overview of closely related results to situate our work within existing research.

In the work [7], in a rectangular domain of the plane xOt for the following equation

$$B^t_{\gamma-1/2}u + (-1)^k\frac{\partial^k}{\partial x^k}\left(x^{\alpha}\frac{\partial^k u}{\partial x^k}\right) = f(x,t),\ \ k \in \mathbb{N},$$

an initial-boundary value problem with local conditions was investigated, where $B^y_q \equiv \frac{\partial^2}{\partial y^2} + \frac{2q+1}{y}\frac{\partial}{\partial y}$ is the Bessel operator [8], $\alpha,\ 2\gamma \in (0,1)$. They proved the existence and uniqueness of the solution to the considered problem by applying the method of separation of variables.

For equation (1.1) with $\alpha = 0$, where the derivative with respect to x is of order $2n$, various initial-boundary value problems involving non-local conditions have been studied in [9, 10].

Several studies, particularly [11, 12, 13], have examined both local and nonlocal problems for partial differential equations containing higher-order derivatives in the spatial variable x and time variable t.

For partial differential equations with multiple variables, [14] analyzed initial-boundary value problems involving both local and nonlocal conditions, while [15] focused on initial-boundary value problems with local conditions for a $4n$th-order partial differential equation in the spatial variable x.

We also note that for the case $n = 1$ in equation (1.1), the authors of [16] investigated an initial-boundary value problem for a degenerate equation with three lines of degeneracy.

The main goal and novelty of our work consists in the formulation and analysis of a new boundary value problem for a degenerate higher-order even partial differential equation, incorporating both local and nonlocal conditions. As this problem was first introduced by Professor A. Urinov, we hereafter refer to it as the Urinov problem (Problem U).

Problem U. Find a function $u(x,t)$ that with the following properties:
1) u_t, $(\partial^j/\partial x^j)\,u$, $(\partial^j/\partial x^j)\left[x^\alpha\left(\partial^{2n}/\partial x^{2n}\right)u\right] \in C\left(\overline{\Omega}\right)$, $j = \overline{0, 2n-1}$,
$\left(\partial^{2n}/\partial x^{2n}\right)\left[x^\alpha\left(\partial^{2n}/\partial x^{2n}\right)u\right]$, $u_{tt} \in C(\Omega)$;
2) in the domain Ω it satisfies equation (1.1);
3) it satisfies the following initial and boundary conditions:

$$u(x,0) = \varphi_1(x), \quad x \in [0,1], \;\; u_t(x,0) = \varphi_2(x), \quad x \in [0,1], \tag{1.2}$$

$$\frac{\partial^{2j}}{\partial x^{2j}}u(0,t) + \frac{\partial^{2j}}{\partial x^{2j}}u(1,t) = 0, \quad \frac{\partial^{2j}}{\partial x^{2j}}\left(x^\alpha\frac{\partial^{2n}}{\partial x^{2n}}u(x,t)\right)\bigg|_{x=0} = 0, \;\; j = \overline{0, n-1}, \;\; t \in [0,T], \tag{1.3}$$

$$\begin{gathered}\frac{\partial^{2j+1}}{\partial x^{2j+1}}u(1,t) = 0, \;\; \frac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^\alpha\frac{\partial^{2n}}{\partial x^{2n}}u(x,t)\right)\bigg|_{x=0} + \frac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^\alpha\frac{\partial^{2n}}{\partial x^{2n}}u(x,t)\right)\bigg|_{x=1} = 0,\\ j = \overline{0, n-1}, \;\; t \in [0,T],\end{gathered} \tag{1.4}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are given sufficiently smooth functions.

2. Study of the spectral problem

When formally applying the Fourier method to the problem U, we obtain the following spectral problem:

$$Mv(x) = \lambda v(x), \;\; 0 < x < 1, \tag{2.1}$$

$$\left.\begin{aligned} &v^{(j)}(x), \;\; \left(x^\alpha v^{(2n)}(x)\right)^{(j)} \in C[0,1], \;\; j = \overline{0, 2n-1},\\ &v^{(2j)}(0) + v^{(2j)}(1) = 0, \quad \left[x^\alpha v^{(2n)}(x)\right]^{(2j)}\bigg|_{x=0} = 0,\\ &v^{(2j+1)}(1) = 0, \quad \left[x^\alpha v^{(2n)}(x)\right]^{(2j+1)}\bigg|_{x=0} + \left[x^\alpha v^{(2n)}(x)\right]^{(2j+1)}\bigg|_{x=1} = 0, \;\; j = \overline{0, n-1}, \end{aligned}\right\} \tag{2.2}$$

where $M \equiv \partial^{2n}/\partial x^{2n}\;\left[x^\alpha \partial^{2n}/\partial x^{2n}\right]$.

Assume that there exists a non-trivial solution $v(x)$ to the problem {(2.1), (2.2)}. Under this assumption, multiplying both parts of equation (2.1) by the function $v(x)$ and integrating the resulting equality on the segment $[0,1]$, and then applying the rule of integration by parts $2n$ times and considering (2.2), we obtain

$$\lambda\int_0^1 v^2(x)\,dx = \int_0^1 x^\alpha\left[v^{(2n)}(x)\right]^2 dx. \tag{2.3}$$

From (2.3) it follows that $\lambda \geq 0$. Let $\lambda = 0$. Then from (2.3), we have $v^{(2n)}(x) = 0$, $0 < x < 1$. Hence, by virtue of conditions $v^{(2j)}(0) + v^{(2j)}(1) = 0$, $v^{(2j+1)}(1) = 0$, $j = \overline{0, n-1}$, we obtain $v(x) \equiv 0$, $0 \leq x \leq 1$. Consequently, problem {(2.1), (2.2)} can have non-trivial solutions only for $\lambda > 0$.

Let us assume that $\lambda > 0$. We will prove the existence of eigenvalues of the problem {(2.1),(2.2)} using the Green's function method. The Green's function $G(x,s)$ of this problem has the following properties:

1) the functions $(\partial^j/\partial x^j)\,G(x,s)$, $j = \overline{0, 2n-1}$ and $(\partial^j/\partial x^j)\left[x^\alpha\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]$, $j = \overline{0, 2n-2}$ are continuous for all x, $s \in [0,1]$;

2) in each of the intervals $[0, s)$ and $(s, 1]$ there exists a continuous derivative $(\partial^{2n-1}/\partial x^{2n-1})\left[x^{\alpha}\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]$, and at $x = s$ it has a jump:

$$\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^{\alpha}\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]\Big|_{x=s-0}^{x=s+0} = 1; \tag{2.4}$$

3) in intervals $(0, s)$ and $(s, 1)$ with respect to the argument x there is a continuous derivative $MG(x,s)$ and equality $MG(x,s) = 0$ is satisfied;
4) at $s \in (0,1)$ by argument x satisfies the conditions

$$\frac{\partial^{2j}G(0,s)}{\partial x^{2j}} + \frac{\partial^{2j}G(1,s)}{\partial x^{2j}} = 0,\ \frac{\partial^{2j}}{\partial x^{2j}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\Big|_{x=0} = 0, j = \overline{0, n-1}, \tag{2.5}$$

$$\frac{\partial^{2j+1}G(1,s)}{\partial x^{2j+1}} = 0,\ \frac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\Big|_{x=0} + \frac{\partial^{2j+1}}{\partial x^{2j+1}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\Big|_{x=1} = 0, j = \overline{0, n-1}. \tag{2.6}$$

Considering the form of the general solution of the equation $MG(x,s) = 0$ in intervals $(0, s)$ and $(s, 1)$, we will seek the function $G(x,s)$ in the following form

$$G(x,s) = \begin{cases} \sum\limits_{j=1}^{2n} \dfrac{a_j x^{4n-\alpha-j}}{(2n-j)!(2n-\alpha-j+1)_{2n}} + \sum\limits_{j=1}^{2n} \dfrac{a_{2n+j}x^{2n-j}}{(2n-j)!},\ 0 \le x \le s, \\ \sum\limits_{j=1}^{2n} \dfrac{b_j x^{4n-\alpha-j}}{(2n-j)!(2n-\alpha-j+1)_{2n}} + \sum\limits_{j=1}^{2n} \dfrac{b_{2n+j}x^{2n-j}}{(2n-j)!},\ s \le x \le 1, \end{cases} \tag{2.7}$$

where a_j and b_j, $j = \overline{1, 4n}$ are unknown functions of the variable s, and $(z)_n = z(z+1)(z+2)...(z+n-1)$ is the Pochhammer's symbol [17].

Satisfying the function (2.7) with properties 1) and 2) of the Green's function, we obtain the following system of equations with respect to $b_j - a_j$, $j = \overline{1, 4n}$:

$$\begin{cases} b_1 - a_1 = 1, \\ \sum\limits_{j=1}^{m_1} \dfrac{s^{m_1-j}}{(m_1-j)!}(b_j - a_j) = 0, \quad m_1 = \overline{2, 2n}, \\ \sum\limits_{j=1}^{2n} \dfrac{s^{2n-\alpha+m_2-j}(b_j - a_j)}{(2n-j)!(2n-\alpha-j+1)_{m_2}} + \sum\limits_{j=1}^{m_2} \dfrac{s^{m_2-j}(b_{2n+j} - a_{2n+j})}{(m_2-j)!} = 0,\ \ m_2 = \overline{1, 2n}. \end{cases}$$

This system has a unique solution [18]:

$$b_j - a_j = \frac{(-1)^{j-1}s^{j-1}}{(j-1)!},\ b_{2n+j} - a_{2n+j} = \frac{(-1)^{j-1}s^{2n+j-1-\alpha}}{(j-1)!(j-\alpha)_{2n}},\quad j = \overline{1, 2n}. \tag{2.8}$$

Obeying (2.7) into the second condition of (2.5), we obtain

$$a_{2j+2} = 0,\quad j = \overline{0, n-1}, \tag{2.9}$$

and we substituting this expression into the first equality of (2.8), we have

$$b_{2j+2} = -\frac{s^{2j+1}}{(2j+1)!},\quad j = \overline{0, n-1}. \tag{2.10}$$

Hence, t he second of the conditions (2.6) and (2.5), we obtain

$$a_{2j+1} + \sum_{i=1}^{2j+1} \frac{b_i}{(2j+1-i)!} = 0,\quad j = \overline{0, n-1} \tag{2.11}$$

and
$$\sum_{i=1}^{2j+2}\frac{b_i}{(2j+2-i)!}=0,\quad j=\overline{0,n-1},\tag{2.12}$$
respectively.

Changing j to $2j+1$ in the first relation (2.8), we get
$$a_{2j+1}=b_{2j+1}-\frac{s^{2j}}{(2j)!},\quad j=\overline{0,n-1},\tag{2.13}$$
and substituting this expression into the system of equations (2.11), we obtain
$$b_1=\frac{1}{2},\quad b_{2j+1}=\frac{s^{2j}}{2\,(2j)!}-\sum_{i=1}^{2j}\frac{b_i}{2\,(2j+1-i)!},\quad j=\overline{1,n-1}.\tag{2.14}$$

From the system of equations (2.12), it is easy to determine the following equalities
$$b_2=-\frac{1}{2},\quad b_{2j+2}=-\sum_{i=1}^{2j+1}\frac{b_i}{(2j+2-i)!},\quad j=\overline{1,n-1}.\tag{2.15}$$

Solving the system of equations $\{(2.14),\,(2.15)\}$, we find $b_j,\ \ j=\overline{1,2n}$ and substituting their values into (2.13), we find $a_{2j+1},\ \ j=\overline{0,n-1}$, and the value of $a_{2j+2},\ \ j=\overline{0,n-1}$, by virtue of (2.8), is found from relations
$$a_{2j+2}=b_{2j+2}+\frac{s^{2j+1}}{(2j+1)!},\quad j=\overline{0,n-1}.\tag{2.16}$$

Now, substituting (2.7) into the first part of the condition (2.5), and as a result, we compose the following system of equations
$$a_{2n+2j}+\sum_{i=1}^{2n}\frac{b_i}{(2n-i)!(2n+1-i-\alpha)_{2j}}+\sum_{i=1}^{2j}\frac{b_{2n+i}}{(2j-i)!}=0,\quad j=\overline{1,n}.\tag{2.17}$$

In this case, the ratio
$$a_{2n+2j}=b_{2n+2j}+\frac{s^{2n+2j-1-\alpha}}{(2j-1)!(2j-\alpha)_{2n}},\quad j=\overline{1,n}\tag{2.18}$$
is determined by substituting $j\sim 2j$ into the second part of ratio (2.8).

We substitute the found relations (2.18) into the system of equations (2.17) and, taking into account (2.10), we obtain
$$\begin{gathered}b_{2n+2j}=-\frac{s^{2n+2j-1-\alpha}}{2\,(2j-1)!(2j-\alpha)_{2n}}-\\-\sum_{i=1}^{2n}\frac{b_i}{2\,(2n-i)!(2n+1-i-\alpha)_{2j}}-\sum_{i=1}^{2j-1}\frac{b_{2n+i}}{2\,(2j-i)!}=0,\quad j=\overline{1,n}.\end{gathered}\tag{2.19}$$

Substituting the results obtained from (2.19) into relation (2.18), we determine $a_{2n+2j},\ \ j=\overline{1,n}$.

Consequently, the Green function satisfying conditions 1)-4) exists, is unique and has the form (2.7), and the coefficients a_j and b_j, $j=\overline{1,4n}$ are determined by the equalities (2.9), (2.10), (2.13), (2.14), (2.15), (2.16), (2.18) and (2.19).

Now, let us prove that it is symmetric for its arguments. It is not easy to prove this fact using formulas (2.9), (2.10), (2.13), (2.14), (2.15), (2.16), (2.18) and (2.19). Therefore, we will use properties 1)-4) of the Green's function.

Let $v(x)$, $h(x) \in C^{2n-1}[0,1]$; $x^{\alpha}v^{(2n)}(x)$, $x^{\alpha}h^{(2n)}(x) \in C^{2n-1}[0,1] \cap C^{2n}(0,1)$, $n \in \mathbb{N}$. Then, the following identity holds true:

$$hM[v] - vM[h] = h(x)\left(x^{\alpha}v^{(2n)}(x)\right)^{(2n)} - v(x)\left(x^{\alpha}h^{(2n)}(x)\right)^{(2n)} =$$
$$\sum_{j=0}^{2n-1}\frac{d}{dx}\left\{(-1)^j\left[h^{(j)}(x)\left(x^{\alpha}v^{(2n)}(x)\right)^{(2n-1-j)} - v^{(j)}(x)\left(x^{\alpha}h^{(2n)}(x)\right)^{(2n-1-j)}\right]\right\}, 0 < x < 1. \tag{2.20}$$

If we assume that $v(x) = G(x,s)$ and $h(x) = G(x,\xi)$, then at all points of segment $(0,1)$, except points $x \neq \xi$, $x \neq s$, the equalities $M[v] = 0$ and $M[h] = 0$ hold. Then equality (2.20) takes the form

$$\sum_{j=0}^{2n-1}\frac{\partial}{\partial x}\left\{(-1)^j\left[\frac{\partial^j}{\partial x^j}G(x,\xi)\frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right) - \right.\right.$$
$$\left.\left.-\frac{\partial^j}{\partial x^j}G(x,s)\frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,\xi)\right)\right]\right\} = 0,\ x \in (0,1)/\{s,\xi\}. \tag{2.21}$$

Without loss of generality, assume that $s < \xi$. Then segment $[0,1]$ is divided into three segments, i.e. $[0,s]$, $[s,\xi]$ and $[\xi,1]$. Integrating the equality (2.21) over these segments, we obtain

$$\sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{\partial^j}{\partial x^j}G(x,\xi)\frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right) - \right.\right.$$
$$\left.\left.-\frac{\partial^j}{\partial x^j}G(x,s)\frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,\xi)\right)\right]\right\}_{x=0}^{x=s-0} +$$
$$+\sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{\partial^j}{\partial x^j}G(x,\xi)\frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right) - \right.\right.$$
$$\left.\left.-\frac{\partial^j}{\partial x^j}G(x,s)\frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,\xi)\right)\right]\right\}_{x=s+0}^{x=\xi-0} +$$
$$+\sum_{j=0}^{2n-1}\left\{(-1)^j\left[\frac{\partial^j}{\partial x^j}G(x,\xi)\frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right) - \right.\right.$$
$$\left.\left.-\frac{\partial^j}{\partial x^j}G(x,s)\frac{\partial^{2n-1-j}}{\partial x^{2n-1-j}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,\xi)\right)\right]\right\}_{x=\xi+0}^{x=1} = 0.$$

Taking into account properties 1) and 4) of the Green's function $G(x,s)$, the last equality takes the form

$$-\left[G(x,\xi)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\right]\Big|_{x=s-0}^{x=s+0} + \left[G(x,s)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,\xi)\right)\right]\Big|_{x=s-0}^{x=s+0} -$$
$$-\left[G(x,\xi)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\right]\Big|_{x=\xi-0}^{x=\xi+0} + \left[G(x,s)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,\xi)\right)\right]\Big|_{x=\xi-0}^{x=\xi+0} = 0.$$

According to 2) property of the Green's function $G(x,s)$, the function $\left(\partial^{2n-1}/\partial x^{2n-1}\right)\left[x^{\alpha}\left(\partial^{2n}/\partial x^{2n}\right)G(x,s)\right]$ is continuous at $x \neq s$. Considering this, from the last, we have

$$\left[G(x,\xi)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\Big|_{x=s-0} - G(x,\xi)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,s)\right)\Big|_{x=s+0}\right] +$$
$$+\left[G(x,s)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,\xi)\right)\Big|_{x=\xi+0} - G(x,s)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^{\alpha}\frac{\partial^{2n}}{\partial x^{2n}}G(x,\xi)\right)\Big|_{x=\xi-0}\right] = 0.$$

Hence, by virtue of equality (2.4), we have $-G(s,\xi)+G(\xi,s)=0$, which was required to be proved.

In a particular case, for $n=1$, Green's function $G(x,s)$ has the form

$$G(x,s)=\begin{cases} -\dfrac{x^{3-\alpha}}{2(2-\alpha)_2}+\dfrac{x^{2-\alpha}s}{(1-\alpha)_2}-\dfrac{x^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{s^{3-\alpha}}{2(2-\alpha)_2}-\dfrac{s^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{1}{2(1-\alpha)_3}, & 0\le x\le s,\\ -\dfrac{s^{3-\alpha}}{2(2-\alpha)_2}+\dfrac{s^{2-\alpha}x}{(1-\alpha)_2}-\dfrac{s^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{x^{3-\alpha}}{2(2-\alpha)_2}-\dfrac{x^{2-\alpha}}{2(1-\alpha)_2}+\dfrac{1}{2(1-\alpha)_3}, & s\le x\le 1,\end{cases}$$

and for $n=2$, it has the following form

$$G(x,s)=\begin{cases} -\dfrac{x^{7-\alpha}}{2\cdot 3!(4-\alpha)_4}+\left(s-\dfrac{1}{2}\right)\dfrac{x^{6-\alpha}}{2!(3-\alpha)_4}+\left(\dfrac{1}{2}-s^2\right)\dfrac{x^{5-\alpha}}{4(2-\alpha)_4}+\\ +\left(\dfrac{s^3}{3!}-\dfrac{s^2}{4}+\dfrac{1}{24}\right)\dfrac{x^{4-\alpha}}{(1-\alpha)_4}+\dfrac{s^{7-\alpha}}{2\cdot 3!(4-\alpha)_4}-\dfrac{s^{6-\alpha}}{2\cdot 2!(3-\alpha)_4}+\\ +\left(x^2+\dfrac{1}{2}\right)\dfrac{s^{5-\alpha}}{4(2-\alpha)_4}+\left(\dfrac{1}{6}-x^2\right)\dfrac{s^{4-\alpha}}{4(1-\alpha)_4}+\dfrac{x^2s^2}{8(1-\alpha)_3}-\dfrac{\alpha^2-9\alpha+12}{16(1-\alpha)_5}x^2-\\ -\dfrac{\alpha^2-9\alpha+12}{16(1-\alpha)_5}s^2+\dfrac{\alpha^4-22\alpha^3+163\alpha^2-430\alpha+328}{32(1-\alpha)_7}, \quad 0\le x\le s,\\ -\dfrac{s^{7-\alpha}}{2\cdot 3!(4-\alpha)_4}+\left(x-\dfrac{1}{2}\right)\dfrac{s^{6-\alpha}}{2!(3-\alpha)_4}+\left(\dfrac{1}{2}-x^2\right)\dfrac{s^{5-\alpha}}{4(2-\alpha)_4}+\\ +\left(\dfrac{x^3}{3!}-\dfrac{x^2}{4}+\dfrac{1}{24}\right)\dfrac{s^{4-\alpha}}{(1-\alpha)_4}+\dfrac{x^{7-\alpha}}{2\cdot 3!(4-\alpha)_4}-\dfrac{x^{6-\alpha}}{2\cdot 2!(3-\alpha)_4}+\\ +\left(s^2+\dfrac{1}{2}\right)\dfrac{x^{5-\alpha}}{4(2-\alpha)_4}+\left(\dfrac{1}{6}-s^2\right)\dfrac{x^{4-\alpha}}{4(1-\alpha)_4}+\dfrac{s^2x^2}{8(1-\alpha)_3}-\dfrac{\alpha^2-9\alpha+12}{16(1-\alpha)_5}s^2-\\ -\dfrac{\alpha^2-9\alpha+12}{16(1-\alpha)_5}x^2+\dfrac{\alpha^4-22\alpha^3+163\alpha^2-430\alpha+328}{32(1-\alpha)_7}, \quad s\le x\le 1.\end{cases}$$

The last two examples demonstrate that the Green's function is symmetric in its arguments.

Now, using the method applied in [19], problem $\{(2.1),(2.2)\}$ can be equivalently reduced to the following integral equation

$$v(x)=\lambda\int_0^1 G(x,s)\,v(s)\,ds. \tag{2.22}$$

Since the kernel $G(x,s)$ is continuous, symmetric and positive ($\lambda>0$), then the integral equation (2.22), and therefore the problem $\{(2.1),(2.2)\}$ has a countable set of eigenvalues $0<\lambda_1\le\lambda_2\le\lambda_3\le \dots\le\lambda_k<\dots,\quad \lambda_k\to+\infty$, and the corresponding eigenfunctions $v_1(x), v_2(x), v_3(x), \dots, v_k(x)\dots$ form an orthonormal system in the space $L_2(0,1)$ [20].

Moreover, it is easy to verify by direct verification that the system of functions $s^{\alpha/2}v_k^{(2n)}(s)/\sqrt{\lambda_k}$, $k=1,2,\dots$ also constitutes an orthonormal system in $L_2(0,1)$.

Lemma 2.1. *Let the function $g(x)$ satisfy conditions (2.2) and $Mg(x)\in C(0,1)\cap L_2(0,1)$. Then, it can be expanded on the segment $[0,1]$ into an absolutely and uniformly convergent series in the system of eigenfunctions of the problem $\{(2.1),(2.2)\}$.*

Proof. Using the rule of integration by parts, the properties of Green's function $G(x,s)$ and the conditions imposed on the function $g(x)$, it is easy to verify that the following equality holds:

$$\int_0^1 G(x,s)\,Mg(s)\,ds=\int_0^1 G(x,s)\left[s^{\alpha}g^{(2n)}(s)\right]^{(2n)}ds=g(x).$$

Since $Mg(x)\in L_2(0,1)$, it follows from the last equality that $g(x)$ is a function that can be representable through the kernel of $G(x,s)$. In addition, the function $G(x,s)$, i.e. the kernel of equation

(2.22), is continuous in $\overline{\Omega}$. Then, based on Theorem 2, p. 153, of the book [20], the statement of Lemma 2.1 is true. □

Lemma 2.2. *The following series converge uniformly on the segment* $[0,1]$*:*

$$\sum_{k=1}^{+\infty}\frac{\left[v_k^{(j)}(x)\right]^2}{\lambda_k},\ \sum_{k=1}^{+\infty}\frac{\left(\left[x^\alpha v_k^{(2n)}(x)\right]^{(j)}\right)^2}{\lambda_k^2},\ j=\overline{0,2n-1}. \tag{2.23}$$

Proof. Taking into account equality (2.1) and the properties of function $G(x,s)$, from (2.22) for $v(x)\equiv v_k(x)$ we obtain

$$v_k^{(j)}(x)=\lambda_k\int_0^1\frac{\partial^j}{\partial x^j}G(x,s)\,v_k(s)\,ds=\int_0^1\left[s^\alpha v_k^{(2n)}(s)\right]^{(2n)}\frac{\partial^j}{\partial x^j}G(x,s)\,ds,\quad j=\overline{0,2n-1}.$$

Hence, applying the rule of integration by parts $2n$ times, and then taking into account conditions (2.2), we have

$$v_k^{(j)}(x)=\int_0^1 s^\alpha v_k^{(2n)}(s)\,\frac{\partial^{2n+j}}{\partial x^j\partial s^{2n}}G(x,s)\,ds,\ j=\overline{0,2n-1},$$

from which, by virtue of $\lambda_k>0$, the equality

$$\frac{v_k^{(j)}(x)}{\sqrt{\lambda_k}}=\int_0^1\left(s^{\alpha/2}\frac{\partial^{2n+j}}{\partial x^j\partial s^{2n}}G(x,s)\right)\left(\frac{s^{\alpha/2}v_k^{(2n)}(s)}{\sqrt{\lambda_k}}\right)ds,\quad j=\overline{0,2n-1} \tag{2.24}$$

follows.

From (2.24) it follows that $v_k^{(j)}(x)/\sqrt{\lambda_k}$ is the Fourier coefficient of the function by the orthonormal system of functions $\left\{s^{\alpha/2}v_k^{(2n)}(s)/\sqrt{\lambda_k}\right\}_{k=1}^{+\infty}$. Therefore, according to Bessel's inequality [20], we have

$$\sum_{k=1}^{+\infty}\frac{\left[v_k^{(j)}(x)\right]^2}{\lambda_k}\le\int_0^1 s^\alpha\left[\frac{\partial^{2n+j}}{\partial x^j\partial s^{2n}}G(x,s)\right]^2 ds,\quad j=\overline{0,2n-1}. \tag{2.25}$$

By $0<\alpha<1$, the integral on the right-hand side of inequality (2.25) is uniformly bounded for $j=\overline{0,2n-1}$, which implies that the first series in (2.23) converges uniformly.

The convergence of the remaining series is proved similarly.

Lemma 2.2 has been proved. □

From here and the rest of the paper g_k denotes the Fourier coefficient of the function $g(x)$ by the system of eigenfunctions $\{v_k(x)\}_{k=1}^{+\infty}$, i.e.

$$g_k=\int_0^1 g(x)\,v_k(x)\,dx,\quad k\in\mathbb{N}. \tag{2.26}$$

Lemma 2.3. *Let* $g^{(j)}(x)\in C[0,1],\ j=\overline{0,2n-1},\ x^{\alpha/2}g^{(2n)}(x)\in C(0,1)\cap L_2(0,1),\ g^{(2j)}(0)+g^{(2j)}(1)=0,\ g^{(2j+1)}(1)=0,\ j=\overline{0,n-1}$. *Then, the following inequality is valid*

$$\sum_{k=1}^{+\infty}\lambda_k g_k^2\le\int_0^1 x^\alpha\left[g^{(2n)}(x)\right]^2dx. \tag{2.27}$$

In particular, the series on the left-hand side converges.

Proof. Using the equation (2.1), we can rewrite (2.26) as follows

$$\lambda_k^{1/2} g_k = \lambda_k^{1/2} \int_0^1 g(x)\, v_k(x)\, dx = \lambda_k^{-1/2} \int_0^1 g(x) \left[x^\alpha v_k^{(2n)}(x)\right]^{(2n)} dx.$$

Hence, applying the rule of integration by parts $2n$ times and considering the properties of the functions $g(x)$ and $v_k(x)$, we obtain

$$\lambda_k^{1/2} g_k = \int_0^1 \left\{x^{\alpha/2} g^{(2n)}(x)\right\} \left\{\lambda_k^{-1/2} x^{\alpha/2} v_k^{(2n)}(x)\right\} dx.$$

From the last, it follows that the number $\lambda_k^{1/2} g_k$ is the Fourier coefficient of the function $x^{\alpha/2} g^{(2n)}(x)$ in the orthonormal system $\left\{x^{\alpha/2} v^{(2n)}(x)/\sqrt{\lambda_k}\,\right\}_{k=1}^{+\infty}$. Then, according to Bessel's inequality [20], we have the validity of (2.27). Lemma 2.3 has been proved. □

Lemma 2.4. *If the function $g(x)$ satisfies the conditions* (2.2) *and $Mg(x) \in C(0,1) \cap L_2(0,1)$, then the inequality holds true*

$$\sum_{k=1}^{+\infty} \lambda_k^2 g_k^2 \leq \int_0^1 \left[Mg(x)\right]^2 dx. \tag{2.28}$$

In particular, the series on the left-hand side converges.

Proof. Considering (2.1), we rewrite (2.26), in the following form

$$\lambda_k g_k = \lambda_k \int_0^1 g(x)\, v_k(x)\, dx = \int_0^1 g(x) \left[x^\alpha v_k^{(2n)}(x)\right]^{(2n)} dx.$$

Hence, applying the rule of integration by parts $4n$ times to the integral on the right-hand side of the last equality and considering the properties of the functions $g(x)$ and $v_k(x)$, we obtain

$$\lambda_k g_k = \int_0^1 \left[x^\alpha g^{(2n)}(x)\right]^{(2n)} v_k(x)\, dx = \int_0^1 [Mg(x)] v_k(x)\, dx.$$

From the last, it follows that the number $\lambda_k g_k$ is the Fourier coefficient of the function $Mg(x)$ with respect to the orthonormal system of functions $\{v_k(x)\}_{k=1}^{+\infty}$. Then, according to Bessel's inequality, inequality (2.28) is valid. Lemma 2.4 has been proved. □

Similarly, the following lemma can be proved:

Lemma 2.5. *If the function $g(x)$ satisfies conditions* (2.2) *and $[Mg(x)]^{(j)} \in C[0,1]$, $j = \overline{0, 2n-1}$, $x^{\alpha/2}[Mg(x)]^{(2n)} \in C(0,1) \cap L_2(0,1)$, $[Mg(x)]^{(2j)}\Big|_{x=0} + [Mg(x)]^{(2j)}\Big|_{x=1} = 0$, $[Mg(x)]^{(2j+1)}\Big|_{x=1} = 0$, $j = \overline{0, n-1}$, then inequality*

$$\sum_{k=1}^{+\infty} \lambda_k^3 g_k^2 \leq \int_0^1 x^\alpha \left\{[Mg(x)]^{(2n)}\right\}^2 dx \tag{2.29}$$

is true, in particular, the series on the left-hand side of (2.29) *converges.*

3. Existence, uniqueness and stability of the solution of problem U

We will seek a solution to the problem U in the form

$$u(x,t)=\sum_{k=1}^{+\infty} u_k(t)\, v_k(x), \tag{3.1}$$

where $v_k(x)$, $k\in\mathbb{N}$ are the eigenfunctions of the problem $\{(2.1),(2.2)\}$, and $u_k(t)$, $k\in\mathbb{N}$ are the unknown functions to be determined.

Substituting (3.1) into equation (1.1) and into the initial conditions (1.2), with respect to $u_k(t)$, $k\in\mathbb{N}$, we obtain the following problem

$$u_k''(t)+\lambda_k u_k(t)=f_k(t),\ \ t\in(0,T),\ \ k\in\mathbb{N},$$

$$u_k(0)=\varphi_{1k},\quad u_k'(0)=\varphi_{2k},$$

where

$$\varphi_{jk}=\int_0^1 \varphi_j(x)\, v_k(x)\,dx,\ \ j=\overline{1,2};\ \ f_k(t)=\int_0^1 f(x,t)\, v_k(x)\,dx,\ \ k\in\mathbb{N}. \tag{3.2}$$

It is known that the solution to the last problem exists, is unique, and is determined by the following formula:

$$u_k(t)=\varphi_{1k}\cos\left(t\sqrt{\lambda_k}\right)+\varphi_{2k}\lambda_k^{-1/2}\,\sin\left(t\sqrt{\lambda_k}\right)+\lambda_k^{-1/2}\int_0^t f_k(\tau)\sin\left[(t-\tau)\sqrt{\lambda_k}\right]d\tau, 0\le t\le T. \tag{3.3}$$

From this, the estimate

$$|u_k(t)|\le|\varphi_{1k}|+\frac{1}{\sqrt{\lambda_k}}|\varphi_{2k}|+\frac{1}{\sqrt{\lambda_k}}\sqrt{\int_0^T f_k^2(\tau)d\tau},\ \ 0\le t\le T \tag{3.4}$$

easily follows.

Theorem 3.1. *Let the function $\varphi_1(x)$ satisfy the conditions of Lemma 2.5, the function $\varphi_2(x)$ satisfy the conditions of Lemma 2.4, and the function $f(x,t)$ satisfy the conditions of Lemma 2.4 with respect to the argument x uniformly with respect to t. Then the series* (3.1)*, whose coefficients are defined by equalities* (3.3)*, determines the solution of the problem U.*

Proof. To do this, we need to prove the uniform convergence in $\overline{\Omega}$ of the series (3.1) and the following series, formally obtained from (3.1):

$$u_t(x,t)=\sum_{k=1}^{+\infty} u_k'(t)\, v_k(x),\ \ \frac{\partial^j u(x,t)}{\partial x^j}=\sum_{k=1}^{+\infty} u_k(t)\, v_k^{(j)}(x),\ \ j=\overline{1,2n-1},$$

$$\frac{\partial^j}{\partial x^j}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial^{2n}}\right)=\sum_{k=1}^{+\infty} u_k(t)\left(x^\alpha v_k^{(2n)}(x)\right)^{(j)},\ \ j=\overline{0,2n-1},$$

and the uniform convergence in any compact set $\Omega_0\subset\Omega$ of the series

$$\frac{\partial^{2n}}{\partial x^{2n}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial^{2n}}\right)=\sum_{k=1}^{+\infty} u_k(t)\left(x^\alpha v_k^{(2n)}(x)\right)^{(2n)}, \tag{3.5}$$

$$u_{tt}(x,t)=\sum_{k=1}^{+\infty} u_k''(t)\, v_k(x). \tag{3.6}$$

Let us consider the series (3.1). By virtue of (3.4) from (3.1), for any $(x,t) \in \overline{\Omega}$, we have

$$|u(x,t)| \leq \sum_{k=1}^{+\infty} |u_k(t)|\,|v_k(x)| \leq \sum_{k=1}^{+\infty} \frac{|v_k(x)|}{\sqrt{\lambda_k}} \left(\sqrt{\lambda_k}\,|\varphi_{1k}| + |\varphi_{2k}| + \sqrt{\int_0^T f_k^2(\tau)d\tau} \right).$$

Hence, applying the Cauchy-Schwarz inequality, we obtain

$$|u(x,t)| \leq \sqrt{\sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k}} \left(\sqrt{\sum_{k=1}^{+\infty} \lambda_k \varphi_{1k}^2} + \sqrt{\sum_{k=1}^{+\infty} \varphi_{2k}^2} + \sqrt{\int_0^T \sum_{k=1}^{+\infty} [f_k(\tau)]^2 d\tau} \right). \tag{3.7}$$

The series on the right-hand side of this inequality, by the condition of Theorem 3.1, according to Lemmas 2.2 and 2.3, converge uniformly. Therefore, the series on the left-hand side, i.e., series (3.1) converges uniformly in $\overline{\Omega}$.

Now, let us consider the series (3.5). By virtue of equation (2.1), in any compact Ω_0 the series in (3.5) can be written as

$$\sum_{k=1}^{+\infty} \lambda_k u_k(t)\, v_k(x). \tag{3.8}$$

To prove the uniform convergence of series (3.8), according to (3.4), it is sufficient to prove the absolute and uniform convergence of the series

$$\sum_{k=1}^{+\infty} \lambda_k \varphi_{1k} v_k(x), \quad \sum_{k=1}^{+\infty} \sqrt{\lambda_k} \varphi_{2k} v_k(x), \quad \sum_{k=1}^{+\infty} \sqrt{\lambda_k} \sqrt{\int_0^T [f_k(\tau)]^2 d\tau}\, v_k(x). \tag{3.9}$$

Similarly, applying Cauchy-Schwarz inequality to each of these series, we have

$$\left| \sum_{k=1}^{+\infty} \lambda_k \varphi_{1k} v_k(x) \right| \leq \sum_{k=1}^{+\infty} \left| \sqrt{\lambda_k^3} \varphi_{1k} \frac{v_k(x)}{\sqrt{\lambda_k}} \right| \leq \left[\sum_{k=1}^{+\infty} \lambda_k^3 \varphi_{1k}^2 \sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k} \right]^{1/2},$$

$$\left| \sum_{k=1}^{+\infty} \sqrt{\lambda_k} \varphi_{2k} v_k(x) \right| \leq \sum_{k=1}^{+\infty} \left| \lambda_k \varphi_{2k} \frac{v_k(x)}{\sqrt{\lambda_k}} \right| \leq \left[\sum_{k=1}^{+\infty} \lambda_k^2 \varphi_{2k}^2 \cdot \sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k} \right]^{1/2},$$

$$\left| \sum_{k=1}^{+\infty} \sqrt{\lambda_k} \sqrt{\int_0^T [f_k(\tau)]^2 d\tau}\, v_k(x) \right| \leq \sum_{k=1}^{+\infty} \left| \sqrt{\lambda_k^2 \int_0^T [f_k(\tau)]^2 d\tau}\, \frac{v_k(x)}{\sqrt{\lambda_k}} \right| \leq$$

$$\leq \left[\int_0^T \sum_{k=1}^{+\infty} \lambda_k^2 [f_k(\tau)]^2 d\tau \cdot \sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k} \right]^{1/2}.$$

The series on the right-hand sides of these inequalities, by the conditions of Theorem 3.1, according to Lemmas 2.2, 2.4, and 2.5, converge uniformly. Then the series on the left-hand sides, i.e., series (3.9), converge absolutely and uniformly in Ω_0. Therefore, the series (3.8), and consequently the series in (3.5) converge uniformly in the any compact set Ω_0. Uniform convergence in Ω_0 of series (3.6) follows from the convergence of series (3.5) and the validity of equation (1.1).

The uniform convergence of the remaining series can be proved similarly. Theorem 3.1 has been proved. □

Theorem 3.2. *Problem U cannot have more than one solution.*

Proof. Suppose that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ of the problem U. We denote their difference by $u(x,t)$. Then the function $u(x,t)$ satisfies equation (1.1) for $f(x,t)\equiv 0$, and conditions (1.2) and (1.3) for $\varphi_1(x)\equiv\varphi_2(x)\equiv 0$.

Let $\forall T_0\in(0,T]$, $\Omega_0=\{(x,t):\ 0<x<1,\ 0<t<T_0\}$. Obviously $\overline{\Omega}_0\subset\overline{\Omega}$. Let us introduce the following function:

$$\omega(x,t)=-\int\limits_t^{T_0}u(x,\xi)d\xi,\quad (x,t)\in\overline{\Omega}_0\,.$$

This function has the following properties:

1) $\omega_t,\ \omega_{tt},\ \frac{\partial^j\omega}{\partial x^j},\ \frac{\partial^j}{\partial x^j}\left(x^\alpha\frac{\partial^{2n}\omega}{\partial x^{2n}}\right)\in C\left(\overline{\Omega}_0\right),\ \ j=\overline{0,2n-1};$

2) it satisfies conditions (1.3) for $t\in[0,T_0]$.

Let us consider equation (1.1) for $f(x,t)\equiv 0$, multiplying it by the function $\omega(x,t)$ and integrating the resulting equality over the domain Ω_0, we have

$$\int\limits_{\Omega_0}\omega(x,t)\left\{u_{tt}(x,t)+\frac{\partial^{2n}}{\partial x^{2n}}\left[x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right]\right\}dtdx=0.$$

Let us rewrite this equality as

$$\int\limits_0^{T_0}dt\int\limits_0^1\omega(x,t)\frac{\partial^{2n}}{\partial x^{2n}}\left[x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right]dx+\int\limits_0^1dx\int\limits_0^{T_0}\omega(x,t)u_{tt}(x,t)dt=0.$$

Now, applying the rule of integration by parts, we obtain the equality

$$\int\limits_0^{T_0}\left[\omega(x,t)\frac{\partial^{2n-1}}{\partial x^{2n-1}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)-\frac{\partial\omega(x,t)}{\partial x}\frac{\partial^{2n-2}}{\partial x^{2n-2}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)+\ldots\right.$$

$$\left.\ldots-\frac{\partial^{2n-1}\omega(x,t)}{\partial x^{2n-1}}\left(x^\alpha\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}\right)\right]_{x=0}^{x=1}dt+\int\limits_0^{T_0}dt\int\limits_0^1x^\alpha\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}dx+$$

$$+\int\limits_0^1\left[\omega(x,t)\frac{\partial u(x,t)}{\partial t}\Big|_{t=0}^{t=T_0}-\int\limits_0^{T_0}\frac{\partial\omega(x,t)}{\partial t}\frac{\partial u(x,t)}{\partial t}\right]dx=0,$$

from which, due to the properties of the functions $\omega(x,t)$ and $u(x,t)$, it follows that

$$\int\limits_0^{T_0}dt\int\limits_0^1x^\alpha\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n}u(x,t)}{\partial x^{2n}}dx-\int\limits_0^1dx\int\limits_0^{T_0}\frac{\partial\omega(x,t)}{\partial t}\frac{\partial u(x,t)}{\partial t}dt=0.$$

Hence, taking into account the equalities $u=\frac{\partial\omega}{\partial t}$ and $\frac{\partial^{2n}u}{\partial x^{2n}}=\frac{\partial^{2n+1}\omega}{\partial x^{2n}\partial t}$, we have

$$\int\limits_0^1x^\alpha dx\int\limits_0^{T_0}\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n+1}\omega(x,t)}{\partial x^{2n}\partial t}dt-\int\limits_0^1dx\int\limits_0^{T_0}u(x,t)\frac{\partial u(x,t)}{\partial t}dt=0.$$

Next, considering the equalities

$$u(x,t)\frac{\partial u(x,t)}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}[u(x,t)]^2,\quad\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\frac{\partial^{2n+1}\omega(x,t)}{\partial x^{2n}\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{\partial^{2n}\omega(x,t)}{\partial x^{2n}}\right]^2,$$

and applying the rule of integration by parts to the integrals over t, taking into account $\omega(x,T_0)=0$, $u\left(x,0\right)=0$, we obtain

$$\int\limits_0^1 u^2(x,T_0)\,dx+\int\limits_0^1 x^\alpha\left[\frac{\partial^{2n}\omega\left(x,t\right)}{\partial x^{2n}}\right]^2_{t=0}dx=0.$$

Hence, it follows that $u\left(x,T_0\right)\equiv 0,\ x\in[0,1]$. Since $\forall T_0\in(0,T]$, then $u\left(x,t\right)\equiv 0,\ (x,t)\in\overline{\Omega}$. Then $u_1\left(x,t\right)\equiv u_2\left(x,t\right),\ (x,t)\in\overline{\Omega}$. Theorem 3.2 is proven. □

Theorem 3.3. *Let the functions $\varphi_1\left(x\right),\ \varphi_2\left(x\right)$ and $f\left(x,t\right)$ satisfy the conditions of Theorem* (3.1). *Then the following estimates are valid for the solution of problem U:*

$$\left\|u\left(x,t\right)\right\|^2_{L_2(0,1)}\le K_0\left[\left\|\varphi_1\left(x\right)\right\|^2_{L_2(0,1)}+\left\|\varphi_2\left(x\right)\right\|^2_{L_2(0,1)}+\left\|f\left(x,t\right)\right\|^2_{L_2(\Omega)}\right],\tag{3.10}$$

$$\left\|u\left(x,t\right)\right\|_{C\left(\overline{\Omega}\right)}\le K_1\left[\left\|\varphi_1^{(2n)}\left(x\right)\right\|_{L_{2,r}(0,1)}+\left\|\varphi_2\left(x\right)\right\|_{L_2(0,1)}+\left\|f\left(x,t\right)\right\|_{L_2(\Omega)}\right],\tag{3.11}$$

where $\left\|\varphi_1\left(x\right)\right\|_{L_{2,r}(0,1)}=\left[\int\limits_0^1 x^\alpha[\varphi_1\left(x\right)]^2dx\right]^{1/2}$ *and* $r=r\left(x\right)=x^\alpha$, *and* K_0 *and* K_1 *are some real positive numbers.*

Proof. Here, taking into account the orthonormality of the system $\left\{v_k\left(x\right)\right\}_{k=1}^{+\infty}$ and inequality (3.4), from (3.1), we have

$$\left\|u\left(x,t\right)\right\|^2_{L_2(0,1)}=\sum_{k=1}^{+\infty}u_k^2\left(t\right)\le$$

$$\le\sum_{k=1}^{+\infty}\left[\left|\varphi_{1k}\right|+\frac{1}{\sqrt{\lambda_k}}\left|\varphi_{2k}\right|+\frac{1}{\sqrt{\lambda_k}}\left\|f_k\left(t\right)\right\|_{L_2(0,T)}\right]^2\le 3\sum_{k=1}^{+\infty}\left[\varphi_{1k}^2+\frac{1}{\lambda_k}\varphi_{2k}^2+\frac{1}{\lambda_k}\left\|f_k(t)\right\|^2_{L_2(0,T)}\right].$$

Hence, applying Bessel's inequality, we get

$$\left\|u\left(x,t\right)\right\|^2_{L_2(0,1)}\le K_0\left(\left\|\varphi_1\left(x\right)\right\|^2_{L_2(0,1)}+\left\|\varphi_2\left(x\right)\right\|^2_{L_2(0,1)}+\sum_{k=1}^{+\infty}\left\|f_k\left(t\right)\right\|^2_{L_2(0,T)}\right),\tag{3.12}$$

where $K_0=3C,\ C=\max\left(1,\ 1/\lambda_1\right)$.

Taking into account the easily verified equality

$$\left\|f\left(x,t\right)\right\|^2_{L_2(\Omega)}=\sum_{n=1}^{+\infty}\left\|f_k\left(t\right)\right\|^2_{L_2(0,T)},$$

from (3.12) we obtain inequality (3.10).

Taking into account the statements of Lemmas 2.2 and 2.3, from (3.7), we obtain the inequality

$$\left\|u\left(x,t\right)\right\|_{C\left(\overline{\Omega}\right)}=\sup_{\overline{\Omega}}\left|u\left(x,t\right)\right|\le K_1\left\{\sqrt{\int\limits_0^1 x^\alpha\left[\varphi_1^{(2n)}\left(x\right)\right]^2dx}+\left\|\varphi_2\left(x\right)\right\|^2_{L_2(0,1)}+\sum_{k=1}^{+\infty}\left\|f_k\left(t\right)\right\|^2_{L_2(0,T)}\right\},$$

where $K_1=\sup\limits_{[0,1]}\sqrt{\sum\limits_{k=1}^{+\infty}v_k^2\left(x\right)/\lambda_k}$.

From here, by the introduced notations, inequality (3.11) follows.

Theorem 3.3 has been proved. □

References

[1] Tikhonov A. N., Samarskii A. A.; Equations of Mathematical Physics. Moscow, Nauka, –1966.

[2] Strutt J.; Theory of sound. Vol. 1, Gostekhizdat, Moscow, USSR. –1955.

[3] Timoshenko S.P., Young D.Kh., Weaver U.; Oscillations in engineering. Moscow, Mechanical Engineering,–1985.

[4] Frankl F.I.; Flowing profiles by a stream of subsonic velocity with a supersonic zone ending with a direct surge of seal. PMM,–1956.–47(2).–P.196-202.

[5] Ionkin N.I.; The Solution of a boundary value problem of thermal conduction theory with a nonlocal boundary condition. Dier. equations.,–1977.–13(2).–P.294-304.

[6] Ionkin N.I.; On the sustainability of one problem theory of thermal conductivity with a nonclassical edge condition. Differ. equations,–1979.–15(7).–P.1279-1283.

[7] Urinov A.K., Azizov M.S.; On the solvability of an initialboundary value problem for a high even order partial differential equation degenerating on the domain boundary. Journal of Applied and Industrial Mathematics,–2023.–17(2).–P.414-426.

[8] Kipriyanov I.A.; Singular elliptic boundary value problems. Moscow, Nauka,–1997.

[9] Urinov A.K., Azizov M.S.; On the solvability of nonlocal initial-boundary value problems for a partial differential equation of high even order. Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Kompyuternye Nauki,–2022.–32(2).–P.240-255.

[10] Azizov M.S.; About an initial-boundary value problem for a partial differential equation of higher even order with the Bessel operator. Bull. Inst. Math.,–2022.–5(1).–P.14-24.

[11] Sabitov K.B.; The Dirichlet problem for higher-order partial dierential equations. Math. Notes,–2015.–97(2).–P.255-267.

[12] Irgashev B.Yu.; On the conditions for the solvability of a problem of Dirichlet type for the equation of high order with discontinuous coecients. Uzbek Mathematical Journal. –2021.–65(4).–P.13-26.

[13] Irgashev B.Yu.; Boundary value problem for high order equation with discontinuous coecients . Uzbek Mathematical Journal. –2020.–3.–P.85-97.

[14] Kasimov Sh.G., Ataev Sh.K.; On solvability of the mixed problem for a partial equation of a fractional order with Laplace operators and nonlocal boundary conditions in the Sobolev classes. Uzbek Mathematical Journal,–2018.–1.–P.73-89.

[15] Kasimov Sh.G., Madrakhimov U.S; On the unique solvability for initial - boundary problems of vibrations of a beam, one end of which is xed and the other is free, in the Sobolev classes in the multidimensional case. Uzbek Mathematical Journal,–2019.–4.–P.89-103.

[16] Urinov A.K., Usmonov D.A.; Initial boundary value problems for the fourth order equation with three lines of degeneracy. Uzbek Mathematical Journal. –2023.–67(1).–P.129-136.

[17] Beytmen G., Erdeyi A.; Higher transcendental functions. Vol. 1. Moskva, Nauka,–1965.

[18] Urinov A.K., Oripov D.D.; On the solvability of an initial boundary problem for a high even order degenerate equation. J. Samara State Tech. Univ., Ser. Phys. Math. Sci.,–2023.–27(4).–P.621-644.

[19] Naimark M.A.; Linear Differential Operators. Moscow, Nauka,–1969.

[20] Mikhlin S.G.; Lectures on Linear Integral Equations. Moscow, Fizmatgiz,–1959.

Karimov K.T.,
Romanovskii Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan,
Fergana State University, Fergana, Uzbekistan,
email: karimovk80@mail.ru

Oripov D.D.,
Fergana State University, Fergana, Uzbekistan,
email: dastonbekoripov94@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.165-168
DOI: 10.29229/uzmj.2025-4-16

About one composition of partial mapping of Euclidean space E_5

Matieva G., Papieva T., Shamshieva G.

Abstract. In the domain $\Omega \subset E_5$ we consider a set of smooth lines such that through each point $X \in \Omega$ there passes exactly one line ω^1 from the given set. The moving frame of the domain Ω is a Frenet frame [1] associated with the line ω^1. The integral lines of the coordinate vector fields form a Frenet net [1]. We define the point F_1^5 on the tangent of the line ω^1 in an invariant manner. As the point X moves within the domain Ω, the point F_1^5 traces out a new domain $\Omega_1^5 \subset E_5$. This defines the partial mapping $f_1^5 : \Omega \to \Omega_1^5$ such that $f_1^5(X) = F_1^5$.

Similarly, we define another partial mapping $f_5^4 : \Omega \to \Omega_5^4$ Next, we consider the composition of these two partial mappings, specifically the inverse mapping $(f_1^5)^{-1}$ and f_5^4 given by:

$f_5^4 \circ (f_1^5)^{-1} : \Omega_1^5 \to \Omega_5^4$ such that $f_5^4 \circ (f_1^5)^{-1}(F_1^5) = F_5^4$, where $(f_1^5)^{-1}$ - is the inverse mapping f_1^5.

Let the line γ, which belongs to the distribution $\Delta_4 = (X, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5)$ be a quasi-double line of the pair of distributions (Δ_4, Δ_4') in the partial mapping f_1^5 (where $\Delta_4' = f_1^5(\Delta_4)$).

We establish necessary and sufficient conditions for the line $f_5^4 \circ (f_1^5)^{-1}(\gamma)$ to be a quasi-double line of the pair (Δ_4, Δ_4') of distributions Δ_4, Δ_4' in the partial mapping $f_5^4 \circ (f_1^5)^{-1}$.

Keywords: Euclidean space, Frenet frame, cyclic Frenet net, partial mapping, quasi-double line, distribution.

MSC (2020): 53A04, 53A15

1. Introduction

In the domain $\Omega \subset E_5$ we consider a set of smooth lines such that through each point $X \in \Omega$ exactly one line of the given set passes. The moving frame $\Re = (X, \overrightarrow{e}_i)(i, j, k = \overline{1,5})$ is the Frenet frame for the line ω^1 of the given set of smooth lines. The derivation formula for the frame $\Re$ take the form:

$$d\overrightarrow{X} = \omega^i \overrightarrow{e}_i, d\overrightarrow{e}_i = \omega_i^k \overrightarrow{e}_k. \tag{1.1}$$

The forms ω^i, ω_i^k satisfy the structure equations of Euclidean space:

$$D\omega^i = \omega^k \wedge \omega_k^i, D\omega_i^k = \omega_i^j \wedge \omega_j^k, \omega_i^j + \omega_j^i = 0. \tag{1.2}$$

Integral lines of the vector fields $\overrightarrow{e}_i$ form the Frenet net Σ_5 [1] for the line ω^1 of the given set of lines. Since frame $\Re$ is constructed on the tangent of the lines of the net Σ_5, the forms ω_i^k are principal forms. In other words:

$$\omega_i^k = \Lambda_{ij}^k \omega^j. \tag{1.3}$$

Taking in to account (1.3) from (1.2) we have:

$$\Lambda_{ij}^k = -\Lambda_{kj}^i. \tag{1.4}$$

If we differentiate externally (1.3), we obtain:

$$D\omega_i^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k D\omega^j.$$

Using equation (1.2) we get:

$$\omega_i^j \wedge \omega_j^k = d\Lambda_{ij}^k \wedge \omega^j + \Lambda_{ij}^k \wedge \omega^\ell \wedge \omega_\ell^j.$$

Taking in to account equation (1.3) the last formula simplifies to:

$$\omega_i^j \wedge \Lambda_{j\ell}^k \omega^\ell = d\Lambda_{ij}^k \wedge \omega^j - \Lambda_{ij}^k \wedge \omega_\ell^j \wedge \omega^\ell$$

or equivalently,

$$\Lambda^k_{j\ell}\omega^j_i \wedge \omega^\ell = d\Lambda^k_{ij} \wedge \omega^j - \Lambda^k_{ij} \wedge \omega^j_\ell \wedge \omega^\ell.$$

From here, we obtain:

$$d\Lambda^k_{ij} \wedge \omega^j - \Lambda^k_{i\ell}\omega^\ell_j \wedge \omega^j - \Lambda^k_{j\ell}\omega^j_i \wedge \omega^\ell = 0$$

or

$$(d\Lambda^k_{ij} - \Lambda^k_{i\ell}\omega^\ell_j - \Lambda^k_{\ell j}\omega^\ell_i) \wedge \omega^j = 0.$$

Using Cartan's lemma [2], [6], we conclude:

$$d\Lambda^k_{ij} - \Lambda^k_{i\ell}\omega^\ell_j - \Lambda^k_{\ell j}\omega^\ell_i = \Lambda^k_{ijm}\omega^m.$$

Which simplifies to:

$$d\Lambda^k_{ij} = (\Lambda^k_{ijm} + \Lambda^k_{i\ell}\Lambda^\ell_{jm} + \Lambda^k_{\ell j}\Lambda^\ell_{im})\omega^m. \tag{1.5}$$

The system of variable $\{\Lambda^k_{ij}, \Lambda^k_{ijm}\}$ forms a second - order geometrical object. The Frenet formulas for the line ω^1 of given set take the form:

$$d_1\overrightarrow{e}_1 = \Lambda^2_{11}\overrightarrow{e}_2, \qquad d_1\overrightarrow{e}_2 = \Lambda^1_{21}\overrightarrow{e}_1 + \Lambda^3_{21}\overrightarrow{e}_3, \qquad d_1\overrightarrow{e}_3 = \Lambda^2_{31}\overrightarrow{e}_2 + \Lambda^4_{31}\overrightarrow{e}_4,$$

$$d_1\overrightarrow{e}_4 = \Lambda^3_{41}\overrightarrow{e}_3 + \Lambda^5_{41}\overrightarrow{e}_5, \qquad d_1\overrightarrow{e}_5 = \Lambda^4_{51}\overrightarrow{e}_4,$$

and

$$\Lambda^3_{11} = -\Lambda^3_{31} = 0, \Lambda^4_{11} = -\Lambda^4_{41} = 0, \Lambda^5_{11} = -\Lambda^5_{51} = 0. \tag{1.6}$$

$$\Lambda^5_{21} = -\Lambda^2_{51} = 0, \Lambda^4_{21} = -\Lambda^2_{41} = 0, \Lambda^5_{31} = -\Lambda^3_{51} = 0. \tag{1.7}$$

Here $k^1_1 = \Lambda^2_{11}, k^1_2 = \Lambda^3_{21}, k^1_3 = \Lambda^4_{31}, k^1_4 = \Lambda^5_{41} = -\Lambda^4_{51}$ - represent the first, second, third and fourth curvature of the line ω^1 respectively (where d_1 - denotes differentiation along the line ω^1).

A pseudofocus [3], [5] F^j_i (where $i \neq j$) of the tangent of the line ω^1 in the Frenet net Σ_5 is defined by the radius-vector:

$$\overrightarrow{F}^j_i = \overrightarrow{X} - \frac{1}{\Lambda^j_{ij}}\overrightarrow{e}_i = \overrightarrow{X} + \frac{1}{\Lambda^i_{jj}}\overrightarrow{e}_i. \tag{1.8}$$

There exist four pseudofocuses on each tangent $(X, \overrightarrow{e}_i)$:
on the straight line $(X, \overrightarrow{e}_1)$ - F^2_1, F^3_1, F^4_1, F^5_1;
on $(X, \overrightarrow{e}_2)$ - F^1_2, F^3_2, F^4_2, F^5_2;
on $(X, \overrightarrow{e}_3)$ - F^1_3, F^2_3, F^4_3, F^5_3;
on $(X, \overrightarrow{e}_4)$ - F^1_4, F^2_4, F^3_4, F^5_4;
on $(X, \overrightarrow{e}_5)$ - F^1_5, F^2_5, F^3_5, F^4_5.

The net Σ_5 in E_5 is called a cycle Frenet net [3] if the frames

$$\Re_1 = (X, \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5), \qquad \Re_2 = (X, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5, \overrightarrow{e}_1), \qquad \Re_3 = (X, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5, \overrightarrow{e}_1, \overrightarrow{e}_2),$$

$$\Re_4 = (X, \overrightarrow{e}_4, \overrightarrow{e}_5, \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3), \qquad \Re_5 = (X, \overrightarrow{e}_5, \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4),$$

are Frenet frames for the lines $\omega^1, \omega^2, \omega^3, \omega^4, \omega^5$ respectively, of the net Σ_5 simultaneously.

If the net Σ_5 be a cycle Frenet net, we denote it by $\widetilde{\Sigma}_5$. [3]. The pseudofocus $F^5_1 \in (X, \overrightarrow{e}_1)$ is defined by the radius-vector:

$$\overrightarrow{F}^5_1 = \overrightarrow{X} - \frac{1}{\Lambda^1_{15}}\overrightarrow{e}_1. \tag{1.9}$$

As the point X moves within the domain $\Omega \subset E_5$, the pseudofocus F^5_1 traces out its own domain $\Omega^5_1 \subset E_5$. Thus, we define the partial mapping $f^5_1 : \Omega \to \Omega^5_1$ such that $f^5_1(X) = F^5_1$. We associate the domain $\Omega^5_1 \subset E_5$ with the moving frame $\Re' = (F^5_1, \overrightarrow{b}_i)(i = \overline{1,5})$, where the vectors $\overrightarrow{b}_i$ have the form [4], [7]:

$$\overrightarrow{b}_1 = \overrightarrow{e}_1 + \frac{B^5_{151}}{(\Lambda^5_{15})^2}\overrightarrow{e}_1 - \frac{\Lambda^i_{11}}{\Lambda^5_{15}}\overrightarrow{e}_i;$$

$$\overrightarrow{b}_2 = \overrightarrow{e}_2 + \frac{B^5_{152}}{(\Lambda^5_{15})^2}\overrightarrow{e}_1 - \frac{\Lambda^i_{12}}{\Lambda^5_{15}}\overrightarrow{e}_i;$$
$$\overrightarrow{b}_3 = \overrightarrow{e}_3 + \frac{B^5_{153}}{(\Lambda^5_{15})^2}\overrightarrow{e}_1 - \frac{\Lambda^i_{13}}{\Lambda^5_{15}}\overrightarrow{e}_i; \quad (1.10)$$
$$\overrightarrow{b}_4 = \overrightarrow{e}_4 + \frac{B^5_{154}}{(\Lambda^5_{15})^2}\overrightarrow{e}_1 - \frac{\Lambda^i_{14}}{\Lambda^5_{15}}\overrightarrow{e}_i;$$
$$\overrightarrow{b}_5 = \overrightarrow{e}_5 + \frac{B^5_{155}}{(\Lambda^5_{15})^2}\overrightarrow{e}_1 - \frac{\Lambda^i_{15}}{\Lambda^5_{15}}\overrightarrow{e}_i.$$

2. Setting and solving the problem

It is considered that the line γ, belongs to the distribution $\Delta_4 = (X, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5)$.

The tangent vector $\overrightarrow{\gamma}$ of this line γ has the form: $\overrightarrow{\gamma} = \gamma^2\overrightarrow{e}_2 + \gamma^3\overrightarrow{e}_3 + \gamma^4\overrightarrow{e}_4 + \gamma^5\overrightarrow{e}_5$. We will find the tangent vector of the line

$$\overline{\gamma} = f^5_1(\gamma) : \overrightarrow{\overline{\gamma}} = \gamma^2\overrightarrow{b}_2 + \gamma^3\overrightarrow{b}_3 + \gamma^4\overrightarrow{b}_4 + \gamma^5\overrightarrow{b}_5.$$

Taking into account formulas (1.10) from here, we find:

$$\overrightarrow{\overline{\gamma}} = (\gamma^2 b^1_2 + \gamma^3 b^1_3 + \gamma^4 b^1_4 + \gamma^5 b^1_5)\overrightarrow{e}_1 + (\gamma^2 + \gamma^3 b^2_3 + \gamma^4 b^2_4 + \gamma^5 b^2_5)\overrightarrow{e}_2 +$$
$$+\gamma^3\overrightarrow{e}_3 + \gamma^4\overrightarrow{e}_4 + (\gamma^2 b^5_2 + \gamma^3 b^5_3 + \gamma^4 b^5_4)\overrightarrow{e}_5,$$

where b^j_i - j-th coordinate of the vector $\overrightarrow{b}_i$.

Definition 2.1. If line tangent of the line $\gamma \subset \Delta_4$ at the point X and the tangent vector of line $\overline{\gamma} = f^5_1(\gamma)$ at the point $F^5_1 = f^5_1(X)$ belong to the same four-dimensional space $(X, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5)$, then the lines γ and $\overline{\gamma}$ are called quasi-double lines of the pair of distributions (Δ_4, Δ'_4) in the partial mapping f^5_1 (where $\Delta'_4 = f^5_1(\Delta_4)$).

From the condition $\overrightarrow{\gamma}, \overrightarrow{\overline{\gamma}} \in (X, \overrightarrow{e}_2, \overrightarrow{e}_3, \overrightarrow{e}_4, \overrightarrow{e}_5)$ we have:

$$\gamma^2 b^1_2 + \gamma^3 b^1_3 + \gamma^4 b^1_4 + \gamma^5 b^1_5 = 0.$$

Substituting the coordinates b^j_i from the (1.10) we get:

$$\gamma^2 B^5_{152} + \gamma^3 B^5_{153} + \gamma^4 B^5_{154} + \gamma^5 B^5_{155} = 0. \quad (2.1)$$

If the coordinates of the tangent vector $\overrightarrow{\gamma}$ of the line γ satisfy the conditions (2.1), then the lines $\gamma, \overline{\gamma}$ are quasi-double lines of the pair of distributions (Δ_4, Δ'_4) in the partial mapping f^5_1. Thus, the following theorem is proved.

Theorem 2.2. *Lines γ and $\overline{\gamma} = f^5_1(\gamma)$ are quasi-double lines of the pair of distributions (Δ_4, Δ'_4) in the partial mapping f^5_1 if and only if the coordinates of the tangent vector $\overrightarrow{\gamma}$ of the line γ satisfy the conditions (2.1).*

The pseudofocus $F^4_5 \in (X, \overrightarrow{e}_5)$ is defined by the radius-vector:

$$\overrightarrow{F}^4_5 = \overrightarrow{X} - \frac{1}{\Lambda^4_{54}}\overrightarrow{e}_5. \quad (2.2)$$

When the point X is moving in the domain $\Omega \subset E_5$, the pseudofocus F^4_5 describes its domain $\Omega^4_5 \subset E_5$. Thus, the partial mapping $f^4_5 : \Omega \to \Omega^4_5$ is defined such that: $f^4_5(X) = F^4_5$.

We will associate with $\Omega^4_5 \subset E_5$ the moving frame $\Re'' = (F^4_5, \overrightarrow{d}_i)$, $(i = \overline{1,5})$ [4], where:

$$\overrightarrow{d}_1 = \overrightarrow{e}_1 - \frac{\Lambda^4_{51}}{\Lambda^4_{54}}\overrightarrow{e}_4 - \frac{D^4_{541}}{(\Lambda^4_{54})^2}\overrightarrow{e}_5;$$

$$\begin{gathered}\overrightarrow{d}_2=-\frac{\Lambda^1_{52}}{\Lambda^4_{54}}\overrightarrow{e}_1+\overrightarrow{e}_2-\frac{\Lambda^4_{52}}{\Lambda^4_{54}}\overrightarrow{e}_4+\frac{D^4_{542}}{(\Lambda^4_{54})^2}\overrightarrow{e}_5;\\ \overrightarrow{d}_3=-\frac{\Lambda^1_{53}}{\Lambda^4_{54}}\overrightarrow{e}_1+\overrightarrow{e}_3-\frac{\Lambda^4_{53}}{\Lambda^4_{54}}\overrightarrow{e}_4+\frac{D^4_{543}}{(\Lambda^4_{54})^2}\overrightarrow{e}_5;\\ \overrightarrow{d}_4=-\frac{\Lambda^1_{54}}{\Lambda^4_{54}}\overrightarrow{e}_1+\frac{D^4_{544}}{(\Lambda^4_{54})^2}\overrightarrow{e}_5;\\ \overrightarrow{d}_5=-\frac{\Lambda^1_{55}}{\Lambda^4_{54}}\overrightarrow{e}_1+[1+\frac{D^4_{545}}{(\Lambda^4_{54})^2}]\overrightarrow{e}_5.\end{gathered}\tag{2.3}$$

In the work [4], necessary and sufficient conditions were found for the lines γ and $\overline{\overline{\gamma}}=f^4_5(\gamma)$ to be a quasi-double lines of the partial mapping f^4_5 (in which case these lines will automatically be quasi-double lines of the pair (Δ_4,Δ''_4) where $\Delta''_4=f^4_5(\Delta)$):

$$\gamma^2\Lambda^1_{52}+\gamma^3\Lambda^1_{53}+\gamma^4\Lambda^1_{54}+\gamma^5\Lambda^1_{55}=0.\tag{2.4}$$

From conditions (2.1) and (2.4) it follows that the line $\overline{\gamma}$ in the domain Ω^5_1 transforms into the line $\overline{\overline{\gamma}}$ in the domain Ω^4_5.

It is considered the composition $f^4_5\circ(f^5_1)^{-1}:\Omega^5_1\to\Omega^4_5$ of the partial mapping $(f^5_1)^{-1}$ and f^4_5, where $(f^5_1)^{-1}$ - is a reverse mapping such that $(f^5_1)^{-1}(F^5_1)=X\in\Omega$. Let $\overline{\gamma}$ is a line which tangent vectors coordinates satisfy the conditions (2.1). Then $(f^5_1)^{-1}(\overline{\gamma})=\gamma$ and $f^4_5(\gamma)=\overline{\overline{\gamma}}\subset\Omega^4_5$, where $\overline{\overline{\gamma}}$ is a line which tangent vectors coordinates satisfy the conditions (2.4).

Thus, the following theorem is proven:

Theorem 2.3. *If the coordinates of the tangent vector of the line $\overline{\gamma}\subset\Omega^5_1$ satisty the condition (2.1) than $f^4_5\circ(f^5_1)^{-1}(\overline{\gamma})=\overline{\overline{\gamma}}$, where $\overline{\overline{\gamma}}\subset\Omega^4_5$ is line, which tangent vector's coordinates satisfy the condition (2.4).*

References

[1] Rashevsky P.K.; Riemann's geometry and tensor analysis. Nauka, Moscow, –1967.–644 p.

[2] Finikov S.O.; The Cartan's method of external forms in differential geometry. M.-L.: Gosttexizdat,–1948.–432 p.

[3] Bazylev V.T.; On multidimensional networks in Euclidean space. Lithuanian Mathematical Collection,–1966.-No.4.–P.475-491.

[4] Matieva G., Abdullayeva Ch.Kh., Papieva T.M.; Introduction to the geometry of quasu-double lines of partial maps of Euclidean spaces. Monograph. – Osh, – 2024. – 172 p.

[5] Matieva G., Abdullayeva C., Ahmedov A.; Existence of immovability lines of a partial mapping of Euclidean space E_5. Journal of Physics: Conference Series.–2019. – 1366(1). – 012060.

[6] Matieva G., Abdullayeva C., Ahmedov A.; About existence of quasi-double lines of the partial mapping of space E_n. Journal of Physics: Conference Series.–2019.–1366(1). – 012061.

[7] Matieva G., Abdullayeva C., Artykova Z.; Existence of quasidouble lines of a pair $(f^5_1),\Delta_{(135)}$ in Euclidean space E_5. Journal of Physics: Conference Series.– 2021.–1988(1).– 01208.

Matieva G.,
Osh state university, Osh, Kyrgyzstan
email: gulbadan_57@mail.ru
Papieva T.M.,
Osh state university, Osh, Kyrgyzstan
email: tpapka73@mail.ru
Shamshieva G.A.,
Osh state university, Osh, Kyrgyzstan
email: gshamshieva@mail.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.169-180
DOI: 10.29229/uzmj.2025-4-17

Impact of almost η-Ricci-Bourguignon solitons on anti-invariant submanifolds of trans-Sasakian manifolds coupled with generalized symmetric non-metric connection of type (α, β)

Mishra R.K., Yadav S.K.

Abstract. We classify almost η-Ricci-Bourguignon solitons on anti-invariant submanifolds of trans-Sasakian manifolds admitting a generalized symmetric non-metric connection of type (α, β). Certain results of such solitons on submanifolds of trans-Sasakian manifolds with respect to a generalized symmetric non-metric connection (GSNM) of type (α, β) are obtained.

Keywords: trans-Sasakian manifold, anti-invariant submanifold, almost η-Ricci Bourguignon soliton, generalized symmetric non-metric connection of type (α, β).

MSC (2020): 53C05, 53C025, 53C40.

1. Introduction

The Ricci-Bourguignon soliton leads to an equivalent soliton to the Ricci-Bourguignon flow, which is explained by [1, 3, 8].

$$\frac{\partial g}{\partial t} = -2(S - \rho R g), \; g(0) = g_0, \tag{1.1}$$

where S is the Ricci curvature tensor, R is the scalar curvature w.r.t. g, and ρ is a real non-zero constant. It should be noticed that for special values of the constant ρ in equation (1.1), we obtain the following situations for the tensor $S - \rho R g$ appearing in the equation. The PDE system (1.1) defines the evolution equation of special interest if
(i) $\rho = \frac{1}{2}$, the Einstein tensor $S - \frac{R}{2} g$ (Einstein soliton),
(ii) $\rho = \frac{1}{2}$, the traceless Ricci tensor $S - \frac{R}{n} g$,
(iii) $\rho = \frac{1}{2(n-1)}$, the Schouten tensor $S - \frac{R}{2(n-1)} g$ (Schouten soliton),
(iv) $\rho = 0$, the Ricci tensor S (Ricci soliton).
In fact, for small t, equation (1.1) has a unique solution for $\rho < \frac{1}{2(n-1)}$.
A pseudo-Riemannian manifold of dimension $n \geq 3$ is called the Ricci-Bourguignon soliton if

$$L_V g(X, Y) + 2S(X, Y) + 2(\mu + \rho r) g(X, Y) = 0, \tag{1.2}$$

where $L_V g$ is the Lie derivative of the Riemannian metric g along the vector field V, the scalar curvature is denoted by r, S is the Ricci curvature tensor, and ρ, μ are scalars. A Ricci-Bourguignon soliton is called expanding if $\mu > 0$, steady if $\mu = 0$, and shrinking if $\mu < 0$.
From equation (1.2), we have defined a more general view, namely η-Ricci -Bourguignon soliton.

$$L_V g(X, Y) + 2S(X, Y) + 2(\mu + \rho r) g(X, Y) + 2C\eta(X)\eta(Y) = 0. \tag{1.3}$$

For more details, see ([19]-[21]).
Friedman and Schouten first introduced the concept of a semi-symmetric linear connection on a differentiable manifold in 1924 [5]. In 1930, the geometric significance of a semi-symmetric linear connection was given by Bartolotti [2]. In 1932, Hayden [7] was first introduced and investigated a metric connection known as a semi-symmetric metric connection with a non-zero torsion on a Riemannian manifold. Yano has conducted a detailed investigation of the semi-symmetric metric connection on a Riemannian manifold [25]. In 1975, Golab [6] was first introduced, a quarter-symmetric linear connection on a differentiable manifold. Rastogi [16] carried out a subsequent systematic investigation into the quarter-symmetric metric connection on a Riemannian manifold. The study of these connections was further studied by various authors ([12, 14, 18, 22]). If the torsion tensor T of a linear connection on a semi-Riemannian manifold M is said to be a generalized symmetric connection, then T is defined as

$$T(u_1, u_2) = \alpha\{(u_2)u_1 - \pi(u_1)u_2\} + \beta\{\pi(u_2)\varphi u_1 - \pi(u_1)\varphi u_2\}, \tag{1.4}$$

where u_1, u_2 are vector fields on M, α and β are smooth functions on M. Here, φ denotes a tensor of type (1,1) and π is a 1-form and satisfies $\pi(u_1) = g(u_1, v)$ for a vector field v in M. If $\nabla^G g = 0$, then the connection is called a generalized symmetric metric connection (shortly, GSM-connection) of type (α, β). The connection in equation (1.4) is referred to as a β-quarter-symmetric connection and α-semi-symmetric connection, respectively, if $\alpha = 0$ and $\beta = 0$. If $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, then the GSM-connection of type (α, β) is called a semi-symmetric connection and a symmetric connection, respectively. The concircular curvature tensor C of type (1,3) on a Riemannian manifold [23] is defined by the equation

$$C(X,Y)Z = R(X,Y)Z + \frac{r}{n(n-1)}\left[g(X,Z)Y - g(Y,Z)X\right], \tag{1.5}$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor of type (1,3) and r is the scalar curvature. Let us consider C^G be the concircular curvature tensor with respect to the generalized symmetric non-metric connection, then we have

$$C^G(X,Y)Z = R^G(X,Y)Z + \frac{r^G}{2n(2n+1)}[g(X,Z)Y - g(Y,Z)X], \tag{1.6}$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where R^G is called the concircular curvature tensor and r^G is said to be the scalar curvature with respect to the GSNM-connection. According to [10] and [13], the M-projective curvature tensor $\overline{M}$ of rank 3 on an n-dimensional manifold M is given by the following equation

$$\overline{M}(X,Y)Z = R(X,Y)Z + \frac{1}{2(n-1)}S(X,Z)Y - S(Y,Z)X + \frac{1}{2(n-1)}[g(X,Z)QY - g(Y,Z)QX],$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where Q denotes the Ricci operator. Hence, the M-projective curvature tensor with respect to the GSNM connection is given by

$$\overline{M}^G(X,Y)Z = R^G(X,Y)Z + \frac{1}{4n}[S^G(X,Z)Y - S^G(Y,Z)X] + \frac{1}{4n}[g(X,Z)Q^G Y - g(Y,Z)Q^G X], \tag{1.7}$$

where Q^G is the Ricci operator with respect to the GSNM connection.
The pseudo-projective curvature tensor on a Riemannian manifold is given by [15].

$$P(X,Y)Z = AR(X,Y)Z + B[S(Y,Z)X - S(X,Z)Y] + cr[g(Y,Z)X - g(X,Z)Y]. \tag{1.8}$$

For all smooth vector fields $X, Y, Z \in \chi(M)$, where A, B, and c are non-zero constants related as

$$c = -\frac{1}{n}\left(\frac{A}{n-1} + B\right).$$

Now we consider P^G as the pseudo-projective curvature tensor with respect to the GSNM connection as

$$P^G(X,Y)Z = AR^G(X,Y)Z + B[S^G(Y,Z)X - S^G(X,Z)Y] + cr^G[g(Y,Z)X - g(X,Z)Y], \tag{1.9}$$

where A, B, c are non-zero constants related as

$$c = -\frac{1}{2n+1}\left(\frac{A}{2n} + B\right). \tag{1.10}$$

In 1977, K.Yano and M.Kon discussed anti-invariant submanifolds of Sasakian space forms [24]. In 1985, H.B Kumar studied anti-invariant submanifolds of almost paracontact manifolds [11]. In 2020, P. Karmakar and A. Bhattacharyya investigated anti-invariant submanifolds of some indefinite almost contact and paracontact manifolds [9].
Let $\phi : M \to \widetilde{M}$ be a differentiable manifold, and let the dimension of the manifold $M, \widetilde{M}$ be n, m respectively. If at each point p of M, $(\phi^*)_p$ is a 1-1 map, that is, if rank $\phi = n$, then ϕ is called an immersion of M *into* M. f ϕ is 1-1, i.e., if $\phi(p') \neq \phi(q')$ for $p' \neq q'$ then ϕ is called an embedding of M into $\widetilde{M}$. The manifold M is called a submanifold of $\widetilde{M}$, if it satisfies the following conditions

(i) $M \subset \widetilde{M}$

(ii) The inclusion map i from M into $\widetilde{M}$ is an embedding of M into $\widetilde{M}$.

A submanifold M is called anti-invariant if $X \in T_x(M) \Rightarrow \phi X \in T_x^{\perp}(M), \forall\ x \in M$, where $T_x(M)$, $T_x^{\perp}(M)$ are respectively the tangent space and the normal space at $x \in M$. Thus, in an anti-invariant submanifold M, we have $\forall\ X, Y \in \chi(M)$,

$$g(X, \phi Y) = 0. \tag{1.11}$$

2. Preliminaries

Let $\widetilde{M}$ be a differentiable manifold of odd dimension with a metric structure (g, ϕ, ξ, η), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and a Riemannian metric g satisfying the following relations-

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \eta\ o\ \phi = 0,\ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y),\ \eta(X) = g(X, \xi)\ \ \forall X, Y \in \chi(\widetilde{M}). \tag{2.3}$$

An odd-dimensional almost contact metric manifold $\widetilde{M}(\phi, \xi, \eta, g)$ is called a trans-Sasakian manifold of type (p, q), where $p,\ q$ are smooth functions on $\widetilde{M}$ if $\forall\ X, Y \in \chi(M)$ [4].

$$(\nabla_X \phi)Y = p\,[g(X, Y)\xi - \eta(Y)X] + p\,[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.4}$$

$$\nabla_X \xi = -\,p\,\phi X + q\,[X - \eta(X)\xi]. \tag{2.5}$$

In a $(2n+1)$-dimensional trans-Sasakian manifold of type (p, q), we have the following relations [4]

$$(\nabla_X \eta)\,Y = -p\,g(\phi X, Y) + q\,[g(X, Y) - \eta(X)\eta(Y)], \tag{2.6}$$

$$\begin{aligned} R(X, Y)\xi = {} & \left(p^2 - q^2\right)[\eta(Y)X - \eta(X)Y] + 2pq\,[\eta(Y)\phi X - \eta(X)\phi Y] \\ & + \left[(Yp)\phi X - (Xp)\phi Y + (Yq)\phi^2 X - (Xq)\phi^2 Y\right], \end{aligned} \tag{2.7}$$

$$\begin{aligned} R(\xi, Y)X = {} & \left(p^2 - q^2\right)[g(X, Y)\xi - \eta(X)Y] + 2pq\,[g(\phi X, Y)\xi + \eta(X)\phi Y] \\ & +(Xp)\phi Y + g(\phi X, Y)(grad\ p) - g(\phi X, \phi Y)(grad\ q) + (Xq)\,[Y - \eta(Y)\xi], \end{aligned} \tag{2.8}$$

$$S(X, \xi) = \left[2n\left(p^2 - q^2\right) - \xi q\right]\eta(X) - (\phi X)p - (2n-1)(Xq), \tag{2.9}$$

$$Q\xi = \left[2n\left(p^2 - q^2\right) - \xi q\right]\xi + \phi(grad\ q) - (2n-1)(grad\ q). \tag{2.10}$$

Lemma 2.1. [17] *In a $(2n+1)$ dimensional trans-Sasakian manifold of type (p, q), if $\phi(grad\ p) = (2n-1)(grad\ q)$, then $\xi q = 0$.*

3. Generalized Symmetric non-metric connection in trans-Sasakian manifolds

In a trans-Sasakian manifold $\widetilde{M}$, let ∇^G be a linear connection, and ∇ be the Levi-Civita connection and $X, Y \in \chi(M)$. We define a linear connection ∇^G on $\widetilde{M}$ by

$$\nabla^G_X Y = \nabla_X Y + u(X, Y), \tag{3.1}$$

where $u(X, Y)$ is a tensor of type $(1, 2)$ and ∇^G represents a GSNM connection on a trans-Sasakian manifold as-

$$u(X, Y) = \frac{1}{2}[(T(X, Y) + T^{'}(X, Y) + T^{'}(Y, X)] \tag{3.2}$$

where T is the torsion tensor of ∇^G and

$$g(T^{'}(X, Y), Z) = g(T(Z, X), Y). \tag{3.3}$$

Plugging (1.4) in (3.3), we get

$$T^{'}(X,Y) = \alpha\left\{\eta(Y)X - g(X,Y)\xi\right\} + \beta\left\{\eta(Y)\phi X - g(\phi X,Y)\xi\right\} \tag{3.4}$$

Substituting (1.4) and (3.4) in (3.3), we get

$$u(X,Y) = \alpha\left\{\eta(Y)X - g(X,Y)\xi\right\} + \beta\eta(Y)\phi X \tag{3.5}$$

Substituting (3.5) in (3.1), we obtain a generalized symmetric non-metric connection ∇^G of type (α,β) in a trans-Sasakian manifold $\widetilde{M}$ as

$$\nabla^G_X Y = \nabla_X Y + \alpha\left\{\eta(Y)X - g(X,Y)\xi\right\} + \beta\eta(Y)\phi X \tag{3.6}$$

Conversely, from (3.7), the torsion tensor with respect to the connection ∇^G is defined as

$$T(X,Y) = \nabla^G_X Y - \nabla^G_Y X - [X,Y] = \alpha\left\{\eta(Y)X - \eta(X)Y\right\} + \beta\left\{\eta(Y)\phi X - \eta(X)\phi Y\right\}. \tag{3.7}$$

Hence, this shows that the connection ∇^G of type (α,β) in a trans-Sasakian manifold $\widetilde{M}$ a generalized symmetric connection. Now, we have

$$\left(\nabla^G_X g\right)(X,Y) = Xg(Y,Z) - g\left(\nabla^G_X Y,Z\right) - g\left(Y,\nabla^G_X Z\right) = -\beta\left\{\eta(Y)g(\phi X,Z) + \eta(Z)g(\phi X,Y)\right\}. \tag{3.8}$$

From (3.8) and (3.9), we obtained that ∇^G is a generalized symmetric non-metric connection of type (α,β) in a trans-Sasakian manifold $\widetilde{M}$.
Now, setting $Y=\xi$ in (3.7) and using (2.5), we obtain

$$\nabla^G_X\xi = (\alpha+q)\left\{X - \eta(x)\xi\right\} + (\beta-p)\phi X. \tag{3.9}$$

In a trans-Sasakian manifold $\widetilde{M}$ with respect to the generalized symmetric non-metric connection of type (α,β), we get

$$(\nabla^G_X\eta)Y = (\alpha+q)\left\{g(X,Y) - \eta(X)\eta(Y)\right\} - p\; g(Y,\phi X) \tag{3.10}$$

In a trans-Sasakian manifold $\widetilde{M}$, we define its curvature tensor with respect to the generalized symmetric non-metric connection of type (α,β) by

$$R^G(X,Y)Z = \nabla^G_X\nabla^G_Y Z - \nabla^G_Y\nabla^G_X Z - \nabla^G_{[X,Y]}Z. \tag{3.11}$$

Using (3.7), we obtain

$$\begin{aligned}\nabla^G_X\nabla^G_Y Z = \nabla^G_X\nabla_Y Z + \alpha\Big\{\Big(\nabla^G_X(\eta(Z)\Big)Y + \eta(Z)\nabla^G_X Y - (\nabla^G_X\left(g(Y,Z)\right)\xi - g(Y,Z)\nabla^G_X\xi\Big\}\\ +\beta\Big\{(\nabla^G_X\eta(Z))\phi Y + \eta(Z)\nabla^G_X(\phi Y)\Big\}.\end{aligned} \tag{3.12}$$

Now, using (3.10) & (3.7), we get

$$\nabla^G_X\left(\eta(Z)\right) = \eta\left(\nabla_X Z\right) - pg(Z,\phi X) + q\left\{g(X,Z) - \eta(X)\eta(Z)\right\}. \tag{3.13}$$

Using (2.1), (2.4) & (3.7), we obtain

$$\nabla^G_X(\phi Y) = \phi\nabla_X Y + p\left\{g(X,Y)\xi - \eta(Y)X\right\} + q\left\{g(\phi X,Y)\xi - \eta(Y)\phi X\right\} - \alpha\; g(X,\phi Y)\xi. \tag{3.14}$$

Now, applying (3.7), (3.12), (3.13), (3.14) in (3.11), we obtain

$$\begin{aligned}R^G(X,Y)Z = R(X,Y)Z &+ \left(\alpha q - \beta p + \alpha^2\right)\left\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\right\}\\ &+ (\alpha p - \beta q - \alpha\beta)\left\{g(Y,Z)\phi X - g(X,Z)\phi Y\right\} + \left(2\alpha q + \alpha^2\right)\left\{g(X,Z)Y - g(Y,Z)X\right\}\\ &+ \alpha p\left\{g(Z,\phi Y)X - g(Z,\phi X)Y\right\} + \left(\alpha q + \alpha^2\right)\left\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\right\}\\ &+ \beta p\left\{g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y\right\} + 2(\alpha\beta + \beta q)\eta(Z)g(\phi X,Y)\xi\\ &+ \alpha\beta\left\{\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y\right\}.\end{aligned} \tag{3.15}$$

$$S^G(Y,Z) = S(Y,Z) + \left\{2n\left(\alpha q - \beta p + \alpha^2\right) - \alpha q - \beta p - \alpha^2\right\}\eta(Y)\eta(Z) + \left\{\alpha q + \beta p + \alpha^2 - 2n\left(2\alpha q + \alpha^2\right)\right\}g(Y,Z) + \left\{(2n-1)\alpha p + \beta q + \alpha\beta\right\}g(\phi Y, Z). \tag{3.16}$$

$$Q^G Y = QY + \left\{2n\left(\alpha q - \beta p + \alpha^2\right) - \alpha q - \beta p - \alpha^2\right\}\eta(Y)\xi + \left\{\alpha q + \beta p + \alpha^2 - 2n\left(2\alpha q + \alpha^2\right)\right\}Y + \left\{(2n-1)\alpha p + \beta q + \alpha\beta\right\}\phi Y. \tag{3.17}$$

$$r^G = r - 8n^2\alpha q - 2n(2n-1)\alpha^2. \tag{3.18}$$

Setting $Z = \xi$ in (3.16), we get

$$S^G(Y,\xi) = S(Y,\xi) - 2n(\alpha q + \beta p)\eta(Y). \tag{3.19}$$

Again, putting $Y = \xi$ in (3.17), we obtain

$$Q^G\xi = Q\xi - 2n(\alpha q + \beta p)\xi. \tag{3.20}$$

Proposition 3.1. *A generalized symmetric non-metric connection of type (α,β) on an anti-invariant submanifold of a trans-sasakian manifold reduces to a generalized symmetric metric connection of type (α,β).*

4. Almost η-Ricci-Bourguignon solitons on anti-invariant submanifolds and trans-Sasakian manifold with respect to a generalized symmetric non-metric connection

Let, (g,ξ,C,μ,ρ) be an almost η-Ricci-Bourguignon soliton on M with respect to a generalized symmetric non-metric connection ∇^G, then from (1.2) we have $\forall\ Y,Z \in \chi(M)$

$$(L^G_\xi g)(Y,Z) + 2Ric^G(Y,Z) + 2\left(\mu + \rho\, r^G\right)g(Y,Z) + 2C\eta(Y)\eta(Z) = 0.$$
$$\Rightarrow g\left(\nabla^G_Y\xi, Z\right) + g\left(\nabla^G_Z\xi, Y\right) + 2Ric^G(Y,Z) + 2\left(\mu + \rho\, r^G\right)g(Y,Z) + 2C\eta(Y)\eta(Z) = 0.$$

Using (3.9) in the above equation, we get

$$Ric^G(Y,Z) = \left(\alpha + q - \mu - \rho\, r^G\right)g(Y,Z) + (\alpha + q - C)\eta(Y)\eta(Z). \tag{4.1}$$

Theorem 4.1. *Let (g,ξ,μ,ρ,C) be almost η-Ricci-Bourguignon solitons on a $(2n+1)$ dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q), with respect to the GSNM connection of type (α,β), then $\widetilde{M}^{2n+1}$ is η-Einstein manifold.*

Setting $Y = Z = \xi$ in (4.1), we get

$$Ric^G(\xi,\xi) = 2(\alpha + q) - \mu - \rho r^G - C. \tag{4.2}$$

Putting $X = \xi$ in (2.9), we obtain

$$S(\xi,\xi) = 2n\left(p^2 - q^2\right) - 2n(\xi q). \tag{4.3}$$

Now, comparing (4.2) and (4.3), we get

$$2(\alpha + q) - \mu - \rho r^G - C = 2n\left(p^2 - q^2\right) - 2n(\xi q).$$

If $\xi q = 0$, then from the above equation, we have

$$\mu = 2(\alpha + q) - \rho r^G - 2n\left(p^2 - q^2\right) - C.$$

In particular, if we take $\mu = 0,\ C = 0$, then, we yield

$$r^G = \frac{2}{\rho}\left\{\alpha + q - n\left(p^2 - q^2\right)\right\}. \tag{4.4}$$

Theorem 4.2. *If a $(2n+1)$-dimensional of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits almost Ricci-Bourguignon soliton with respect to the generalized symmetric non-metric connection of type (α,β), provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r^G>\frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\ r^G=\frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\quad or\ \ r^G<\frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},$$

respectively.

Corollary 4.3. *If a $(2n+1)$-dimensional of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits almost Ricci-Bourguignon soliton with respect to the generalized symmetric metric connection of type $\alpha,\beta)$, provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r^G>\frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\ r^G=\frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\quad or\ \ r^G<\frac{2}{\rho}\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},$$

respectively.

Now, if $(\alpha,\beta)=(1,0)$, then from (4.4), we get

$$r^G=\frac{2}{\rho}\left\{1+q-n\left(p^2-q^2\right)\right\}.$$

Corollary 4.4. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits an almost Ricci-Bourguignon soliton with respect to the generalized symmetric metric connection of type (α,β), provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r^G>\frac{2}{\rho}\left\{1+q-n\left(p^2\ -\ q^2\right)\right\},\ r^G=\frac{2}{\rho}\left\{1+q-n\left(p^2\ -\ q^2\right)\right\},\quad or\ \ r^G<\frac{2}{\rho}\left\{1+q-n\left(p^2\ -\ q^2\right)\right\},$$

respectively.

Now, if $(\alpha,\beta)=(0,1)$, then from (4.4), we get

$$r^G=\frac{2}{\rho}\left\{q-n\left(p^2-q^2\right)\right\}.$$

Corollary 4.5. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits an almost Ricci-Bourguignon soliton with respect to the generalized quarter-symmetric metric connection of type (α,β), provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r^G>\frac{2}{\rho}\left\{q-n\left(p^2\ -\ q^2\right)\right\},\ r^G=\frac{2}{\rho}\left\{q-n\left(p^2\ -\ q^2\right)\right\},\quad or\ \ r^G<\frac{2}{\rho}\left\{q-n\left(p^2\ -\ q^2\right)\right\},$$

respectively.

Corollary 4.6. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits a traceless Ricci soliton with respect to the generalized symmetric metric connection of type (α,β), provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then the soliton is expanding, steady, or shrinking according as*

$$\begin{aligned}&r^G>2(2n+1)\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\\&r^G=2(2n+1)\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\ or\\&r^G<2(2n+1)\left\{\alpha+q-n\left(p^2\ -\ q^2\right)\right\},\ respectively.\end{aligned}$$

Corollary 4.7. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) admits the Schouten soliton with respect to the generalized symmetric metric connection of type (α,β), provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then the soliton is expanding, steady, or shrinking according as*

$$r^G > 4n\left\{\alpha + q - n\left(p^2 - q^2\right)\right\},$$
$$r^G = 4n\left\{\alpha + q - n\left(p^2 - q^2\right)\right\}, \text{ or}$$
$$r^G < 4n\left\{\alpha + q - n\left(p^2 - q^2\right)\right\}, \text{ respectively.}$$

5. Almost η - Ricci-Bourguignon solitons on Ricci flat anti-invariant submanifolds

In this section, we characterize an almost η-Ricci-Bourguignon soliton on a Ricci flat $(2n+1)$ anti-invariant submanifold with respect to the generalized symmetric non-metric connection of type (α,β). Let (g,ξ,μ,ρ,C) be an almost η-Ricci-Bourguignon soliton on M, then from (1.3) we have $\forall\ Y,Z \in \chi(M)$.

$$L_\xi g(Y,Z) + 2S(Y,Z) + 2(\mu+\rho r)g(Y,Z) + 2C\eta(Y)\eta(Z) = 0$$
$$\Rightarrow g(\nabla_Y\xi,Z) + g(\nabla_Z\xi,Y) + 2S(Y,Z) + 2(\mu+\rho r)g(Y,Z) + 2C\eta(Y)\eta(Z) = 0.$$

Using (2.5), and then applying (1.11), in the above equation, we get

$$S(Y,Z) = (\mu+\rho r - q)g(Y,Z) + (C+q)\eta(Y)\eta(Z). \tag{5.1}$$

Setting $Z=\xi$ in (5.1), we obtain

$$S(Y,\xi) = (\mu+\rho r + C)\eta(Y). \tag{5.2}$$

Now, if $\xi q = 0$ and M is Ricci flat with respect to ∇^G, then from (3.16) we have

$$\begin{aligned} S(Y,Z) = &\left\{\alpha q + \beta p + \alpha^2 - 2n\left(\alpha q - \beta p + \alpha^2\right)\right\}\eta(Y)\eta(Z) \\ &- \left\{\alpha q + \beta p + \alpha^2 - 2n\left(2\alpha q + \alpha^2\right)\right\}g(Y,Z) - \left\{(2n-1)\alpha p + \beta q + \alpha\beta\right\}g(\phi Y,Z). \end{aligned}$$

Using (1.11) in the above equation, we obtain

$$S(Y,Z) = \left\{\alpha q + \beta p + \alpha^2 - 2n\left(\alpha q - \beta p + \alpha^2\right)\right\}\eta(Y)\eta(Z) - \left\{\alpha q + \beta p + \alpha^2 - 2n\left(2\alpha q + \alpha^2\right)\right\}g(Y,Z).$$

Setting $Z=\xi$ in the above equation, we obtain

$$S(Y,\xi) = 2n(\alpha q + \beta p)\eta(Y). \tag{5.3}$$

Equating (5.2) and (5.3), we obtain

$$\mu = 2n(\alpha q + \beta p) - \rho r - C. \tag{5.4}$$

Now, if $\mu = 0$, $C = 0$, then from (5.4), we get

$$r = \frac{2n}{\rho}(\alpha q + \beta p). \tag{5.5}$$

Theorem 5.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is Ricci flat with respect to the generalized symmetric non-metric connection of type (α,β), provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{2n}{\rho}(\alpha q + \beta p),\ r = \frac{2n}{\rho}(\alpha q + \beta p),\ \text{or } r < \frac{2n}{\rho}(\alpha q + \beta p),\ \text{respectively.}$$

Now, from (5.5), if $(\alpha,\beta)=(0,1)$, and $(\alpha,\beta)=(1,0)$, then we get $r=\frac{1}{\rho}(2np)$ and $r=\frac{1}{\rho}(2nq)$, respectively. We get the following results.

Corollary 5.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is Ricci flat with respect to the generalized quarter- symmetric non-metric connection of type $(0,1)$, provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r>\frac{1}{\rho}(2np),\ r=\frac{1}{\rho}(2np),\ or\ r<\frac{1}{\rho}(2np),\ respectively.$$

Corollary 5.3. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is Ricci flat with respect to the generalized semi- symmetric metric connection of type $(1,0)$, provided $\phi(grad\ p)=(2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r>\frac{1}{\rho}(2nq),\ r=\frac{1}{\rho}(2nq),\ or\ r<\frac{1}{\rho}(2nq),\ respectively.$$

6. Almost η-Ricci-Bourguignon solitons on concircularly flat anti-invariant submanifolds

In this section, we have discussed an almost η-Ricci-Bourguignon soliton with respect to the GSNM connection of type (α,β), α-semi-symmetric connection and β-quarter-symmetric connection on $(2n+1)$-dimensional concircularly flat anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q).
Since M is concircularly flat with respect to ∇^G, from (1.6), we have

$$R^G(X,Y)Z=\frac{r^G}{2n(2n+1)}\left[g(Y,Z)X-g(X,Z)Y\right].$$

Contracting the above equation, we get

$$S^G(Y,Z)=\frac{r^G}{2n+1}g(Y,Z). \tag{6.1}$$

Let $\xi q=0$, hence using (3.16), (3.18) and (1.11) in (6.1), we obtain

$$\begin{aligned}S(Y,Z)=&\left[\frac{r^G}{2n+1}-\left\{\alpha q+\beta p+\alpha^2-2n\left(2\alpha q+\alpha^2\right)\right\}\right]g(Y,Z)\\&-\left\{2n\left(\alpha q-\beta p+\alpha^2\right)-\alpha q-\beta p-\alpha^2\right\}\eta(Y)\eta(Z).\end{aligned} \tag{6.2}$$

Setting $Z=\xi$ in (6.2), we obtain

$$S(Y,\xi)=\left\{\frac{r^G}{2n+1}+2n(\alpha q+\beta p)\right\}\eta(Y). \tag{6.3}$$

Comparing (5.2) and (6.3), we get

$$\mu=\frac{r^G}{2n+1}+2n(\alpha q+\beta p)-\rho r-C. \tag{6.4}$$

Now, if $\mu=0$, $C=0$, then from (6.4), we get

$$r=\frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p)-2n(2n-1)\alpha^2-8n^2\alpha q\right]. \tag{6.5}$$

Theorem 6.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is concircularly flat with respect to the generalized symmetric non-metric connection of type (α,β), provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right],\ \textit{or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right],\ \textit{respectively.}$$

Now, from (6.5), if $(\alpha,\beta) = (0,1)$, and $(\alpha,\beta) = (1,0)$, then we get $r = \frac{1}{[\rho(2n+1)-1]}[2n(2n+1)p]$, and $r = \frac{1}{[\rho(2n+1)-1]}[2n(2n+1)q - 2n(2n-1) - 8n^2q]$. We get the following results-

Corollary 6.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is concircularly flat with respect to the generalized quarter-symmetric non-metric connection of type $(0,1)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}[2n(2n+1)p],$$
$$r = \frac{1}{[\rho(2n+1)-1]}[2n(2n+1)p],\ \textit{or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}[2n(2n+1)p],\ \textit{respectively.}$$

Corollary 6.3. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is concircularly flat with respect to the generalized semi-symmetric metric connection of type $(1,0)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2q\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2q\right],\ \textit{or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2q\right],\ \textit{respectively.}$$

7. Almost η-Ricci-Bourguignon solitons on M-projectively flat anti-invariant submanifolds

This section deals with the study of almost η-Ricci-Bourguinon soliton with respect to the generalized symmetric non-metric connection of type (α,β), α-semi-symmetric connection, and β-quarter-symmetric connection on a (2n + 1)-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q), are investigated.
Since M is M-projectively flat with respect to a generalized symmetric non-metric connection ∇^G. From equation (1.7), we get

$$R^G(X,Y)Z = \frac{1}{4n}\left[S^G(Y,Z)X - S^G(X,Z)Y\right] + \frac{1}{4n}\left[g(Y,Z)Q^GX - g(X,Z)Q^GY\right]. \tag{7.1}$$

Contracting (7.1) with respect to X, we get

$$S^G(Y,Z) = \frac{r^G}{2n+1}g(Y,Z).$$

Which is the same as equation (6.1). Hence, we get the following results:

Theorem 7.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is M-projectively flat with respect to the generalized symmetric non-metric connection of type (α,β), provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right], \text{ respectively.}$$

Corollary 7.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is M-projectively flat with respect to the generalized quarter-symmetric non-metric connection of type $(0,1)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right], \text{ respectively.}$$

Corollary 7.3. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is M-projectively flat with respect to the generalized semi-symmetric metric connection of type $(1,0)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$r > \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right],$$
$$r = \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right], \text{ or}$$
$$r < \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right], \text{ respectively.}$$

8. Almostη-Ricci-Bourguignon solitons on pseudo-projectively flat anti-invariant submanifolds

This section deals with the study of almost η-Ricci-Bourguignon soliton, α-semi-symmetric contraction and β-quarter-symmetric connection on a $(2n+1)$-dimensional pseudo-projectively flat anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) with respect to the generalized symmetric non-metric connection of type (α,β).
Let M be pseudo-projectively flat with respect to a generalized symmetric non-metric connection ∇^G. Now, from equation (1.9), we get

$$AR^G(X,Y)Z = -\,B\left[S^G(Y,Z)X - S^G(X,Z)Y\right] - c\,r^G\left[g(Y,Z)X - g(X,Z)Y\right]. \tag{8.1}$$

Contracting (8.1) with respect to X, we have

$$(A+2nB)S^G(Y,Z) = -2cg(Y,Z). \tag{8.2}$$

Applying (1.10), (1.11), (3.16) and (3.18) in (8.2), we get

$$\begin{aligned} S(Y,Z) = &\left[\frac{r^G}{2n+1} - \left\{\alpha q + \beta p + \alpha^2 - 2n\left(2\alpha q + \alpha^2\right)\right\}\right] g(Y,Z) \\ &- \left\{2n\left(\alpha q - \beta p + \alpha^2\right) - \alpha q - \beta p - \alpha^2\right\}\eta(Y)\eta(Z). \end{aligned} \tag{8.3}$$

The above equation is the same as the equation (6.2). Hence, we get the following results:

Theorem 8.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is pseudo-projectively flat with respect to the generalized symmetric non-metric connection of type (α,β), provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$
\begin{aligned}
r &> \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right], \\
r &= \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right], \ \textit{or} \\
r &< \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)(\alpha q+\beta p) - 2n(2n-1)\alpha^2 - 8n^2\alpha q\right], \ \textit{respectively.}
\end{aligned}
$$

Corollary 8.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is pseudo-projectively flat with respect to the generalized quarter-symmetric non-metric connection of type $(0,1)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$
\begin{aligned}
r &> \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right], \\
r &= \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right], \ \textit{or} \\
r &< \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)p\right], \ \textit{respectively.}
\end{aligned}
$$

Corollary 8.3. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (p,q) is pseudo-projectively flat with respect to the generalized semi-symmetric metric connection of type $(1,0)$, provided $\phi(grad\ p) = (2n-1)(grad\ q)$, then an almost Ricci-Bourguignon soliton (g,ξ,μ,ρ,C) on M is expanding, steady, or shrinking according as*

$$
\begin{aligned}
r &> \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right], \\
r &= \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right], \ \textit{or} \\
r &< \frac{1}{[\rho(2n+1)-1]}\left[2n(2n+1)q - 2n(2n-1) - 8n^2 q\right], \ \textit{respectively.}
\end{aligned}
$$

9. Conclusion

In 1981, the notion of Ricci-Bourguignon flow as a generalization of Ricci flow [8] was introduced by J. P. Bourguignon [1], and the short-time existence and uniqueness of the solution of this geometric flow have been proved in [3]. In this study, we discuss the geometric properties of an almost η-Ricci-Bourguignon soliton on an anti-invariant submanifold of trans-Sasakian manifolds admitting a generalized symmetric non-metric connection of type (α,β), and we deduce several conditions at which the soliton is expanding, steady, and shrinking.

Acknowledgment

The authors would like to thank the anonymous reviewers and the Editor for their valuable suggestions to improve our manuscript.

References

[1] Bourguignon J. P.; Ricci curvature and Einstein metrics. Global differential geometry and global analysis. Lecture notes in Math.838(1981),42-63.

[2] Bartolotti E.; Sulla geometria della variata a connection affine. Ann. Mat. 1930, 4, 53101.

[3] Catino G., Cremaschi L., Djadli Z., Mantegazza C., Mazzieri L.; The Ricci-Bourguignon flow. Pacific J.Math. 287(2015). 337-370.

[4] De U. C., Shaikh A. A.; Complex Manifolds and Contact Manifolds, Narosa Publishing House, 2009.

[5] Friedmann A., Schouten J.A.; ber die Geometrie der halbsymmetrischen bertragungen. Math Z 21, 211223 (1924).

[6] Golab S.; On semi-symmetric and quarter-symmetric linear connection. Tensor 1975, 29, 249254.

[7] Hayden H.A.; Subspaces of space with torsion. Proc. Lond. Math. Soc. 1932, 34, 2750.

[8] Hamilton R.S.; The Ricci flow on surfaces, Contemp. Mathematics, 71, 237-261, (1988).

[9] Karmakar P., Bhattacharyya A.; Anti-invariant submanifolds of some indefinite almost contact and para-contact manifolds. Bull. Calcutta Math. Soc. 112, 2 (2020), 95-108.

[10] Ojha R. H.; M-projectively flat Sasakian manifolds, Indian J. Pure Appl. Math. 17, 4(1986), 481-484.

[11] Pandey H.B.; A.Kumar, Anti-invariant submanifolds of almost para-contact manifolds, Indian J. Pure Appl. Math. 20, 11 (1989), 1119-1125.

[12] Prakasha D.G.; On ϕ-symmetric Kenmotsu manifolds with regard to quarter-symmetric metric connection. Int. Electron. J. Geom. 2011, 4, 8896

[13] Prakasha D.G., Mirji K.; On the M-projective curvature tensor of a (k, μ)- contact metric manifold. Ser. Math. Inform. 32, 1 (2017), 117-128.

[14] Prakasha D.G., Vanli A.T., Bagewadi C.S., Patil, D.A.; Some classes of Kenmotsu manifolds with regard to semi-symmetric metric connection. Acta Math. Sin. Engl. Ser. 2013, 29, 13111322.

[15] Prasad B.; A pseudo projective curvature tensor on a Riemannian manifold, Bull. Calcutta Math. Soc. 94, 3 (2002), 163-166.

[16] Rastogi S.C.; On a quarter-symmetric metric connection. CR Acad. Sci. Bulg. 1978, 31, 811814.

[17] Shukla S. S., Singh D. D.; On (ϵ)-trans-Sasakian manifolds, Int. J. Math. Anal. 4, 49 (2010), 2401-2414.

[18] Sular S., Ozgur C., De U.C.; Quarter-symmetric metric connection in a Kenmotsu manifold. SUT J. Math. 2008, 44, 297306.

[19] Traore M., Tastan H.M., Gerdan Aydn S.; On almost η-Ricci-Bourguignon solitons, Miskolc. Math. Notes, 25(1), 493508, (2024).

[20] Traore M., Tastan H.M., On sequential warped product η-Ricci-Bourguignon solitons, Filomat. 38(19) 67856797, (2024).

[21] Traore M., Tastan H.M., Gerdan Aydn S.; Some characterizations on Gradient Almost η-Ricci-Bourguignon Solitons. Bol. Soc. Parana. Mat. 43. (2025). 1-12.

[22] Tripathi M.M.; On a semi-symmetric metric connection in a Kenmotsu manifold. J. Pure Math. 1999, 16, 6771.

[23] Yano K.; Concircular geometry I. concircular transformations, Proc.Imp. Acad. Tokyo 16, 6 (1940), 195-200.

[24] Yano K., Kon M.; Anti-invariant submanifolds of Sasakian space forms I, Tohoku Math. J. 29, 1 (1977), 9-23.

[25] Yano K.; On semi-symmetric metric connections, Revue Roumaine De Math. Pures Appl. 1970, 15, 15791586.

Mishra Rajesh Kumar,
Department of Applied Sciences and Humanities,
United College of Engineering and Research,
Naini, Prayagraj, INDIA
email: rrsimt.rajesh@gmail.com

Yadav Sunil Kumar,
Department of Applied Sciences and Humanities,
United College of Engineering and Research,
Naini, Prayagraj, INDIA
email: prof_sky16@yahoo.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.181-193
DOI: 10.29229/uzmj.2025-4-18

Description of periodic and weakly periodic ground states for the modified SOS model on the Cayley tree

Rahmatullaev M.M., Askarov J.N.

Abstract. This paper investigates the periodic and weakly periodic ground states on the Cayley tree of order two and three. Utilizing techniques from statistical mechanics, algebraic graph theory, and group theory, we determine the conditions under which these ground states can exist. Our results demonstrate that the unique structure of the Cayley tree supports a variety of periodic and weakly periodic ground states, each with distinct physical properties. These findings enhance our understanding of phase transitions and critical phenomena in physical systems.

Keywords: Cayley tree, configuration, modified SOS model, translation-invariant ground state, periodic ground state, weakly periodic ground state.

MSC (2020): 82B26; 60K35

1. Introduction

It is considered that a phase diagram of the Gibbs measures for a Hamiltonian has been close to the phase diagram of isolated (stable) ground states of this Hamiltonian. A periodic ground state is compatible with a periodic Gibbs measure at the low temperatures, (see [5, 9, 11, 16, 17, 18]). It encourages us to investigate the problem of description of periodic and weakly periodic ground states. The notion of a weakly periodic ground state is introduced in [18].

For the Ising model with competing interactions, weakly periodic ground states are described in [9, 18]. The authors of [18] also explore weakly periodic ground states for the normal subgroups of index 2 and 4. For the Ising model with competing interactions in [17] ground states are described constructively on a Cayley tree of order $k \geq 1$. In [12] translation-invariant ground states for the Ising model with translation-invariant external field and some periodic ground states for the Ising model with periodic external field are described. In [11], ground states are investigated for the Ising model with competing interactions and a nonzero external field on the Cayley tree of order two.

In [4], for the three-state Potts model with competing interactions on the Cayley tree of order $k = 2$, all periodic ground states are studied. Weakly periodic ground states for the Potts model for the normal subgroups of index 2 are examined in [10, 13]. For the Potts model with competing interactions, such states for the normal subgroups of index 4 are explored in [14]. In [3], periodic ground states are studied for the Potts model with competing interactions and countable spin values on a Cayley tree of order three.

In [7], translation-invariant ground states for the solid-on-solid (SOS) model with a translation-invariant external field, as well as several periodic ground states for the SOS model with a periodic external field, are delineated. The authors of [8] examines periodic and weakly periodic ground states for the SOS model with competing interactions on Cayley trees of orders two and three. Additionally, periodic and weakly periodic ground states for the SOS model with competing interactions on a Cayley tree of order two are examined for the normal subgroups of index 2 in [1].

Modified models in statistical mechanics extend classical frameworks to capture richer behaviors on complex structures. A recent study by Akin [2] investigates a modified q-state Potts model on the Cayley tree, introducing cosine-modulated interactions that lead to phase transitions exclusively in the antiferromagnetic region which is a notable departure from classical models. Using the cavity method and recurrence relations, the work constructs Gibbs measures and analyzes phase transitions via fixed-point dynamics. These methods, rooted in the self-similarity of Cayley trees, align with foundational approaches by Rozikov [15], and contribute to the growing literature on modified models on graphs [2].

In this paper, we study periodic and weakly periodic ground states for the modified SOS model with competing interactions on the Cayley trees of order two and three. The main concepts are presented in the second section. In the third section, periodic and weakly periodic ground states are investigated on the Cayley tree of order two. Furthermore, in the fourth section, periodic and weakly periodic ground states are meticulously examined on the Cayley tree of order three.

2. Preliminaries

Let $\Gamma^k = (V, L)$ be the Cayley tree of order k, i.e. an infinite tree such that exactly $k+1$ edges are incident to each vertex. Here V is the set of vertices and L is the set of edges of Γ^k. Let G_k denote the free product of $k+1$ cyclic groups $\{e; a_i\}$ of order two with generators $a_1, a_2, a_3, ...a_{k+1}$, i.e., $a_i^2 = e$ (see [6, 15]).

There exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order k and the group G_k, (see [6, 15]).

For an arbitrary vertex $x^0 \in V$, we put

$$W_n = \{x \in V : \ d(x, x^0) = n\}, \quad V_n = \{x \in V : \ d(x, x^0) \leq n\}, \tag{2.1}$$

where $d(x, y)$ is the distance between x and y on the Cayley tree, i.e., the number of edges of the path between x and y. For each $x \in G_k$, let $S(x)$ denote the set of direct successors of x, i.e., if $x \in W_n$ then

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

For each $x \in G_k$, let $S_1(x)$ denote the set of all neighbors of x, i.e., $S_1(x) = \{y \in G_k : \langle x, y\rangle \in L\}$. The set $S_1(x)\backslash S(x)$ is a singleton. Let $x_\downarrow$ denote the (unique) element of this set.

Let assume that the spin values belong to the set $\Phi = \{-1, 0, 1\}$. A function $\sigma : x \in V \to \sigma(x) \in \Phi$ is called configuration on V. The set of all configurations coincides with the set $\Omega = \Phi^V$.

Consider the quotient group $G_k/G_k^* = \{H_1, H_2, ..., H_r\}$, where G_k^* is a normal subgroup of index r with $r \geq 1$.

Definition 2.1. A configuration σ is called *G_k^*-periodic*, if $\sigma(x) = \sigma_i$ for all $x \in G_k$ with $x \in H_i$. A G_k-periodic configuration is called translation-invariant.

The period of a periodic configuration is the index of the corresponding normal subgroup.

Definition 2.2. A configuration σ is called G_k^*-weakly periodic, if $\sigma(x) = \sigma_{ij}$ for all $x \in G_k$ with $x_\downarrow \in H_i$ and $x \in H_j$.

The Hamiltonian of the modified SOS model with competing interactions has a form:

$$H(\sigma) = J_1 \sum_{\langle x,y\rangle \in L} |\sigma(x) - \sigma(y)| \cos[\pi(\sigma(x) - \sigma(y))] + J_2 \sum_{\substack{x,y \in V:\\ d(x,y)=2}} |\sigma(x) - \sigma(y)| \cos[\pi(\sigma(x) - \sigma(y))], \tag{2.2}$$

where $J_1, J_2 \in \mathbb{R}$.

Unlike the classical SOS model, which depends solely on absolute spin differences $|\sigma(x) - \sigma(y)|$, our modified Hamiltonian includes an oscillatory factor $\cos\big(\pi(\sigma(x) - \sigma(y))\big)$, introducing parity-sensitive interactions. This modification promotes alternating spin configurations and reflects competing, antiferromagnetic-like behavior. As a result, the model enables the study of richer phase structures and boundary effects, especially on non-amenable graphs like the Cayley tree.

3. Ground States on the Cayley Tree of Order Two

In this section, we study periodic ground states corresponding to the normal subgroups of index two and four for the model (2.2). Moreover, H_A-weakly periodic ground states are examined. Note that H_A is a normal subgroup of index two in G_k (see [6]). For the classical SOS model, the ground states were studied in the papers [1, 7, 8].

Let M be the set of all unit balls with vertices in V, i.e. $M = \{\{x\} \cup S_1(x) : \forall x \in V\}$. A restriction of a configuration σ to the ball $b \in M$ is a bounded configuration and it is denoted by σ_b. We let c_b denote the center of the unit ball b.

We define the energy of the configuration σ_b on b by the following formula

$$U(\sigma_b) = \frac{1}{2} J_1 \sum_{x \in S_1(c_b)} |\sigma(x) - \sigma(c_b)| \cos[\pi(\sigma(x) - \sigma(c_b))] + J_2 \sum_{\substack{x,y \in b;\\ d(x,y)=2}} |\sigma(x) - \sigma(y)| \cos[\pi(\sigma(x) - \sigma(y))], \tag{3.1}$$

where $J_1, J_2 \in \mathbb{R}$.

We consider the case $k = 2$. It is easy to prove the following:

Lemma 3.1. *For all $b \in M$ and any $\sigma \in \Omega$ we have*

$$U(\sigma_b) \in \{U_0, U_1, U_2, ..., U_{11}\},$$

where

$$U_0 = 0, \quad U_1 = -\frac{3J_1}{2}, \quad U_2 = -2J_2, \quad U_3 = -J_1 - 2J_2,$$

$$U_4 = 3J_1, \quad U_5 = \frac{3J_1}{2} - 2J_2, \quad U_6 = 2J_1 + 4J_2, \quad U_7 = -\frac{J_1}{2} - 2J_2,$$

$$U_8 = J_1 + 4J_2, \quad U_9 = \frac{J_1}{2}, \quad U_{10} = -J_1, \quad U_{11} = -\frac{3J_1}{2} + 4J_2.$$

Definition 3.2. A configuration φ is called a ground state of the Hamiltonian (2.2), if

$$U(\varphi_b) = \min\{U_0, U_1, U_2, ..., U_{11}\},$$

for all $b \in M$.

For a fixed $m = 0, 1, 2, ..., 11$, we set

$$A_m = \{(J_1, J_2) \in \mathbb{R}^2 : U_m = \min\{U_0, U_1, U_2, ..., U_{11}\}\}, \tag{3.2}$$

and we find the sets:

$$A_0 = A_9 = A_{10} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 = 0\},$$

$$A_1 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad 0 \leq J_2 \leq \frac{J_1}{4}\},$$

$$A_2 = A_7 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 \geq 0\},$$

$$A_3 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad J_2 \geq \frac{J_1}{4}\},$$

$$A_4 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad \frac{J_1}{4} \leq J_2 \leq -\frac{3J_1}{4}\},$$

$$A_5 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad J_2 \geq \frac{-3J_1}{4}\},$$

$$A_6 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad J_2 \leq \frac{J_1}{4}\},$$

$$A_8 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 \leq 0\},$$

$$A_{11} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad J_2 \leq 0\}$$

and $\bigcup_{m=0}^{11} A_m = \mathbb{R}^2$.

We put

$$C_i = \{\sigma_b : U(\sigma_b) = U_i\}, i = 0, ..., 11$$

and

$$B^{(t)} = | \{x \in S_1(c_b) : \varphi_b(x) = t\} |,$$

for $t \in \Phi$.

The following theorem describes translation-invariant ground states.

Theorem 3.3. *If $(J_1, J_2) \notin A_0$, then there is no translation-invariant ground state for the model (2.2) on the Cayley tree of order two.*

Proof. Let $\sigma(x) = l, l \in \Phi$, $\forall x \in V$. According to (3.1), we have $U_0 = 0$, $\forall b \in M$ and then we can easily see from (3.2) that

$$A_0 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 = 0\}.$$

This finishes the proof of Theorem 3.3. □

Remark 3.4. If $\cos[\pi(\sigma(x) - \sigma(c_b))] = -1$, where $x \in S_1(c_b)$ and $\cos[\pi(\sigma(x) - \sigma(y))] = -1$, where $x, y \in b; d(x, y) = 2$, then Hamiltonian (2.2) coincides with Hamiltonian of SOS model with competing interactions which is studied in (see [1, 8]).

3.1. H_A**-Periodic Ground States.** Let $A \subset \{1, 2, 3\}$ and $H_A = \{x \in G_2 : \sum_{i\in A} w_x(a_i) - \text{even}\}$, where $w_x(a_i)$ is the number of a_i in the word x.

Note that H_A is a normal subgroup of index two in G_k (see [6]). Let $G_2/H_A = \{H_A, G_2 \setminus H_A\}$ be a quotient group. We set $H_1 := H_A, H_2 := G_2 \setminus H_A$.

We shall study H_A-periodic ground states. Note that each H_A-periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_1, \text{ if } x \in H_1, \\ \sigma_2, \text{ if } x \in H_2, \end{cases} \tag{3.3}$$

where $\sigma_i \in \Phi = \{-1, 0, 1\}, i = 1, 2$.

In the sequel, we write (σ_1, σ_2) for such an H_A-periodic configuration (3.3).

Theorem 3.5. *Let $|A| = p, p \in \{1, 2, 3\}$. For the modified SOS model with competing interactions given by (2.2) the following statements hold on the Cayley tree of order two:*
a) If $(J_1, J_2) \in A_{p^2-7p+13}$ and $|\sigma_1 - \sigma_2| = 1$, then the number of H_A-periodic ground states is 4 and in the forms $(1, 0), (0, 1), (-1, 0), (0, -1)$.
b) If $(J_1, J_2) \in A_{10-2p}$ and $|\sigma_1 - \sigma_2| = 2$, then the number of H_A-periodic ground states is 2 and in the forms $(1, -1), (-1, 1)$.

Proof. We prove the theorem for $p = 1$.
a) Let us consider the following configuration

$$\varphi(x) = \begin{cases} m, \text{ if } \ x \in H_1, \\ j, \text{ if } \ x \in H_2, \end{cases}$$

where $|m - j| = 1, m, j \in \Phi$.
1) Assume that $c_b \in H_1$

$$\varphi_b(c_b) = m, B^{(m)} = 2, B^{(j)} = 1.$$

Consequently, $\varphi_b \in C_7$.
2) Let $c_b \in H_2$, then one has

$$\varphi_b(c_b) = j, B^{(j)} = 2, B^{(m)} = 1.$$

Consequently, $\varphi_b \in C_7$.
We conclude that, if $(J_1, J_2) \in A_7$ then the periodic configuration φ is an H_A-periodic ground state.
b) Let us consider the following configuration

$$\varphi(x) = \begin{cases} m, \text{ if } \ x \in H_1, \\ j, \text{ if } \ x \in H_2, \end{cases}$$

where $|m - j| = 2$.
1) Assume that $c_b \in H_1$

$$\varphi_b(c_b) = m, B^{(m)} = 2, B^{(j)} = 1.$$

Consequently, $\varphi_b \in C_8$.
2) Let $c_b \in H_2$, then one has

$$\varphi_b(c_b) = j, B^{(j)} = 2, B^{(m)} = 1.$$

Consequently, $\varphi_b \in C_8$.
We conclude that, if $(J_1, J_2) \in A_8$ then the periodic configuration φ is an H_A-periodic ground state. □

Other cases are proved similar. This finishes the proof of Theorem 3.5.

Remark 3.6. If $|A| = p, p \in \{1, 3\}$ and $|\sigma_1 - \sigma_2| = 1$, the number of H_A-periodic ground states obtained in the Theorem 3.5 is equal to the Theorem 2 of [1]. However, the sets containing these ground states differ. If $|A| = 2$ and $|\sigma_1 - \sigma_2| = 1$, the paper [1] does not have the H_A-periodic ground state, but it is found that the H_A-periodic ground state exists by contrast. Furthermore, if $|A| = p, p \in \{1, 2, 3\}$ and $|\sigma_1 - \sigma_2| = 2$, the number of H_A-periodic ground states obtained in Theorem 3.5 is equal to the Theorem 2 of [1]. However, the sets containing these ground states differ. The H_A-periodic ground state in $p = 3$ is $G_2^{(2)}$-periodic ground state, where $G_2^{(2)} = \{x \in G_2 : |x|-\text{even}\}$. If $|\sigma_1 - \sigma_2| = 0$, then H_A-periodic ground state is translation-invariant ground state. Furthermore, the differences in the translation-invariant ground states studied in Theorem 3.3 and the paper [1] shows that the SOS model does not overlap with model (2.2).

3.2. $G_2^{(4)}$**-Periodic Ground States.** Let $A \subset \{1, 2, 3\}$, $G_2^{(4)} = \{x \in G_2 : \sum_{j \in A} w_j(x) - \text{even}, |x| - \text{even}\}$.

Note that $G_2^{(4)}$ is a normal subgroup of index four in G_2 (see[6]). In fact, there are other forms of normal subgroup of index four as well. But for the sake of convenience we consider the following quotient group $G_2/G_2^{(4)} = \{H_1, H_2, H_3, H_4\}$, where

$$H_1 := G_2^{(4)},$$

$$H_2 := \{x \in G_2 : \sum_{j \in A} w_j(x) - \text{even}, |x| - \text{odd}\},$$

$$H_3 := \{x \in G_2 : \sum_{j \in A} w_j(x) - \text{odd}, |x| - \text{even}\},$$

$$H_4 := \{x \in G_2 : \sum_{j \in A} w_j(x) - \text{odd}, |x| - \text{odd}\}.$$

We study $G_2^{(4)}$-periodic ground states. We note that each $G_2^{(4)}$-periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_1, \text{ if } x \in H_1, \\ \sigma_2, \text{ if } x \in H_2, \\ \sigma_3, \text{ if } x \in H_3, \\ \sigma_4, \text{ if } x \in H_4, \end{cases} \tag{3.4}$$

where $\sigma_i \in \Phi = \{-1, 0, 1\}, i = 1, 2, 3, 4$.

In the sequel, we write $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ for such a $G_2^{(4)}$-periodic configuration (3.4).

Theorem 3.7. *For the modified SOS model with competing interactions given by (2.2) the following statements holds on a Cayley tree of order two:*
1. Let $|A| = 1$.
1.a) *If* $(J_1, J_2) \in A_1 \cap A_{11}$ *and* $|\sigma_1 - \sigma_2| = 1$, *then the configuration* $\varphi = (\sigma_1, \sigma_2, -\sigma_1, -\sigma_2)$ *is* $G_2^{(4)}$-*periodic ground state.They have the following forms* $(\pm 1, 0, \mp 1, 0)$ *and* $(0, \pm 1, 0, \mp 1)$.
1.b) *If* $(J_1, J_2) \in A_2 \cap A_3$ *and* $|\sigma_1 - \sigma_2| = 1$, *then the configuration* $\varphi = (\sigma_1, \sigma_2, -\sigma_2, -\sigma_1)$ *is* $G_2^{(4)}$-*periodic ground state. They have the following forms* $(\pm 1, 0, 0, \mp 1)$ *and* $(0, \pm 1, \mp 1, 0)$.
1.c) *If* $(J_1, J_2) \in A_5 \cap A_7$ *and* $|\sigma_1 - \sigma_2| = 1$, *then the configuration* $\varphi = (\sigma_1, -\sigma_1, \sigma_2, -\sigma_2)$ *is* $G_2^{(4)}$-*periodic ground state. They have the following forms* $(\pm 1, \mp 1, 0, 0)$ *and* $(0, 0, \pm 1, \mp 1)$.
1.d) *Other* $G_2^{(4)}$-*periodic ground states correspond to* H_A-*periodic ground states.*

2. Let $|A| = 2$.

2.a) *If $(J_1, J_2) \in A_2 \cap A_3$ and $|\sigma_1 - \sigma_2| = 1$, then the configuration $\varphi = (\sigma_1, -\sigma_1, \sigma_2, -\sigma_2)$ is $G_2^{(4)}$-periodic ground state. They have the following forms $(\pm1, \mp1, 0, 0)$ and $(0, 0, \pm1, \mp1)$.*

2.b) *If $(J_1, J_2) \in A_5 \cap A_7$ and $|\sigma_1 - \sigma_2| = 1$, then the configuration $\varphi = (\sigma_1, \sigma_2, -\sigma_2, -\sigma_1)$ is $G_2^{(4)}$-periodic ground state. They have the following forms $(\pm1, 0, 0, \mp1)$ and $(0, \pm1, \mp1, 0)$.*

2.c) *Other $G_2^{(4)}$-periodic ground states correspond to H_A-periodic ground states.*

3. Let $|A| = 3$.

3.a) *All $G_2^{(4)}$-periodic ground states correspond to $G_2^{(2)}$-periodic ground states.*

Proof. 1.a) Let us consider the following configuration

$$\varphi_1(x) = \begin{cases} 1, \text{ if } & x \in H_1, \\ 0, \text{ if } & x \in H_2, \\ -1, \text{ if } & x \in H_3, \\ 0, \text{ if } & x \in H_4. \end{cases}$$

1) Assume that $c_b \in H_1$

$$\varphi_b(c_b) = 1, B^{(1)} = 0, B^{(-1)} = 0, B^{(0)} = 3.$$

Consequently, $\varphi_b \in C_1$.

2) Let $c_b \in H_2$, then one has

$$\varphi_b(c_b) = 0, B^{(0)} = 0, B^{(1)} = 2, B^{(-1)} = 1.$$

Consequently, $\varphi_b \in C_{11}$.

3) Let $c_b \in H_3$, then one has

$$\varphi_b(c_b) = -1, B^{(-1)} = 0, B^{(1)} = 0, B^{(0)} = 3.$$

Consequently, $\varphi_b \in C_1$.

4) Let $c_b \in H_4$, then one has

$$\varphi_b(c_b) = 0, B^{(0)} = 0, B^{(-1)} = 2, B^{(1)} = 1.$$

Consequently, $\varphi_b \in C_{11}$.

We conclude that, if $(J_1, J_2) \in A_1 \cap A_{11}$, then corresponding periodic configuration φ_1 is a $G_2^{(4)}$-periodic ground state. Other cases are proved similarly.

b) and c) are proved similar to a). This finishes the proof of Theorem 3.7. □

Remark 3.8. If $|\sigma_1 - \sigma_2| = 1$ and (J_1, J_2) takes a value different from the sets in the above Theorem 3.7, the $G_2^{(4)}$-periodic ground states coincide with the case a) of Theorem 3.5. If $|\sigma_1 - \sigma_2| \neq 1$, the $G_2^{(4)}$-periodic ground states coincide with H_A-periodic ground states which is studied the case b) of Theorem 3.5.

3.3. **Weakly Periodic Ground States.** Weakly periodic ground states have been studied and shown to be exist for other models, including the Ising and Potts models. For the Ising model with competing interactions, weakly periodic ground states are described in [9, 18]. In [18], the authors also explore weakly periodic ground states for the normal subgroups of index 2 and 4. Such states for the Potts model with competing interactions for the normal subgroups of index 2 are examined in [10, 13]. For this model weakly periodic ground states for the normal subgroups of index 4 are explored in [14]. It should be noted that the set of all weakly periodic ground states include all periodic ground states. In the above works, non-periodic weakly periodic ground states were found for the considered models. Therefore, the question arises whether there are non-periodic weakly periodic ground states for model (2.2) as well. The answer to this question is given in this subsection.

Let $A \in \{1, 2, 3\}$. In this subsection, we describe H_A-weakly periodic ground states, where H_A is a normal subgroup of index two in G_2 (see [6]). Let $G_2/H_A = \{H_A, G_2 \setminus H_A\}$ be the quotient group.

We set $H_1 := H_A, H_2 := G_2 \setminus H_A$. Due to the Definition 2.2, we infer that each H_A-weakly periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_{11}, \text{ if } x_\downarrow \in H_1, \ x \in H_1, \\ \sigma_{12}, \text{ if } x_\downarrow \in H_1, \ x \in H_2, \\ \sigma_{21}, \text{ if } x_\downarrow \in H_2, \ x \in H_1, \\ \sigma_{22}, \text{ if } x_\downarrow \in H_2, \ x \in H_2, \end{cases} \tag{3.5}$$

where $\sigma_{ij} \in \Phi, i, j = 1, 2$.

In the sequel, we write $\sigma = (\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22})$ for such a weakly periodic configuration $\sigma(x), x \in G_k$.

Theorem 3.9. *Let $k = 2$ and $|A| = j, j = 1, 2$. If $(J_1, J_2) \notin A_0$, then for the modified SOS model (2.2) there is no H_A-weakly periodic (non periodic) ground state.*

Proof. Let $|A| = 1$. If $\sigma_{11} = \sigma_{12} = \sigma_{21} = \sigma_{22}$, then corresponding configurations are translation-invariant. Translation-invariant ground states for this case are studied in Theorem 3.3. It is easy to see that in the case $\sigma_{11} = \sigma_{21}$ and $\sigma_{12} = \sigma_{22}$ the H_A-weakly periodic configuration (3.5) are periodic configuration which are studied in Theorem 3.5.

Now we consider the cases $\sigma_{11} \neq \sigma_{21}$ or $\sigma_{12} \neq \sigma_{22}$.
Let

$$\varphi(x) = \begin{cases} -1, & x_\downarrow \in H_1, \ x \in H_1, \\ 0, & x_\downarrow \in H_1, \ x \in H_2, \\ 0, & x_\downarrow \in H_2, \ x \in H_1, \\ 1, & x_\downarrow \in H_2, \ x \in H_2. \end{cases}$$

Let $c_b \in H_1$, then we have the following possible cases:
a) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = -1$, $B^{(-1)} = 1$, $B^{(0)} = 2$, $B^{(1)} = 0$, $\varphi_b \in C_3$.
b) if $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = -1$, $B^{(-1)} = 2$, $B^{(0)} = 1$, $B^{(1)} = 0$, $\varphi_b \in C_7$.
c) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = 0$, $B^{(-1)} = 2$, $B^{(0)} = 0$, $B^{(1)} = 1$, $\varphi_b \in C_{11}$.
Let $c_b \in H_2$, then we have the following possible cases:
a) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = 0$, $B^{(-1)} = 1$, $B^{(0)} = 0$, $B^{(1)} = 2$, $\varphi_b \in C_{11}$.
b) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 0$, $B^{(0)} = 2$, $B^{(1)} = 1$, $\varphi_b \in C_3$.
c) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 0$, $B^{(0)} = 1$, $B^{(1)} = 2$, $\varphi_b \in C_7$.
We conclude that the configuration φ is a ground state if

$$(J_1, J_2) \in A_3 \cap A_7 \cap A_{11} = \{(J_1, J_2) \in R^2 : J_1 = J_2 = 0\}.$$

Therefore, if $J_1 \neq 0$ and $J_2 \neq 0$ then the weakly periodic configuration φ is not a weakly periodic ground state. All possible configurations can be checked similarly.
By similar way we can prove that all H_A-weakly periodic (non periodic) configurations are not ground states. This finishes the proof of Theorem 3.9. □

Remark 3.10. Due to computational complexity, weakly periodic ground states for the normal subgroups of index 4 have not been considered. Furthermore, all H_A-weakly periodic ground states correspond to H_A-periodic ground state. If $|A| = 1$, H_A-weakly periodic ground states are explored in the [1]. All weakly periodic ground states for the SOS model with competing interactions found to be H_A-periodic ground states but not translation-invariant.

4. Ground States on the Cayley Tree of Order Three

In this section, we are going to continue investigation related to periodic ground states corresponding to the normal subgroups of index two and four for the modified SOS model on the Cayley tree of order three. Moreover, H_A-weakly periodic ground states are examined. Let $k = 3$. It is easy to prove the following lemma.

Lemma 4.1. *For all $b \in M$ and any $\sigma \in \Omega$ we have*

$$U(\sigma_b) \in \{U_0, U_1, U_2, ..., U_{17}\},$$

where

$$U_0 = 0, \quad U_1 = -\frac{J_1}{2} - 3J_2, \quad U_2 = -2J_1, \quad U_3 = -2J_1 + 6J_2,$$

$$U_4 = -\frac{3J_1}{2} - 3J_2, \quad U_5 = -J_1 - 2J_2, \quad U_6 = -J_1 - 4J_2, \quad U_7 = -2J_1 + 8J_2,$$

$$U_8 = J_1 + 6J_2, \quad U_9 = \frac{5J_1}{2} - 3J_2, \quad U_{10} = 3J_1 + 6J_2, \quad U_{11} = 4J_1,$$

$$U_{12} = -2J_2, \quad U_{13} = -\frac{3J_1}{2} + J_2, \quad U_{14} = \frac{J_1}{2} + J_2,$$

$$U_{15} = -\frac{3J_1}{2} + J_2, \quad U_{16} = J_1 - 4J_2, \quad U_{17} = 2J_1 + 8J_2.$$

Definition 4.2. A configuration φ is called a ground state for the Hamiltonian (2.2), if $U(\varphi_b) = \min\{U_0, U_1, U_2, ..., U_{17}\}$, for all $b \in M$.

For a fixed $m = 0, 1, 2, ..., 17$, we set

$$A_m = \{(J_1, J_2) \in \mathbb{R}^2 : U_m = \min\{U_0, U_1, U_2, ..., U_{17}\}\}. \tag{4.1}$$

It is easy to check that

$$A_0 = A_1 = A_5 = A_8 = A_{12} = A_{13} = A_{14} = A_{15} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 = 0\},$$

$$A_2 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad 0 \leq J_2 \leq \frac{J_1}{6}\},$$

$$A_3 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad J_2 = 0\},$$

$$A_4 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad \frac{J_1}{6} \leq J_2 \leq \frac{J_2}{2}, J_2 \geq 0\},$$

$$A_6 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad J_2 \geq \frac{J_1}{10}\},$$

$$A_7 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \geq 0, \quad J_2 \leq 0\},$$

$$A_9 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad -\frac{J_1}{2} \leq J_2 \leq -\frac{3J_1}{2}\},$$

$$A_{10} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad \frac{5J_1}{2} \leq J_2 \leq \frac{J_1}{6}\},$$

$$A_{11} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad \frac{J_1}{6} \leq J_2 \leq -\frac{J_1}{2}\},$$

$$A_{16} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad J_2 \geq 0, \quad J_2 \geq -\frac{3J_1}{2}\},$$

$$A_{17} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 \leq 0, \quad J_2 \leq \frac{J_1}{2}\}$$

and $\bigcup_{m=0}^{17} A_m = \mathbb{R}^2$.

4.1. H_A**-Periodic Ground States.** Let $A \subset \{1,2,3,4\}$, $H_A = \{x \in G_3 : \sum_{j\in A} w_j(x) - \text{even}\}$, where $w_j(x)$ is the number of letters a_j in the word x. It is obvious that H_A is a normal subgroup of index two. Let $G_3/H_A = \{H_A, G_3 \setminus H_A\}$ be the quotient group. We set $H_1 := H_A, H_2 := G_3 \setminus H_A$.

We study H_A-periodic ground states. Note that each H_A-periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_1, \text{ if } x \in H_1, \\ \sigma_2, \text{ if } x \in H_2, \end{cases} \tag{4.2}$$

where $\sigma_i \in \Phi = \{-1,0,1\}, i = 1,2$.

Theorem 4.3. *If $(J_1, J_2) \notin A_0$, then there is no translation-invariant ground state for the model (2.2) on the Cayley tree of order three.*

Proof. Let $\sigma(x) = l, l \in \Phi$, $\forall x \in V$. According to (3.1), we have $U_0 = 0$, $\forall b \in M$ and then we can see easily from (4.1) that

$$A_0 = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = 0, \quad J_2 = 0\}.$$

This finishes the proof of Theorem 4.3. □

In the sequel, we write (σ_1, σ_2) for such an H_A-periodic configuration (4.2).

Theorem 4.4. *Let $|A| = p, p \in \{2,3,4\}$. For the modified SOS model with competing interactions given by (2.2) the following statements hold on the Cayley tree of order three:*
a) If $(J_1, J_2) \in A_{10-2p}$ and $|\sigma_1 - \sigma_2| = 1$, then the number of H_A-periodic ground states is 4 *and in the forms* $(1,0), (0,1), (-1,0), (0,-1)$.
b) If $(J_1, J_2) \in A_{4p^2-27p+55}$ and $|\sigma_1 - \sigma_2| = 2$, then the number of H_A-periodic ground states is 2 *and in the forms* $(1,-1), (-1,1)$.

Proof. Let $p = 2$. Now we consider the following configuration

$$\varphi(x) = \begin{cases} m, \text{ if } x \in H_1, \\ j, \text{ if } x \in H_2, \end{cases}$$

where $|m - j| = 1$ or $|m - j| = 2, m, j \in \Phi$.
a) Let $|m - j| = 1$.
1) Assume that $c_b \in H_1$, then

$$\varphi_b(c_b) = m, B^{(m)} = 2, B^{(j)} = 2.$$

Hence, $\varphi_b \in C_6$.
2) Assume that $c_b \in H_2$, then

$$\varphi_b(c_b) = j, B^{(j)} = 2, B^{(m)} = 2.$$

Hence, $\varphi_b \in C_6$.
We conclude that if $(J_1, J_2) \in A_6$, then the periodic configuration φ is an H_A-periodic ground state.
b) Let $|m - j| = 2$.
1) Assume that $c_b \in H_1$, then

$$\varphi_b(c_b) = m, B^{(m)} = 2, B^{(j)} = 2.$$

Hence, $\varphi_b \in C_{17}$.
2) Assume that $c_b \in H_2$, then

$$\varphi_b(c_b) = j, B^{(j)} = 2, B^{(m)} = 2.$$

Hence, $\varphi_b \in C_{17}$.
We conclude that if $(J_1, J_2) \in A_{17}$, then the periodic configuration φ is an H_A-periodic ground state.
By the similar way, we can prove easily other cases of this theorem. This finishes the proof of the Theorem 4.4. □

Remark 4.5. It is known from [8] there exist the H_A-periodic ground states for the SOS model with competing interactions when $|A| = 1$. By contrast, by Theorem 4.4 the H_A-periodic ground state does not exist when $|A| = 1$ for modified SOS model with competing interactions. H_A-periodic ground state does not exist in [8] when $|A| = p, p \in \{2, 3\}$ and $|\sigma_1 - \sigma_2| = 1$ but the existence of H_A-periodic ground states in the Theorem 4.4 is proved. If $|A| = 4$ and $|\sigma_1 - \sigma_2| = 1$, the number of H_A-periodic ground states obtained in the Theorem 4.4 is similar to the Theorem 2.2 of [8], but the sets containing the ground states are different. In addition, if $|A| = p, p \in \{2, 3, 4\}$ and $|\sigma_1 - \sigma_2| = 2$ too, the number of H_A-periodic ground states obtained in the Theorem 4.4 is similar to the Theorem 2.2 of [8], but the sets containing the ground state are different. The H_A-periodic ground state in $p = 4$ is $G_3^{(2)}$-periodic ground state which is studied in Theorem 4.4, where $G_3^{(2)} = \{x \in G_3 : |x| - \text{even}\}$. If $|\sigma_1 - \sigma_2| = 0$, then H_A-periodic ground state is translation-invariant ground state but there is no translation-invariant ground state due to Theorem 4.3.

4.2. $G_3^{(4)}$**-Periodic Ground States.** Let $A \subset \{1, 2, 3, 4\}$, $G_3^{(4)} = \{x \in G_3 : \sum_{j\in A} w_j(x) - \text{even}, |x| - \text{even}\}$. Note that $G_3^{(4)}$ is a normal subgroup of index four in G_3 (see [6]). In fact, there are other forms of normal subgroup of index four as well. But for the sake of convenience we consider the following quotient group $G_3/G_3^{(4)} = \{H_1, H_2, H_3, H_4\}$, where

$$H_1 := G_3^{(4)},$$

$$H_2 := \{x \in G_3 : \sum_{j\in A} w_j(x) - \text{even}, |x|\text{-odd}\},$$

$$H_3 := \{x \in G_3 : \sum_{j\in A} w_j(x) - \text{odd}, |x| - \text{even}\},$$

$$H_4 := \{x \in G_3 : \sum_{j\in A} w_j(x) - \text{odd}, |x| - \text{odd}\}.$$

We study $G_3^{(4)}$- periodic ground states. We note that each $G_3^{(4)}$-periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_1, \text{ if } x \in H_1, \\ \sigma_2, \text{ if } x \in H_2, \\ \sigma_3, \text{ if } x \in H_3, \\ \sigma_4, \text{ if } x \in H_4, \end{cases} \tag{4.3}$$

where $\sigma_i \in \Phi = \{-1, 0, 1\}, i = 1, 2, 3, 4$.

In the sequel, we write $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ for such a $G_3^{(4)}$-periodic configuration (4.3).

Theorem 4.6. *Let $|\sigma_1 - \sigma_2| = 1$. For the modified SOS model with competing interactions given by (2.2) the following statements hold on the Cayley tree of order three:*
a) If $(J_1, J_2) \in A_2 \cap A_3$ and $|A| = 1$ or $|A| = 3$, then the configuration $\varphi = (\sigma_1, \sigma_2, -\sigma_1, -\sigma_2)$ is $G_3^{(4)}$-periodic ground state. They have the following forms $(\pm1, 0, \mp1, 0)$ and $(0, \pm1, 0, \mp1)$.
b) If $(J_1, J_2) \in A_2 \cap A_7$ and $|A| = 2$, then the configuration $\varphi = (\sigma_1, \sigma_2, -\sigma_1, -\sigma_2)$ is $G_3^{(4)}$-periodic ground state. They have the following forms $(\pm1, 0, \mp1, 0)$ and $(0, \pm1, 0, \mp1)$.
c) If $(J_1, J_2) \in A_6 \cap A_{16}$ and $|A| = 2$, then the configurations $\varphi = (\sigma_1, -\sigma_1, \sigma_2, -\sigma_2)$ and $\varphi^* = (\sigma_1, -\sigma_2, \sigma_2, -\sigma_1)$ are $G_3^{(4)}$-periodic ground states. They have the following forms $(\pm1, \mp1, 0, 0)$, $(0, 0, \pm1, \mp1)$ and $(\pm1, 0, 0, \mp1)$, $(0, \pm1, \mp1, 0)$.
d) All $G_3^{(4)}$-periodic ground states except the case a),b),c) are H_A-periodic ground states.

Proof. a) Let us consider the following configuration

$$\varphi_1(x) = \begin{cases} 1, & \text{if } x \in H_1, \\ 0, & \text{if } x \in H_2, \\ -1, & \text{if } x \in H_3, \\ 0, & \text{if } x \in H_4. \end{cases}$$

1) Assume that $c_b \in H_1$

$$\varphi_b(c_b) = 1, B^{(1)} = 0, B^{(-1)} = 0, B^{(0)} = 4.$$

Consequently, $\varphi_b \in C_2$.
2) Let $c_b \in H_2$, then one has

$$\varphi_b(c_b) = 0, B^{(0)} = 0, B^{(1)} = 3, B^{(-1)} = 1.$$

Consequently, $\varphi_b \in C_3$.
3) Let $c_b \in H_3$, then one has

$$\varphi_b(c_b) = -1, B^{(-1)} = 0, B^{(1)} = 0, B^{(0)} = 4.$$

Consequently, $\varphi_b \in C_2$.
4) Let $c_b \in H_4$, then one has

$$\varphi_b(c_b) = 0, B^{(0)} = 0, B^{(-1)} = 3, B^{(1)} = 1.$$

Consequently, $\varphi_b \in C_3$.
We conclude that, if $(J_1, J_2) \in A_2 \cap A_3$ then the periodic configuration φ_1 is a $G_3^{(4)}$- periodic ground state. Other cases are proved similarly. This finishes the proof of Theorem 4.6. □

Remark 4.7. When $|\sigma_1 - \sigma_2| = 1$ and (J_1, J_2) takes a value different from the sets in the above Theorem 4.6, the $G_3^{(4)}$-periodic ground states coincide with the case a) of Theorem 4.4. When $|\sigma_1 - \sigma_2| \neq 1$, the $G_3^{(4)}$-periodic ground states coincide with H_A-periodic ground states which is studied the case b) of Theorem 4.4. If $|A| = 4$, $G_3^{(4)}$-periodic ground states correspond to $G_3^{(2)}$-periodic ground states.

4.3. **Weakly Periodic Ground States.** In this subsection we describe H_A-weakly periodic ground states, where H_A is a normal subgroup of index two.

Theorem 4.8. *Let $k = 3$. The following statements hold:*
1. *Let $|A| = 2$.*
1.*a) If $(J_1, J_2) \in A_{10} \cap A_{17}$, then for the modified SOS model with competing interactions there are H_A-weakly periodic (non periodic) ground states which are the following forms:*

$$\varphi_{1,2} = \pm \begin{cases} 1, & if \quad x_\downarrow \in H_1, \ x \in H_1, \\ -1, & if \quad x_\downarrow \in H_1, \ x \in H_2, \\ -1, & if \quad x_\downarrow \in H_2, \ x \in H_1, \\ 1, & if \quad x_\downarrow \in H_2, \ x \in H_2. \end{cases}$$

1.*b) If $(J_1, J_2) \in A_4 \cap A_6$, then for the modified SOS model with competing interactions there are H_A-weakly periodic (non periodic) ground states which are the following forms:*

$$\varphi_{3,4} = \pm \begin{cases} 1, & if \quad x_\downarrow \in H_1, \ x \in H_1, \\ 0, & if \quad x_\downarrow \in H_1, \ x \in H_2, \\ 0, & if \quad x_\downarrow \in H_2, \ x \in H_1, \\ 1, & if \quad x_\downarrow \in H_2, \ x \in H_2, \end{cases} \qquad \varphi_{5,6} = \pm \begin{cases} 0, & if \quad x_\downarrow \in H_1, \ x \in H_1, \\ 1, & if \quad x_\downarrow \in H_1, \ x \in H_2, \\ 1, & if \quad x_\downarrow \in H_2, \ x \in H_1, \\ 0, & if \quad x_\downarrow \in H_2, \ x \in H_2. \end{cases}$$

2. *Let $|A| = p, p \in \{1, 3\}$. If $(J_1, J_2) \notin A_0$, then for the modified SOS model with competing interactions there is no H_A-weakly periodic (non periodic) ground state.*

Proof. 1.a) Let us prove the case $|A| = 2$ without loss of generality which is H_A-weakly periodic (non periodic) ground state on the set $A_{10} \cap A_{17}$.
Let

$$\varphi_1 = \begin{cases} 1, & \text{if} \quad x_\downarrow \in H_1, \ x \in H_1, \\ -1, & \text{if} \quad x_\downarrow \in H_1, \ x \in H_2, \\ -1, & \text{if} \quad x_\downarrow \in H_2, \ x \in H_1, \\ 1, & \text{if} \quad x_\downarrow \in H_2, \ x \in H_2. \end{cases}$$

Let $c_b \in H_1$, then we have the following possible cases:
a) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 3$, $B^{(0)} = 0$, $B^{(1)} = 1$, $\varphi_b \in C_{10}$.
b) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 2$, $B^{(0)} = 0$, $B^{(1)} = 2$, $\varphi_b \in C_{17}$.
c) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = -1$, $B^{(-1)} = 1$, $B^{(0)} = 0$, $B^{(1)} = 3$, $\varphi_b \in C_{10}$.
d) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = -1$, $B^{(-1)} = 2$, $B^{(0)} = 0$, $B^{(1)} = 2$, $\varphi_b \in C_{17}$.
Let $c_b \in H_2$, then we have the following possible cases:
a) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = -1$,$B^{(-1)} = 1$, $B^{(0)} = 0$, $B^{(1)} = 3$, $\varphi_b \in C_{10}$.
b) If $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = -1$, $B^{(-1)} = 2$, $B^{(0)} = 0$, $B^{(1)} = 2$, $\varphi_b \in C_{17}$.
c) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = -1$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 3$, $B^{(0)} = 0$, $B^{(1)} = 1$, $\varphi_b \in C_{10}$.
d) If $c_{b\downarrow} \in H_2$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = 1$, $B^{(-1)} = 2$, $B^{(0)} = 0$, $B^{(1)} = 2$, $\varphi_b \in C_{17}$.
We conclude that the configuration φ_1 is a weakly periodic ground state if $(J_1, J_2) \in A_{10} \cap A_{17}$. Remaining cases can be proved analogously. This finishes the proof of Theorem 4.8. □

Remark 4.9. Theorem 4.8 says that if $|A| = p, p \in \{1, 3\}$, then for the modified SOS model with competing interactions there is no H_A-weakly periodic (non periodic) ground state. Furthermore, in the paper [8], it is proved that there are no weakly periodic (non periodic) ground states for the SOS model when $|A| = 1$.

Acknowledgements. We are grateful to the reviewer for a careful reading of the manuscript and especially for valuable remarks, which have improved the readability of the paper. We also thank Professor N.N. Ganikhodjaev for helpful discussions during the preparation of this work. The first author (RMM) thanks the State Grant F-FA-2021-425 of the Republic of Uzbekistan for support.

References

[1] Abraev B. U.; On weakly periodic ground states for the SOS model. Scientific Bulletin of Namangan State University, –2020.–2.–P.14–21.

[2] Akin H.; q-state modified Potts model on a Cayley tree and its phase transition in antiferromagnetic region. Preprint on SSRN. https://ssrn.com/abstract=5175125.

[3] Botirov G. I., Rahmatullaev M. M.; Ground states for Potts model with a countable set of spin values on a Cayley tree. Algebra, complex analysis, and pluripotential theory, Springer Proc. Math. Stat., –2018.–264.–P.59–71.

[4] Botirov G. I, Rozikov U. A.; Potts model with competing interactions on the Cayley tree: The contour method. Theor. Math. Phys.,–2007.–153.–P.1423–1433.

[5] Fernandez R.; Contour ensembles and the description of Gibbsian probability distributions at low temperature.–1998.

[6] Ganikhodjaev N. N.; Group representation and automorphisms of the Cayley tree. Dokl. Akad. nauk Resp. Uzbekistan,–1994.–4. –P.3–5.

[7] Rahmatullaev M. M., Abdusalomova M. R., Rasulova M. A.; Ground states for the SOS model with an external field on the Cayley tree. Uzbek Mathematical Journal, –2020.–2.–P.145–156.

[8] Rahmatullaev M. M., Abraev B. U.; On Ground states for the SOS model with competing interactions. Journal of Siberian Federal University. Mathematics Physics,–2022.–15.–P.160–173.

[9] Rakhmatullaev M. M.; Description of weak periodic ground states of Ising model with competing interactions on Cayley tree. Appl. Math. Inf. Sci.,–2010. –4.–P.237–251.

[10] Rakhmatullaev M. M.; Weakly periodic Gibbs measures and ground states for the Potts model with competing interactions on the Cayley tree. Theor. Math. Phys.,–2013.–176. –P.1236–1251.

[11] Rakhmatullaev M. M., Rasulova M. A., Asqarov J. N.; Ground States and Gibbs Measures of Ising Model with Competing Interactions and an External Field on a Cayley Tree. Journal of Statistical Physics, –2023.–190. –P.116–130.

[12] Rahmatullaev M. M., Rasulova M. A.; Ground states for the Ising model with an external field on the Cayley tree. Uzbek Mathematical Journal,–2018.–3.–P.147–155.

[13] Rakhmatullaev M. M., Rasulova M. A.; Existence of weakly periodic ground states for the Potts model with competing interactions on the Cayley tree. Dokl. Akad. nauk Resp. Uzbekistan, –2013.–10.

[14] Rahmatullaev M. M., Rasulova M. A.; Periodic and Weakly Periodic Ground States for the Potts Model with Competing Interactions on the Cayley Tree. Siberian Advances in Mathematics, –2016.–26. –P.215–229.

[15] Rozikov U. A.; Gibbs measures on Cayley trees. World scientific,–2013.

[16] Rozikov U. A.; On q-Component Models on Cayley Tree: Contour Method. Letters in Mathematical Physics, –2005. –71.–P.27–38.

[17] Rozikov U. A.; A Constructive Description of Ground States and Gibbs Measures for Ising Model with Two-Step Interactions on Cayley Tree. Journal of Statistical Physics, –2006.–122. –P.217–235.

[18] Rozikov U. A., Rakhmatullaev M. M.; Weakly periodic ground states and Gibbs measures for the Ising model with competing interactions on the Cayley tree. Theor. Math. Phys., –2009.–160.–P.1292–1300.

Rahmatullaev M.M.,
V.I.Romanovskiy Institute of mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
email: mrahmatullaev@rambler.ru

Askarov J.N.,
Namangan state technical university,
Namangan, Uzbekistan.
email:javokhiraskarov1997@mail.ru

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.194-200
DOI: 10.29229/uzmj.2025-4-19

Optimal quadrature formulas in the Sobolev space of complex-valued functions

Shadimetov Kh. M., Tursunmuradov S.T.

Abstract. In this work, the extremal function of the error functional in the Sobolev space of complex-valued functions is derived. Using the obtained extremal function, the squared norm of the error functional of quadrature formulas is computed. By minimizing the squared norm of the error functional with respect to the coefficients of the quadrature formulas, a system is derived for determining the optimal coefficients of the considered quadrature formula in the Sobolev space. In addition, an analogue of I. Babuška's theorem is proved.

Keywords: Extremal function, error functional, Sobolev space, quadrature formula, optimal coefficient.

MSC (2020): 65D30, 65D32

1. Introduction

One of the important tasks of computational mathematics is the development of new methods for constructing optimal quadrature, cubature, interpolation and difference formulas, and the assessment of their errors in various functional spaces. The construction of optimal quadrature and cubature formulas was first considered in the works of S.M. Nikolsky [1], A. Sard [2], S.L. Sobolev [3, 4]. In the works of M.D. Ramazanov [5, 6, 7] a new algorithm for constructing cubature formulas with a boundary layer was constructed. In the work of M.D. Ramazanov and Kh.M. Shadimetov [8] weighted optimal cubature formulas in the Sobolev space of periodic functions were constructed. In recent years, a number of works have been published on this topic (see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). In this paper, the variational method will be used to construct optimal quadrature formulas in the Sobolev space of complex-valued functions.

Let us consider quadrature formulas of the form

$$\int_0^1 p(x)\,\varphi(x)\,dx \cong \sum_{\beta=0}^{N} C[\beta]\,\varphi[\beta], \tag{1.1}$$

with the error functional

$$\ell_N(x) = \varepsilon_{[0,1]}(x)\,p(x) - \sum_{\beta=0}^{N} C[\beta]\,\delta(x-h\beta). \tag{1.2}$$

Here $p(x)$ is a weight function. It is required that the integral of this function exists in some sense, i.e.

$$\int_0^1 p(x)\,dx < \infty,$$

$\varphi(x) \in H_2^{(m)}(0,1)$, and $H_2^{(m)}(0,1)$ is the Sobolev space of complex-valued functions with the inner product and norm, respectively

$$(\psi,\varphi)_{H_2^{(m)}} = \sum_{k=0}^{m} \binom{m}{k} \frac{1}{\omega^{2k}} \int_0^1 \frac{d^k}{dx^k}\overline{\psi}(x)\,\frac{d^k}{dx^k}\varphi(x)dx, \tag{1.3}$$

$$\|\varphi\|_{H_2^{(m)}} = \left(\sum_{k=0}^{m} \binom{m}{k} \frac{1}{\omega^{2k}} \int_0^1 \frac{d^k}{dx^k}\overline{\varphi}(x)\,\frac{d^k}{dx^k}\varphi(x)dx \right)^{\frac{1}{2}}, \tag{1.4}$$

ω a real number and $\omega \neq 0$, $\overline{\psi}(x)$ is the complex conjugate function of the function $\psi(x)$, $C[\beta]$, $\beta = 0, 1, ..., N$, are coefficients of the quadrature formula (1.1), $\varepsilon_{[0,1]}(x)$ is the characteristic function of the segment $[0,1]$, $\delta(x)$ is the Dirac delta-function, $\binom{m}{k} = \frac{m!}{k!(m-k)!}$, $[\beta] = h\beta$, $h = \frac{1}{N}$, $N = 1, 2, ...$.

The next difference

$$(\ell_N, \varphi) = \int_0^1 p(x)\varphi(x)\,dx - \sum_{\beta=0}^{N} C[\beta]\varphi[\beta] = \int_R \ell_N(x)\varphi(x)\,dx \tag{1.5}$$

is called the error of the quadrature formula (1.1).

From the Cauchy-Schwarz inequality

$$|(\ell_N, \varphi)| \leq \left\|\ell_N|H_2^{(m)*}\right\| \cdot \left\|\varphi|H_2^{(m)}\right\|,$$

it is clear that the absolute value of the error of the quadrature formula (1.1) is estimated using the norm of the error functional (1.2)

$$\left\|\ell_N|H_2^{(m)*}\right\| = \sup_{\varphi,\ \|\varphi\|\neq 0} \frac{|(\ell_N, \varphi)|}{\left\|\varphi|H_2^{(m)}\right\|} \tag{1.6}$$

in the conjugate space $H_2^{(m)*}$. It follows that the error estimate of the quadrature formula (1.1) on complex-valued functions in the Sobolev space $H_2^{(m)}(0,1)$ reduces to finding the norm of the error functional (1.2) in the conjugate space $H_2^{(m)*}(0,1)$. It is evident that the norm of the error functional (1.2) depends on the coefficients $C[\beta]$ of the quadrature formula (1.1). By minimizing the norms of the error functional $\ell_N(x)$ with respect to the coefficients $C[\beta]$, we obtain **the optimal coefficients** of the quadrature formula. The resulting formula is called **the optimal quadrature formula**.

Thus, in order to construct an optimal quadrature formula in the Sobolev space, $H_2^{(m)}(0,1)$ it is necessary to find the following quantity

$$\inf_{C[\beta]} \left\|\ell_N|H_2^{(m)*}\right\|. \tag{1.7}$$

We denote this value by $\left\|\overset{\circ}{\ell}_N|H_2^{(m)*}\right\|$, which we call the norm of the error functional of the optimal quadrature formula, i.e.

$$\left\|\overset{\circ}{\ell}_N|H_2^{(m)*}\right\| = \inf_{\overset{\circ}{C}[\beta]} \left\|\ell_N|H_2^{(m)*}\right\|. \tag{1.8}$$

The coefficients $C[\beta]$ for which the exact lower bound is achieved as in (1.8) are called the optimal coefficients of the quadrature formula and are denoted by $\overset{\circ}{C}[\beta]$, $\beta = 0, 1, \ldots, N$.

So, we need to solve the following problems sequentially.

Problem 2. Find the norm of the error functional $\ell_N(x)$ of the quadrature formula (1.1) on the space $H_2^{(m)}(0,1)$.

Problem 3. Find the coefficients $\overset{\circ}{C}[\beta]$, $\beta = 0, 1, \ldots, N$ that satisfy (1.8).

In the next section, we find the extremal function that helps us to calculate the norm of error functional.

2. An extremal function of quadrature formulas in Sobolev space $H_2^{(m)}(0,1)$ - complex-valued functions

The explicit form of the norm of the error functional $\ell_N(x)$ in the space $H_2^{(m)*}(0,1)$ is obtained by means of the so-called extremal function of this functional [1, 2].

Definition 2.1. (S.L. Sobolev) A function $\psi_{\ell,H}(x) \in H_2^{(m)}(0,1)$ is called an extremal function of a given functional $\ell_N(x)$ if

$$(\ell, \psi_{\ell,H}) = \left\| \ell_N | H_2^{(m)*} \right\| \cdot \left\| \psi_{\ell,H} | H_2^{(m)} \right\|. \tag{2.1}$$

The following is true.

Theorem 2.2. *In the Sobolev space $H_2^{(m)}(0,1)$, the extremal function of the error functional $\ell_N(x)$ is given by the formula*

$$\psi_{\ell,H}(x) = \ell_N(x) * \varepsilon_{m,\omega}(x) = \int_0^1 \overline{p}(y)\, \varepsilon_m(x-y)\, dy - \sum_{\beta=0}^{N} \overline{C}[\beta]\, \varepsilon_{m,\omega}(x - h\beta), \tag{2.2}$$

where

$$\varepsilon_{m,\omega}(x) = \frac{\omega e^{-\omega|x|}}{2^{2m-1}(m-1)!} \sum_{k=0}^{m} \frac{(2m-k-2)!(2\omega)^k |x|^k}{k!\,(2m-k-1)!}. \tag{2.3}$$

$\overline{\ell}_N(x)$ is the complex conjugate functional of the functional $\ell_N(x)$, $\overline{p}(y)$ is the complex conjugate function of the function $p(y)$, $\overline{C}[\beta]$ is complex conjugate coefficient of $C[\beta]$.

Proof. To find an extremal function $\psi_{\ell,H}(x)$, we use the well-known Riesz theorem on the general form of a linear continuous functional. The space $H_2^{(m)}(0,1)$ is a Hilbert space, according to the Riesz theorem for the error functional $\ell_N(x)$ and for any function $\varphi(x) \in H_2^{(m)}(0,1)$ there is a unique function $\psi_{\ell,H}(x) \in H_2^{(m)}(0,1)$ for which the following equality holds

$$(\ell_N, \varphi) = (\psi_{\ell,H}, \varphi)_{H_2^{(m)}}, \tag{2.4}$$

and

$$\left\| \ell_N | H_2^{(m)*} \right\| = \left\| \psi_{\ell,H} | H_2^{(m)} \right\|. \tag{2.5}$$

The right side of (2.4) is determined by formula (1.3).

Now to find $\psi_{\ell,H}(x)$ we solve equation (2.4). Taking into account (1.3), after integrating by parts the right side of (2.4), we get

$$(\ell_N, \varphi) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{\omega^{2k}} \int_0^1 \overline{\psi}_{\ell,H}^{(2k)}(x)\, \varphi(x)\, dx + \sum_{s=1}^{m} \sum_{k=s}^{m} \binom{m}{k} \frac{(-1)^{k-s}}{\omega^{2k}} \overline{\psi}_{\ell,H}^{(2k-s)}(x)\, \varphi^{(s-1)}(x) \Big|_0^1. \tag{2.6}$$

From here, taking account the arbitrariness $\varphi(x)$ and uniqueness of the extremal function, we obtain the following boundary value problem

$$\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{\omega^{2k}} \overline{\psi}_{\ell,H}^{(2k)}(x) = \ell_N(x), \tag{2.7}$$

$$\sum_{s=1}^{m} \sum_{k=s}^{m} \binom{m}{k} \frac{(-1)^{k-s}}{\omega^{2k}} \overline{\psi}_{\ell,H}^{(2k-s)}(x) \Big|_0^1 = 0. \tag{2.8}$$

On the other hand, by Sobolev's theorem (see [1] theorem 11), $C_0^\infty(0,1)$ is the space of functions that are infinitely differentiable and finite on the interval [0,1] is dense in L_p for $1 \le p \le \infty$. It follows that the space $C_0^\infty(0,1)$ is dense in the Sobolev space $H_2^{(m)}(0,1)$. Further, by virtue of these results it follows that in equality (2.6) the last double sum vanishes. Indeed, from the finiteness of φ we have $\varphi^{(s-1)}(0) = 0$, $\varphi^{(s-1)}(1) = 0$. Then

$$\sum_{s=1}^{m} \sum_{k=s}^{m} \binom{m}{k} \frac{(-1)^{k-s}}{\omega^{2k}} \overline{\psi}_{\ell,H}^{(2k-s)}(x)\, \varphi^{(s-1)}(x) \Big|_0^1 = 0.$$

As a result, to find $\overline{\psi}_{\ell,H}(x)$ it is enough to solve equation (2.7). Now we will deal with solving equation (2.7). We write equation (2.7) in the form

$$\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{\omega^{2k}} \frac{d^{2k}}{dx^{2k}} \overline{\psi}_{\ell,H}(x) = \ell_N(x), \tag{2.9}$$

The operator L_m has the form

$$L_m = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{\omega^{2k}} \frac{d^{2k}}{dx^{2k}} = \left(1 - \frac{d^2}{\omega^2 dx^2}\right)^{[m]} = \left(1 - \frac{d^2}{\omega^2 dx^2}\right)^{[m-1]} \left(1 - \frac{d^2}{\omega^2 dx^2}\right). \tag{2.10}$$

If we find a fundamental solution of the operator L_m, i.e.

$$L_m \varepsilon_{m,\omega}(x) = \delta(x), \tag{2.11}$$

then the solution to equation (2.9) is given by the following formula for the convolution of two functions

$$\overline{\psi}_{\ell,H}(x) = \ell_N(x) * \varepsilon_{m,\omega}(x). \tag{2.12}$$

Lemma 2.1. A fundamental solution of the operator L_m has the form

$$\varepsilon_{m,\omega}(x) = \frac{\omega e^{-\omega|x|}}{2^{2m-1}(m-1)!} \sum_{k=0}^{m-1} \frac{(2m-k-2)!(2\omega)^k |x|^k}{k!\,(m-k-1)!}. \tag{2.13}$$

Proof. It is not difficult to verify that

$$\left(1 - \frac{d^2}{\omega^2 dx^2}\right) \varepsilon_{m,\omega}(x) = \varepsilon_{m-1,\omega}(x). \tag{2.14}$$

Then

$$\left(1 - \frac{d^2}{\omega^2 dx^2}\right)^{[m]} \varepsilon_{m,\omega}(x) = \left(1 - \frac{d^2}{\omega^2 dx^2}\right)^{[m-1]} \left(1 - \frac{d^2}{\omega^2 dx^2}\right) \varepsilon_{m,\omega}(x) = \left(1 - \frac{d^2}{\omega^2 dx^2}\right)^{[m-1]} \varepsilon_{m-1,\omega}(x). \tag{2.15}$$

In the same way, we have

$$\left(1 - \frac{d^2}{\omega^2 dx^2}\right)^{[m]} \varepsilon_{m,\omega}(x) = \left(1 - \frac{d^2}{\omega^2 dx^2}\right)^{[m-1]} \varepsilon_{m-(m-1),\omega}(x) = \left(1 - \frac{d^2}{\omega^2 dx^2}\right) \varepsilon_{1,\omega}(x). \tag{2.16}$$

In case $m = 1$ expression (2.13) takes the following form

$$\varepsilon_{1,\omega}(x) = \frac{\omega e^{-\omega|x|}}{2}. \tag{2.17}$$

Now, calculating the generalized derivatives of the function, $\varepsilon_{1,\omega}(x)$ we have

$$\varepsilon_{1,\omega}'(x) = -\frac{\omega^2}{2} e^{-\omega|x|} \operatorname{sign}(x), \qquad \varepsilon_{1,\omega}''(x) = \frac{\omega^3}{2} e^{-\omega|x|} \text{-} \omega^2 \delta(x). \tag{2.18}$$

Due to (2.17), (2.18), expression (2.16) takes the following form

$$\left(1 - \frac{d^2}{\omega^2 dx^2}\right)^{[m]} \varepsilon_{m,\omega}(x) = \left(1 - \frac{d^2}{\omega^2 dx^2}\right) \varepsilon_{1,\omega}(x) = \varepsilon_{1,\omega}(x) - \frac{\varepsilon_{1,\omega}''(x)}{\omega^2} = \frac{\omega e^{-\omega|x|}}{2} - \frac{\omega}{2} e^{-\omega|x|} + \delta(x) = \delta(x).$$

So, we have proved Lemma 2.1 completely. □

Next, using formulas (1.2), (2.12) and calculating the convolutions, we obtain

$$\overline{\psi}_{\ell,H}(x)=\ell_N(x)*\varepsilon_{m,\omega}(x)=\int\limits_{-\infty}^{\infty}\ell_N(y)\,\varepsilon_{m,\omega}(x-y)\,dy=\int\limits_{-\infty}^{\infty}\Big(\varepsilon_{[0,1]}(y)\,p(y)$$

$$-\sum_{\beta=0}^{N}C[\beta]\,\delta(y-h\beta)\Big)\varepsilon_{m,\omega}(x-y)\,dy=\int\limits_{0}^{1}p(y)\,\varepsilon_{m,\omega}(x-y)\,dy-\sum_{\beta=0}^{N}C[\beta]\,\varepsilon_{m,\omega}(x-h\beta).$$

From this, we have

$$\psi_{\ell,H}(x)=\int\limits_{0}^{1}\overline{p}(y)\,\varepsilon_{m,\omega}(x-y)\,dy-\sum_{\beta=0}^{N}\overline{C}[\beta]\,\varepsilon_{m,\omega}(x-h\beta).$$

Theorem 2.2 is completely proved. □

3. The square of the norm of the error functional for quadrature formulas

Now, using the expression of the extremal function, $\overline{\psi}_{\ell,H}(x)$ we calculate the norm of the error functional for the quadrature formula (1.1). For this, using formulas (2.1), (2.2) and (2.5), we obtain

$$\left\|\ell_N|H_2^{(m)*}\right\|^2=(\ell_N,\psi_{\ell,H})=\int\limits_{-\infty}^{\infty}\Big(\varepsilon_{[0,1]}(x)\,p(x)-\sum_{\gamma=0}^{N}C[\gamma]\,\delta(x-h\gamma)\Big)$$

$$\times\Big(\int\limits_{0}^{1}\overline{p}(y)\,\varepsilon_{m,\omega}(x-y)\,dy-\sum_{\beta=0}^{N}\overline{C}[\beta]\,\varepsilon_{m,\omega}(x-h\beta)\Big)\,dx=\int\limits_{0}^{1}\int\limits_{0}^{1}p(x)\overline{p}(y)\,\varepsilon_{m,\omega}(x-y)\,dxdy$$

$$-\sum_{\gamma=0}^{N}C[\gamma]\int\limits_{0}^{1}\overline{p}(y)\,\varepsilon_{m,\omega}(h\gamma-y)\,dy-\sum_{\beta=0}^{N}\overline{C}[\beta]\int\limits_{0}^{1}p(x)\,\varepsilon_{m,\omega}(x-h\beta)\,dx+\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}\overline{C}[\beta]\,C[\gamma]\varepsilon_{m,\omega}(h\beta-h\gamma)\,.$$

So, we have proven the following theorem.

Theorem 3.1. *The square of the norm of the error functional of quadrature formulas of the type* (1.1) *in the Sobolev space* $H_2^{(m)*}(0,1)$ *is expressed by the following formula*

$$\left\|\ell_N|H_2^{(m)*}\right\|^2=\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}\overline{C}[\beta]\,C[\gamma]\varepsilon_{m,\omega}(h\gamma-h\beta)-\sum_{\gamma=0}^{N}C[\gamma]\int\limits_{0}^{1}\overline{p}(y)\,\varepsilon_{m,\omega}(h\gamma-y)\,dy-$$
$$-\sum_{\beta=0}^{N}\overline{C}[\beta]\int\limits_{0}^{1}p(x)\,\varepsilon_{m,\omega}(x-h\beta)\,dx+\int\limits_{0}^{1}\int\limits_{0}^{1}p(x)\overline{p}(y)\,\varepsilon_{m,\omega}(x-y)\,dxdy. \tag{3.1}$$

$C[\gamma]$ are the coefficients of the quadrature formula and $\overline{C}[\beta]$ are the complex conjugate value to them, $\varepsilon_{m,\omega}(x)$ is determined by equality (2.3), $\overline{p}(y)$ is the complex conjugate function to the function $p(y)$.

4. System of equations for determining the optimal coefficients of quadrature formulas

We proceed to minimize the error of quadrature formulas with respect to the coefficients $C[\gamma]$ and $\overline{C}[\beta]$. To do this, by calculating the partial derivatives of the square of the norm of the error functional ℓ_N with respect to $C[\gamma]$ and $\overline{C}[\beta]$. Then we obtain

$$\frac{\partial\left\|\ell_N|H_2^{(m)*}\right\|^2}{\partial C[\gamma]}=\sum_{\beta=0}^{N}\overline{C}[\beta]\,\varepsilon_{m,\omega}(h\gamma-h\beta)-\int\limits_{0}^{1}\overline{p}(y)\,\varepsilon_{m,\omega}(h\gamma-y)\,dy, \tag{4.1}$$

$$\frac{\partial\left\|\ell_N|H_2^{(m)*}\right\|^2}{\partial\overline{C}\left[\beta\right]}=\sum_{\gamma=0}^{N}C\left[\gamma\right]\varepsilon_{m,\omega}\left(h\gamma-h\beta\right)-\int_0^1 p\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx. \quad (4.2)$$

Equating these partial derivatives to zero, we have

$$\sum_{\beta=0}^{N}\overline{C}\left[\beta\right]\varepsilon_{m,\omega}\left(h\gamma-h\beta\right)=\int_0^1\overline{p}\left(y\right)\varepsilon_{m,\omega}\left(h\gamma-y\right)dy,\ \gamma=0,1,...,N, \quad (4.3)$$

$$\sum_{\gamma=0}^{N}C\left[\gamma\right]\varepsilon_{m,\omega}\left(h\beta-h\gamma\right)=\int_0^1 p\left(x\right)\varepsilon_{m,\omega}\left(h\beta-x\right)dx,\ \beta=0,1,...,N. \quad (4.4)$$

From these systems it is clear that systems (4.3) and (4.4) are obtained from each other. Therefore, it is enough for us to solve one of them. In what follows, we will solve system (4.4). The solution of this system are the optimal coefficients for the quadrature formula (1.1), which we denoted by $\overset{\circ}{C}[\beta]$, $\beta=0,1,...,N$.

If we know the optimal coefficients of the quadrature formulas, then by virtue of (4.3) the optimal square of the norm of the error functional of the optimal quadrature formula is determined by the equality

$$\left\|\overset{\circ}{\ell}_N|H_2^{(m)*}\right\|^2=\int_0^1\int_0^1 p\left(x\right)\overline{p}\left(y\right)\varepsilon_{m,\omega}\left(x-y\right)dxdy-\sum_{\beta=0}^{N}\overline{\overset{\circ}{C}}\left[\beta\right]\int_0^1 p\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx. \quad (4.5)$$

If we use system (4.4) then we have

$$\left\|\overset{\circ}{\ell}_N|H_2^{(m)*}\right\|^2=\int_0^1\int_0^1 p\left(x\right)\overline{p}\left(y\right)\varepsilon_{m,\omega}\left(x-y\right)dxdy-\sum_{\beta=0}^{N}\overset{\circ}{C}\left[\beta\right]\int_0^1\overline{p}\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx. \quad (4.6)$$

It follows that

$$\sum_{\beta=0}^{N}\overline{\overset{\circ}{C}}\left[\beta\right]\int_0^1 p\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx=\sum_{\beta=0}^{N}\overset{\circ}{C}\left[\beta\right]\int_0^1\overline{p}\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx \quad (4.7)$$

or

$$\sum_{\beta=0}^{N}\int_0^1\left[\overline{\overset{\circ}{C}}\left[\beta\right]p\left(x\right)-\overset{\circ}{C}\left[\beta\right]\overline{p}\left(x\right)\right]\varepsilon_{m,\omega}\left(x-h\beta\right)dx=0.$$

Thus, we have proved the following

Theorem 4.1. *The square of the norm of the error functional of optimal quadrature formulas in the space $H_2^{(m)*}\left(0,1\right)$ is determined by the formula*

$$\left\|\overset{\circ}{\ell}_N|H_2^{(m)*}\right\|^2=\int_0^1\int_0^1 p\left(x\right)\overline{p}\left(y\right)\varepsilon_{m,\omega}\left(x-y\right)dxdy-\sum_{\beta=0}^{N}\overset{\circ}{C}\left[\beta\right]\int_0^1\overline{p}\left(x\right)\varepsilon_{m,\omega}\left(x-h\beta\right)dx.$$

From system (4.3) we obtain the following theorem, an analogue of the theorem of I. Babuška.

Theorem 4.2. *Let the error functional $\ell_N\left(x\right)$ be defined on the space $H_2^{(m)*}\left(0,1\right)$ and be optimal, i.e., among all functionals of the form*

$$\varepsilon_{[0,1]}\left(x\right)p\left(x\right)-\sum_{\beta=0}^{N}C\left[\beta\right]\delta\left(x-h\beta\right)$$

it has the smallest norm in the space $H_2^{(m)*}(0,1)$. Then there exists a solution to the equation

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{\omega^{2k}}\frac{d^{2k}}{dx^{2k}}\overline{\psi}_{\ell,H}(x)=\ell_N(x),$$

which vanishes at points $h\beta,\ \beta=0,1,...,N$ and belongs to $H_2^{(m)}(0,1)$.

References

[1] Nikolsky S.M., Quadrature formulas, Nauka, Moscow,—1988, 256 p.

[2] Sard A., Best approximate integration formulas, best approximate formulas, American Journal of Mathematics,—1949, LXXI, pp. 80–91.

[3] Sobolev S.L., Introduction to the theory of cubature formulas, Nauka, Moscow,—1974, 808 p.

[4] Sobolev S.L., Vaskevich V.L., Cubature formulas, Publishing House of the Institute of Mathematics, SB RAS, Novosibirsk,—1996, 484 p.

[5] Ramazanov M.D., Optimization of the arrangement of nodes of lattice cubature formulas, Reports of the Russian Academy of Sciences,—1995, Vol. 343, No. 4, pp. 456–457.

[6] Ramazanov M.D., Optimization of the norm of the error functional of lattice cubature formulas in Wiener spaces, Reports of the Russian Academy of Sciences,—1998, Vol. 361, No. 6, pp. 743–745.

[7] Ramazanov M.D., Theory of lattice cubature formulas with limited boundary layers, Ufa,—2007, 178 p.

[8] Ramazanov M.D., Shadimetov Kh.M., Weighted optimal cubature formulas in the periodic Sobolev space, Reports of the Russian Academy of Sciences,—1999, Vol. 358, No. 4, pp. 453–455.

[9] Shadimetov Kh.M., Optimal lattice quadrature and cubature formulas in Sobolev spaces, Fan va texnologiya, Tashkent,—2019, 224 p.

[10] Shadimetov Kh.M., Hayotov A.R. Optimal approximation of error functionals of quadrature and interpolation formulas in spaces of differentiable functions, Mukhr Press, Tashkent,—2022, 248 p.

[11] Shadimetov Kh.M., Akhmedov D.M. Numerical Integration Formulas for Hypersingular Integrals. Numer. Math. Theor. Meth. Appl. Vol. 17, No. 3, pp. 805-826, doi: 10.4208/nmtma.OA-2024-0028

[12] Shadimetov Kh.M., Akhmedov D.M. An optimal approximate solution of the I kind Fredholm singular integral equations. Filomat 38:30 (2024), 1076510796 https://doi.org/10.2298/FIL2430765S

[13] Azamov S.S., Tursunmuradov S.T., Minimization of the norm of the error functional of one quadrature formula, Uzbek Mathematical Journal,—2024, Vol. 68, No. 4, pp. 17–25.

[14] Akhmedov D.M., Abdikayumova G.A. Construction of optimal quadrature formulas with derivatives for Cauchy type singular integrals in the Sobolev space. Uzbek Mathematical Journal, 2020, Volume 64, Issue 1, pp.4-9, DOI: 10.29229/uzmj.2020-1-1

[15] Akhmedov D.M., Atamuradova B.M. Construction of optimal quadrature formulas for Cauchy type singular integrals in the $W_2^{(1,0)}(01)$ space. Uzbek Mathematical Journal, 2022, Volume 66, Issue 2, pp. 5-9, DOI:10.29229/uzmj.2022-2-1

[16] Akhmedov D.M. Optimal approximation of the Hadamard hypersingular integrals. Uzbek Mathematical Journal, 2023, Volume 67, Issue 3, pp. 5-12, DOI: 10.29229/uzmj.2023-3-1

[17] Azamov S.S. On the construction of optimal quadrature formulas with equally spaced nodes. Uzbek Mathematical Journa 2023, Volume 67, Issue 3, pp. 4148.

[18] Shadimetov Kh.M., Azamov S.S. Discrete Analogues of High Order Differential Operators. Journal of Mathematical Sciences United States, 2024, 284(2), pp. 253265.

Shadimetov Kh. M.
Tashkent state transport university,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
email: kholmatshadimetov@mail.ru
Tursunmurodov S.T.
Tashkent state transport university

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.201-216
DOI: 10.29229/uzmj.2025-4-20

Differential game with a "life-line" for nonlinear motion dynamics of players

Turgunboeva M.A., Soyibboev U.B.

Abstract. We investigate the interception problem in a differential game with non-inertial players (a pursuer and an evader) who move in dynamic flow fields with various influences. Throughout the paper, we solve the pursuit and "life-line" game problems. To solve the pursuit, the strategy of parallel pursuit (**Π**-strategy for short) is defined and used. With the help of the **Π**-strategy and applying the Grönwall-Bellman inequality, sufficient pursuit condition is determined. In order to solve the "life-line" game to the advantage of the pursuer, we build the set of meeting points of the players and prove that this set monotonically decreases with regard to inclusion relative to time. The "life-line" game to the advantage of the evader is solved by constructing evader's attainability domain where it reaches without being caught for an arbitrary control of the pursuer.

Keywords: Differential game, Caratheodory's conditions, Lipschitz's condition, players, geometric constraint, pursuit, "life-line", **Π**-strategy, Grönwall-Bellman inequality
MSC (2020): 49N70, 49N75, 91A23, 91A24

1. Introduction

Differential games are a special kind of problems for dynamic systems particularly moving objects. In 1965, this theory was studied systematically by R. Isaacs and published in the form of monograph [21], in which numerous examples were examined and theoretical questions were only touched upon. The foundation of the modern theory of the differential game was settled by mathematicians R. Isaacs [21], L.D. Bercovitz [8], R.J. Elliot, N.J. Kalton [14], A. Friedman [15], O. Hajek [17], Y. Ho, A. Bryson, S. Baron [18], L.S. Pontryagin [27], N.N. Krasovskii [22], L.A. Petrosyan [26], B.N. Pshenichnii [28], A.A. Chikrii [12].

In the differential games theory, in accordance with the basic approaches proposed by L.S. Pontryagin [27] and N.N. Krasovskii [22], a differential game is explored as a control problem from the viewpoint of either the chasing player (pursuer) or the escaping player (evader). Under this framework, the game can be stated as either a pursuit or an evasion problem. Pursuit-evasion differential game have been extensively studied in the literature [8, 12, 16, 19, 20, 32] with significant contributions addressing theoretical foundations, optimal strategies, and real-world applications.

The book [21] by R. Isaacs covers several game problems that were explored thoroughly and proposed for further investigation. One of these is named "life-line" problem that was initially stated and examined for specific cases in [21] (Problem 9.5.1). When control functions of both players meet geometric restrictions, the stated game has been rather comprehensively considered in [26] by L.A. Petrosyan. The Π-strategy, which was introduced in [26, 28] for a simple pursuit game with geometric restrictions, functioned as the starting point for the development of the effective method in pursuit games with multiple pursuers (see [3, 4, 11, 12], [29]–[30]).

There are numerous studies on nonlinear differential games that have found key conditions for successful capture and the optimality of capture time. For example, in work [33], a differential game of the stationary nonlinear system was studied, and the optimality of capture time was analyzed for a specific case on a plane, where the pursuer applied a counter-strategy. Similarly, A. Azamov, in [2], considered the pursuit differential game, where the dynamics were governed by a nonlinear system of differential equations of a specific form, through a positional counter-strategy on a plane, and also presented clear examples that illustrate the explicit characteristics of the game. In [24], a two-player nonlinear differential game with an integral quality criterion was investigated at the time interval divided into two segments. Necessary and sufficient conditions were obtained for the existence of a saddle point for a general two-person zero-sum differential game when one or both players use suboptimal control laws of specified form (referred to as piecewise control laws). Additionally, the work of K.A. Shchelchkov [34] was concerned with the problems of stabilization to zero under disturbance in terms of a differential pursuit game described by a nonlinear autonomous system of differential equations. The sufficient conditions for the existence of a neighborhood of zero from each point of which a capture occurs in the indicated sense were derived.

Some optimal control problem formulations have taken into account the effect of an external flow field. For example, in [23], the authors considered the problem of optimal guidance of a Dubin's vehicle [13] to a specified position under the influence of an external flow. The minimum-time guidance problem for an isotropic rocket in the presence of wind has been studied in [7]. The problem of minimizing the expected time to steer a Dubin's vehicle to a target set in a stochastic wind field has also been discussed in [1]. However, the same level of attention in the literature has not been devoted to pursuit-evasion or "life-line" games with two (or more) competing agents under the influence of external disturbances (e.g., winds or currents). In papers [35]–[36], a multi-pursuer and one-evader for the pursuit-evasion game in an external dynamic flow field is considered. Due to the generality of the external flow, Isaacs's approach is not readily applicable [21]. Instead, in [36], a different approach is used and, the optimal trajectories of the players through a reachable set method are found.

This work studies the differential game with a "life-line" when players (a pursuer and an evader) move in dynamic flow fields with various influences. Throughout the paper, we solve the pursuit problem and the game with a "life-line". The obtained results are based on Krasovskii-Pontryagin's formalization ([22, 27]), Pshenichnii-Chikrii's method of resolving functions ([12, 28]), the **Π**-strategy ([3]–[6], [11, 26], [29]–[31]) and the properties of the multi-valued mapping [10].

2. Statement of problems

Let two controllable players P and E be given in Euclidean space $\mathbb{R}^n$. The first player P called a Pursuer chases the second player E called an Evader. Suppose, x signifies the position of the Pursuer, and y signifies that of the Evader in $\mathbb{R}^n$. Then the players perform their motions in accordance with the equations

$$P: \quad \dot{x} = u + F_P(t, x), \qquad x(0) = x_0, \tag{2.1}$$

$$E: \quad \dot{y} = v + F_E(t, y), \qquad y(0) = y_0, \tag{2.2}$$

appropriately, where $x, y, u, v \in \mathbb{R}^n$, $n \geq 2$, $t \in \mathbb{R}_+ := [0, +\infty)$; $F_P : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ ($F_E : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$) is an effective flow field for the Pursuer (for the Evader); x_0, y_0 are the players' initial positions. It is considered that $x_0 \neq y_0$.

In (2.1), the parameter u denotes as the Pursuer's control, and it is hereafter selected as a measurable function $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ complying with

$$|u(t)| \leq \alpha \text{ for almost all } t \geq 0, \tag{2.3}$$

where α is a positive constant.

Likewise, in (2.2), the parameter v denotes as the Evader's control, and it is henceforth chosen as a measurable function $v(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ complying with

$$|v(t)| \leq \beta \text{ for almost all } t \geq 0, \tag{2.4}$$

where β is a non-negative constant.

In the Theory of Differential Games, inequalities (2.3) and (2.4) are generally called *geometric constraints* (briefly, **G**-constraints) for the control functions.

Henceforward, the considered game (2.1)–(2.4) is referred as the *nonlinear differential game (2.1)–(2.4)* or briefly, *NDG (2.1)–(2.4)*.

Definition 2.1. A measurable function $u : \mathbb{R}_+ \to \mathbb{R}^n$, $|u(t)| \leq \alpha$, $t \geq 0$, (respectively, $v : \mathbb{R}_+ \to \mathbb{R}^n$, $|v(t)| \leq \beta$, $t \geq 0$) is called an *admissible control* of the Pursuer (respectively, Evader).

Let $U_{\mathbf{G}}$ (respectively, $V_{\mathbf{G}}$) denote the set of all admissible controls of the Pursuer (respectively, Evader).

Assumption 2.2. (Caratheodory's conditions) Let the functions $F_P(t,x)$ and $F_E(t,y)$ be defined on the domain $D := \mathbb{R}_+ \times \mathbb{R}^n$ and let they satisfy the conditions given below: 1) $F_P(t,x)$ and $F_E(t,y)$ are continuous in x and y for each fixed t; 2) $F_P(t,x)$ and $F_E(t,y)$ are measurable functions in t for each fixed x and y; 3) for each compact subset Q of D, there can be found Lebesgue-integrable functions $h_P(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ and $h_E(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, where $\sup_{t\geq 0} h_P(t) = h_P^*$, $0 \leq h_P^* < \infty$, $\sup_{t\geq 0} h_E(t) = h_E^*$, $0 \leq h_E^* < \infty$, such that $|F_P(t,x)| \leq h_P(t)$ and $|F_E(t,y)| \leq h_E(t)$ for all $(t,x),(t,y) \in Q$.

In equations (2.1), (2.2), the functions $F_P(t,x)$ and $F_E(t,y)$ represent the exogenous dynamic flows, but they may also represent the endogenous drift owing to the nonlinear dynamics of the players [35]–[36]. It is reasonable to suppose that the magnitude of these flows (e.g. winds or currents) is bounded from above by some functions $h_P(t)$ and $h_E(t)$ in the third condition of Assumption 2.2. B.T. Samatov et al. [31] considered the intercept problem when objects move in the same type external dynamic flow field. Unlike this work, in our study, the players are assumed to move within different influence zones, and the pursuit problem is solved for the "life-line" game also. In other words, our work can be regarded as a logical continuation of the study [31].

Assumption 2.3. (Lipshitz's conditions) For each compact subsets Q_P and Q_E of D, there exist Lebesgue-integrable functions $k_P(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ and $k_E(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, where $\sup_{t\geq 0} k_P(t) = k_P^*$, $0 \leq k_P^* < \infty$, $\sup_{t\geq 0} k_E(t) = k_E^*$, $0 \leq k_E^* < \infty$, such that

$$|F_P(t,x_1) - F_P(t,x_2)| \leq k_P(t)|x_1 - x_2|,$$

$$|F_E(t,y_1) - F_E(t,y_2)| \leq k_E(t)|y_1 - y_2|$$

for all $(t,x_1),(t,x_2) \in Q_P$ and $(t,y_1),(t,y_2) \in Q_E$.

Proposition 2.4. *If Assumptions 2.2 and 2.3 are valid, then*

$$|F_P(t,x) - F_E(t,y)| \leq p(t)|x-y| + q(t) \tag{2.5}$$

is true for any $x, y \in \mathbb{R}^n$, *where*

$$p(t) = k_P(t) + k_E(t), \quad q(t) = h_P(t) + h_E(t),$$

$$\sup_{t\geq 0} p(t) = p = k_P^* + k_E^* < \infty, \quad \sup_{t\geq 0} q(t) = q = h_P^* + h_E^* < \infty. \tag{2.6}$$

Proof. Indeed, by using inequalities in Assumptions 2.2 and 2.3, the left side of (2.5) can be estimated as follows:

$$|F_P(t,x) - F_E(t,y)| = |F_P(t,x) - F_P(t,y) + F_P(t,y) - F_E(t,x) + F_E(t,x) - F_E(t,y)|$$

$$\leq |F_P(t,x) - F_P(t,y)| + |F_P(t,y)| + |F_E(t,x)| + |F_E(t,x) - F_E(t,y)|$$

$$\leq k_P(t)|x-y| + h_P(t) + h_E(t) + k_E(t)|x-y| = (k_P(t)+k_E(t))|x-y| + h_P(t) + h_E(t) = p(t)|x-y| + q(t),$$

which is the desired result.

□

In NDG (2.1)–(2.4), the objective of Pursuer P is to catch Evader E (a pursuit game) at some moment T_*, $0 < T_* < +\infty$, i.e. to reach the equality

$$x(T_*) = y(T_*),$$

where $x(t)$ and $y(t)$ are trajectories generated during the game. The notion of "trajectories generated during the game" requires clarification. Evader E tries to avoid the meeting with Pursuer P (an evasion game), i.e. to guarantee the relation $x(t) \neq y(t)$ for all $t \geq 0$, and if it is impossible, to prolong the moment of the meeting as far as possible. Naturally, this is the preliminary problems setting.

Definition 2.5. If $u(\cdot) \in U_{\mathbf{G}}$ and $v(\cdot) \in V_{\mathbf{G}}$, then Caratheodory's differential equations

$$\dot{x} = u(t) + F_P\big(t, x\big), \quad x(0) = x_0,$$

$$\dot{y} = v(t) + F_E\big(t, y\big), \quad y(0) = y_0$$

give rise to the unique trajectories $x(t) = x\big(t; x_0, u(\cdot)\big)$ and $y(t) = y\big(t; y_0, v(\cdot)\big)$ correspondingly. In the given case, $x(t)$ is called the Pursuer's trajectory, and $y(t)$ is called the Evader's trajectory.

Definition 2.6. A control function $\boldsymbol{u}(t, x, y, v) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be a strategy of Pursuer P if: 1) $\boldsymbol{u}(t, x, y, v)$ is continuous with regard to x, y, v for each fixed t; 2) $\boldsymbol{u}(t, x, y, v)$ is Lebesgue measurable with regard to t for each fixed (x, y, v) and is Borel measurable with regard to v for each fixed (t, x, y); 3) $\boldsymbol{u}\big(t, x(\cdot), y(\cdot), v(\cdot)\big) \in U_{\mathbf{G}}$ for all $v(\cdot) \in V_{\mathbf{G}}$; 4) there exists such a Lebesgue-integrable function $w(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ that

$$\big|\boldsymbol{u}\big(t, x_1, y, v(t)\big) - \boldsymbol{u}\big(t, x_2, y, v(t)\big)\big| \le w(t)|x_1 - x_2|$$

for any $\big(t, x_1, y, v(t)\big)$, $\big(t, x_2, y, v(t)\big)$ with $t \ge 0$, $x_1, x_2, y \in \mathbb{R}^n$, $v(\cdot) \in V_{\mathbf{G}}$.

Write

$$z(t) = x(t) - y(t), \ z(0) = z_0, \ z_0 = x_0 - y_0. \tag{2.7}$$

Definition 2.7. A strategy $\boldsymbol{u}(t, x, y, v)$ is said to be a parallel pursuit strategy or briefly, **Π**-strategy if for all $v(\cdot) \in V_{\mathbf{G}}$, the function $z(t)$ is representable as

$$z(t) = \mathcal{A}\big(t, x(t), y(t), v(\cdot)\big) z_0, \tag{2.8}$$

where $\big(x(t), y(t)\big)$ is the solution of the system of differential equations

$$\begin{cases} \dot{x} = \boldsymbol{u}\big(t, x, y, v(t)\big) + F_P\big(t, x\big), & x(0) = x_0, \\ \dot{y} = v(t) + F_E\big(t, y\big), & y(0) = y_0, \end{cases} \tag{2.9}$$

and $\mathcal{A}\big(t, x(t), y(t), v(\cdot)\big)$ is a scalar monotonically decreasing continuous function with respect to t, $t \ge 0$, and it is generally called an approach function of the players P and E in the pursuit game.

Remark 2.8. It is necessary to state that similar to the definition of **Π**-strategy for the case of simple motions of the players given in works [3], [4], [26], [28]–[31], the following properties are met: a) the vector $z(t)$ in (2.8) joining the positions of the players changes its position in the parallel way to itself during the pursuit; b) depending on the property of the approach function $\mathcal{A}\big(t, x(t), y(t), v(\cdot)\big)$ in Definition 2.7, the distance between the players $|z(t)| = |x(t) - y(t)|$ strictly decreases.

Definition 2.9. We say that **Π**-strategy guarantees that Pursuer P wins on the time interval $\big[0, T_{\mathbf{G}}\big]$ in NDG (2.1)–(2.4) if for all $v(\cdot) \in V_{\mathbf{G}}$: a) there exists a time moment $T_* \in \big[0, T_{\mathbf{G}}\big]$ at which the vector function $z(t)$, which is defined by the solutions $x(t)$ and $y(t)$ of system (2.9), meets $z(T_*) = 0$; b) $\boldsymbol{u}\big(t, x(\cdot), y(\cdot), v(\cdot)\big) \in U_{\mathbf{G}}$ on $[0, T_*]$. In the given case, the number $T_{\mathbf{G}}$ is called a guaranteed time of the pursuit.

3. The obtained results

The given section is devoted to give solutions of the pursuit and "life-line" game problems for NDG (2.1)–(2.4). First of all, in the pursuit game, a **Π**-strategy is set up for Pursuer P, and a sufficient pursuit condition is demonstrated. By this strategy, an explicit formula for a set of all the meeting points of the players is generated. Then in the "life-line" game, a reachability domain of Evader E is constructed.

3.1. **The pursuit game solution.** Write the following functions:

$$\omega = \omega(t, x, y, v) := v + F_E(t, y) - F_P(t, x), \tag{3.1}$$

$$r(\omega) = \langle \omega, \hat{z}_0 \rangle + \sqrt{\langle \omega, \hat{z}_0 \rangle^2 + \alpha^2 - |\omega|^2}, \tag{3.2}$$

where $\hat{z}_0 = \frac{z_0}{|z_0|}$, and $\langle \omega, \hat{z}_0 \rangle$ means the scalar product of the vectors ω and $\hat{z}_0$ in $\mathbb{R}^n$.

Definition 3.1. For $\alpha > |\omega|$, the control function

$$\boldsymbol{u}(\omega) := \omega - r(\omega)\hat{z}_0 \tag{3.3}$$

is called the **Π**-strategy of Pursuer P in the pursuit game.

It should be mentioned that the function $r(\omega)$ is mainly called a *resolving function* [12].

Lemma 3.2. *If $\alpha > |\omega|$, then the function $r(\omega)$ is continuous and non-negative in ω, $\omega \in \mathbb{R}^n$, and it is bounded as*

$$\alpha - |\omega| \le r(\omega) \le \alpha + |\omega|. \tag{3.4}$$

Proof. As $\alpha > |\omega|$, the function $r(\omega)$ is monotonically increasing with regard to $\langle \omega, \hat{z}_0 \rangle$ (see (3.2)). Thus, applying $-|\omega| \le \langle \omega, \hat{z}_0 \rangle \le |\omega|$ to $r(\omega)$ yields (3.4), which ends the proof. □

Lemma 3.3. *If $\alpha > |\omega|$, then the function $\boldsymbol{u}(\omega)$ is continuous in ω, $\omega \in \mathbb{R}^n$, and it satisfies*

$$|\boldsymbol{u}(\omega)| = \alpha.$$

Proof. Squaring both sides of (3.3) gives

$$|\boldsymbol{u}(\omega)|^2 = |\omega|^2 + r(\omega)\big[r(\omega) - 2\langle \omega, \hat{z}_0 \rangle\big],$$

and replacing (3.2) into the last expression leads to $|\boldsymbol{u}(\omega)|^2 = \alpha^2$, which is our claim. □

Lemma 3.4. *If $\alpha > |\omega|$, then there is a Lebesgue-integrable function $g(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies*

$$\big|\boldsymbol{u}(\omega_1) - \boldsymbol{u}(\omega_2)\big| \le g(t)|x_1 - x_2|$$

for any $\omega_1, \omega_2 \in \mathbb{R}^n$, where

$$\omega_1 = \omega(t, x_1, y, v) := v + F_E(t, y) - F_P(t, x_1), \quad \omega_2 = \omega(t, x_2, y, v) := v + F_E(t, y) - F_P(t, x_2).$$

Proof. Write $c = \langle \omega, \hat{z}_0 \rangle$, $b = \alpha^2 - |\omega|^2$ and introduce a function $\psi(c) = c + \sqrt{c^2 + b}$ from (3.2). Since $\frac{d\psi(c)}{dc} = 1 + \frac{1}{\sqrt{c^2+b}}$, we can assert that the function $\psi(c)$ is continuous on $\big[c_1, c_2\big]$ and is differentiable at each point of the interval $\big(c_1, c_2\big)$. Thus, on the basis of the Lagrange theorem, there is such a point $c^* \in \big(c_1, c_2\big)$ that

$$\begin{aligned}\psi(c_2) - \psi(c_1) &= \psi\big(c^*\big)\big(c_2 - c_1\big) = \psi\big(c^*\big)\big(\langle \omega_2, \hat{z}_0 \rangle - \langle \omega_1, \hat{z}_0 \rangle\big)\\ &= \psi\big(c^*\big)\big\langle \omega_2 - \omega_1, \hat{z}_0 \big\rangle \le \psi\big(c^*\big)\big|\omega_1 - \omega_2\big|.\end{aligned} \tag{3.5}$$

Now, with the help of (3.1), (3.3), (3.5) and by Assumptions 2.3, the function $\boldsymbol{u}(\omega)$ can be estimated for any $\omega_1, \omega_2 \in \mathbb{R}^n$ as follows:

$$\big|\boldsymbol{u}(\omega_1) - \boldsymbol{u}(\omega_2)\big| = \big|v + F_E(t, y) - F_P\big(t, x_1\big) - v - F_E(t, y) + F_P\big(t, x_2\big) - r\big(\omega_1\big)\hat{z}_0 + r\big(\omega_2\big)\hat{z}_0\big|$$

$$\le \big|F_P\big(t, x_1\big) - F_P\big(t, x_2\big)\big| + \big|r\big(\omega_1\big) - r\big(\omega_2\big)\big| = \big|F_P\big(t, x_1\big) - F_P\big(t, x_2\big)\big| + \big|\psi(c_1) - \psi(c_2)\big|$$

$$\le \big|F_P\big(t, x_1\big) - F_P\big(t, x_2\big)\big| + \psi\big(c^*\big)\big|\omega_1 - \omega_2\big| = \big(\psi\big(c^*\big) + 1\big)\big|F_P\big(t, x_1\big) - F_P\big(t, x_2\big)\big| \le g(t)|x_1 - x_2|,$$

where $g(t) = \big(\psi\big(c^*\big) + 1\big)k_P(t)$. This completes the proof. □

Lemma 3.5. *(The Grönwall-Bellman inequality* [25, pp. 13, Theorem 1.3.2]*) Let $\eta(t)$ be a real valued continuous function, and let $\phi(t)$ be a non-negative integrable function in respect to t, $t \geq 0$. If the integral inequality*

$$|z(t)| \leq \eta(t) + \int_0^t \phi(s)|z(s)|ds$$

is valid, then

$$|z(t)| \leq \eta(t) + \int_0^t \phi(s)\eta(s)\exp\left(\int_s^t \phi(\tau)d\tau\right)ds$$

holds.

Theorem 3.6. *Let $\alpha > \beta + q + p|z_0|$. Then $\mathbf{\Pi}$-strategy (3.3) guarantees that Pursuer P wins on the time interval $\left[0, T_{\mathbf{G}}\right]$ in the pursuit game, where*

$$T_{\mathbf{G}} = \begin{cases} \frac{1}{p}\ln\frac{\alpha-\beta-q}{\alpha-\beta-q-p|z_0|}, & \textit{if} \;\; p > 0, \\ \frac{|z_0|}{\alpha-\beta-q}, & \textit{if} \;\; p = 0. \end{cases}$$

Proof. Suppose, Evader E makes use of an arbitrary control function $v(\cdot) \in V_{\mathbf{G}}$ and Pursuer P realizes $\mathbf{\Pi}$-strategy (3.3). Then by means of (2.9), (3.1), (3.3), we derive the system of Caratheodory's differential equations

$$\begin{cases} \dot{x} = v(t) + F_E\big(t, y(t)\big) - r\big(\omega\big(t, x(t), y(t), v(t)\big)\big)\hat{z}_0, & x(0) = x_0, \\ \dot{y} = v(t) + F_E\big(t, y(t)\big), & y(0) = y_0, \end{cases} \tag{3.6}$$

where the equations in (3.6) pose the unique trajectories $x(t) := x\big(t; x_0, \boldsymbol{u}(\cdot)\big)$ and $y(t) := y\big(t; y_0, v(\cdot)\big)$ of the players P and E, respectively. On the basis of (2.7), it proceeds from (3.6) that

$$\dot{z} = -r\big(\omega\big(t, x(t), y(t), v(t)\big)\big)\hat{z}_0, \;\; z(0) = z_0. \tag{3.7}$$

Integrating equation (3.7), we attain the solution

$$z(t) = \mathcal{A}\big(t, x(t), y(t), v(\cdot)\big)z_0, \tag{3.8}$$

where

$$\mathcal{A}\big(t, x(t), y(t), v(\cdot)\big) = 1 - \frac{1}{|z_0|}\int_0^t r\big(\omega\big(s, x(s), y(s), v(s)\big)\big)ds. \tag{3.9}$$

By reason of (2.3), (2.4), (3.1), (3.4) and by Proposition 2.4, the function (3.9) is maximized as follows:

$$\mathcal{A}\big(t, x(t), y(t), v(\cdot)\big) = 1 - \frac{1}{|z_0|}\int_0^t r\big(\omega\big(s, x(s), y(s), v(s)\big)\big)ds \leq 1 - \frac{1}{|z_0|}\int_0^t \big(\alpha - |\omega(s)|\big)ds$$

$$= 1 - \frac{1}{|z_0|}\left(\alpha t - \int_0^t \big|v(s) + F_E\big(s, y(s)\big) - F_P\big(s, x(s)\big)\big|ds\right)$$

$$\leq 1 - \frac{1}{|z_0|}\left((\alpha - \beta)t - \int_0^t \big|F_P\big(s, x(s)\big) - F_E\big(s, y(s)\big)\big|ds\right)$$

$$\leq 1 - \frac{1}{|z_0|}\left((\alpha - \beta)t - \int_0^t \big[p(s)|x(s) - y(s)| + q(s)\big]ds\right)$$

$$\leq 1 - \frac{1}{|z_0|}\left((\alpha-\beta-q)t - \int_0^t p|z(s)|ds\right),$$

or to sum up,

$$\mathcal{A}\big(t, x(t), y(t), v(\cdot)\big) \leq 1 - \frac{(\alpha-\beta-q)t}{|z_0|} + \int_0^t \frac{p}{|z_0|}|z(s)|ds. \tag{3.10}$$

Combining (3.8) and (3.10) we obtain

$$|z(t)| \leq |z_0| - (\alpha-\beta-q)t + \int_0^t p|z(s)|ds. \tag{3.11}$$

In the right side of (3.11), taking as $\eta(t) = |z_0| - \big(\alpha-\beta-q\big)t$, $\phi(t) = p$ and applying Lemma 3.5 to (3.11) give rise to

$$|z(t)| \leq \mathcal{K}(t), \tag{3.12}$$

where $\mathcal{K}(t) = |z_0| - (\alpha-\beta-q-p|z_0|)\left(e^{pt}-1\right)/p$ if $p > 0$, $\mathcal{K}(t) = |z_0| - (\alpha-\beta-q)t$ if $p = 0$. Since $\alpha > \beta + q + p|z_0|$, substituting the value of $T_{\mathbf{G}}$ (see the theorem) into $\mathcal{K}(t)$ yields $\mathcal{K}\left(T_{\mathbf{G}}\right) = 0$. For this reason and because of (3.12), there is a time value $T_* \in [0, T_{\mathbf{G}}]$ satisfying $z(T_*) = 0$, which is the desired conclusion.

Now let's confirm the admissibility of $\mathbf{\Pi}$-strategy (3.3) for all $t \in [0, T_*]$. To this end, we have to show $\alpha > |\omega(t)|$ on the interval $[0, T_*]$ as specified by Definition 3.1. By means of Proposition 2.4 and by (3.1), we obtain the following estimations from the condition of the theorem:

$$\alpha > \beta + q + p|z_0| \geq |v(t)| + q(t) + p(t)|z(t)|$$

$$\geq |v(t)| + \big|F_E\big(t, y(t)\big) - F_P\big(t, x(t)\big)\big| \geq \big|v(t) + F_E\big(t, y(t)\big) - F_P\big(t, x(t)\big)\big| = |\omega(t)|.$$

This ends the proof of the theorem. □

Remark 3.7. If the players P and E move in the same compact subset of $\mathbb{R}^n$, then in (2.5), it is supposed that $p(t) = \min\{k_P(t), k_E(t)\}$ and $q(t) = h_P(t) + h_E(t)$, where $k_P(t)$, $k_E(t)$, $h_P(t)$, $h_E(t)$ are given functions in Assumptions 2.2 and 2.3.

3.2. **The set of meeting points of the players.** In the theory of differential games, after solving the pursuit game problem, it is highly significant that the set of all points, which the players P and E meet, is explicitly constructed.

Let $\mathcal{D}(x, y)$ designate a domain consisting of such all points d that Pursuer P starting its motion from the position x is able to first get through to the point d before Evader E starting its motion from the position y, i.e.:

$$\mathcal{D}(x, y) = \big\{d \mid \beta|d-x| \geq \alpha|d-y|\big\}. \tag{3.13}$$

For $\alpha \neq \beta$, the boundary of the domain in (3.13) is represented as

$$\partial\mathcal{D}(x, y) = \big\{d \mid \beta|d-x| = \alpha|d-y|\big\},$$

which is usually said as the *sphere of Apollonius*.

If Theorem 3.6 is satisfied, then, with the help of $\mathbf{\Pi}$-strategy (3.3), Pursuer P can catch Evader E on some point in $\mathbb{R}^n$. For NDG (2.1)–(2.4), we are going to define a meeting domain of the players.

As known, the pair $\big(y_0, v(\cdot)\big)$, $v(\cdot) \in V_{\mathbf{G}}$, produces the Evader's motion trajectory $y(t) := y\big(t; y_0, v(\cdot)\big)$, and the pair $\big(x_0, \boldsymbol{u}(\cdot)\big)$, $\boldsymbol{u}(\cdot) \in U_{\mathbf{G}}$, creates the Pursuer's motion trajectory $x(t) := x\big(t; x_0, \boldsymbol{u}(\cdot)\big)$ for every $t \in \big[0, T_*\big]$, $0 < T_* \leq T_{\mathbf{G}}$, where T_* is the players' meeting time, viz,

$x(T_*) = y(T_*)$ holds at this time. Accordingly, for $(x(t), y(t))$ at each $t \in [0, T_*]$, it makes sense to write the multi-valued mapping

$$\mathcal{D}\big(x(t), y(t)\big) = \big\{d \mid \beta|d - x(t)| \geq \alpha|d - y(t)|\big\} \tag{3.14}$$

on the interval $[0, T_*]$. Mention that

$$\mathcal{D}(x_0, y_0) = \big\{d \mid \beta|d - x_0| \geq \alpha|d - y_0|\big\}.$$

It is apparent that $y(t) \in \mathcal{D}\big(x(t), y(t)\big)$ is accurate on account of $|z(t)| \geq 0$ on the interval $[0, T_*]$.

Proposition 3.8. *It is true to write the multi-valued mapping (3.14) as*

$$\mathcal{D}\big(x(t), y(t)\big) = x(t) + \mathcal{A}\big(t, x(t), y(t), v(\cdot)\big)\big[\mathcal{D}(x_0, y_0) - x_0\big], \tag{3.15}$$

where $\mathcal{A}\big(t, x(t), y(t), v(\cdot)\big)$ is the approach function of the players in (3.9), and

$$\mathcal{D}(x_0, y_0) = x_0 - \mathcal{C}(z_0) + \mathcal{R}(z_0)\mathcal{B},\ \mathcal{C}(z_0) = \left(\frac{\alpha^2}{\alpha^2 - \beta^2}\right) z_0,\ \mathcal{R}(z_0) = \frac{\alpha\beta|z_0|}{\alpha^2 - \beta^2}, \tag{3.16}$$

$\mathcal{B} = \big\{b \in \mathbb{R}^n \mid |b| \leq 1\big\}$.

Set

$$\mathcal{H}\big(t, x(t), y(t)\big) = \frac{\alpha^2}{\alpha^2 - \beta^2} \int_0^t \big(F_P\big(s, x(s)\big) - F_E\big(s, y(s)\big)\big) ds$$

$$- \left(\frac{\alpha\beta}{\alpha^2 - \beta^2} \int_0^t \big|F_P\big(s, x(s)\big) - F_E\big(s, y(s)\big)\big| ds\right) \mathcal{B} - \int_0^t F_P\big(s, x(s)\big) ds. \tag{3.17}$$

Then, write the multi-valued mapping

$$\mathcal{D}^*\big(t, x(t), y(t)\big) = \mathcal{D}\big(x(t), y(t)\big) + \mathcal{H}\big(t, x(t), y(t)\big) \tag{3.18}$$

Theorem 3.9. *$\mathcal{D}^*\big(t_2, x(t_2), y(t_2)\big) \subset \mathcal{D}^*\big(t_1, x(t_1), y(t_1)\big)$ for $t_1 < t_2$ from any $t_1,\ t_2 \in [0, T_*]$.*

Proof. First off, let us introduce the following notation for convenience in calculations:

$$\xi(t) = \xi(t, x(t), y(t)) := F_P(t, x(t)) - F_E(t, y(t)). \tag{3.19}$$

Inequality (2.4) can be immediately transformed into the form

$$|v(t)|^2 \leq \frac{\beta^2}{\alpha^2 - \beta^2} \left(\alpha^2 - |v(t)|^2\right). \tag{3.20}$$

Then considering (3.1) and (3.19), inequality (3.20) may be rewritten as

$$|\omega(t) + \xi(t)|^2 \leq \frac{\beta^2}{\alpha^2 - \beta^2} \left(\alpha^2 - |\omega(t) + \xi(t)|^2\right),$$

or from here it is derived that

$$|\omega(t)|^2 + 2\langle\omega(t), \xi(t)\rangle + |\xi(t)|^2 \leq \frac{\beta^2}{\alpha^2 - \beta^2} \left(\alpha^2 - |\omega(t)|^2 - 2\langle\omega(t), \xi(t)\rangle - |\xi(t)|^2\right). \tag{3.21}$$

In accordance with (3.2), it can be readily verified that the following equality holds:

$$\alpha^2 - |\omega(t)|^2 = r(\omega(t)) \left(r(\omega(t)) - 2\langle\omega(t), \hat{z}_0\rangle\right) \tag{3.22}$$

Replacing the right-side term of (3.22) into (3.21) leads to the inequality

$$|\omega(t)|^2 + 2\left\langle \omega(t),\ \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 \right\rangle + \frac{\alpha^2}{\alpha^2-\beta^2}|\xi(t)|^2$$

$$\leq \frac{\beta^2}{\alpha^2-\beta^2} r^2(\omega(t)) - 2\left\langle \omega(t),\ \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t) \right\rangle,$$

or

$$|\omega(t)|^2 + 2\left\langle \omega(t),\ \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t) \right\rangle + \frac{\alpha^2}{\alpha^2-\beta^2}|\xi(t)|^2 \leq \frac{\beta^2}{\alpha^2-\beta^2} r^2(\omega(t)). \tag{3.23}$$

We convert both sides of (3.23) into quadratic forms:

$$|\omega(t)|^2 + 2\left\langle \omega(t),\ \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t) \right\rangle + \left| \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t) \right|^2$$

$$\leq \frac{\beta^2}{\alpha^2-\beta^2} r^2(t,\omega(t)) - \frac{\alpha^2}{\alpha^2-\beta^2}|\xi(t)|^2 + \frac{\alpha^4}{(\alpha^2-\beta^2)^2}|\xi(t)|^2 + \frac{\beta^4}{(\alpha^2-\beta^2)^2} r^2(\omega(t))$$

$$+2\left(\frac{\alpha\beta}{\alpha^2-\beta^2}\right)^2 \langle r(\omega(t))\hat{z}_0, \xi(t)\rangle.$$

or we get

$$\left|\omega(t) + \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t)\right|^2$$

$$\leq \left(\frac{\alpha\beta}{\alpha^2-\beta^2}\right)^2 \left(r^2(\omega(t)) + 2\langle r(\omega(t))\hat{z}_0, \xi(t)\rangle + |\xi(t)|^2\right),$$

that is,

$$\left|\omega(t) + \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t)\right| \leq \frac{\alpha\beta}{\alpha^2-\beta^2}|r(\omega(t))\hat{z}_0 + \xi(t)|. \tag{3.24}$$

It is evident that for any vector $\psi \in \mathbb{R}^n$, with $|\psi| = 1$, the following relation is true:

$$\left\langle \omega(t) + \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t), \psi \right\rangle \leq \left|\omega(t) + \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t)\right|$$

Applying the latter inequality to (3.24), we obtain

$$\left\langle \omega(t) + \frac{\beta^2}{\alpha^2-\beta^2} r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t), \psi \right\rangle \leq \frac{\alpha\beta}{\alpha^2-\beta^2}|r(\omega(t))\hat{z}_0 + \xi(t)|. \tag{3.25}$$

The left-side term of (3.25) may be rewritten as:

$$\left\langle \omega(t) + \left(\frac{\alpha^2}{\alpha^2-\beta^2} - 1\right) r(\omega(t))\hat{z}_0 + \frac{\alpha^2}{\alpha^2-\beta^2}\xi(t), \psi \right\rangle = \langle \omega(t) - r(\omega(t))\hat{z}_0, \psi\rangle +$$

$$+\frac{\alpha^2}{\alpha^2-\beta^2}\langle r(\omega(t))\hat{z}_0 + \xi(t), \psi\rangle = \langle \boldsymbol{u}(\omega(t)), \psi\rangle + \frac{\alpha^2}{\alpha^2-\beta^2}\langle r(\omega(t))\hat{z}_0 + \xi(t), \psi\rangle.$$

From the last equality and from (3.25), it is achieved that

$$\langle \boldsymbol{u}(\omega(t)), \psi\rangle + \frac{\alpha^2}{\alpha^2-\beta^2}\langle r(\omega(t))\hat{z}_0 + \xi(t), \psi\rangle - \frac{\alpha\beta}{\alpha^2-\beta^2}|r(\omega(t))\hat{z}_0 + \xi(t)| \leq 0. \tag{3.26}$$

The multi-valued mapping $\mathcal{D}\big(x(t), y(t)\big)$ is, by and large, regarded as the ball with center and radius changing in time. Thus, a support function $c\Big(\mathcal{D}\big(x(t), y(t)\big), \psi\Big)$ of $\mathcal{D}\big(x(t), y(t)\big)$ can be defined for arbitrary $\psi \in \mathbb{R}^n$, $|\psi| = 1$ (see [10, pp. 68]), and this enables to determine a support function

$$c\Big(\mathcal{D}^*\big(t, x(t), y(t)\big), \psi\Big) = \sup_{d \in \mathcal{D}^*(t,x(t),y(t))} \langle d, \psi\rangle$$

of the multi-valued mapping $\mathcal{D}^*\big(t, x(t), y(t)\big)$ as well. Now, we compute the t-derivative of $c\Big(\mathcal{D}^*\big(t, x(t), y(t)\big), \psi\Big)$ by the properties of a support function (see [10, Property 1, pp. 34; Property 3, pp. 35; Theorem 1, pp. 67]). To do this, from (2.1), (3.3), (3.9), (3.15)–(3.19), (3.26) it is derived that

$$\begin{aligned}
\frac{d}{dt}c\Big(\mathcal{D}^*\big(t, x(t), y(t)\big), \psi\Big) &= \frac{d}{dt}c\Big(\mathcal{D}\big(x(t), y(t)\big) + \mathcal{H}\big(t, x(t), y(t)\big), \psi\Big) \\
&= \frac{d}{dt}c\Big(x(t) + \mathcal{A}\big(t, x(t), y(t), v(\cdot)\big)\big[\mathcal{D}(x_0, y_0) - x_0\big], \psi\Big) \\
&+\frac{d}{dt}c\left(\frac{\alpha^2}{\alpha^2-\beta^2}\int_0^t \xi(s)ds - \left(\frac{\alpha\beta}{\alpha^2-\beta^2}\int_0^t |\xi(s)|ds\right)\mathcal{B} - \int_0^t F_P\big(s, x(s)\big)ds, \psi\right) \\
&= \langle \boldsymbol{u}(\omega(t)), \psi\rangle + \langle F_P(t, x(t)), \psi\rangle + \frac{\alpha^2}{\alpha^2-\beta^2}\langle r(\omega(t))\hat{z}_0, \psi\rangle - \frac{\alpha\beta}{\alpha^2-\beta^2} r(\omega(t)) \\
&+ \frac{\alpha^2}{\alpha^2-\beta^2}\langle \xi(t), \psi\rangle - \frac{\alpha\beta}{\alpha^2-\beta^2}|\xi(t)| - \langle F_P(t, x(t)), \psi\rangle \\
&= \langle \boldsymbol{u}(\omega(t)), \psi\rangle + \frac{\alpha^2}{\alpha^2-\beta^2}\langle r(\omega(t))\hat{z}_0 + \xi(t), \psi\rangle - \frac{\alpha\beta}{\alpha^2-\beta^2}\big(|r(\omega(t))\hat{z}_0| + |\xi(t)|\big) \\
&\le \langle \boldsymbol{u}(\omega(t)), \psi\rangle + \frac{\alpha^2}{\alpha^2-\beta^2}\langle r(\omega(t))\hat{z}_0 + \xi(t), \psi\rangle - \frac{\alpha\beta}{\alpha^2-\beta^2}|r(\omega(t))\hat{z}_0 + \xi(t)|.
\end{aligned}$$

In consequence, we get

$$\frac{d}{dt}c\Big(\mathcal{D}^*\big(t, x(t), y(t)\big), \psi\Big) \le \langle \boldsymbol{u}(\omega(t)), \psi\rangle + \frac{\alpha^2}{\alpha^2-\beta^2}\langle r(\omega(t))\hat{z}_0 + \xi(t), \psi\rangle$$
$$- \frac{\alpha\beta}{\alpha^2-\beta^2}|r(\omega(t))\hat{z}_0 + \xi(t)|.$$

Referring to (3.26), we conclude that the relation $\frac{d}{dt}c\Big(\mathcal{D}^*\big(t, x(t), y(t)\big), \psi\Big) \le 0$ holds for all $t \in [0, T_*]$ and for any $\psi \in \mathbb{R}^n$, $|\psi| = 1$. The proof is now complete. □

Lemma 3.10. *For an arbitrary control $v(\cdot) \in V_{\mathbf{G}}$, the following inclusions are satisfied for all $t \in [0, T_*]$:*
1) $\mathcal{D}\big(x(t), y(t)\big) \subset \mathcal{D}(x_0, y_0) - \mathcal{H}\big(t, x(t), y(t)\big)$;
2) $y(t) \in \mathcal{D}(x_0, y_0) - \mathcal{H}\big(t, x(t), y(t)\big)$.

Proof. 1) We should say that Theorem 3.9 implies

$$\mathcal{D}^*\big(t, x(t), y(t)\big) \subset \mathcal{D}^*\big(0, x(0), y(0)\big).$$

For this reason, from the views of $\mathcal{D}^*\big(t, x(t), y(t)\big)$ and $\mathcal{H}\big(t, x(t), y(t)\big)$ in (3.17)–(3.18) the following arise:

$$\mathcal{D}\big(x(t), y(t)\big) + \mathcal{H}\big(t, x(t), y(t)\big) \subset \mathcal{D}^*\big(0, x(0), y(0)\big) = \mathcal{D}(x_0, y_0),$$

or we can write

$$\mathcal{D}\big(x(t), y(t)\big) \subset \mathcal{D}(x_0, y_0) - \mathcal{H}\big(t, x(t), y(t)\big). \tag{3.27}$$

2) It is evident from (3.14) that $y(t) \in \mathcal{D}\big(x(t), y(t)\big)$ for $t \in [0, T_*]$, and accordingly, we see that $y(t) \in \mathcal{D}(x_0, y_0) - \mathcal{H}\big(t, x(t), y(t)\big)$ is valid owing to (3.27). □

On the strength of Theorem 3.9 and Lemma 3.10, we define the set of meeting points of the players in the pursuit game.

Definition 3.11. For an arbitrary control $v(\cdot) \in V_{\mathbf{G}}$, we call

$$\mathcal{D}_P(x_0, y_0, T_{\mathbf{G}}) = \bigcup_{t=0}^{T_{\mathbf{G}}} \Big(\mathcal{D}(x_0, y_0) - \mathcal{H}\big(t, x(t), y(t)\big)\Big) \tag{3.28}$$

the set of meeting points of the players, where $T_{\mathbf{G}}$ is defined in Theorem 3.6.

3.3. **The "life-line" game solution.** Consider a closed subset $\mathcal{L}$, referred to as a "life-line", in the space $\mathbb{R}^n$. The Pursuer aims to capture the Evader before the Evader reaches the set $\mathcal{L}$, meaning that there exists a time $\widehat{T} > 0$ such that their positions coincide, i.e., $x(\widehat{T}) = y(\widehat{T})$. Meanwhile, the Evader's goal is to either to reach the subset $\mathcal{L}$ before being captured or to carry on the inequality $x(t) \neq y(t)$ for all $t \geq 0$. Notably, the Pursuer's motion is not restricted by the subset $\mathcal{L}$. Additionally, it is supposed that the initial positions x_0 and y_0 are given under the conditions $x_0 \neq y_0$ and $y_0 \neq \mathcal{L}$.

Definition 3.12. It is said that $\mathbf{\Pi}$-strategy (3.3) guarantees that Pursuer P wins on the time interval $\big[0, T_{\mathbf{G}}\big]$ in the "life-line" game, if there exists a time $\widehat{T} \in \big[0, T_{\mathbf{G}}\big]$ such that:

(i): $x\Big(\widehat{T}\Big) = y\Big(\widehat{T}\Big)$;

(ii): $y(t) \neq \mathcal{L}$ for all $t \in \Big[0, \widehat{T}\Big]$.

Definition 3.13. We say that a control $\boldsymbol{v}_{\mathcal{L}}(\cdot) \in V_{\mathbf{G}}$ guarantees that Evader E wins in the "life-line" game if, for any control $u(\cdot) \in U_{\mathbf{G}}$:

(i): there exists a finite time $T_{\mathcal{L}}$ such that $y(T_{\mathcal{L}}) \in \mathcal{L}$ and $x(t) \neq y(t)$ for all $t \in \big[0, T_{\mathcal{L}}\big)$;

(ii): $x(t) \neq y(t)$ for all $t \geq 0$.

Theorem 3.14. *Let $\alpha > \beta + q + p|z_0|$ and $\mathcal{L} \cap \mathcal{D}_P\big(x_0, y_0, T_{\mathbf{G}}\big) = \varnothing$. Then $\mathbf{\Pi}$-strategy (3.3) guarantees that the Pursuer wins on the time interval $\big[0, T_{\mathbf{G}}\big]$ in the "life-line" game, where $T_{\mathbf{G}}$ is the guaranteed time of the pursuit.*

Proof. The proof arises instantly from Theorems 3.6 and 3.9 and from Lemma 3.10. □

Our next concern will be solving the "life-line" game to the advantage of Evader E.
First off, define the following set:

$$\overline{\mathcal{D}}(x_0, y_0) = \big\{d \mid \beta|d - x_0| \geq (\alpha + p|z_0| + q)|d - y_0|\big\}. \tag{3.29}$$

Lemma 3.15. *In accord with the definitions of $\overline{\mathcal{D}}(x_0, y_0)$ and $\mathcal{D}(x_0, y_0)$, the inclusion*

$$\overline{\mathcal{D}}(x_0, y_0) \subset \mathcal{D}(x_0, y_0)$$

is satisfied.

Proof. Due to $p \geq 0$, $q \geq 0$ (see Proposition 2.4) and owing to $\frac{\alpha + p|z_0| + q}{\beta} \geq \frac{\alpha}{\beta}$, we have

$$\frac{\alpha + p|z_0| + q}{\beta}|d - y_0| \geq \frac{\alpha}{\beta}|d - y_0|. \tag{3.30}$$

Combining the inequality in (3.29) with (3.30) yields

$$|d - x_0| \geq \frac{\alpha + p|z_0| + q}{\beta}|d - y_0| \geq \frac{\alpha}{\beta}|d - y_0|.$$

From here, it follows

$$|d - x_0| \geq \frac{\alpha}{\beta}|d - y_0|.$$

The lemma is proved. □

Definition 3.16. The set

$$\mathcal{D}_E\big(y_0, T_{\mathcal{L}}, \boldsymbol{v}_{\mathcal{L}}\big) = \big\{d_E \mid d_E = y\big(y_0, T_{\mathcal{L}}, \boldsymbol{v}_{\mathcal{L}}\big)\big\}\,, \tag{3.31}$$

is said to be a reachability domain of the Evader in the "life-line" game.

Theorem 3.17. *Let $\alpha > \beta + q + p|z_0|$ and $\mathcal{L} \cap \mathcal{D}_E\big(y_0, T_{\mathcal{L}}, \boldsymbol{v}_{\mathcal{L}}\big) \neq \varnothing$. Then there exists a control $\boldsymbol{v}_{\mathcal{L}}(\cdot) \in V_{\mathbf{G}}$ which guarantees that Evader E wins in the "life-line" game.*

Proof. In accordance with the second condition in the theorem, there is at least one point $d_E \in \mathcal{D}_E\big(y_0, T_{\mathcal{L}}, \boldsymbol{v}_{\mathcal{L}}\big) \cap \mathcal{L}$ that

$$d_E = y\big(y_0, T_{\mathcal{L}}, \boldsymbol{v}_{\mathcal{L}}\big).$$

From Definition 3.16, if the Evader employs $\boldsymbol{v}_{\mathcal{L}}$, then it gets to the point d_E at the time $T_{\mathcal{L}}$.

Let

$$T_{\mathcal{L}} := T_{\mathcal{L}}(d) = \frac{|d - y_0|}{\beta}, \quad d \in \overline{\mathcal{D}}(x_0, y_0), \tag{3.32}$$

and

$$\boldsymbol{v}_{\mathcal{L}} := \boldsymbol{v}_{\mathcal{L}}(d) = \beta\frac{d - y_0}{|d - y_0|}, \quad d \in \overline{\mathcal{D}}(x_0, y_0). \tag{3.33}$$

Now, we prove that the condition ***(i)*** of Definition 3.13 is satisfied, more precisely, the Evader remains uncaught. Let us suppose the opposite, that is, there exists some control $\tilde{u}(\cdot) \in U_{\mathbf{G}}$ of the Pursuer giving rise to $x(t) = y(t)$ at a time $\widetilde{T}$ less than $T_{\mathcal{L}}$, i.e., $\widetilde{T} < T_{\mathcal{L}}$. Due to denotations (2.7), we generate the initial value problem

$$\dot{z}(t) = \tilde{u}(t) - \boldsymbol{v}_{\mathcal{L}} + F_P\big(t, x(t)\big) - F_E\big(t, y(t)\big),\ z(0) = z_0,$$

and integrate both sides of this equation, we obtain

$$z(t) = z_0 + \int_0^t \tilde{u}(s)ds - \int_0^t \boldsymbol{v}_{\mathcal{L}}ds + \int_0^t \big(F_P\big(s, x(s)\big) - F_E\big(s, y(s)\big)\big)ds. \tag{3.34}$$

Consequently, equation (3.34) allows us to write

$$z\big(\widetilde{T}\big) = z_0 + \int_0^{\widetilde{T}} \tilde{u}(s)ds - \int_0^{\widetilde{T}} \boldsymbol{v}_{\mathcal{L}}ds + \int_0^{\widetilde{T}} \big(F_P\big(s, x(s)\big) - F_E\big(s, y(s)\big)\big)ds = 0. \tag{3.35}$$

In essence, depending on how the control $\tilde{u}(\cdot) \in U_{\mathbf{G}}$ is chosen, the Pursuer can chase the Evader along different motion trajectories. In particular, the distance between the players may first increase and then decrease, or in the second case, it may continuously decrease from the initial distance $|z_0|$ at the start of the game. Therefore, the time $\widetilde{T}$ will be less in the second case rather than the first one. For this reason, it suffices to consider the second case, i.e., $|z(t)| \leq |z_0|$ for all $t \in [0, \widetilde{T}]$, to prove the condition ***(i)*** of Definition 3.13. Hence, by dint of (2.3), (2.5), (2.6), we carry out the following estimates in (3.35):

$$\left|z_0 - \int_0^{\widetilde{T}} \boldsymbol{v}_{\mathcal{L}}ds\right| \leq \int_0^{\widetilde{T}} |\tilde{u}(s)|ds + \int_0^{\widetilde{T}} \big|F_P\big(s, x(s)\big) - F_E\big(s, y(s)\big)\big|ds$$

$$\leq \int_0^{\widetilde{T}} |\tilde{u}(s)|ds + \int_0^{\widetilde{T}} \big(p(s)|z(s)| + q(s)\big)ds \leq (\alpha + p|z_0| + q)\widetilde{T},$$

or

$$\Big|z_0 - \boldsymbol{v}_{\mathcal{L}}\widetilde{T}\Big| \leq (\alpha + p|z_0| + q)\widetilde{T}. \tag{3.36}$$

Taking the square of both sides of (3.36) and taking account of $|\boldsymbol{v}_{\mathcal{L}}| = \beta$, we get

$$\big((\alpha + p|z_0| + q)^2 - \beta^2\big)\widetilde{T}^2 + 2\widetilde{T}\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle - |z_0|^2 \geq 0.$$

From the latter, the solution is

$$\widetilde{T} \geq \frac{1}{(\alpha + p|z_0| + q)^2 - \beta^2}\Big(\sqrt{\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle^2 + |z_0|^2\big((\alpha + p|z_0| + q)^2 - \beta^2\big)} - \langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle\Big). \tag{3.37}$$

In light of the assumption $T_{\mathcal{L}} > \widetilde{T}$ it is derived from (3.32) and (3.37) that

$$\frac{|d - y_0|}{\beta} > \frac{1}{(\alpha + p|z_0| + q)^2 - \beta^2}\Big(\sqrt{\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle^2 + |z_0|^2\big((\alpha + p|z_0| + q)^2 - \beta^2\big)} - \langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle\Big). \tag{3.38}$$

We obtain the following inequality from (3.29):

$$\beta|d - x_0| \geq (\alpha + p|z_0| + q)|d - y_0|.$$

From this and from $z_0 = x_0 - y_0$ (see (2.7)) it is taken that

$$\beta^2|z_0 - (d - y_0)|^2 \geq (\alpha + p|z_0| + q)^2|d - y_0|^2,$$

and this reduces to the following form after using (3.33) and making some computations:

$$|z_0|^2 \geq \frac{|d - y_0|^2}{\beta^2}\big((\alpha + p|z_0| + q)^2 - \beta^2\big) + \frac{2|d - y_0|}{\beta}\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle. \tag{3.39}$$

Now, firstly, multiplying both sides of (3.39) by $\big((\alpha + p|z_0| + q)^2 - \beta^2\big)$ and then adding $\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle^2$ to both sides gives

$$\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle^2 + \big((\alpha + p|z_0| + q)^2 - \beta^2\big)|z_0|^2 \geq \left[\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle + \frac{|d - y_0|}{\beta}\big((\alpha + p|z_0| + q)^2 - \beta^2\big)\right]^2,$$

or

$$\frac{1}{(\alpha + p|z_0| + q)^2 - \beta^2}\left(\sqrt{\langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle^2 + |z_0|^2\big((\alpha + p|z_0| + q)^2 - \beta^2\big)} - \langle \boldsymbol{v}_{\mathcal{L}}, z_0\rangle\right) \geq \frac{|d - y_0|}{\beta}. \tag{3.40}$$

In the light of (3.38) and (3.40), we meet a contradiction. This concludes the proof. □

Corollary 3.18. *In accord with the definitions of (3.28) and (3.31), it is confirmed that*

$$\mathcal{D}_E\big(y_0, T_{\mathcal{L}}, \boldsymbol{v}_{\mathcal{L}}\big) \subset \mathcal{D}_P\big(x_0, y_0, T_{\mathbf{G}}\big)$$

is accurate.

4. Examples

Example 1. Consider the differential game

$$P: \quad \dot{x} = u + x + z_0\cos^2 t, \quad x(0) = x_0, \quad |u(t)| \leq \alpha, \tag{4.1}$$

$$E: \quad \dot{y} = v + y - z_0\sin^2 t, \quad y(0) = y_0, \quad |v(t)| \leq \beta, \tag{4.2}$$

respectively, where $x, y, u, v, z_0 \in \mathbb{R}^n$, $n \geq 2$. $z_0 = x_0 - y_0$.

For Proposition 2.4, we can take as $p = 1$ and $q = |z_0|$. Thus, the function $\mathcal{K}(t)$ in (3.12) will be as follows: $\mathcal{K}(t) = |z_0| - (\alpha - \beta - 2|z_0|)\,[\exp(t) - 1]$. Then we will give the following result.

Theorem 4.1. *Let $\alpha > \beta + 2|z_0|$. Then $\mathbf{\Pi}$-strategy (3.3) guarantees that Pursuer P wins on the time interval $\big[0, T_{\mathbf{G}}\big]$ in the pursuit game (4.1)–(4.2), where $T_{\mathbf{G}} = \ln\frac{\alpha-\beta-|z_0|}{\alpha-\beta-2|z_0|}$.*

Example 2. Consider the differential game

$$P: \quad \dot{x} = u + sin(2e^{-t}x), \qquad x(0) = x_0, \quad |u(t)| \le \alpha, \tag{4.3}$$

$$E: \quad \dot{y} = v + cos\left(\frac{1}{t^2+1}y\right), \quad y(0) = y_0, \quad |v(t)| \le \beta, \tag{4.4}$$

where $x, y, u, v \in \mathbb{R}^n$. Here it is obvious that

$$|F_P(t,x)| = \left|\sin\left(2e^{-t}x\right)\right| \le 1 = h_P^*,$$

$$|F_E(t,y)| = \left|\cos\left(\frac{1}{t^2+1}y\right)\right| \le 1 = h_E^*.$$

To compute Lipschitz constants for $F_P(t,x) = \sin\left(2e^{-t}x\right)$ and $F_E(t,y) = \cos\left(\frac{1}{t^2+1}y\right)$, we use the following statement.

Lemma 4.2. *([9]) Let $f : [a,b] \times \mathbb{D} \to \mathbb{R}^m$ be continuous for some domain $\mathbb{D} \subset \mathbb{R}^n$. Suppose that $[\partial f/\partial x]$ exists and is continuous on $[a,b] \times \mathbb{D}$. If, for a convex subset $W \subset \mathbb{D}$, there exists a constant $L \ge 0$ such that $\left|\frac{\partial f}{\partial x}\right| \le L$ on $[a,b] \times W$, then the Lipschitz condition*

$$|f(t,x) - f(t,y)| \le L|x-y|$$

is satisfied for all $t \in [a,b]$, $x, y \in W$.

According to this property, we get

$$\left|\frac{\partial F_P(t,x)}{\partial x}\right| = \left|2e^{-t}\cos\left(2e^{-t}x\right)\right| \le 2 = L_1, \quad \left|\frac{\partial F_E(t,y)}{\partial y}\right| = \left|-\frac{1}{t^2+1}\sin\left(\frac{1}{t^2+1}y\right)\right| \le 1 = L_2.$$

From this and from Assumption 2.3 it follows that

$$|F_P(t,x_1) - F_P(t,x_2)| \le k_P(t)|x_1 - x_2| = L_1|x_1 - x_2|,$$

$$|F_E(t,y_1) - F_E(t,y_2)| \le k_E(t)|x_1 - x_2| = L_2|y_1 - y_2|.$$

Consequently, for Proposition 2.4, we find $p = k_P^* + k_E^* = 2 + 1 = 3$ and $q = h_P^* + h_E^* = 1 + 1 = 2$. Thus, the function $\mathcal{K}(t)$ in (3.12) will be as follows: $\mathcal{K}(t) = |z_0| - (\alpha - \beta - 2 - 3|z_0|)\left[\exp(3t) - 1\right]/3$. Then we will give the following result.

Theorem 4.3. *Let $\alpha > \beta + 3|z_0| + 2$. Then $\mathbf{\Pi}$-strategy (3.3) guarantees that Pursuer P wins on the time interval $\left[0, T_{\mathbf{G}}\right]$ in the pursuit game (4.3)–(4.4), where $T_G = \frac{1}{3}\ln\frac{\alpha-\beta-2}{\alpha-\beta-3|z_0|-2}$.*

References

[1] Anderson R.P., Bakolas E., Milutinović D., and Tsiotras P.; Optimal Feedback Guidance of a Small Aerial Vehicle in a Stochastic Wind. Journal of Guidance, Control, and Dynamics,–2013.–36.–No4.–P.975–985. https://doi.org/10.2514/1.59512

[2] Azamov A.; A class of nonlinear differential games. Mathematical notes of the Academy of Sciences of the USSR,–1981.–30.–issue 4.–P.805–808. https://doi.org/10.1007/BF01137812

[3] Azamov A.; About the quality problem for the games pursuit with the restriction. Serdica Bulgarian math.,–1986.–12.–No.2.–P.38–43.

[4] Azamov A., Samatov B.T.; The Π-Strategy: Analogies and Applications. GTM 2010: The fourth international conference on Game Theory and Management, St. Petersburg University,–2010.–No.4.–P.33–47.

[5] Azamov A.A., Samatov B.T., Soyibboev U.B.; The game with a “life-line” for simple garmonic motions of objects. International Game Theory Review,–2024.–IGTR.151.–P.2450009-1–2450009-29. DOI: 10.1142/S0219198924500099

[6] Azamov A.A., Turgunboeva M.A.; The differential game with inertial players under integral constraints on controls. Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes,–2025.–21.–1.–P.122–138.

[7] Bakolas E.; Optimal Guidance of the Isotropic Rocket in the Presence of Wind. Journal of Optimization Theory and Applications.,–2014.–162.–No.3.–P.954–974. doi:10.1007/s10957–013–0504–4

[8] Berkovitz L.D.; Differential game of generalized pursuit and evasion. SIAM J. Contr.,–1986.–24.–No.3.–P.361–373. DOI 10.1137/0324021

[9] Birkhoff G., Rota G.C.; Ordinary differential equations. John Wiley and Sons, New York.–1989.

[10] Blagodatskikh V.I.; Introduction to Optimal Control. Vysshaya Shkola, Moscow.–1979 (in Russian).

[11] Blagodatskikh A.I., Petrov N.N.; Conflict interaction of groups of controlled objects. Izhevsk: Udmurt State University.–2009 (in Russian).

[12] Chikrii A.A.; Conflict-Controlled Processes. Kluwer, Dordrecht.–1997. DOI 10.1007/978-94-017-1135-7

[13] Dubins L.E.; On Curves of Minimal Length with a Constraint on Average Curvature, and with Prescribed Initial and Terminal Positions and Tangents. American Journal of Mathematics,–1957.–79.–No.3.–P.497–516. doi:10.2307/2372560

[14] Elliot R.J., Kalton N.J.; The existence of value in differential games of pursuit and evasion. Journal of Differential Equations,–1972.–12.–No.3.–P.504–523. DOI 10.1016/0022-0396(72)90022-8

[15] Friedman A.; Differential Games (2nd ed.). Pure and Applied Mathematics, Wiley Interscience, New York.–1971.–25.

[16] Grigorenko N.L.; Mathematical Methods of Control for Several Dynamic Processes. Izdat. Mos. Univ., Moscow.–1990.

[17] Hajek O.; Pursuit Games: An Introduction to the Theory and Applications of Differential Games of Pursuit and Evasion. Dove. Pub. New York.–2008.

[18] Ho Y., Bryson A., Baron S.; Differential games and optimal pursuit-evasion strategies. IEEE Trans Autom Control,–1965.–10.–P.385-389. DOI 10.1109/TAC.1965.1098197

[19] Ibragimov G.I.; The optimal pursuit problem reduced to an infinite system of differential equations. J. Appl. Maths Mekhs.–2013.–77.–No.5.–P.470–476. http://dx.doi.org/10.1016/j.jappmathmech.2013.12.002.

[20] Ibragimov G.I.; Optimal pursuit time for a differential game in the Hilbert Space l_2. Science Asia,–2013.–39.–No.S.–P.25–30. DOI 10.2306/scienceasia1513-1874.2013.39S.025

[21] Isaacs R; Differential Games. John Wiley and Sons, New York,–1965.

[22] Krasovskii N.N.; Control of a Dynamical System. Nauka, Moscow.–1985 (in Russian).

[23] McNeely R.L., Iyer R.V., Chandler P.R.; Tour Planning for an Unmanned Air Vehicle Under Wind Conditions. Journal of Guidance, Control, and Dynamics,–2007.–3.–No.5.–P.1299–1306. doi:10.2514/1.26055

[24] Natarajan T., Pierre D.A., Naadimuthu G., Lee E.S.; Piecewise suboptimal control laws for differential games. Journal of Mathematical Analysis and Applications,–1984.–104.–issue 1.–P.189–211. https://doi.org/10.1016/0022-247X(84)90042-8

[25] Pachpatte B.G.; Inequalities for Differential and Integral Equations. Mathematics in Science and Engineering Academic, London.–1998.–197.

[26] Petrosyan L.A.; Differential games of pursuit (Series on Optimization). World Scientific Pob., Singapore,–1993.

[27] Pontryagin L.S.; Selected Works. CRC Press, New York,–2004.

[28] Pshenichnii B.N.; Simple pursuit by several objects. Cybernetics and System Analysis,–1976,–12.–No.5.–P.484–485.

[29] Samatov B.T.; Problems of group pursuit with integral constraints on controls of the players I. Cybernetics and Systems Analysis,–2013.–49.–No.5.-P.756–767.

[30] Samatov B.T.; Problems of group pursuit with integral constraints on controls of the players II. Cybernetic and Systems Analysis,–2013.–49.–No.6.–P.907–921.

[31] Samatov B.T., Sotvoldiyev A.I.; Intercept problem in dynamic flow field. Uzbek Mathematical Journal.–2019.–No.2.–P.103–112.

[32] Satimov N.Yu.; Methods to Solve Pursuit Problems in Theory Differential Games. NUU Press, Tashkent,–2003. (in Russian)

[33] Satimov N.; Pursuit problems in nonlinear differential games. Cybernetics.–1973.–9.–issue 3.–P.469–475. https://doi.org/10.1007/BF01069203

[34] Shchelchkov K.A.; On the problem of controlling a nonlinear system by a discrete control under disturbance, Differential Equations,–2024.–60.–issue 1.–P.127–135. https://doi.org/10.1134/S0012266124010105

[35] Sun W., Tsiotras P.; Pursuit evasion game of two players under an external flow field. Proceedings of the American Control Conference. Los Angeles, CA, USA, P.–2015.–P.5617–5622. DOI: 10.1109/ACC.2015.7172219

[36] Sun W., Tsiotras P., Lolla T., Subramani D.N., Lermusiaux P.F.J.; Multiple-pursuit/one-evader pursuit-evasion game in dynamical flow fields. J. of Guidance, Control, and Dynamics.–2017.–40.–No.7.–P.1627–1637. https://doi.org/10.2514/1.G002125

Turgunboeva M.A.,
Department of Mathematics,
Namangan State University,
Namangan, Uzbekistan
email: turgunboyevamohisanam95@gmail.com

Soyibboev U.B.,
Department of Applied Mathematics,
Andijan State University,
Andijan, Uzbekistan
email: ulmasjonsoyibboev@gmail.com

Uzbek Mathematical Journal
2025, Volume 69, Issue 4, pp.217-226
DOI: 10.29229/uzmj.2025-4-21

Vanishing of the second cohomology groups of certain solvable Lie algebras

Urazmatov G.

Abstract. In this article, we consider cohomology groups of maximal solvable Lie algebras. We give necessary and sufficient conditions for cohomological rigidity of certain solvable Lie algebras.

Keywords: Lie algebras, nilpotent Lie algebras, solvable Lie algebras, cohomology groups.
MSC (2020): 17B30, 17B40, 17B50

1. Introduction

An extensive study of Lie algebras has yielded many beautiful results and generalizations. In the classical theory of finite-dimensional Lie algebras, it is known that any Lie algebra over a field of characteristic zero decomposes into a semidirect sum of its solvable radical and its semisimple subalgebra, according to Levi's theorem. Thanks to the results of Malcev and Mubarakzjanov, the study of non-nilpotent solvable Lie algebras is reduced to the analysis of nilpotent algebras and their derivations [14], [11]. Consequently, the focus of research on finite-dimensional Lie algebras has shifted to nilpotent algebras and the representations of semisimple algebras. Numerous studies have been dedicated to solvable Lie algebras with given nilradicals [10], [13], [4], [6], [15]. Maximal solvable Lie algebras, which form an important class of Lie algebras, can be analyzed through their cohomological properties. This research explores the conditions under which the second cohomology group of such algebras becomes trivial, offering insights into their deeper structure and potential applications in representation theory and algebraic topology.

The investigation of cohomology groups of Lie algebras has been a central focus in the fields of mathematical physics and algebraic geometry. Cohomology problems have been explored in many papers [1], [2], [3], [5]. In particular, the second cohomology group, $H^2(\mathcal{L}, K)$, where $\mathcal{L}$ is a Lie algebra and K is a field, plays a significant role in understanding the structure and classification of Lie algebras. This article specifically examines when the second cohomology group of certain maximal solvable Lie algebras vanishes. The goal of this work is to establish specific criteria that determine when the second cohomology groups of these algebras are trivial, under a variety of necessary and sufficient conditions. F.Leger and E.Luks [9] provided some necessary conditions for cohomological rigidity in certain solvable algebras. We build on their work by generalizing these conditions and providing additional sufficient conditions for cohomological rigidity under specific restrictions.

Unless otherwise stated, any Lie algebra considered in this work is finite-dimensional and $\mathcal{R} = \mathcal{N} \rtimes \mathcal{T}$. Here, $\mathcal{T}$ is the maximal torus of $\mathcal{N}$, $\mathcal{N} = \mathcal{N}_{\alpha_1} \oplus \mathcal{N}_{\alpha_2} \oplus \cdots \oplus \mathcal{N}_{\alpha_n}$, $\mathcal{N}_{\alpha_i}$ with representing the root subspaces with respect to the maximal torus in $\mathcal{N}$. Additionally, $\dim \mathcal{N}_\alpha = 1$ for all $\alpha \in W$, $rank(\mathcal{N}) = s$ and zero is not in W.

2. Preliminaries

Definition 2.1. A vector space with a bilinear bracket $(\mathcal{L}, [-,-])$ over a field of $\mathbb{F}$ is called a Lie algebra if for any $x, y, z \in \mathcal{L}$ the following identities hold:

$$[x, y] = -[y, x]$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

For a given Lie algebra $\mathcal{L}$ we define the *descending central sequence* and the *derived sequence* in the following recursive way:

$$\mathcal{L}^1 = \mathcal{L}, \ \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}], \ \text{and} \ \mathcal{L}^{[1]} = \mathcal{L}, \ \mathcal{L}^{[k+1]} = [\mathcal{L}^{[k]}, \mathcal{L}^{[k]}], \quad k \geq 1, \ \text{respectively.}$$

Definition 2.2. A Lie algebra $\mathcal{L}$ is called nilpotent (respectively, solvable) if there exists $s \in \mathbb{N}$ (respectively, $k \in \mathbb{N}$) such that $\mathcal{L}^s = 0$ (respectively, $\mathcal{L}^{[k]} = 0$.)

For a given Lie algebra $\mathcal{R}$, let $C^k(\mathcal{R}, M)$ be the space of all alternating F-linear homogeneous mappings $\wedge^n\mathcal{R} \to M,\ k \geq 0$ and $C^0(\mathcal{R}, M) = M$. Let $d^k : C^k(\mathcal{R}, M) \to C^{k+1}(\mathcal{R}, M)$ be an F-homomorphism defined by

$$\begin{aligned}(d^k f)(x_1, \dots, x_{k+1}) : &= \sum_{i=1}^{k+1}(-1)^{i+1}[x_i, f(x_1, \dots, \widehat{x}_1, \dots, x_{k+1})]+ \\ &+ \sum_{1\leq i<j\leq k+1}(-1)^{i+j} f([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots \widehat{x}_j, \dots, x_{k+1}),\end{aligned}$$

where $d^k f \in Z^k(R, M)$ and $x_i \in \mathcal{R}$. Since the derivative operator $d = \sum_{i\geq 0} d^i$ satisfies the property $d \circ d = 0$, the k-th cohomology group well defined and

$$H^k = Z^k(\mathcal{R}, M)/B^k(\mathcal{R}, M),$$

where elements of $Z^k(R, M) := Kerd^k$ and $B^k(\mathcal{R}, M) :=\ Imd^{k-1}$ are called k-cocycles and k-coboundaries, respectively.

Hoschild-Serre factorization theorem simplifies computations of cohomology groups for semidirect sums of algebras [12].

Theorem 2.3. *If $\mathcal{R} = \mathcal{N} \oplus \mathcal{Q}$ is a solvable Lie algebra such that $\mathcal{Q}$ is Abelian and operators $ad_{\mathcal{R}}t$ ($t \in \mathcal{Q}$) are diagonal, then the adjoint cohomology $H^p(\mathcal{R}, \mathcal{R})$ satisfies the following isomorphism*

$$H^p(\mathcal{R}, \mathcal{R}) \cong \sum_{a+b=p} H^a(\mathcal{Q}, \mathbb{K}) \otimes H^b(\mathcal{N}, \mathcal{R})^{\mathcal{Q}},$$

where

$$H^b(\mathcal{N}, \mathcal{R})^{\mathcal{Q}} = \{\varphi \in H^b(\mathcal{N}, \mathcal{R})|\ (t.\varphi) = 0,\ t \in \mathcal{Q}\} \tag{2.1}$$

is the space of $\mathcal{Q}$-invariant cocycles of $\mathcal{N}$ with values in $\mathcal{R}$, the invariance being defined by:

$$(t.\varphi)(z_1, z_2, \dots, z_b) = [t, \varphi(z_1, z_2, \dots, z_b)] - \sum_{s=1}^{b} \varphi(z_1, \dots, [t, z_s], \dots, z_b).$$

Let consider

$$H^a(\mathcal{Q}, \mathbb{K}) = \frac{Ker\ (d^a)}{Im\ (d^{a-1})},$$

where $d^a : C^a(\mathcal{Q}, \mathbb{K}) \to C^{a+1}(\mathcal{Q}, \mathbb{K})$. Since $\varphi : \mathcal{Q} \times \dots \times \mathcal{Q} \to \mathbb{K}$ and $[t_i, t_j] = 0,\ t_i, t_j \in \mathcal{Q}$ we get $(d^a\varphi)(t_1, \dots, t_{a+1}) = 0$. It implies that $Im\ d^a = 0$ and $Ker\ d^a = C^a(\mathcal{Q}, \mathbb{K})$, i.e., $H^a(\mathcal{Q}, \mathbb{K}) = \wedge^a\mathcal{Q}$. Therefore, the cohomology groups $H^k(\mathcal{R}, \mathcal{R})$ vanish if and only if the space of $\mathcal{Q}$-invariant cocycles $H^b(\mathcal{N}, \mathcal{R})^{\mathcal{Q}}$ vanish. On the other hand, $H^p(\mathcal{R}, \mathcal{R}) = 0$ implies that $H^b(\mathcal{N}, \mathcal{R})^{\mathcal{Q}} = 0$ for all $0 \leq b \leq p$.

Definition 2.4. A torus on a Lie algebra $\mathcal{L}$ is a commutative subalgebra of $Der(\mathcal{L})$ (the set of all derivations of $\mathcal{L}$) consisting of semisimple endomorphisms. A torus is said to be maximal if it is not strictly contained in any other torus. We denote by $\mathcal{T}_{max}$ a maximal torus of a Lie algebra $\mathcal{L}$.

We should note that if $dim\mathcal{T} = dim(\mathcal{N}/\mathcal{N}^2)$, then $\mathcal{N}$ is called nilpotent Lie algebra of maximal rank.

Definition 2.5. A solvable Lie algebra $\mathcal{R}_{\mathcal{T}} = \mathcal{N} \rtimes \mathcal{T}$ is said to be of maximal rank, if $dim\mathcal{T} = dim(\mathcal{N}/N^2)$.

The dimension of a maximal torus of a nilpotent Lie algebra is denoted by $rank(\mathcal{N})$.

Denote by $W = \{\alpha \in \mathcal{T}^* : \mathcal{N}_\alpha \neq 0\}$ the roots system of $\mathcal{N}$ associated to $\mathcal{T}$, and by $\Psi_1 = \{\alpha_1, \dots, \alpha_s\}$ the set of primitive roots such that any non-primitive root can be expressed by a linear combination of them. In fact, any root $\alpha \in W$ we have $\alpha = \sum_{\alpha_i\in\Psi_1} p_i\alpha_i$ with $p_i \in \mathbb{Z}$.

We should note that $\mathcal{Q}$ in the Theorem 2.3. is nothing else but torus of $\mathcal{N}$. In the following steps, we consider maximal solvable extensions of nilpotent Lie algebras which are constructed by their maximal torus, i.e. $\mathcal{Q} = \mathcal{T}_{max}$. We have the following remark.

Remark 2.6. It should be noted that if $\mathcal{H}(\mathcal{R},\mathcal{R})^p = 0$, where $\mathcal{R} = \mathcal{N} \rtimes \mathcal{T}$ is a solvable Lie algebra such that $\mathcal{T}$ is Abelian and operators $ad_{\mathcal{R}}t$ $(t \in \mathcal{T})$ are diagonal, then $\mathcal{H}(\mathcal{R},\mathcal{R})^i = 0$ for any $0 \le i \le p-1$.

Let consider $b = 1$ on (2.1):

$$H^1(\mathcal{N},\mathcal{R})^{\mathcal{T}} = \{\varphi \in H^1(\mathcal{N},\mathcal{R}) |\ (t.\varphi) = 0,\ t \in \mathcal{T}\}.$$

Hence,

$$H^1(\mathcal{N},\mathcal{R}) = H^1(\mathcal{N}_{\alpha_1} \oplus \cdots \oplus \mathcal{N}_{\alpha_n}, \mathcal{N}_{\alpha_1} \oplus \cdots \oplus \mathcal{N}_{\alpha_n} \oplus \mathcal{T}).$$

Let $\varphi \in H^1(\mathcal{N},\mathcal{R})^{\mathcal{T}}$, then

$$(t.\varphi)(n_\alpha) = [t, \varphi(n_\alpha)] - \varphi([t, n_\alpha]) = [t, \varphi(n_\alpha)] + \alpha(t)\varphi(n_\alpha) = 0.$$

It implies that

$$[\varphi(n_\alpha), t] = \alpha(t)\varphi(n_\alpha) \Rightarrow \varphi(\mathcal{N}_\alpha) \subseteq \mathcal{N}_\alpha.$$

Conditions of the triviality of the first cohomology groups of maximal solvable Lie algebras are obtained in [8].

Theorem 2.7. [8] *A solvable Lie algebra is complete if and only if it is the maximal solvable extension of a d-locally diagonalizable nilpotent Lie algebra.*

Let $\mathcal{R} = \mathcal{N} \rtimes \mathcal{T}$ be maximal solvable extensions of nilpotent Lie algebra of maximal rank. Then by applying Theorem 4.9. in [7], we can take multiplications table of $\mathcal{R}$ as follows:

$$\mathcal{R}: \begin{cases} [\mathcal{N},\mathcal{N}]; \\ [n_{\alpha_i}, t_i] = \alpha_i n_{\alpha_i}, & 1 \le i \le k, \\ [n_{\alpha_i}, t_j] = \alpha_{i,j} n_{\alpha_i}, & 1 \le j \le k, \quad k+1 \le j \le n, \end{cases}$$

where $\mathcal{N} = \mathcal{N}_{\alpha_1} \oplus \cdots \oplus \mathcal{N}_{\alpha_k} \oplus \cdots \oplus \mathcal{N}_{\alpha_n}$ and $\mathcal{N}_{\alpha_i}$ are root subspaces with respect to the maximal torus on $\mathcal{N}$. Since any Lie algebra is also a Lie superalgebra, we can rewrite the following corollary (see, Corollary 4.11. in [7]).

Corollary 2.8. [7] *A maximal solvable extension of a nilpotent Lie algebra of maximal rank has trivial center and it admits only inner derivations.*

Due to Theorem 2.3 and Corollary 2.8, we can obtain the following remark.

Remark 2.9. Let $\mathcal{R}$ be a maximal solvable extension of a nilpotent Lie algebra of maximal rank. $H^2(\mathcal{R},\mathcal{R}) = 0$ if and only if $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$.

For the elements $\varphi \in Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$, we have the following:

$$0 = (t.\varphi)(n_{\alpha_i}, n_{\alpha_j}) = [t, \varphi(n_{\alpha_i}, n_{\alpha_j})] - \varphi([t, n_{\alpha_i}], n_{\alpha_j}) - \varphi(n_{\alpha_i}, [t, n_{\alpha_j}]) \Rightarrow$$
$$[t, \varphi(n_{\alpha_i}, n_{\alpha_j})] = \varphi([t, n_{\alpha_i}], n_{\alpha_j}) + \varphi(n_{\alpha_i}, [t, n_{\alpha_j}]) = (\alpha_i + \alpha_j)(t)\varphi(n_{\alpha_i}, n_{\alpha_j}).$$

It implies that $\varphi(n_{\alpha_i}, n_{\alpha_j}) \in \mathcal{N}_{\alpha_i+\alpha_j}$. Since $B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} \subseteq Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$, for any element $\psi \in B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$ we have $\psi(n_{\alpha_i}, n_{\alpha_j}) \in \mathcal{N}_{\alpha_i+\alpha_j}$.

Now, we present a proposition mentioned in the paper by F. Leger and E. Luks in [9]. We will formulate this proposition using our notation:

Let $\mathcal{N}$ be a nilpotent Lie algebra over a field F, of characteristic not 2, $\mathcal{T}$ is a subalgebra of $Der(\mathcal{N})$, $\mathcal{R}$ is semi-direct sum $\mathcal{R} = \mathcal{N} \rtimes \mathcal{T}$ and we shall assume the following properties (i)-(iv) hold.
(**i**) $\mathcal{T}$ is diagonalizable over F and $dim\mathcal{T} = dim(\mathcal{N}/\mathcal{N}^2)$.
Let W denote the set of weights of $\mathcal{T}$ in $\mathcal{N}$ and for each α in W, denote by $\mathcal{N}_\alpha$ the weight space for α.
(**ii**) For $\alpha \in W$, $dim\mathcal{N}_\alpha = 1$ and if $\alpha, \beta, \alpha+\beta$ are all in W, $[\mathcal{N}_\alpha, \mathcal{N}_\beta] = \mathcal{N}_{\alpha+\beta}$. We fix $\alpha_1, \alpha_2, ..., \alpha_n$ in W and $e_1, e_2, ..., e_n$ in $\mathcal{N}$ so that $\mathcal{N} = \sum F e_i + \mathcal{N}^2$ (vector space direct sum) and $t \cdot e_j = \alpha_j(t)e_j$ for t in $\mathcal{T}$. The weights $\alpha_1, \alpha_2, ..., \alpha_n$ will be called primitive. Every weight in W has the form $\sum_{i=1}^{n} r_i\alpha_i$, where the r_i are non-negative integers.

(**iii**) If characteristic of F is $p \neq 0$ we assume further, for every α in W, that $0 \leq r_i < p/2$ for each i. Since each weight in $\mathcal{N}^2$ is the sum of two weights, it follows from (iii) that zero is not in W.
(**iv**) If $\alpha, \beta, \gamma, \delta, \alpha+\gamma, \beta+\delta$ are all in W with α, β primitive and unequal and with $\alpha+\gamma = \beta+\delta$, then there is some μ in W such that $\delta = \alpha+\mu,\ \gamma = \beta+\mu$ and at least one of the following is satisfied:

Case 1. $\alpha+\beta$ is not in W.

Case 2. $\alpha+\beta$ is in W but $\alpha+2\beta$ is not in W and $\mu = \beta+\nu$ for some ν in W.

Case 3. $\alpha+\beta$ is in W but $2\alpha+\beta$ is not in W and $\mu = \alpha+\nu$ for some ν in W.

Proposition A. Suppose $\mathcal{T}$, $\mathcal{N}$ are as (i)-(iv). Let $\mathcal{R}$ be an $\mathcal{T}+\mathcal{N}$ module such that the representation of $\mathcal{T}$ on $\mathcal{R}$ is toroidal and the weights of $\mathcal{T}$ in $\mathcal{R}$ are in W. Then $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$.

A linear Lie algebra K is called *toroidal* if it can be diagonalized over an algebraic closure of the base field; a representation ρ of K is called *toroidal* if $\rho(K)$ is *toroidal.* From this, we can conclude that in our case, the representation of $\mathcal{T}$ on $\mathcal{R}$ is toroidal.

3. Main part.

Theorem 3.1. *Let $\mathcal{N}$ be a nilpotent Lie algebra of maximal rank such that $\mathcal{N}^3 = 0$. Then, $\mathcal{R} = \mathcal{N} \rtimes \mathcal{T}$ is cohomologically rigid, i.e., $H^2(\mathcal{R},\mathcal{R}) = 0$.*

Proof. Due to $\mathcal{N}^3 = 0$, we can express non-zero products using only primitive roots:

$$\mathcal{N} : \left\{ \ [n_{\alpha_i}, n_{\alpha_j}] = A_{i,j} n_{\alpha_i+\alpha_j}, \quad 1 \leq i \neq j \leq s, \right.$$

where $n_{\alpha_k} \in \mathcal{N}_{\alpha_k}$. If $A_{i,j} = 0$ for all $1 \leq i \neq j \leq s$, then $\mathcal{N}$ is abelian. In this case, it is clear that $\mathcal{R}$ can be written as a direct sum of two dimensonal solvable Lie algebras $\mathcal{R}_i$, which have one dimensional abelian nilradical. Since, they are all rigid, we can assume $\mathcal{R}$ is cohomologically rigid(see, Proposition 2, in [3]). Assume $A_{i,j} \neq 0$, for some $1 \leq i \neq j \leq s', (s' \leq s)$, then without loss of generalities we can take as $A_{i,j} = 1$ for $1 \leq i \neq j \leq s'$:

$$\mathcal{N} : \left\{ \ [n_{\alpha_i}, n_{\alpha_j}] = n_{\alpha_i+\alpha_j}, \quad 1 \leq i \neq j \leq s'. \right.$$

Remark 2.9 implies that it is enough to show that $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$. Let suppose $\varphi \in Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$. Since $\varphi(n_{\alpha_i}, n_{\alpha_j}) \in \mathcal{N}_{\alpha_i+\alpha_j}$, we can denote as:

$$\left\{ \ \varphi(n_{\alpha_i}, n_{\alpha_j}) = D_{i,j} n_{\alpha_i+\alpha_j}, \quad 1 \leq i \neq j \leq s', \right.$$

where $D_{i,j}$ are scalars.

We now show that $\varphi \in B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$ too. In other words, we prove existence of $f \in C^1(\mathcal{N},\mathcal{R})$ such that $\varphi = d^1 f$, satisfying the following identity:

$$\varphi(e_\alpha, e_\beta) = [f(e_\alpha), e_\beta] + [e_\alpha, f(e_\beta)] - f([e_\alpha, e_\beta]) \quad \text{for all} \quad \alpha, \beta \in W.$$

Let denote $f(n_\alpha) = \mu(n_\alpha) n_\alpha$, where $\mu(n_\alpha)$ are scalars. Then

$$\left\{ \ \varphi(n_{\alpha_i}, n_{\alpha_j}) = \Big(\mu(n_{\alpha_i}) + \mu(n_{\alpha_j}) - \mu(n_{\alpha_i+\alpha_j})\Big) n_{\alpha_i+\alpha_j} = D_{i,j} n_{\alpha_i+\alpha_j}, \quad 1 \leq i \neq j \leq s', \right.$$

Then we come to

$$\left\{ \ \mu(n_{\alpha_i}) + \mu(n_{\alpha_j}) - \mu(n_{\alpha_i+\alpha_j}) - D_{i,j} = 0, \quad 1 \leq i \neq j \leq s', \right. \tag{3.1}$$

system with variables $\mu(n_{\alpha_i}), \mu(n_{\alpha_j}), \mu(n_{\alpha_i+\alpha_j}),\ \ 1 \leq i \neq j \leq s'$.
System (3.1) has the following solution:

$$\begin{cases} \mu(n_{\alpha_i}) = \mu(n_{\alpha_j}) = 1, & 1 \leq i \neq j \leq s', \\ \mu(n_{\alpha_i+\alpha_j}) = 2 - D_{i,j}, & 1 \leq i \neq j \leq s'. \end{cases}$$

Therefore, there exists $f \in C^1(\mathcal{N},\mathcal{R})$ such that $d^1 f = \varphi$, i.e., $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$. By Remark 2.9 we obtain that $H^2(\mathcal{R},\mathcal{R}) = 0$. $\square$

We now consider the case $\mathcal{N}^4 = 0$.

Theorem 3.2. *Let $\mathcal{N}$ be a nilpotent Lie algebra of maximal rank satisfying $\mathcal{N}^4 = 0$ and $rank(\mathcal{N}) = 2$. Then $\mathcal{R}$ is cohomologically rigid, i.e., $H^2(\mathcal{R},\mathcal{R}) = 0$.*

Proof. Without loss of generalities we can take the following products for $\mathcal{N}$:

$$\mathcal{N}: \begin{cases} [n_{\alpha_1}, n_{\alpha_2}] = A_1 n_{\alpha_1+\alpha_2}, \\ [n_{\alpha_1+\alpha_2}, n_{\alpha_1}] = A_2 n_{2\alpha_1+\alpha_2}, \\ [n_{\alpha_1+\alpha_2}, n_{\alpha_2}] = A_3 n_{\alpha_1+2\alpha_2}. \end{cases}$$

where $n_\alpha \in \mathcal{N}_\alpha$. If $A_1 = 0$, then we come to the case $\mathcal{N}$ is abelian, then similar to the proof of Theorem 1, we can prove that $H^2(\mathcal{R},\mathcal{R}) = 0$. Let $A_1 \neq 0$, then we can assume $A_1 = 1$. Remark 2.9 implies that it is enough to show $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$. Let suppose $\varphi \in Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$.

Take the following denotions:

$$\begin{cases} \varphi(n_{\alpha_1}, n_{\alpha_2}) = B_1 n_{\alpha_1+\alpha_2}, \\ \varphi(n_{\alpha_1+\alpha_2}, n_{\alpha_1}) = (1-\delta_{A_2,0}) B_2 n_{2\alpha_1+\alpha_2}, \\ \varphi(n_{\alpha_1+\alpha_2}, n_{\alpha_2}) = (1-\delta_{A_3,0}) B_3 n_{\alpha_1+2\alpha_2}, \end{cases} \tag{3.2}$$

where B_1, B_2, B_3 are scalars. We now show $\varphi \in B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$ too. In other words, we prove the existence of $f \in C^1(\mathcal{N},\mathcal{R})$ such that $\varphi = d^1 f$, satisfying the following identity:

$$\varphi(e_\alpha, e_\beta) = [f(e_\alpha), e_\beta] + [e_\alpha, f(e_\beta)] - f([e_\alpha, e_\beta]) \quad \text{for all} \quad \alpha, \beta \in W.$$

Let denote $f(n_\alpha) = \mu(n_\alpha) n_\alpha$, where $\mu(n_\alpha)$ are scalars. Then the system (3.2) has the following non-zero solution:

$$\begin{cases} \mu(n_{\alpha_1}) = \mu(n_{\alpha_2}) = 1, \\ \mu(n_{\alpha_1+\alpha_2}) = 2 - B_1, \\ \mu(n_{2\alpha_1+\alpha_2}) = \begin{cases} 3 - B_1 - \frac{B_2}{A_2}, \text{ if } A_2 \neq 0, \\ 0, \text{ otherwise}, \end{cases} \\ \mu(n_{\alpha_1+2\alpha_2}) = \begin{cases} 3 - B_1 - \frac{B_3}{A_3}, \text{ if } A_3 \neq 0, \\ 0, \text{ otherwise}. \end{cases} \end{cases}$$

Therefore, there exists $f \in C^1(\mathcal{N},\mathcal{R})$ such that $d^1 f = \varphi$, i.e., $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$. □

Now, we consider the case $rank(\mathcal{N}) \geq 3$ and $\mathcal{N}^4 = 0$.

Lemma 3.3. *Let $\mathcal{N}$ be a nilpotent Lie algebra of maximal rank satisfying $\mathcal{N}^4 = 0$ and $rank(\mathcal{N}) \geq 3$. If $\alpha_{i_0} + \alpha_{j_0} + \alpha_{k_0} \in W$, then the element $n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}} \in \mathcal{N}_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}$ can be represented in at least two different ways.*

Proof. It is enough to show that the following equality holds:

$$\alpha_{i_0} + (\alpha_{j_0} + \alpha_{k_0}) = \alpha_{j_0} + (\alpha_{i_0} + \alpha_{k_0}), \tag{3.3}$$

if $\alpha_{i_0} + \alpha_{j_0} + \alpha_{k_0} \in W$ for unequal primitive roots $\alpha_{i_0}, \alpha_{j_0}, \alpha_{k_0}$.

Suppose contrary

$$\alpha_{i_0} + (\alpha_{j_0} + \alpha_{k_0}) \neq \alpha_{j_0} + (\alpha_{i_0} + \alpha_{k_0}) \tag{3.4}$$

for the triple $\{i_0, j_0, k_0\}$. Because of $\alpha_{i_0} + \alpha_{j_0} + \alpha_{k_0} \in W$ and (3.4), without loss of generalities, we can suppose

$$[n_{\alpha_{j_o}+\alpha_{k_0}}, n_{\alpha_{i_0}}] \neq 0, \quad [n_{\alpha_{i_o}+\alpha_{k_0}}, n_{\alpha_{j_0}}] = 0, \quad [n_{\alpha_{i_o}+\alpha_{j_0}}, n_{\alpha_{k_0}}] = 0.$$

(if this $[n_{\alpha_{i_o}+\alpha_{j_0}}, n_{\alpha_{k_0}}]$ product is nonzero too, then we can write the equality(3.3) as

$$\alpha_{i_0} + (\alpha_{j_0} + \alpha_{k_0}) = \alpha_{k_0} + (\alpha_{i_0} + \alpha_{j_0}),$$

which contradicts our assumption). However, this Jacoby identity

$$\underbrace{[n_{\alpha_{i_o}}, [n_{\alpha_{j_o}}, n_{\alpha_{k_o}}]]}_{\neq 0} + \underbrace{[n_{\alpha_{j_o}}, [n_{\alpha_{k_o}}, n_{\alpha_{i_o}}]]}_{0} + \underbrace{[n_{\alpha_{k_o}}, [n_{\alpha_{i_o}}, n_{\alpha_{j_o}}]]}_{0} = 0$$

leads to contradiction.

□

Now, we provide criteria for the triviality of the second cohomology groups of maximal solvable extensions in the case $\mathcal{N}^4 = 0$ and $rank(\mathcal{N}) \geq 3$. We consider it in two cases:

- $\alpha_i + \alpha_j + \alpha_k \notin W$ for all unequal primitive roots $\alpha_i, \alpha_j, \alpha_k$.
- $\alpha_i + \alpha_j + \alpha_k \in W$ for some unequal primitive roots $\alpha_i, \alpha_j, \alpha_k$.

Theorem 3.4. *Let $\mathcal{N}$ be a nilpotent Lie algebra of maximal rank satisfying $\mathcal{N}^4 = 0$ and $rank(\mathcal{N}) \geq 3$. If $\alpha_i + \alpha_j + \alpha_k \notin W$ for all unequal primitive roots $\alpha_i, \alpha_j, \alpha_k$, then $H^2(\mathcal{R}, \mathcal{R}) = 0$.*

Proof. In the multiplication table of $\mathcal{N}$, without loss of generalities, we can write the following products:

$$\mathcal{N} : \begin{cases} [n_{\alpha_i}, n_{\alpha_j}] = A_{i,j} n_{\alpha_i+\alpha_j}, \quad 1 \leq i \neq j \leq s, \\ [n_{\alpha_i+\alpha_j}, n_{\alpha_i}] = A_{i+j,i} n_{2\alpha_i+\alpha_j} \quad 1 \leq i \neq j \leq s, \\ [n_{\alpha_i+\alpha_j}, n_{\alpha_j}] = A_{i+j,j} n_{\alpha_i+2\alpha_j} \quad 1 \leq j \neq i \leq s. \end{cases} \tag{3.5}$$

Because of $\varphi(n_{\alpha_i}, n_{\alpha_j}) \in \mathcal{N}_{\alpha_i+\alpha_j}$ property for $\varphi \in Z^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}}$, we can denote as:

$$\begin{cases} \varphi(n_{\alpha_i}, n_{\alpha_j}) = (1 - \delta_{A_{i,j},0}) B_{i,j} n_{\alpha_i+\alpha_j}, \quad 1 \leq i \neq j \leq s, \\ \varphi(n_{\alpha_i+\alpha_j}, n_{\alpha_i}) = (1 - \delta_{A_{i,j},0})(1 - \delta_{A_{i+j,i},0}) B_{i+j,i} n_{2\alpha_i+\alpha_j}, \quad 1 \leq i \neq j \leq s, \\ \varphi(n_{\alpha_i+\alpha_j}, n_{\alpha_j}) = (1 - \delta_{A_{i,j},0})(1 - \delta_{A_{i+j,j},0}) B_{i+j,j} n_{\alpha_i+2\alpha_j}, \quad 1 \leq i \neq j \leq s. \end{cases} \tag{3.6}$$

We now show $\varphi \in B^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}}$ too. In other words, we prove the existence of $f \in C^1(\mathcal{N}, \mathcal{R})$ such that $\varphi = d^1 f$, satisfying the following identity:

$$\varphi(e_\alpha, e_\beta) = [f(e_\alpha), e_\beta] + [e_\alpha, f(e_\beta)] - f([e_\alpha, e_\beta]) \quad \text{for all} \quad \alpha, \beta \in W.$$

Let denote $f(n_\alpha) = \mu(n_\alpha) n_\alpha$, where $\mu(n_\alpha)$ are scalars. Then, we come the following system of linear equations ($1 \leq i \neq j \leq s$):

$$\begin{cases} A_{i,j}\Big(\mu(n_{\alpha_i}) + \mu(n_{\alpha_j}) - \mu(n_{\alpha_i+\alpha_j})\Big) = (1 - \delta_{A_{i,j},0}) B_{i,j}, \\ A_{i+j,i}\big(\mu(n_{\alpha_i+\alpha_j}) + \mu(n_{\alpha_i}) - \mu(n_{2\alpha_i+\alpha_j})\big) = (1 - \delta_{A_{i,j},0})(1 - \delta_{A_{i+j,i},0}) B_{i+j,i}, \\ A_{i+j,j}\big(\mu(n_{\alpha_i+\alpha_j}) + \mu(n_{\alpha_j}) - \mu(n_{\alpha_i+2\alpha_j})\big) = (1 - \delta_{A_{i,j},0})(1 - \delta_{A_{i+j,j},0}) B_{i+j,j}. \end{cases} \tag{3.7}$$

Then the system (3.7) has the following solution:

$$\begin{cases} \mu(n_{\alpha_i}) = \mu(n_{\alpha_j}) = 1, \quad 1 \leq i \neq j \leq s, \\ \mu(n_{\alpha_i+\alpha_j}) = \begin{cases} 2 - \frac{B_{i,j}}{A_{i,j}}, \text{ if } \ A_{i,j} \neq 0 \ \ 1 \leq i \neq j \leq s, \\ 0, \text{ otherwise}, \quad 1 \leq i \neq j \leq s, \end{cases} \\ \mu(n_{2\alpha_i+\alpha_j}) = \begin{cases} 3 - \frac{B_{i,j}}{A_{i,j}} - \frac{B_{i+j,i}}{A_{i+j,i}}, \text{ if } \ A_{i+j,i} \neq 0 \ \ 1 \leq i \neq j \leq s, \\ 0, \text{ otherwise}, \quad 1 \leq i \neq j \leq s, \end{cases} \\ \mu(n_{\alpha_i+2\alpha_j}) = \begin{cases} 3 - \frac{B_{i,j}}{A_{i,j}} - \frac{B_{i+j,j}}{A_{i+j,j}}, \text{ if } \ A_{i+j,j} \neq 0 \ \ 1 \leq i \neq j \leq s, \\ 0, \text{ otherwise}, \quad 1 \leq i \neq j \leq s. \end{cases} \end{cases}$$

Therefore, there exists $f \in C^1(\mathcal{N}, \mathcal{R})$ such that $d^1 f = \varphi$, i.e., $H^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}} = 0$. Then by Remark 2.9 we obtain that $H^2(\mathcal{R}, \mathcal{R}) = 0$. □

An example that satisfies the conditions of Theorem 3.4:

$$\mathcal{N}: \left\{ [e_1, e_2] = e_4, \quad [e_1, e_3] = e_5, \quad [e_2, e_3] = e_6, \quad [e_5, e_1] = e_7. \right.$$

$\mathcal{N}$ can be decomposed to the following root subspaces:

$$\mathcal{N}_{\alpha_1} = \{e_1\},\ \mathcal{N}_{\alpha_2} = \{e_2\},\ \mathcal{N}_{\alpha_3} = \{e_3\},\ \mathcal{N}_{\alpha_1+\alpha_2} = \{e_4\},\ \mathcal{N}_{\alpha_1+\alpha_3} = \{e_5\},\ \mathcal{N}_{\alpha_2+\alpha_3} = \{e_6\},\ \mathcal{N}_{2\alpha_1+\alpha_3} = \{e_7\}.$$

Its maximal solvable extension $\mathcal{R}(\mathcal{N}) = \mathcal{N} \rtimes \mathcal{T}_{max}$ has the following products:

$$\mathcal{R}(\mathcal{N}): \left\{ \begin{array}{l} [e_i, t_1] = e_i,\ i = 1, 4, 5, \quad [e_7, t_1] = 2e_7, \\ [e_i, t_2] = e_i,\ i = 2, 4, 6, \quad [e_i, t_3] = e_i,\ i = 3, 6, 7, \quad [\mathcal{N}, \mathcal{N}]. \end{array} \right.$$

This algebra satisfies the conditions of Theorem 3.4, i.e., $\alpha_1 + \alpha_2 + \alpha_3 \notin W$ and one can check that $H^2(\mathcal{R}(\mathcal{N}), \mathcal{R}(\mathcal{N})) = 0$.

An example that does not satisfy the conditions of Theorem 3.4:

$$\mathcal{M}: \left\{ \begin{array}{llll} [e_1, e_2] = e_4, & [e_2, e_3] = e_6, & [e_4, e_3] = e_8, & [e_6, e_1] = e_8, \\ [e_1, e_3] = e_5, & [e_5, e_1] = e_7, & [e_5, e_2] = 2e_8. & \end{array} \right.$$

$\mathcal{M}$ can be decomposed to the following root subspaces:

$$\mathcal{M}_{\alpha_1} = \{e_1\},\ \ \mathcal{M}_{\alpha_2} = \{e_2\},\ \ \mathcal{M}_{\alpha_3} = \{e_3\},\ \ \mathcal{M}_{\alpha_1+\alpha_2} = \{e_4\},\ \mathcal{M}_{\alpha_1+\alpha_3} = \{e_5\},\ \ \ \mathcal{M}_{\alpha_2+\alpha_3} = \{e_6\},$$

$$\mathcal{M}_{2\alpha_1+\alpha_3} = \{e_7\},\ \ \mathcal{M}_{\alpha_1+\alpha_2+\alpha_3} = \{e_8\}.$$

Its maximal solvable extension $\mathcal{R}(\mathcal{M}) = \mathcal{M} \rtimes \mathcal{T}_{max}$ has the following products:

$$\mathcal{R}(\mathcal{M}): \left\{ \begin{array}{l} [e_i, t_1] = e_i,\ i = 1, 4, 5, 8, \quad [e_7, t_1] = 2e_7, \\ [e_i, t_2] = e_i,\ i = 2, 4, 6, 8, \quad [e_i, t_3] = e_i,\ i = 3, 5, 6, 7, 8, \quad [\mathcal{M}, \mathcal{M}]. \end{array} \right.$$

This algebra does not satisfy the conditions of Theorem 3.4, i.e., $\alpha_1+\alpha_2+\alpha_3 \in W$, and one can verify that $\dim H^2(\mathcal{R}(\mathcal{M}), \mathcal{R}(\mathcal{M})) = 1$.

We should note that in Theorem 3.4, the case $\alpha_{i_p} + \alpha_{j_p} + \alpha_{k_p} \notin W$ for all unequal primitive roots $\alpha_{i_p}, \alpha_{j_p}, \alpha_{k_p}$ has been considered. It has been proven that all the maximal solvable Lie algebras in this case are cohomologically rigid. Therefore, we present the following theorem in the case where $\alpha_{i_p} + \alpha_{j_p} + \alpha_{k_p} \in W$ for some roots $\alpha_{i_p}, \alpha_{j_p}, \alpha_{k_p}$ in Ψ_1, where $1 \le p \le C_s^3$, and s is the number of different roots in Ψ_1.

Theorem 3.5. *Let $\mathcal{N}$ be a nilpotent Lie algebra of maximal rank satisfying $\mathcal{N}^4 = 0$ and rank$(\mathcal{N}) \ge 3$. Then $H^2(\mathcal{R}, \mathcal{R}) = 0$ if and only if for any unequal primitive roots $\alpha_{i_p}, \alpha_{j_p}, \alpha_{k_p}$ such that $\alpha_{i_p}+\alpha_{j_p}+\alpha_{k_p} \in W$, the equalities $\alpha_{i_p} + (\alpha_{j_p} + \alpha_{k_p}) = \alpha_{j_p} + (\alpha_{i_p} + \alpha_{k_p})$ imply $\alpha_{i_p} + \alpha_{j_p} \notin W$ for $1 \le p \le C_s^3$.*

Proof. By Lemma 2.3, from $\alpha_{i_p} + \alpha_{j_p} + \alpha_{k_p} \in W$, we can write

$$\alpha_{i_p} + (\alpha_{j_p} + \alpha_{k_p}) = \alpha_{j_p} + (\alpha_{i_p} + \alpha_{k_p}). \tag{3.8}$$

Because of Remark 2.9, instead of $H^2(\mathcal{R}, \mathcal{R})$, we consider $H^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}}$. Let $H^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}} = 0$, we now show that for any unequal primitive roots $\alpha_{i_p}, \alpha_{j_p}, \alpha_{k_p}$ such that $\alpha_{i_p} + \alpha_{j_p} + \alpha_{k_p} \in W$, these equalities

$$\alpha_{i_p} + (\alpha_{j_p} + \alpha_{k_p}) = \alpha_{j_p} + (\alpha_{i_p} + \alpha_{k_p}), \quad 1 \le p \le C_s^3,$$

imply

$$\alpha_{i_p} + \alpha_{j_p} \notin W, \quad 1 \le p \le C_s^3.$$

Assume contrary, let $H^2(\mathcal{N}, \mathcal{R})^{\mathcal{T}} = 0$, but there exists a triple of primitive roots $\{\alpha_{i_0}, \alpha_{j_0}, \alpha_{k_0}\}$ such that $\alpha_{i_0} + \alpha_{j_0} + \alpha_{k_0} \in W$, and this equality

$$\alpha_{i_0} + (\alpha_{j_0} + \alpha_{k_0}) = \alpha_{j_0} + (\alpha_{i_0} + \alpha_{k_0})$$

implies
$$\alpha_{i_0}+\alpha_{j_0}\in W.$$
Then, without loss of generality, we can write the products of $\mathcal{N}$ as follows:
$$\mathcal{N}:\begin{cases}[n_{\alpha_{i_0}},n_{\alpha_{j_0}}]=n_{\alpha_{i_0}+\alpha_{j_0}},\\ [n_{\alpha_{i_0}},n_{\alpha_{k_0}}]=n_{\alpha_{i_0}+\alpha_{k_0}},\\ [n_{\alpha_{j_0}},n_{\alpha_{k_0}}]=n_{\alpha_{j_0}+\alpha_{k_0}},\\ [n_{\alpha_{i_0}+\alpha_{j_0}},n_{\alpha_{k_0}}]=n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\\ [n_{\alpha_{i_0}+\alpha_{k_0}},n_{\alpha_{j_0}}]=An_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\quad A\neq 0,\\ [n_{\alpha_{j_0}+\alpha_{k_0}},n_{\alpha_{i_0}}]=(A-1)n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\\ [\mathcal{N}_\alpha,\mathcal{N}_\beta],\quad \{\alpha,\beta\}\neq\{\alpha_{i_0},\alpha_{j_0}\}\neq\{\alpha_{i_0},\alpha_{k_0}\}\neq\{\alpha_{j_0},\alpha_{k_0}\},\\ [\mathcal{N}_{\alpha+\beta},\mathcal{N}_\gamma],\quad \{\alpha,\beta\}\neq\{\alpha_{i_0},\alpha_{j_0}\}\neq\{\alpha_{i_0},\alpha_{k_0}\}\neq\{\alpha_{j_0},\alpha_{k_0}\},\\ [\mathcal{N}_{2\alpha},\mathcal{N}_\beta],\ [\mathcal{N}_\alpha,\mathcal{N}_{2\beta}].\end{cases}$$
Let us consider the following map, defined as:
$$\begin{cases}\varphi(n_{\alpha_{i_0}+\alpha_{k_0}},n_{\alpha_{j_0}})=n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\\ \varphi(n_{\alpha_{j_0}+\alpha_{k_0}},n_{\alpha_{i_0}})=n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\\ \varphi(n_\alpha,n_\beta)=0,\quad\text{otherwise.}\end{cases}\tag{3.9}$$
For 2-cocycles, we have the following identity:
$$\varphi(a,[b,c])-\varphi([a,b],c)+\varphi([a,c],b)+[a,\varphi(b,c)]-[\varphi(a,b),c]+[\varphi(a,c),b]=0,\tag{3.10}$$
for any elements $a,b,c\in\mathcal{N}$.

Now, we show that φ, defined as in (3.9), belongs to $Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$. Let us consider the identity (3.10) for the triple $\{n_\alpha,n_\beta,n_\gamma\}$ in the case where one of α,β,γ is non-primitive. Then, due to $\mathcal{N}^4=0$ and the way φ is defined, all terms in (3.10) simultaneously equal zero. Therefore, it remains to check the triples $\{n_\alpha,n_\beta,n_\gamma\}$ where all α,β,γ are primitive. If at least one element in the triple $\{n_\alpha,n_\beta,n_\gamma\}$ differs from the triple $\{n_{\alpha_{i_0}},n_{\alpha_{j_0}},n_{\alpha_{k_0}}\}$, then, due to the way φ is defined, all six summands in the 2-cocycle identity are simultaneously equal to zero.

Now, it is enough to check 3.10 for the triples $\{n_{\alpha_{i_0}},n_{\alpha_{j_0}},n_{\alpha_{k_0}}\}$. By applying the 2-cocycle identity to the triple $\{n_{\alpha_{i_0}},n_{\alpha_{j_0}},n_{\alpha_{k_0}}\}$, we obtain the following result:
$$\underbrace{[n_{\alpha_{i_0}},\varphi(n_{\alpha_{j_0}},n_{\alpha_{k_0}})}_{0}-\underbrace{[\varphi(n_{\alpha_{i_0}},n_{\alpha_{j_0}}),n_{\alpha_{k_0}}]}_{0}+\underbrace{[\varphi(n_{\alpha_{i_0}},n_{\alpha_{k_0}}),n_{\alpha_{j_0}}]}_{0}+$$
$$+\underbrace{\varphi(n_{\alpha_{i_0}},[n_{\alpha_{j_0}},n_{\alpha_{k_0}}])}_{-n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}}-\underbrace{\varphi([n_{\alpha_{i_0}},n_{\alpha_{j_0}}],n_{\alpha_{k_0}})}_{0}+\underbrace{\varphi([n_{\alpha_{i_0}},n_{\alpha_{k_0}}],n_{\alpha_{j_0}})}_{n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}}=0.$$

Therefore, $\varphi\in Z^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$. Let assume $\varphi\in B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$, then suppose there exists $f\in C^1(\mathcal{N},\mathcal{R})$ such that $\varphi=d^1f$, and we have the following identity:
$$\varphi(n_\alpha,n_\beta)=[f(n_\alpha),n_\beta]+[n_\alpha,f(n_\beta)]-f([n_\alpha,n_\beta])\quad\text{for all}\quad \alpha,\beta\in W.$$
$$\begin{cases}\varphi(n_{\alpha_{i_0}},n_{\alpha_{j_0}})=(\mu(n_{\alpha_{i_0}})+\mu(n_{\alpha_{j_0}})-\mu(n_{\alpha_{i_0}+\alpha_{j_0}}))n_{\alpha_{i_0}+\alpha_{j_0}}=0,\\ \varphi(n_{\alpha_{i_0}},n_{\alpha_{k_0}})=(\mu(n_{\alpha_{i_0}})+\mu(n_{\alpha_{k_0}})-\mu(n_{\alpha_{i_0}+\alpha_{k_0}}))n_{\alpha_{i_0}+\alpha_{k_0}}=0,\\ \varphi(n_{\alpha_{i_0}+\alpha_{j_0}},n_{\alpha_{k_0}})=(\mu(n_{\alpha_{i_0}+\alpha_{j_0}})+\mu(n_{\alpha_{k_0}})-\mu(n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}))n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}=0,\\ \varphi(n_{\alpha_{i_0}+\alpha_{k_0}},n_{\alpha_{j_0}})=A(\mu(n_{\alpha_{i_0}+\alpha_{k_0}})+\mu(n_{\alpha_{j_0}})-\mu(n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}))n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}}=n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}},\end{cases}$$
Then we come to
$$\mu(n_{\alpha_{i_0}})+\mu(n_{\alpha_{j_0}})+\mu(n_{\alpha_{k_0}})=\mu(n_{\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0}})=-\tfrac{1}{A}+\mu(n_{\alpha_{i_0}})+\mu(n_{\alpha_{j_0}})+\mu(n_{\alpha_{k_0}}),\quad A\neq 0.$$

Thus, $\varphi \notin B^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$ and it is a contradiction for triviality of $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}}$.

Let assume for any unequal primitive roots $\alpha_{i_p}, \alpha_{j_p}, \alpha_{k_p}$ such that $\alpha_{i_p}+\alpha_{j_p}+\alpha_{k_p} \in W$, the equalities $\alpha_{i_p}+(\alpha_{j_p}+\alpha_{k_p}) = \alpha_{j_p}+(\alpha_{i_p}+\alpha_{k_p})$ imply $\alpha_{i_p}+\alpha_{j_p} \notin W, \quad 1 \leq p \leq C_s^3$. Let us show that all the conditions of Proposition A hold true for $\mathcal{N}$.

(i) holds true, because of $\mathcal{N}$ is a nilpotent Lie algebra of maximal rank i.e., $dim\mathcal{T} = dim(\mathcal{N}/\mathcal{N}^2)$;
(ii) holds true, because in our case $\dim \mathcal{N}_\alpha = 1$ for all $\alpha \in W$;
(iii) holds true, because we are considering zero is not in W case;
(iv) holds true, because, if we take as

$$\alpha = \alpha_{i_p}, \quad \beta = \alpha_{j_p}, \quad \gamma = \alpha_{j_p}+\alpha_{k_p}, \quad \delta = \alpha_{i_p}+\alpha_{k_p}, \quad 1 \leq p \leq C_s^3,$$

then $\alpha+\gamma = \beta+\delta$, which is what we need to show.

Finally, the condition $\alpha+\beta \notin W$ implies that we are in **Case1.** Therefore, by Proposition A, we obtain $H^2(\mathcal{N},\mathcal{R})^{\mathcal{T}} = 0$. $\square$

An example that satisfies the conditions of Theorem 3.5:

$$\mathcal{L}: \left\{ [e_1,e_2]=e_4, \quad [e_1,e_3]=e_5, \quad [e_4,e_3]=e_6, \quad [e_5,e_2]=e_6. \right.$$

$\mathcal{L}$ can be decomposed to the following root subspaces:

$$\mathcal{L}_{\alpha_1} = \{e_1\}, \quad \mathcal{L}_{\alpha_2} = \{e_2\}, \quad \mathcal{L}_{\alpha_3} = \{e_3\}, \quad \mathcal{L}_{\alpha_1+\alpha_2} = \{e_4\}, \ \mathcal{L}_{\alpha_1+\alpha_3} = \{e_5\}, \mathcal{L}_{\alpha_1+\alpha_2+\alpha_3} = \{e_6\}.$$

Its maximal solvable extension $\mathcal{R}(\mathcal{L}) = \mathcal{L} \rtimes \mathcal{T}_{max}$ has the following products:

$$\mathcal{R}(\mathcal{L}): \left\{ [e_i,t_1]=e_i, \ i=1,4,5,6, \quad [e_i,t_2]=e_i, \ i=2,4,6, \quad [e_3,t_i]=e_i, \ i=3,5,6, \quad [\mathcal{L},\mathcal{L}]. \right.$$

If we take as $\alpha_1 = \alpha_{i_0}, \ \alpha_2 = \alpha_{j_0}, \ \alpha_3 = \alpha_{k_0}$, then $\alpha_{i_0}+\alpha_{j_0}+\alpha_{k_0} \in W$. Due to Lemma 3.3, we can write $\alpha_{j_0}+(\alpha_{k_0}+\alpha_{i_0}) = \alpha_{k_0}+(\alpha_{i_0}+\alpha_{j_0})$, and we have $\alpha_{j_0}+\alpha_{k_0} \notin W$. One can check that $H^2(\mathcal{R}(\mathcal{L}),\mathcal{R}(\mathcal{L})) = 0$.

An example that does not satisfy the conditions of Theorem 3.5:

Let consider the algebra $\mathcal{M}$ as we already mentioned above:

$$\mathcal{M}: \left\{ \begin{array}{llll} [e_1,e_2]=e_4, & [e_2,e_3]=e_6, & [e_4,e_3]=e_8, & [e_6,e_1]=e_8, \\ [e_1,e_3]=e_5, & [e_5,e_1]=e_7, & [e_5,e_2]=2e_8. & \end{array} \right.$$

$\mathcal{M}$ can be decomposed to the following root subspaces:

$$\mathcal{M}_{\alpha_1} = \{e_1\}, \quad \mathcal{M}_{\alpha_2} = \{e_2\}, \quad \mathcal{M}_{\alpha_3} = \{e_3\}, \quad \mathcal{M}_{\alpha_1+\alpha_2} = \{e_4\}, \ \mathcal{M}_{\alpha_1+\alpha_3} = \{e_5\}, \quad \mathcal{M}_{\alpha_2+\alpha_3} = \{e_6\},$$

$$\mathcal{M}_{2\alpha_1+\alpha_3} = \{e_7\}, \quad \mathcal{M}_{\alpha_1+\alpha_2+\alpha_3} = \{e_8\}.$$

This algebra does not satisfy the conditions of Theorem 3.5. Indeed, $\alpha_1+\alpha_2+\alpha_3 \in W$ and for all identity $\alpha_i+(\alpha_j+\alpha_k) = \alpha_j+(\alpha_k+\alpha_i), 1 \leq i \neq j \neq k \leq 3$ we have $\alpha_i+\alpha_j \in W$. One can check that for its maximal solvable extension, we have $dimH^2(\mathcal{R}(\mathcal{M}),\mathcal{R}(\mathcal{M})) = 1$.

References

[1] Ancochea Bermúdez J.M. On the rigidity of solvable Lie algebras, Deformation Theory of Algebras and Structures and Applications., 247 (1988), 403-445.

[2] Ancochea Bermúdez J.M., Campoamor-Stursberg R. On a complete rigid Leibniz non-Lie algebra in arbitrary dimension, Linear Algebra Appl., 438 (2013), 3397-3407.

[3] Ancochea Bermúdez J. M. Campoamor-Stursberg R. Cohomologically rigid solvable Lie algebras with a nilradical of arbitrary characteristic sequence, Linear Algebra Appl., 488 (2016), 135-147.

[4] Ancochea Bermúdez J. M., Campoamor-Stursberg R., Garca Vergnolle L. Classification of Lie algebras with naturally graded quasi-filiform nilradicals, J. Geom. Phys, 61 (2011), 2168-2186.

[5] Ancochea Bermúdez J.M., Goze M. Nonrationality of rigid Lie algebras, Proc. Amer. Math. Soc., 127 (1999), 2611-2618.

[6] Ancochea Bermúdez J.M., Goze M. On the classification of rigid Lie algebras, J. Algebra., 245 (2001), 68-91.

[7] B.A. Omirov, I.S. Rakhimov, G.O. Solijanova, The description of solvable Lie superalgebras of maximal rank, Linear Algebra Appl., 695 (2024), 136-162.

[8] B.A. Omirov, G.O. Solijanova, On the uniqueness of maximal solvable extensions of nilpotent Lie superalgebras, arXiv:2402.03012, pp. 21.

[9] F. Leger, E. Luks, Cohomology theorems for Borel-Like Solvable Lie algebras in arbitrary characteristic, Canad.J. Math., 24 (1972), 1019-1026.

[10] G.M. Mubarakzjanov, On solvable Lie algebras(Russian), Izv. Vyssh. Uchebn. Zaved. Mat., 32(1) (1963), 114-123.

[11] G.M. Mubarakzyanov, Classification of real structures of Lie algebra of fifth order, Izv. Vyssh. Uchebn. Zaved. Mat., 32(3) (1963), 99-106.

[12] G. Hochschild and J.P. Serre, Cohomology of Lie algebras, Ann.of Math., 57 (1953), 591-603.

[13] J.C. Ndogmo and P. Winternitz, Solvable Lie algebras with abelian nilradicals, J.Phys. A., 27 (1994), 405-423.

[14] Malcev A.I., Solvable Lie algebras, Trans. Amer. Math. Soc., 1950.

[15] Šnobl L., On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras, J. Phys. A., (2010).

Urazmatov G.Kh.,
V.I.Romanovskiy Institute of Mathematics,
Uzbekistan Academy of Sciences,
Tashkent, Uzbekistan
e-mail: gulmurod0405@gmail.com

Contents

Computer imposition: *A.F. Aliyev*

The journal was registered by the Press and Information Agency of the Republic of Uzbekistan on December 22, 2006. Register. No 0044.

Handed over to the set on 11/03/2022. Signed for printing on 12/04/2023
Format 60×84 1/16. Literary typeface. Offset printing.

V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences 9 University st. 100174

Printed in a printing house "MERIT-PRINT"
Tashkent city, Yakkasaray district, Sh. Rustaveli street, 91